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ROTATIONAL ENTROPY FOR ANNULUS ENDOMORPHISMS

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We study how some dynamical properties on a homeomorphism of the annulus affects its rotation set. We also introduce a topological invariant, designated "rotational entropy," that basically measures the rotational complexity of an annulus mapping.

0. Introduction. Poincaré associated to each orientation preserving homeomorphism of the circle a number, designated *rotation number*, that quanitifies the asymptotic behavior of different orbits. A natural extension of Poincaré rotation number for annuli and tori has been considered by Franks and Hermann among others.

For homeomorphisms of the annulus different orbits may rotate at different speeds. Furthermore there are orbits whose rotational behavior is so chaotic that one cannot associate a single number to its "wrapping." In general, the rotation of each orbit is captured by its *rotation interval*. The union of all rotation intervals of a map is designated the *rotation set* of the given mapping.

Franks in [**12**] proved that any orbit of an annulus homeomorphism, isotopic to the identity, with finitely many periods has rotation number. Therefore examples with chaotic rotations are expected to be dynamically complicated. Topological entropy is a topological invariant that roughly tells how many different orbits a map has. However we may have a positive entropy homeomorphism with trivial rotation set. Bowen's definition of topological entropy suggests a natural way of measuring the chaotic rotation of a given homeomorphism. We designate such a number "rotational entropy."

In this paper we start by reviewing a construction of an orientation preserving annulus homeomorphism (cf. [**4**, **15**]) with nontrivial rotation intervals. We imbed in the annulus a horseshoe in such a way that many orbits in the invariant Cantor set have "unpredictable" rotation; the rotation of the whole Cantor set is nevertheless obtained by the rotation of just one orbit. In the proof of this last assertion the *shadowing property* plays an important role.

In §3 we define "rotational entropy" and prove that it is a topological invariant closely related to the rotation set. More precisely, we prove that for homeomorphisms of the annulus isotopic to the identity:

Finitely many periods implies zero rotational entropy.

Furthermore

The rotation shadowing property and every orbit with rotation number imply zero rotational entropy.

I am grateful to R. Swanson for pointing out a mistake in the proof of Theorem 3.1 and for suggesting a version for Proposition 3.3. I am also grateful to the referee for several valuable comments. This work was partially done at the Dept. de Matemática Pura da Universidade do Porto.

1. Definitions. To each orbit we associate a sequence that quantifies its rotation. This sequence might not converge. The set of its limit points is called the *rotation interval*.

Let A be $S^1 \times [0, 1]$ and f an orientation preserving homeomorphism of A. We consider $\mathbb{R} \times [0, 1]$ the covering space of A and the standard covering map $p: \mathbb{R} \times [0, 1] \to A$ is defined by

$$p(x, y) = (\exp(2\pi x i), y).$$

Let $F: \mathbb{R} \times [0, 1] \to \mathbb{R} \times [0, 1]$ be a lift of f. We notice that F satisfies

$$F(x+1, y) = F(x, y) + (1, 0), \qquad \forall (x, y) \in \mathbb{R} \times [0, 1].$$

Small italic letters $a, b, \ldots$ will be used for points in A, small italic letters with a bar on top $\overline{a}, \overline{b}, \ldots$ will be used for points in $[0, 1] \times [0, 1]$ so that $p(\overline{a}) = a$, $p(\overline{b}) = b, \ldots$. The two projection maps defined in $\mathbb{R} \times [0, 1]$ will be denoted by π_i $(i = 1, 2)$ and $F_i^n(\overline{x}) = \pi_i(F^n(\overline{x}))$, $i = 1, 2$.

The most important invariant associated to a circle map is its rotation number. It measures the average amount orbits are rotated under iterations of the map **[17]**. It is natural to associate to each point x in A the rotation sequence $\{\theta_n(x)\}_{n\geq 1}$ of x given by

$$\theta_n(x) = \frac{F_1^n(\overline{x}) - \overline{x}_1}{n}.$$

Let

$$R_x = \text{the set of all limit points of } (\theta_n(x))_{n\geq 1}.$$

LEMMA 1.1. *R_x is a closed interval.*

Proof. Omitted, see [**4**].

DEFINITION 1.1. R_x is said to be the rotation interval of x. If the limit of the sequence $(\theta_n(x))_{n\geq 1}$ exists, then R_x is one number and it is the rotation number of x with respect to f, usually denoted by $\rho(x)$.

J. Franks [**12**] showed that if an annulus homeomorphism, isotopic to the identity, has finitely many periods then every orbit has rotation number. Next we review the construction of an annulus homeomorphism with many orbits that have nontrivial rotation (cf. [**4**], [**15**]).

EXAMPLE 1.1. Denote $A = \{(r, \theta): 1 \leq r \leq 2\}$ and $Q \subset A$, a small square with edge $\frac{1}{4}$. We consider a horseshoe map that contracts linearly in the vertical direction by a factor of $\delta < \frac{1}{2}$ and expands horizontally by a factor $\frac{1}{\delta}$. See Figure 1.

We can extend it to a diffeomorphism f of A (for details see [**11**]). There exists an invariant hyperbolic set $\Lambda \subset Q$ where f is topologically conjugate to a shift σ in two symbols. See Figure 2.

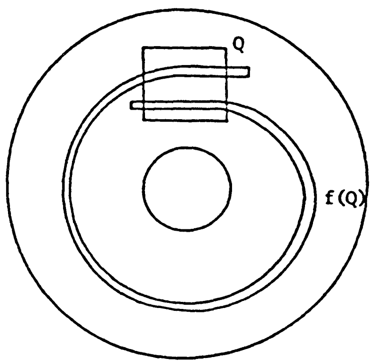

FIGURE 1

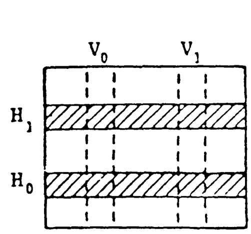

FIGURE 2

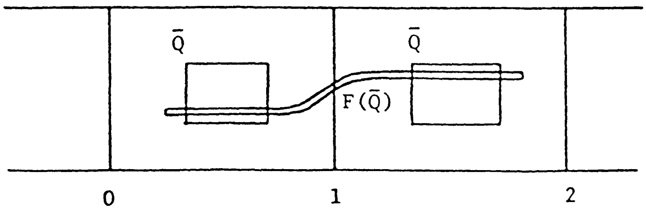

FIGURE 3

To each $x \in \Lambda$ we associate an infinite sequence $(a_0, a_1, \dots)$—the positive itinerary of x— so that:

$$a_j = s \quad \text{iff} \quad f^j(x) \in V_s \qquad \text{(see [11])}.$$

Now $\sigma(a_0, a_1, \dots) = (b_0, b_1, \dots)$ where $b_i = a_{i+1}$. It follows that

If $a_1 = 0$ then $|F_1(\overline{x}) - \overline{x}_1| \le \frac{1}{4}$,

If $a_1 = 1$ then $|F_1(\overline{x}) - \overline{x}_1 - 1| \le \frac{1}{4}$ (see Figure 3).

Therefore we conclude that $|F_1^n(\overline{x}) - \overline{x}_1 - k_n| \le \frac{1}{4}$, where k_n is the number of 1's that appears in $\{a_1, \dots, a_n\}$ and

$$\lim_n \frac{F_1^n(\overline{x}) - \overline{x}_1}{n} = \lim_n \frac{k_n}{n}.$$

In order to find a point in Λ without rotation number simply choose a sequence $(a_0, a_1, \dots)$ such that k_n/n does not converge.

DEFINITION 1.2. For any subset $E \subseteq A$ the rotation set of f restricted to E, denoted by $R(f|_E)$, is the union of all rotation intervals of orbits of points in E, i.e., $R(f|_E) = \bigcup_{x \in E} R_x$. If $E = A$ then $R(f)$ is called the rotation set of f.

2. The shadowing property. In this section f denotes an annulus diffeomorphism and U an invariant region where f is hyperbolic. We prove that if U is chain transitive then the rotation set of f restricted to U is obtained by the rotation interval of one of its points.

Let h be a homeomorphism defined on a compact metric space (X, D).

DEFINITION 2.1. (i) A sequence $\{x_i\}_{i \in \mathbb{Z}}$ in X is a δ-pseudo orbit of h if $\forall i \ D(h(x_i), x_{i+1}) < \delta$.

(ii) We say that the sequence $\{x_i\}_{i \in \mathbb{Z}}$ is ε-traceable if $\exists x \in X$ so that

$$D(h^n(x), x_n) < \varepsilon, \qquad \forall n \in \mathbb{Z}.$$

(iii) The homeomorphism h satisfies the shadowing property (also called the pseudo orbit tracing property) if for all $\varepsilon > 0$ there is $\delta > 0$ s.t.

any δ-pseudo orbit of h is ε-traceable.

It is a known fact that the "shadowing property" is topologically invariant for homeomorphisms defined on compact metric spaces. Furthermore for any integer $n \geq 1$

h satisfies the shadowing property iff h^n also does.

Robinson in **[18]** showed that $f|_U$ satisfies the shadowing property. We next state a theorem by Morimoto basically saying that "the shadowing property lifts":

THEOREM 2.1 ([**16**]). *Let M and $\widetilde{M}$ be metric spaces and $\pi: \widetilde{M} \to M$ be a locally isometric covering map of $\widetilde{M}$ onto M. Assume that M is compact and locally connected. Let ψ and φ be homeomorphisms of $\widetilde{M}$ and M, respectively, such that $\pi \circ \psi = \varphi \circ \pi$. If Γ is a φ-invariant set in M then $\varphi|_\Gamma$ satisfies the shadowing property if and only if $\psi|_{\pi^{-1}(\Gamma)}$ does.*

LEMMA 2.1. *$R(f|_U)$ contains its supremum and infimum.*

Proof. Omitted, see **[4]** and **[13]**.

The set U is chain transitive, meaning that for any $x, y \in U$ and $\delta > 0$ there exists a finite δ-pseudo orbit (usually called the δ-chain) from x to y (cf. **[12]**).

THEOREM 2.2. *There exists $w \in U$ s.t. $R(f|_U) = R_w$.*

Proof. Let $\alpha = \inf R(f|_U)$ and $\beta = \sup R(f|_U)$. There exist x and y in U and sequences $(n_i)_i$ and $(m_j)_j$ such that $\theta_{n_i}(x)$ converges to α and $\theta_{m_j}(y)$ converges to β. Furthermore U is chain transitive, implying that for any $\delta > 0$ there exist two finite δ-chains "connecting" the two points x and y. We follow the orbit of x until $\theta_{n_i}(x)$ gets very close to α, use the chain property on U to jump to the orbit of y and follow it until $\theta_{m_j}(y)$ gets much closer to β than $\theta_{n_i}(x)$ was to α. Once more we use the chain property on U to jump to the orbit of x. This way we constructed a δ-pseudo orbit on U, $\{\omega_i\}_i$. This pseudo orbit can be lifted to $\{\hat{\omega}_i\}_i$ which is a δ-pseudo orbit of F. The lift $F|_U$ also satisfies the shadowing property; then $\{\hat{\omega}_i\}_i$ is

ε-shadowed by the orbit of some point $\overline{z} \in p^{-1}(U)$. Obviously the rotation interval of the orbit of z is $[\alpha, \beta]$. □

An Axiom A annulus diffeomorphism is a diffeomorphism with an Axiom A extension to S^2. The nonwandering set of an Axiom A diffeomorphism can be split into finitely many basic sets where the diffeomorphism is topologically transitive and therefore it satisfies the assumptions of Theorem 2.2.

The rotation set of each "basic set" is a closed interval. It was proved in **[4]** (cf. **[6]**) that the rotation interval of any orbit is contained in the convex hull of the rotation set of its ω-limit set. Consequently we have the following corollary of the theorem.

COROLLARY 2.1. *The rotation set of any Axiom* A *annulus diffeomorphism is a finite union of closed intervals. Furthermore if every point has rotation number then the rotation set is finite.*

The last result of this section asserts that the "shadowing property condition" and the "existence of rotation number in the strong sense" for all orbits of an annulus map restricts tremendously the admissible rotation sets. Under these strong conditions maps of the annulus behave rotationally like maps of the circle.

DEFINITION 2.2. $x \in A$ has rotation number (in the strong sense) if

$$\exists \alpha \in \mathbb{R} \text{ s.t. } \forall \varepsilon > 0 \; \exists n_0 \in \mathbb{N} \; \forall n \in \mathbb{Z} \; |n| \geq n_0 \; |\theta_n(x) - \alpha| < \varepsilon .$$

(We denote α by $\rho_s(x)$.)

It is easy to construct a homeomorphism of the annulus with shadowing property such that every point has rotation number in the strong sense and whose rotation set is *any* given rational number.

THEOREM 2.3. *If a homeomorphism f of A satisfies the shadowing property and every point has rotation number in the strong sense, then the rotation set of f reduces to a rational number.*

Proof. We start by proving that the rotation set of f reduces to a single number. By Morimoto's theorem (2.1) we may assume that F satisfies the shadowing property. Let $\alpha, \beta \in R(f)$ with $\alpha < \beta$. Define

$$\Lambda_1 = \left\{ z \in A \colon \rho_s(z) \leq \frac{\alpha + \beta}{2} \right\} \quad \text{and} \quad \Lambda_2 = A \setminus \Lambda_1 .$$

Since A is connected then Λ_1 and Λ_2 are not both closed. We assume that Λ_1 is not closed. Let $z \in \overline{\Lambda_1} \backslash \Lambda_1$. Since F has shadowing there exists a $\delta > 0$ so that any δ-pseudo orbit of F can be 1-traceable. We choose $z_0 \in \Lambda_1$ s.t.

$$|\overline{z}_0 - \overline{z}| < \delta .$$

Consider the sequence $(a_i)_{i \in \mathbb{Z}}$ defined by:

$$\begin{aligned} a_i =& F^i(\overline{z}) \quad \text{if } i \geq 0 , \\ a_i =& F^i(\overline{z}_0) \quad \text{if } i < 0 . \end{aligned}$$

The sequence $(a_i)_{i \in \mathbb{Z}}$ is a δ-pseudo orbit of F, it is 1-traced by $\omega \in \mathbb{R} \times [0, 1]$. Hence

$$\limsup_{n \to +\infty} \frac{|F_1^n(\omega) - F_1^n(\overline{z})|}{n} = 0 ,$$

and

$$\limsup_{n \to -\infty} \frac{|F_1^n(\omega) - F_1^n(\overline{z}_0)|}{n} = 0 ,$$

Therefore

$$\lim_{n \to +\infty} \frac{F_1^n(\omega)}{n} = \lim_{n \to +\infty} \frac{F_1^n(\overline{z})}{n} > \frac{\alpha + \beta}{2}$$

and

$$\lim_{n \to -\infty} \frac{F_1^n(\omega)}{n} = \lim_{n \to -\infty} \frac{F_1^n(\overline{z}_0)}{n} \leq \frac{\alpha + \beta}{2} .$$

Consequently ω does not have rotation number in the strong sense; establishing that the rotation set reduces to a number. It remains to show that the rotation set is a rational number. (∂A stands for the boundary of A.)

LEMMA 2.2. *A homeomorphism of S^1 without periodic points does not satisfy the shadowing property.*

LEMMA 2.3. *If f is as in the theorem then $f|_{\partial A}$ has rational rotation number.*

We notice that Lemma 2.3 is not a trivial consequence of Lemma 2.2. If $f|_{\partial A}$ has no periodic points, we know from Lemma 2.2 that there exists $\varepsilon > 0$ s.t. for all $\delta > 0$ there is one δ-pseudo orbit not ε-traceable in the boundary of A. Nevertheless it could be traced by some orbit in the interior of A.

Now we assume Lemma 2.2 and Lemma 2.3 to prove the theorem; the proof of the lemmas is done below.

Proof of Lemma 2.2. Every homeomorphism of the circle without periodic points is topologically conjugate either to a rigid rotation by an irrational angle or to a Denjoy map **[10]**. Since the shadowing property is topologically invariant it is sufficient to show that neither of these maps satisfies the shadowing condition.

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and R_α be the rigid rotation by the angle $2\pi\alpha$.

Case 1. R_α does not satisfy the shadowing property.

Let $\varepsilon = 1$. For all $0 < \delta < \varepsilon$ we consider the following δ-pseudo orbit under R_α

$$\{R_{n\alpha + n\delta/2}(0)\}_{n\in\mathbb{Z}}.$$

This pseudo orbit is not 1-traceable.

Case 2. A Denjoy map does not have the shadowing property.

Let g be a Denjoy map of S^1 with irrational rotation number α and G a lift to $\mathbb{R}$. Let C be the minimal Cantor set; its complement is a disjoint and countable union of open intervals $\{I_n\}_{n\in\mathbb{Z}}$, s.t. $\mathrm{length}(I_0) \geq \mathrm{length}(I_n)$, $\forall n \in \mathbb{Z}$.

We choose a sequence $\{I_n\}_{n\in\mathbb{Z}}$ so that

$$G(I_i) = I_{i+1} \quad \text{and} \quad d(I_i, I_{i+1}) \geq \frac{\alpha}{2}, \quad \forall i \geq 0.$$

Let $\varepsilon = \mathrm{length}(I_0)/4$ and $0 < \delta < \min\{\varepsilon, \frac{\alpha}{4}\}$. The length of I_i goes rapidly to zero with i. Choose m s.t. for every n with $|n| \geq m$ we have

$$\sum_{|j|\geq m} \mathrm{length}(I_j) \leq \frac{\delta}{8}.$$

Let z_0 be the middle point of I_0. We define the δ-pseudo orbit under G as follows:

$$z_n = G(z_{n-1}) + \frac{\delta}{2} \quad \text{if } n \geq m \quad \text{and} \quad z_n = G^n(z_0) \quad \text{if } n \leq m.$$

This pseudo orbit is not ε-traceable. In fact, if there exists w s.t. $\{G^n(w)\}_n$ ε-traces $\{z_n\}_n$ then $w \in I_0$ (by the choice of ε). For each $n \in \mathbb{N}$, $G^n(w) \in I_n$. On the other hand if j is large enough we have

$$|G^{m+j}(w) - z_{m+j}| \geq \frac{j\delta}{2}. \qquad \square$$

Proof of Lemma 2.3. Suppose f preserves each boundary component. (If not we apply the same argument to f^2.)

If f is conjugate to an irrational rotation R_α then define a δ-pseudo orbit as in the proof of Lemma 2.2 (Case 1), i.e.,

$$\{R_{n\alpha+n\delta/i}(0)\} \quad \text{for some } i \geq 2.$$

Since an irrational rotation on the circle does not satisfy the shadowing condition then there exists a point w in the interior of A that 1-traces the sequence

$$\{R_{n\alpha+n\delta/i}(0)\}_n.$$

Consequently $\rho_s(w) = \frac{\delta}{i} + \alpha$. By changing i we have infinitely many possible rotation numbers. This contradicts the fact that $R(f)$ is constant.

A similar argument holds if f is conjugate to a Denjoy map. Therefore $f|_{\partial A}$ has rational rotation number. □

The proof above also shows that $f|_{\partial A}$ (or $f^2|_{\partial A}$) cannot be conjugate to a rational rotation.

It is very easy to notice that any homeomorphism of S^1 with shadowing does not have a whole interval of periodic points. However there are examples of maps of the circle with shadowing and with infinitely many periodic points.

3. Rotational entropy. In this section we introduce the definition of rotation entropy based on Bowen's interpretation of the topological entropy. Let f be an annulus endomorphism and F one of its lifts.

DEFINITION 3.1. A subset E of A is called (n, ε)-rotational spanning iff for all $x \in A$ there exists $y \in E$ so that

$$|\overline{x} - \overline{y}| < \varepsilon \qquad \text{and} \qquad \left|\frac{F_1^j(\overline{x}) - F_1^j(\overline{y})}{j}\right| < \varepsilon, \qquad j = 1, 2, \ldots, n-1.$$

PROPOSITION 3.1. *There exists a minimal (n, ε)-rotational spanning set and it is finite.*

Proof. For fixed n and ε, given $x \in A$ there exists $0 < \delta < \varepsilon$ so that

$$\left|\frac{F_1^j(\overline{x}) - F_1^j(\overline{y})}{j}\right| < \varepsilon, \qquad j = 1, 2, \ldots, n-1 \text{ and } y \in \beta(x, \delta).$$

Since A is compact, there are k balls with centers $x_1, x_2, \ldots, x_k$ that cover the whole annulus. The set $\{x_1, x_2, \ldots, x_k\}$ is (n, ε)-rotational spanning. □

We denote by $E^r_{n,\varepsilon}$ a minimal (n, ε)-rotational spanning and by $\#E^r_{n,\varepsilon}$ its cardinality. The following two facts are straightforward:
If $n \le m$ then $\#E^r_{n,\varepsilon} \le \#E^r_{m,\varepsilon}$.
If $\varepsilon_1 \le \varepsilon_2$ then $\#E^r_{n,\varepsilon_2} \le \#E^r_{n,\varepsilon_1}$.

DEFINITION 3.2. The ε-rotational entropy, $h_{r,\varepsilon}$, is given by

$$\limsup \frac{1}{n} \log(\#E^r_{n,\varepsilon}).$$

Obviously if $\varepsilon_1 \le \varepsilon_2$ then $0 \le h_{r,\varepsilon_2} \le h_{r,\varepsilon_1}$.

DEFINITION 3.3. The rotational entropy of f, $h_r(f)$, is the limit of $h_{r,\varepsilon}$ as ε approaches zero.

As in [7] we also can define rotational entropy by considering rotational separated sets. The following proposition states several properties holding for the rotational entropy.

PROPOSITION 3.2. *Let f be an annulus endomorphism then:*

(i) *If B is a compact invariant subset of f whose rotation set is reduced to a number then $h_r(f|_B) = 0$.*

(ii) *The rotational entropy is a topological invariant.*

(iii) *$h_r(f^m) = m \cdot h_r(f)$, for any $m \in \mathbb{N}$.*

(iv) *X_1 and X_2 are closed and invariant subsets whose union is A then $h_r(f) = \max\{h_r(f|_{X_1}), h_r(f|_{X_2})\}$.*

Proof. (i) In [4] it was shown that if the rotation set of a compact invariant subset of f, B, is reduced to one point then the rotation sequence converges uniformly to a constant function ρ defined on B, i.e. given $\varepsilon > 0$ there exists n_0 such that for all $n \ge n_0$ and all $x \in B$

$$\left| \frac{F_1^n(\overline{x})}{n} - \rho(x) \right| < \frac{\varepsilon}{4}.$$

For all $x \in B$ there exists $0 < \delta < \varepsilon$ such that for every $y \in \beta(x, \delta)$, i.e., the ball with center x and radius δ,

$$\left| \frac{F_1^j(\overline{x}) - F_1^j(\overline{y})}{j} \right| < \frac{\varepsilon}{4}, \quad j = 1, 2, \ldots, n_0 - 1 \text{ and } |\rho(x) - \rho(y)| < \frac{\varepsilon}{4}.$$

Since B is compact there exist $x_1, \ldots, x_k$ and δ_{x_i}'s so that the union $\bigcup_1^k \beta(x_i, \delta_{x_i}) \supset B$, therefore the set $\{x_1, \ldots, x_k\}$, is (n, ε)-rotational spanning. More precisely, for every $x \in A$ there exists x_i

so that $x \in \beta(x_i, \delta_i)$ and

$$\left|\frac{F_1^n(\overline{x})}{n} - \rho(x)\right| < \frac{\varepsilon}{4}, \quad \left|\frac{F_1^n(\overline{x}_i)}{n} - \rho(x_i)\right| < \frac{\varepsilon}{4} \quad \text{and}$$

$$|\rho(x) - \rho(x_i)| < \frac{\varepsilon}{4}.$$

Therefore we have for $j \geq n_0$

$$\left|\frac{F_1^j(\overline{x}) - F_1^j(\overline{x}_i)}{j}\right| < \left|\frac{F_1^j(\overline{x})}{j} - \rho(x)\right| + \left|\frac{F_1^j(\overline{x}_i)}{j} - \rho(x_i)\right| + |\rho(x) - \rho(x_i)|$$

$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon.$$

Then $\#(E^r_{n,\varepsilon}) \leq k$ if $n \geq n_0$ which implies zero rotational entropy. □

(ii) We set $h_{r,n_0}(f) = \lim_{\varepsilon\to 0} \limsup_{n\geq n_0} \frac{1}{n} \log \#(E^r_{n\geq n_0,\varepsilon})$, where $E^r_{n\geq n_0,\varepsilon}$ is a minimal set satisfying the following condition:

for any $x \in A$ there exists $y \in E^r_{n\geq n_0,\varepsilon}$ such that $|\overline{x} - \overline{y}| < \varepsilon$ and

$$\left|\frac{F_1^j(\overline{x}) - F_1^j(\overline{y})}{j}\right| < \varepsilon, \qquad j = n_0, \ldots, n-1.$$

LEMMA 3.1. $h_{r,n_0}(f) = h_r(f)$.

Proof. If the set $E^r_{n,\varepsilon}$ is a minimal (n, ε)-rotational spanning set then it is $(n \geq n_0, \varepsilon)$-rotational spanning; therefore $\#(E^r_{n,\varepsilon}) \geq \#(E^r_{n\geq n_0,\varepsilon})$. This implies that

$$h_r(f) \geq h_{r,n_0}(f).$$

To prove the other inequality, one notices that for $\varepsilon > 0$ there exists $\delta < \varepsilon$ so that

$$|\overline{x} - \overline{y}| < \delta \text{ implies } \left|\frac{F_1^j(\overline{x}) - F_1^j(\overline{y})}{j}\right| < \varepsilon, \qquad i \leq j < n_0.$$

Consequently we have $\#(E^r_{n,\varepsilon}) \leq \#(E^r_{n\geq n_0,\delta})$ and

$$\lim_{\delta\to 0} \limsup_{n\geq n_0} \frac{1}{n} \log \#(E^r_{n\geq n_0,\delta}) \geq \lim_{\varepsilon\to 0} \limsup_{n} \frac{1}{n} \log \#(E^r_{n,\varepsilon})$$

proving that $h_{r,n_0}(f) \geq h_r(f)$. □

Proof of Proposition 3.2 (ii). Let f, g be endomorphisms of the annulus and h be a conjugating homeomorphism so that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{R}\times[0,1] \arrow[rrr, "F"] \arrow[dr, "p"'] \arrow[ddd, "H"'] & & & \mathbb{R}\times[0,1] \arrow[dl, "p"] \arrow[ddd, "H"] \\
& A \arrow[r, "f"] \arrow[d, "h"'] & A \arrow[d, "h"] & \\
& A \arrow[r, "g"'] & A & \\
\mathbb{R}\times[0,1] \arrow[rrr, "G"'] \arrow[ur, "p"] & & & \mathbb{R}\times[0,1] \arrow[ul, "p"']
\end{tikzcd}$$

Since H is $(1, 0)$-translation invariant, i.e., for any $y \in \mathbb{R} \times [0, 1]$ $H(y + (1, 0)) = H(y) + (1, 0)$, there exists n_1 so that for all $n \geq n_1$

$$\frac{|H(z) - z_1 - H(w) + w_1|}{n} < \frac{\varepsilon}{2} \quad \text{for all } z, w \in \mathbb{R} \times [0, 1].$$

We designate by E a (n, ε)-rotational spanning set for f. For any $x \in A$ there exists $y \in E$ so that

$$\left|\frac{F_1^j(\overline{x}) - F_1^j(\overline{y})}{j}\right| < \varepsilon, \qquad 1 \leq j < n.$$

We have

$$\begin{aligned}
&\frac{|H_1(F^j(\overline{x})) - H_1(F^j(\overline{y}))|}{j} \\
&\quad \leq \frac{|H_1(F^j(\overline{x})) - F_1^j(\overline{x}) - H_1(F^j(\overline{y})) + F_1^j(\overline{y})|}{j} + \frac{|F_1^j(\overline{x}) - F_1^j(\overline{y})|}{j} \\
&\quad \leq \frac{\varepsilon}{2} + \varepsilon < 2\varepsilon \quad \text{if } j \leq n_1.
\end{aligned}$$

Therefore we have

$$\frac{|G_1^j(H(\overline{x})) - G_1^j(H(\overline{y}))|}{j} < 2\varepsilon \quad \text{if } j = n_1,\ n_1 + 1, \dots, n - 1.$$

The set $H(E)$ is $(n \geq n_1, 2\varepsilon)$-rotational spanning for g. Consequently Lemma 3.1 implies that $h_r(g) \leq h_r(f)$. The other inequality follows similarly. □

(iii) Any (nm, ε)-rotational spanning set for f is $(n, m\varepsilon)$-rotational spanning for f^m. The following inequalities are straightforward:

$$\#E^r_{nm,\varepsilon}(f) \geq \#E^r_{n,m\varepsilon}(f^m),$$

$$\frac{1}{nm}\log\#E^r_{nm,\varepsilon}(f) \geq \frac{1}{nm}\log\#E^r_{n,m\varepsilon}(f^m),$$

$$\limsup_n \frac{1}{nm}\log\#E^r_{nm,\varepsilon}(f) \geq \frac{1}{m}\limsup_n \frac{1}{n}\log\#E^r_{n,m\varepsilon}(f^m).$$

This implies that $mh_r(f) \geq h_r(f^m)$. Now we prove the other inequality. If E is a (n, ε)-rotational spanning set for f^m, then for all $y \in A$ there exists $x \in E$ so that

$$|\overline{x} - \overline{y}| < \varepsilon \text{ and } \left|\frac{F_1^{jm}(\overline{x}) - F_1^{jm}(\overline{y})}{j}\right| < \varepsilon, \qquad j = 1, \dots, n-1.$$

Next we prove that E is $(nm, \frac{3\varepsilon}{m})$-rotational spanning for f. For any k less than nm there exists a nonnegative integer less than m, s, so that $k = jm + s$. Let $M = \sup_{x\in\mathbb{R}\times I}|F_1(x) - x_1|$.

$$\begin{aligned}\left|\frac{F_1^k(\overline{x}) - F_1^k(\overline{y})}{k}\right| &\leq \left|\frac{F_1^k(\overline{x}) - F_1^{jm}(\overline{y})}{k}\right| + \left|\frac{F_1^{jm}(\overline{x}) - F_1^{jm}(\overline{y})}{k}\right| \\ &\quad + \left|\frac{F_1^{jm}(\overline{x}) - F_1^k(\overline{y})}{k}\right| \\ &\leq \frac{sM}{k} + \frac{\varepsilon j}{k} + \frac{sM}{k} < \frac{3\varepsilon}{m} \quad \text{if } k \geq k_0.\end{aligned}$$

Therefore E is $(mn \geq k_0, \frac{3\varepsilon}{m})$-rotational spanning for f. Consequently

$$h_r(f^m) \geq m \cdot \lim_\varepsilon \limsup_m \frac{1}{nm}\log\#E^r_{nm\geq k_0, \frac{3\varepsilon}{m}}(f) = mh_r(f). \qquad \square$$

(iv) It follows easily from the definition that $h_r(f) \geq h_r(f|_{X_i})$, $i = 1, 2$. Now let E_i be a (n, ε)-rotational spanning for $f|_{X_i}$, $i = 1, 2$. The union $E_1 \cup E_2$ is (n, ε)-rotational spanning for f. Therefore

$$\#E^r_{n,\varepsilon} \leq \#(E_1 \cup E_2) \leq \#E_1 + \#E_2.$$

This implies that $\log\#E^r_{n,\varepsilon} \leq \log(\#E_1 + \#E_2)$. Then

$$\begin{aligned}\lim_\varepsilon \limsup_n \frac{1}{n}\log\#E^r_{n,\varepsilon} &\leq \lim_\varepsilon \limsup_n \frac{1}{n}\log(\#E_1 + \#E_2) \\ &= \lim_\varepsilon \max\left\{\limsup_n \frac{1}{n}\log\#E_1, \limsup_n \frac{1}{n}\log\#E_2\right\} \\ &= \max\{h_r(f|_{X_1}), h_r(f|_{X_2})\}. \qquad \square\end{aligned}$$

REMARK 3.1. (1) The proof of proposition 3.2(iv) also implies that for any $\varepsilon > 0$

$$h_{r,\varepsilon}(f) = \max\{h_{r,\varepsilon}(f|_{X_1}), h_{r,\varepsilon}(f|_{X_2})\}.$$

(2) $h_r(f) = \sup\{h_r(f|_K)\colon\ K \text{ is compact and } f(K) \subseteq K\}$. Furthermore if $\{Y_i\}$ is a countable family of closed and invariant subsets of A whose union is A then

$$\begin{aligned} h_r(f) &= \sup_k h_r\left(f|_{\bigcup_{i=1} Y_i}\right) = \sup_k(\max\{h_r(f|_{Y_1}), \dots, h_r(f|_{Y_j})\}) \\ &= \sup_k\{h_r(f|_{Y_k})\}. \end{aligned}$$

(3) If $f = f_1 \times f_2$ where $f_1\colon S^1 \to S^1$ and $f_2\colon [0, 1] \to [0, 1]$ are homeomorphisms then $h_r(f) = 0$.

(4) The rotational entropy of f is obtained by the rotational entropy of f restricted to its nonwandering set, $\Omega(f)$ (cf. [**5**]).

We recall that a δ-chain from x to y is a finite sequence $a_0, \dots, a_n$ in A s.t. $a_0 = x$, $a_n = y$ and $\operatorname{dist}(f(a_i), a_{i+1}) < \delta$ for $i = 0, \dots, n-1$. We say that a subset B of A is chain transitive iff for any $\delta > 0$ and any $x, y \in B$ there exists a δ-chain in A from x to y.

PROPOSITION 3.3. *If f is an annulus homeomorphism, isotopic to the identity, and $\mathfrak{I}$ represents the collection of all invariant, closed and chain transitive subsets of $\Omega(f)$, then*

$$h_r(f|_{\Omega(f)}) = \sup_{\Lambda \in \mathfrak{I}} h_r(f|_\Lambda).$$

Proof. This proof follows Bowen's well-known result about entropy of factors (cf. [**8**]). Obviously $h_r(f|_{\Omega(f)}) \geq \sup_{\Lambda\in\mathfrak{I}} h_r(f|_\Lambda)$.

Now we prove the other inequality. Let $\sim$ be an equivalence relation in $\Omega(f)$ defined as follows:

$x \sim y$ iff there exists a chain transitive subset of $\Omega(f)$ containing x and y.

Each equivalence class is a closed, invariant, maximal chain transitive subset of $\Omega(f)$. In fact if E represents one equivalence class and the closure of E is also chain transitive then it is equal to E. Furthermore if $y \in E$ we show that for any $\delta > 0$ there exists a δ-chain from y to $f(y)$ and a δ-chain from $f(y)$ to y. Obviously $\{y, f(y)\}$ is a δ-chain from y to $f(y)$.

Since $y \in \Omega(f)$ there exist $n \geq 2$ and $0 < \delta_0 < \delta$ s.t.

$$f^n(\beta(y, \delta_0)) \cap \beta(y, \delta_0) \neq \emptyset \quad \text{and} \quad f^2(\beta(y, \delta_0)) \subset \beta(f^2(y), \delta).$$

Therefore, if $z \in f^n(\beta(y, \delta_0)) \cap \beta(y, \delta_0)$, the sequence $\{f(y), f^{2-n}(z), \ldots, f^{-1}(z), y\}$ is a δ-chain from $f(y)$ to y.

The mapping f induces the identity in the quotient space $\Omega(f)/\sim$. We denote by π the projection map, to each $x \in \Omega(f)$ $\pi(x)$ represents the maximal chain transitive subset of $\Omega(f)$ containing x. The map π is perfect (i.e., it is continuous, onto, closed and given $\omega \in \Omega(f) \sim$ $\pi^{-1}(\omega)$ is closed) then the quotient space $\Omega(f)/\sim$ is metrizable (cf. **[11]**).

Let $s = \sup_{\Lambda \in \mathfrak{I}} h_r(f|_\Lambda)$. Given $\varepsilon > 0$ and $\delta > 0$, for each $\Lambda \in \mathfrak{I}$ we choose N_Λ so that

$$s + \delta \geq h_{r,\varepsilon}(f|_\Lambda) + \delta \geq \frac{1}{N_\Lambda} \log \# E^r_{(N_\Lambda, \varepsilon)}$$

where $E^r_{(N_\Lambda, \varepsilon)}$ represents a (N_Λ, ε)-rotationally spanning subset of Λ. The continuity of f implies the existence of $0 < \alpha_\Lambda < \varepsilon$ so that given x and y in $\beta(\Lambda, \alpha_\Lambda)$ satisfying $|x - y| < \alpha_\Lambda$ then

$$\left| \frac{F_1^i(\overline{x}) - F_1^i(\overline{y})}{i} \right| < \varepsilon, \qquad i = 1, 2, \ldots, N_\Lambda - 1.$$

The set $E^r_{(N_\Lambda, \varepsilon)}$ is a $(N_\Lambda, 2\varepsilon)$-rotationally spanning set for $\beta(\Lambda, \alpha_\Lambda)$. The ball $\beta(\Lambda, \alpha_\Lambda)$ is an open neighborhood of Λ; therefore

$$(\Omega(f) \setminus \beta(\Lambda, \alpha_\Lambda)) \cap \left(\bigcap_{r>0} \pi^{-1}(B_r(\pi(\Lambda))) \right) = \emptyset.$$

$(B_r(\pi(\Lambda))$ represents the ball of radius r and center $\pi(\Lambda)$ in the quotient space $\Omega(f)/\sim$.)

The compactness of $\Omega(f)$ implies that $\beta(\Lambda, \alpha_\Lambda) \supset \pi^{-1}(B_{r_\Lambda}(\pi(\Lambda)))$, for some small $r_\Lambda > 0$. The collection $\{B_{r_\Lambda}(\pi(\Lambda))\}_{\Lambda \in \mathfrak{I}}$ covers $\Omega(f)/\sim$. Letting $\eta > 0$ be a Lebesgue number of $\{B_{r_\Lambda}(\pi(\Lambda))\}_{\Lambda \in \mathfrak{I}}$, we choose a finite set of points in $\Omega(f)/\sim$, $S = \{\omega_1, \ldots, \omega_s\}$ so that

$$\bigcup_{i=1}^{s} B_\eta(\omega_i) = \Omega(f)/\sim .$$

Let $n > 0$; for each $\omega_i \in S$ $B_\eta(\omega_i) \subset B_{r_{\Lambda_i}}(\pi(\Lambda_i))$, for some positive number r_{Λ_i} and $\Lambda_i \in \mathfrak{I}$. Furthermore

$$\pi^{-1}(B_{r_{\Lambda_i}}(\pi(\Lambda_i))) \subset \beta(\Lambda_i, \alpha_{\Lambda_i})$$

and let $E^r_{(N_{\Lambda_i}, \varepsilon)}$ be a $(N_{\Lambda_i}, 2\varepsilon)$-rotationally spanning set of $\beta(\Lambda_i, \alpha_{\Lambda_i})$ s.t.

$$s + \delta \geq \frac{1}{N_{\Lambda_i}} \log \# E^r_{(N_{\Lambda_i}, \varepsilon)}.$$

We denote by q_i the smallest positive integer s.t. $q_i N_{\Lambda_i} \geq n$ and $(\alpha_0, \alpha_1, \ldots, \alpha_{q_i-1}) \in \prod_{k=1}^{q_i} E_k$, where for simplicity of notation we consider $E_k = E^r_{(N_{\Lambda_k}, \varepsilon)}$ and $N_i = N_{\Lambda_i}$.

DEFINITION 3.4. We say that the sequence $(\alpha_0, \alpha_1, \ldots, \alpha_{q_i-1})$ (n, ε)-rotationally traces the n-rotation sequence of a point $y \in \Omega(f)$ iff

$$|\overline{f^{jN}(y)} - \alpha_j| < \varepsilon \quad \text{and} \quad \left| \frac{F_1^{jN+k}(\overline{y}) - \overline{F_1^k(\alpha_j)}}{k} \right| < \varepsilon,$$

$$k = 1, 2, \ldots, N_i - 1, \ j = 0, 1, \ldots, q_i - 1 \ (q_i N_i \geq n).$$

(Notation:
(1) $\overline{f^{jN_i}(y)}$ is so that $0 \leq \overline{f^{jN_i}(y)} \leq 1$ and $p(\overline{f^{jN_i}(y)}) = f^{jN_i}(y)$.
(2) $\overline{F_1^k(\alpha_j)} \in p^{-1}(f^k(p(\alpha_j))$ s.t.

$$\inf\{|F_1^{jN_i+k}(\overline{y}) - z| \colon z \in p^{-1}(f^k(p(\alpha_j)))\} = |F_1^{jN_i+k}(\overline{y}) - \overline{F_1^k(\alpha_j)}|.)$$

We designate by A_{ω_i} the set of all admissible sequences $(\alpha_0, \alpha_1, \ldots, \alpha_{q_i-1})$ associated to ω_i and

$$\mathcal{N}_{\omega_i, (\alpha_0, \ldots, \alpha_{q_i-1})} = \{y \in \Omega(f) \colon (\alpha_0, \ldots, \alpha_{q_i-1})(n, 2\varepsilon)\text{-rotationally traces the } n\text{-rotation sequences of } y\}.$$

Obviously

$$\begin{aligned} \log(\#(A_{\omega_i}) &= \sum_{k=1}^{q_i} \log \# E_k \leq q_i \cdot n_{\Lambda_i}(s + \delta) \\ &\leq \left(n + \max_{i=1, \ldots, s} N_i\right) \cdot (s + \delta). \end{aligned}$$

We notice that the upper bound is independent of ω_i. We also have that

$$\bigcup_{\omega_i \in S} \ \bigcup_{(\alpha_0, \ldots, \alpha_{q_i-1}) \in A_{\omega_i}} \mathcal{N}_{\omega_i, (\alpha_0, \alpha_1, \ldots, \alpha_{q_i-1})} = \Omega(f).$$

In fact, if $y \in \Omega(f)$ there exists i so that $\pi(y) \in B_\eta(\omega_i)$ and for any positive integer k, $\pi(f^k(y)) \in B_\eta(\omega_i)$. As seen before we have

$B_\eta(\omega_i) \subset B_{r_{\Lambda_i}}(\pi(\Lambda_i))$, $\pi^{-1}(B_{r_{\Lambda_i}}(\pi(\Lambda_i))) \subset \beta(\Lambda_i, \alpha_{\Lambda_i})$ and $E^r_{(N_{\Lambda_i}, \varepsilon)}$ is a $(N_{\Lambda_i}, 2\varepsilon)$-rotationally spanning set of $\beta(\Lambda_i, \alpha_{\Lambda_i})$ s.t.

$$\varepsilon + \delta \geq \frac{1}{N_i} \log \# E^r_{(N_{\Lambda_i}, \varepsilon)} .$$

Let $\alpha_0 \in E^r_{(N_{\Lambda_i}, \varepsilon)}$ be so that

$$|\overline{y} - \alpha_0| < 2\varepsilon \text{ and } \left| \frac{F_1^k(\overline{y}) - F_1^k(\alpha_0)}{k} \right| < 2\varepsilon, \qquad k = 1, 2, \ldots, N_i - 1 .$$

Since $\pi(f^{N_{\Lambda_i}}(y)) \in B_\eta(\omega_i)$ we repeat the same procedure to define α_1. This implies the existence of a sequence in A_{ω_i} that $(n, 2\varepsilon)$-rotationally traces the n-rotation sequence of y. On the other hand if E is a minimal $(n, 4\varepsilon)$-rotationally spanning subset of $\Omega(f)$ then for each $\omega_i \in S$ we have that

$$\#(E \cap \mathcal{N}_{\omega_i, (\alpha_0, \alpha_1, \ldots, \alpha_{q_i - 1})}) \leq 1$$

implying that $\log(\# E) \leq \log(\# S) + (n + \max_{i=1,\ldots,s} N_i) \cdot (s + \delta)$. Consequently

$$\begin{aligned} h_{r,4\varepsilon}(f|_{\Omega(f)}) &= \limsup_n \frac{1}{n} \log(\# E) \\ &\leq \limsup_n \frac{1}{n} \left(\log(\# S) + \left(n + \max_{i=1,\ldots,s} N_i \right) \cdot (s + \delta) \right) \\ &= s + \delta . \end{aligned}$$

Since δ can be chosen arbitrarily small we have

$$h_{r,4\varepsilon}(f|_{\Omega(f)}) \leq \sup_{\Lambda \in \mathfrak{I}} h_r(f|_\Lambda) ;$$

therefore

$$h_r(f|_{\Omega(f)}) \leq \sup_{\Lambda \in \mathfrak{I}} h_r(f|_\Lambda) . \qquad \square$$

REMARK 3.2. The topological entropy of f is denoted by $h(f)$. If F represents a lift of f, F is uniformly continuous and the following definition due to Bowen makes sense:

$$h(F) = \sup_{k \text{ compact}} h(F|_K) .$$

It is a known fact that $h(f) = h(F)$. Therefore $h_r(f) \leq h(F) = h(f)$.

THEOREM 3.1. *If f is a homeomorphism of the annulus, isotopic to the identity, and with finitely many periods then $h_r(f) = 0$.*

Proof. If we assume $h_r(f|_{\Omega(f)}) > 0$ then there exists a chain transitive subset of A, Λ, such that $h_r(f|_\Lambda) > 0$. By Proposition 3.2(i) the rotation set of f restricted to Λ is nontrivial. Therefore there exist $\alpha, \beta \in R(f|_\Lambda)$ with $\alpha \neq \beta$. The theorem follows from the fact that $h_r(f|_{\Omega(f)}) = h_r(f)$ (cf. **[5]**) and from the following result by Franks:

"*If α and β are two distinct numbers in the rotation set of a chain transitive invariant subset of A then for any rational number p/q between α and β there exists a periodic point of period q.*" □

The "rotation shadowing property" is a definition due to Barge and Swanson [3]. It is a weaker condition than the "shadowing property" and it holds for any endomorphism of the circle. Barge and Swanson conjectured that the "rotation shadowing property" is a generic property.

DEFINITION 2.2. f satisfies the rotation shadowing property if for any $\varepsilon > 0$ there exists $\delta > 0$ so that for any δ-pseudo orbit $\{z_i\}$ in A there exists $z \in A$:

$$\limsup \frac{|\hat{z}_i - F^i(\hat{z})|}{i} < \varepsilon$$

where $\{\hat{z}_i\}$ satisfies $|f(\hat{z}_i) - \hat{z}_{i+1}| < \delta$, $p(\hat{z}_i) = z_i$ and $p(\hat{z}) = z$. The pseudo orbit $\{z_i\}$ is said to be ε-rotationally traceable.

THEOREM 3.2. *If f satisfies the rotational shadowing property and every orbit has rotation number then $h_r(f) = 0$.*

Proof. If we assume $h_r(f) > 0$ then there exists some chain transitive set with positive rotational entropy therefore with nontrivial rotation. The rotation shadowing assumption implies the existence of orbits without rotation number. □

We conclude with the following conjecture:

For any annulus homeomorphism, isotopic to the identity, positive rotational entropy is equivalent to the existence of an orbit with nontrivial rotational interval.

This conjecture would imply that any orbit of an annulus homeomorphism with zero topological entropy has rotation number.

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