ON THE ALGEBRAIC PART OF AN ALTERNATING LINK

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A simple method is given for determining the algebraic part of an alternating link. It is proved that the only alternating diagrams of elementary links are the "obvious" ones.

1. Introduction. For each classical link L in the 3-sphere, whose complement is irreducible and geometrically atoroidal, it is explained in [**B-S**] how the pair (S^3, L) admits a decomposition, unique up to isotopy, into an algebraic part $(A, L \cap A)$ and a *non-algebraic part* $(N, L \cap N)$. The algebraic part A is constituted, in a manner explained in §2 below, from so-called "elementary tangles" in (S^3, L); N is just the closure in S^3 of $S^3 - A$. As proved in [**B-S**], if the complement of L is not a Seifert manifold, the submanifold A of S^3 may be characterized in more general terms as follows: (i) ∂A meets L transversely; (ii) if $\pi: \widetilde{X} \to S^3$ is the 2-fold covering of S^3 branched along L, we may choose the characteristic variety V of $\widetilde{X}$ so that $\pi^{-1}(A)$ is precisely the union of the closed-up Seifert fibered components of $\widetilde{X} - V$. If the complement of L is a Seifert manifold, A is either empty or equal to S^3 (see [**B-S**] for full details).

Since the characteristic variety V of $\widetilde{X}$ consists of incompressible tori and $S^3 - L$ is atoroidal, each component of $\partial A = \partial N$ is a 2-sphere meeting the link L transversely in four points. The link L is said to be *algebraic*, or *arborescent*, if the algebraic part is the whole of S^3.

The main purpose of this article is to describe and justify a simple method for determining the algebraic part of (S^3, L) in the case where the link L is presented as an alternating link. It is proved in [**M1**] that if L admits a connected, alternating diagram which is prime (in the 2-dimensional sense), then $S^3 - L$ is irreducible and geometrically atoroidal. Therefore, in this paper, we shall be concerned with precisely those links admitting connected, prime, alternating diagrams. The method for finding the algebraic part of (S^3, L) is based largely on Menasco's "visibility" results for Conway spheres [**M1**], and has already been investigated by F. Bonahon and L. Siebenmann.

Indeed, Bonahon and Siebenmann have previously observed that the method certainly achieves the desired objective, except possibly in the awkward case where the pair (S^3, L) contains no Conway sphere. It is explained in §4 below how the Kauffman 2-variable polynomial may be used to deal with this awkward case; in fact, the needed result is a special case of Theorem 2 of [T2], but we feel it appropriate to give in this paper an independent and more natural proof of this result. The paper is concluded with some observations on the status of the Tait flyping conjecture, and a curious classification result for Conway spheres in alternating links. I am indebted to Francis Bonahon for illuminating discussions.

2. Background. There now follows a rapid review of the underlying concepts. The definitions of this section are gleaned more or less directly from [B-S]; a minor variance of convention is that the term "Conway sphere" refers here to a *pairwise essential* 2-sphere meeting the link or tangle transversely in four points, and that "Conway disks" are likewise assumed to be pairwise essential.

DEFINITION 2.1. A *tangle* is a pair (X, T), where X is a (closed-up) punctured 3-sphere with at least one boundary component, and T is a proper 1-submanifold of X, such that for each (2-sphere) component F of ∂X, $\partial T \cap F$ consists of four points. Two tangles are *equivalent* if they are homeomorphic as pairs.

This definition generalizes the notion of tangle given, say, in [L]. In keeping with our basic assumption in this paper that links are prime, we shall assume that X does not admit a 3-ball in its interior meeting T in a single knotted arc.

A tangle (X, T) with n boundary components admits a regular projection onto a 2-sphere (extended plane) with n open disks removed, Δ say (we shall refer to Δ loosely as an "n-punctured 2-sphere"); as with links, a *tangle diagram* is obtained from the projection of T in Δ by introducing an overcrossing-undercrossing structure (e.g. Fig. 1(i)). The type of the tangle can be recovered by a suitable small vertical perturbation near each crossing of the diagram; the particular tangle representative thus obtained from a diagram D will be denoted $l(D)$. Of course, we take the boundary components of X to be 2-spheres for which the boundary components of Δ are equatorial curves. We shall say that the diagram D *represents* (X, T) if the tangles $(X, l(D))$ and (X, T) are equivalent, and that tangle diagrams D_1, D_2 are *equivalent* if the associated tangles $(X, l(D_1))$,
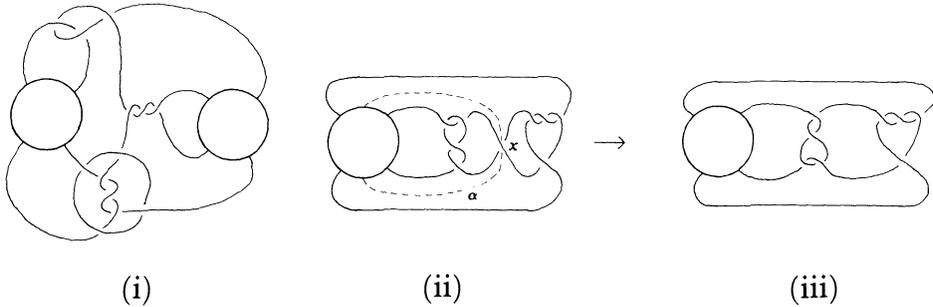
(i)   (ii)   (iii)

FIGURE 1

$(X, l(D_2))$ are equivalent. We shall use the notation $l(D)$ described in the previous paragraph for link diagrams also.

Let D be a tangle diagram on a 2-sphere with n open disks removed, denoted Δ as above. We shall assume throughout this paper that there is not any simple closed curve in Δ meeting D in just one point (this point would be a "nugatory" crossing); however, we shall need to consider *inessential* crossings of D, defined as follows.

DEFINITION 2.2. A crossing-point x of a tangle diagram D on an n-punctured 2-sphere Δ is *inessential* if there exists an arc $\alpha \subset \Delta$, with $\alpha \cap D = x$, such that both ends of α lie on the same boundary component of Δ and α separates the four arcs of D incident upon x into two pairs. A tangle diagram is *reduced* if it contains no inessential crossings.

An example of an inessential crossing is illustrated in Fig. 1(ii). Clearly, any inessential crossing may be removed by twisting in such a way that part of the diagram is inverted, without disturbing the type of the tangle (Fig. 1(iii)). If an inessential crossing is removed in this way from an alternating diagram, the resulting diagram is also alternating, with one less crossing.

DEFINITION 2.3. A *trivial tangle with one boundary component* (resp. *with two boundary components*) is a tangle homeomorphic to that illustrated in Fig. 2(i) (resp. Fig. 2(ii)). Thus a trivial tangle with two boundary components is homeomorphic to $(S^2, 4 \text{ points}) \times I$. A *hollow elementary tangle* is a tangle homeomorphic to the tangle with three boundary components illustrated in Fig. 2(iii). An *elementary tangle* is a tangle obtained from a hollow elementary tangle (X, T) by glueing a trivial tangle of one boundary component to each of 0, 1 or 2 boundary components of (X, T). If all three boundary components of a hollow elementary tangle are plugged in this manner with trivial tangles of one boundary component, the resulting object is called an
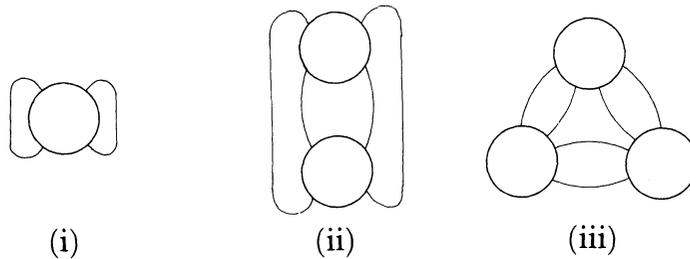
FIGURE 2

elementary link pair. See Fig. 6 for examples of elementary tangles, and an example of an elementary link.

As is well known, given a suitable coordinate system, the attaching map of a tangle of one boundary component is determined up to isotopy by an element of $\mathbb{Q} \cup \{\infty\}$; a trivial tangle together with this glueing information is called a *rational tangle*. Note that an elementary link, as defined above, is merely a Montesinos link with at most three rational substituents.

DEFINITION 2.4. A *Conway sphere* in a tangle (X, T) is a 2-sphere F in $\mathrm{int}(X)$, meeting T transversely in four points, such that the closure of neither component of $(X, T) - (F, F \cap T)$ is a trivial tangle with one or two boundary components.

In the presence of the assumption in this paper that (X, T) does not contain in its interior a 3-ball meeting T in a single knotted arc, a 2-sphere in $\mathrm{int}(X)$ meeting T transversely in four points bounds a trivial tangle with one boundary component in (X, T) if and only if it admits a pairwise compression disk; also, the closure of some component of $(X, T) - (F, F \cap T)$ is a trivial tangle with two boundary components if and only if F is pairwise parallel to the boundary of X. For definitions of "pairwise essential", "pairwise parallel", see [B-S]. A *Conway sphere* in a link pair (S^3, L) is defined likewise; we merely omit the condition regarding trivial tangles with two boundary components. We shall say that two Conway spheres are *parallel* if they are parallel in the pairwise sense, i.e. if they bound a trivial tangle with two boundary components.

DEFINITION 2.5. A *Conway disk* in a tangle pair (X, T) is a disk D properly embedded in X, ∂D being contained in some component W say of ∂X, with the following properties: ∂D does not meet T, $\mathrm{int}(D)$ meets T in two points, each component of $W - \partial D$ meets T in two points, and D is not pairwise parallel to W (i.e. there does not

exist an isotopy, fixing ∂D, of D into W, through disks all meeting T in two points).

The preimage in the twofold cover $\tilde{X}$ of X branched along T of a Conway sphere is an essential torus, and the preimage in $\tilde{X}$ of a Conway disk is an essential annulus, both of whose boundary components lie on the same boundary component of $\tilde{X}$.

Note that elementary tangles and links do not admit Conway spheres: the double branched cover of such a tangle or link cannot contain an essential torus, as it admits a Seifert fibration over a planar surface, where the sum of the numbers of boundary components and singular fibers is at most three. Also, we can see immediately by looking at double branched covers that a trivial tangle with one or two boundary components cannot admit a Conway disk: the double branched cover of a trivial tangle with one boundary component is a solid torus, and the double branched cover of a trivial tangle with two boundary components is a product of a torus with an interval. However, each elementary tangle (X, T), which is not a trivial tangle with one or two boundary components, admits a Conway disk (illustrated as a dotted line in Fig. 6(i), (ii)), visibly separating the tangle into two trivial tangles, (X_1, T_1), (X_2, T_2) say; this disk cannot be pairwise parallel to the boundary of (X, T), for otherwise (X, T) would be equivalent to one of the (X_i, T_i) and would thus be trivial. Conversely, it is easy to see that any tangle manufactured by glueing together two trivial tangles along a Conway disk has a diagram of the general form of Fig. 6(i) or (ii); hence such a tangle is elementary.

Next, we make the observation that if D is a Conway disk in a tangle (X, T) for which some closed-up component (Y, U) of $(X, T) - (D, D \cap T)$ is *non-trivial*, then (X, T) contains a Conway sphere, namely any 2-sphere in $\text{int}(Y)$ pairwise parallel to ∂Y. Because of this, the only Conway disks of interest to us in alternating diagrams will be those occurring in elementary tangles.

From the above discussion, we have the following characterizations of non-trivial elementary tangles.

PROPOSITION 2.6. *The following statements are equivalent:*
  (i) (X, T) *is a non-trivial elementary tangle;*
 (ii) (X, T) *may be cut along a Conway disk into two trivial tangles;*
(iii) (X, T) *admits a Conway disk, but does not admit a Conway sphere.*   □

DEFINITION 2.7. A link or tangle pair is *algebraic* if it is elementary, or if it can be cut along a collection of Conway spheres into elementary tangles.

Of course, the double branched cover of an algebraic link or tangle pair is a graph-manifold [**B-S**]; the decomposition just described corresponds to cutting the graph-manifold along incompressible tori into pieces, each of which is homeomorphic to $(S^2 - 3 \text{ open disks}) \times S^1$ plugged with 0, 1 or 2 solid tori. The double branched cover of an elementary link is merely $(S^2 - 3 \text{ open disks}) \times S^1$ plugged with three solid tori.

DEFINITION 2.8. Let $F_1, \ldots, F_n$ be a maximal, finite collection of pairwise disjoint and non-parallel Conway spheres in (S^3, L) (such a collection exists by Haken finiteness: see [**B-S**], §4.2). Then the *algebraic part* of (S^3, L) is the union of those closed-up components of $S^3 - \bigcup_i F_i$ which are elementary.

As already mentioned, in determining the algebraic part of a link pair (S^3, L), it is necessary to consider separately the cases when (S^3, L) admits a Conway sphere, and when it does not. In the first of these cases, clearly the ability to recognize Conway spheres and elementary tangles is pertinent; this is dealt with in the next section.

3. Recognizing Conway spheres and elementary tangles in alternating diagrams. First, let us recall Menasco's "visibility" result concerning Conway spheres.

Let L be a link which is $l(D)$ for some alternating diagram D. Then it is shown in [**M1**], by elegant and elementary means, that each Conway sphere for L is parallel to one whose intersection with the 2-sphere S_+ of [**M1**] (i.e. the projection 2-sphere perturbed slightly so as to contain the overpasses) consists of either one or two circles, as in Fig. 3(i, ii). The method of proof is equally valid for tangle diagrams. Let us say that a Conway sphere is *visible for a diagram* D if it is parallel to one conforming to Fig. 3(i), and let us say that it is *hidden* otherwise. We may assume that each shaded disk in Fig. 3(ii) represents a portion of diagram with at least one crossing; otherwise the Conway sphere will be parallel to one conforming to Fig. 3(i). For example, the alternating diagram obtained by substituting a single crossing for each of these shaded disks is the usual 6-crossing diagram of the Borromean rings.

Let $(X, l(D))$ be a tangle corresponding to an alternating diagram D in a punctured 2-sphere Δ, and let F be a Conway sphere in

(i)   (ii)   (iii)
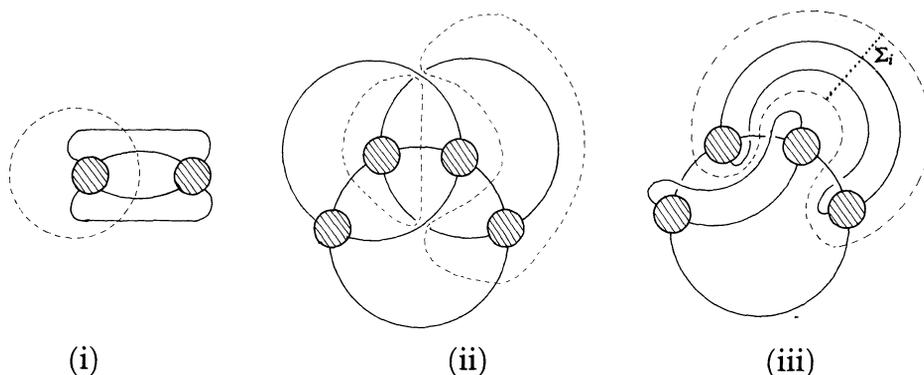
FIGURE 3

X. If F is visible, then cutting X along F clearly results in two tangles $l(D_1)$, $l(D_2)$, where D_1, D_2 are simply the alternating tangle diagrams obtained by cutting Δ along $F \cap \Delta$. On the other hand, if F is hidden, there is an isotopy of X which transforms D to a non-alternating diagram with two extra crossings, in which the image of F under this isotopy is now visible (Fig. 3(iii)). The two tangles (X_1, T_2), (X_2, T_2) obtained by cutting X along F might not admit alternating diagrams, but we can cut each (X_i, T_i) along a disk Σ_i meeting T_i in two points, indicated in Fig. 3(iii) by a dotted line, into two (possibly trivial) tangles which visibly admit alternating diagrams. We shall see, in Corollary 3.4 below, that this disk Σ_i is indeed a Conway disk for (X_i, T_i).

In the case of the Borromean rings, Fig. 3(iii) exhibits a decomposition of this (3-sphere, link)-pair into two elementary tangles, each presented as a diagram with four crossings. Therefore the "Borromean rings" is an algebraic link according to Definition 2.7; it is the simplest example of an alternating link which is algebraic, but which does not admit an alternating, "algebraic" diagram in the sense of [C] (recall from [C] that a diagram is *algebraic* if it has Conway basic polyhedron 1*).

At this point, the reader might wonder how we can tell whether our 2-sphere F, meeting the link or tangle in four points, really is a Conway sphere (the same question arises for purported Conway disks). According to Definition 2.4, this amounts to deciding whether F abuts a non-trivial tangle on both sides. This question has been dealt with by geometric means in Menasco's paper [M2], but here we provide an alternative approach, using properties of the Kauffman polynomial proved in [T3]. The reader is referred to [T3] for the definition of *adequate link diagram*.
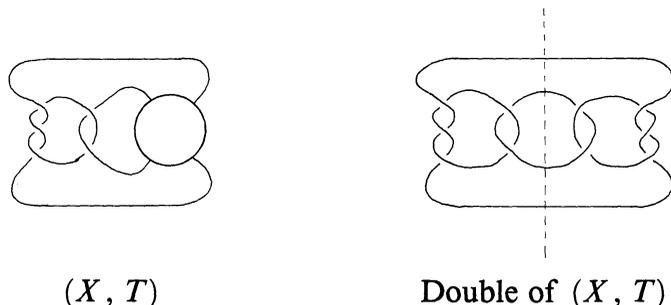
(X, T) Double of (X, T)

FIGURE 4

The next theorem provides a stronger result than is needed for the applications; it may be of independent interest.

THEOREM 3.1. *Let* D *be a reduced alternating diagram, with* n *crossings, of a tangle* (X, T) *with one boundary component. Then no diagram representing that tangle has fewer than* n *crossings.*

Proof. We can form the "double" of any diagram of T, as illustrated in Fig. 4, to obtain a diagram of a (3-sphere, link)-pair which is a pure double of (X, T). The type of this link is evidently independent of the particular diagram of T. If we double a reduced alternating tangle diagram in this way, a (non-alternating) adequate link diagram is produced, which has minimal crossing-number for the link it represents by Corollary 3.4 of [T3].   □

REMARK. Clearly the proof of Theorem 3.1 applies equally to any tangle diagram whose double is an adequate link diagram.

COROLLARY 3.2. *A reduced alternating diagram with at least one crossing cannot represent a trivial tangle with one or two boundary components.*

Proof. Since trivial tangles admit diagrams with no crossing, the result for tangles with one boundary component follows directly from Theorem 3.1. Now let (X, T) be a trivial tangle with two boundary components. Let (X_1, T_1) be the result of glueing some tangle (Y, U) to one of the boundary components of (X, T). Then, since (X, T) is equivalent to $(S^2, 4$ points$) \times I$, (Y, U) is equivalent to (X_1, T_1). Suppose that D is a reduced, alternating diagram with n crossings of (X, T), in a twice-punctured 2-sphere Δ. Then we can adjoin a diagram with no crossing of a trivial tangle with one boundary component to one of the boundary components of (Δ, D), to obtain

a reduced diagram D_1 with n crossings. According to the above argument, D_1 represents a trivial tangle with one boundary component; hence by Theorem 3.1 $n = 0$.   □

It follows in particular that any alternating diagram of a trivial tangle with one boundary component is a 4-string plait diagram, and that any alternating diagram of a trivial tangle with two boundary components is a diagram of a 4-string geometric braid in $S^2 \times I$.

COROLLARY 3.3. *Let D be a reduced alternating diagram of a tangle (or link) $(X, l(D))$, let F be a hidden Conway sphere for $(X, l(D))$, and let (X_1, T_1), (X_2, T_2) be the tangles obtained by cutting $(X, l(D))$ along F. Then*

(i) *Each tangle (X_1, T_1), (X_2, T_2) is non-trivial; moreover, if some (X_i, T_i) $(i = 1, 2)$ has one boundary component, then the diagram of (X_i, T_i) illustrated in Fig. 3(iii) has minimal number of crossings for diagrams representing (X_i, T_i).*

(ii) *The disk Σ_i in (X_i, T_i) indicated by a dotted line in Fig. 3(iii) is a Conway disk for (X_i, T_i).*

Proof. (i) For $i = 1, 2$ let D_i be the (non-alternating) diagram of (X_i, T_i), obtained as in Fig. 3(iii). If (X_i, T_i) has one boundary component, then the double of D_i is an adequate link diagram; hence, by the remark following Theorem 3.1, D_i has minimal crossing-number for (X_i, T_i), and in particular that tangle cannot be trivial. On the other hand, if (X_i, T_i) has two boundary components, the method of proof of Corollary 3.2 shows that (X_i, T_i) is non-trivial.

(ii) Suppose Σ_i were pairwise parallel to the boundary of (X_i, T_i). Then there would be a 2-sphere G, made up from Σ_i with a small open collar of its boundary removed, together with a disk close to and parallel to one of the components of $\partial X_i - \partial \Sigma_i$, such that G and ∂X_i bounded a trivial tangle of two components (Fig. 3(iii)). But the tangle bounded by G and ∂X_i has a reduced, alternating diagram with at least two crossings, so this tangle cannot be trivial.   □

From Corollary 3.2, one may determine whether an alternating diagram of a tangle with one or two boundary components represents a trivial tangle by repeatedly removing inessential crossings until a reduced alternating diagram is obtained. Then the tangle is trivial if and only if that reduced diagram has no crossings.

Corollaries 3.3 and 3.4 enable us to deal effectively with hidden Conway spheres. Using the notation of Corollary 3.3, one looks at the (alternating) subtangles into which (X_i, T_i) is decomposed by cutting along the Conway disk Σ_i. If both these subtangles are seen to be trivial, then (X_i, T_i) is elementary by Proposition 2.6(ii); otherwise, (X_i, T_i) contains a visible Conway sphere, i.e. one such as G in Fig. 3(iii), and cutting along this sphere decomposes (X_i, T_i) into two alternating, non-trivial tangles.

Let us pause for a moment, to see where we stand in our quest for a determination of the algebraic part of (S^3, L), in the case where (S^3, L) admits a Conway sphere. Given an alternating link or tangle Diagram D, we have described a method for detecting 2-spheres meeting the link or tangle $l(D)$ in four points, and a method for determining whether that 2-sphere is pairwise essential. We have also dealt fully with the consequences of cutting along a hidden Conway sphere. Therefore, apart from the matter of recognizing elementary tangles resulting from cuts along *visible* Conway spheres, the procedure is now clear. One cuts (S^3, L) along a Conway sphere, then cuts the resulting pieces likewise, and so on, continuing the process until one has obtained a collection of non-trivial tangle pieces of (S^3, L), none of which admits a Conway sphere. All that remains is to be able to determine which pieces of this collection are elementary; from Proposition 2.6, this is equivalent to determining which pieces admit Conway disks.

Now, Menasco's analysis of Conway spheres for alternating diagrams may easily be adapted to deal with Conway disks. It turns out that, as with spheres, there are "visible" and "hidden" Conway disks (Fig. 5). However, since a hidden Conway disk always abuts a non-trivial tangle, we shall not need to consider these: observe that in Fig. 5(iii) one of the tangles obtained by cutting along the Conway disk has a diagram of the same sort as that occurring in Fig. 3(iii). The upshot is that Conway disks in reduced alternating diagrams of *elementary* tangles are always visible; moreover, since any alternating diagram of a trivial tangle with one boundary component is a 4-string plait diagram, any alternating diagram of a non-trivial elementary tangle consists of 4-string plaits and boundary components set out in the obvious fashion (Fig. 6). Therefore elementary tangles are recognizable from their alternating diagrams, and, in the case where (S^3, L) admits a Conway sphere, the process of determining the algebraic part of (S^3, L) is now complete.
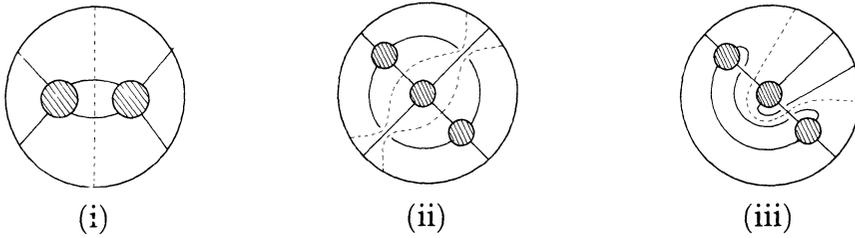
(i)   (ii)   (iii)

FIGURE 5

Typical elementary tangles   A typical elementary link
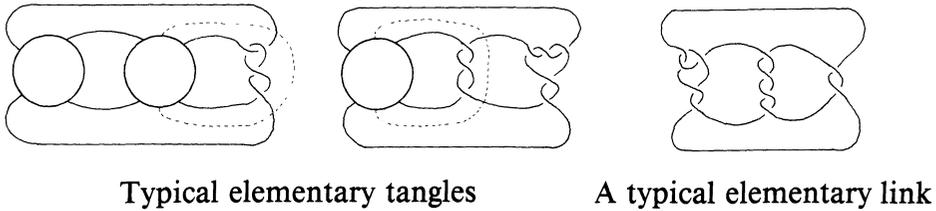
FIGURE 6

4. The case where (S^3, L) does not admit a Conway sphere. We shall assume throughout this section that $L = l(D)$ for some reduced alternating link diagram D, and that there is no Conway sphere in (S^3, L). From Definition 2.8, we see at once that either L is an elementary link, or else the algebraic part of (S^3, L) is empty. The aim of this section is to show that, as with elementary tangles, the only alternating diagrams of elementary links are the "obvious" ones.

Let us recall that an elementary link is merely a Montesinos link with at most three rational substituents. In [L-T], the notion of a *reduced Montesinos diagram* of a Montesinos link is described, and it is proved that a Montesinos link cannot be projected with fewer crossings than in such a diagram. It then follows from Theorems 1 and 2 of [T1] that if L admits a non-alternating reduced Montesinos diagram, then L cannot admit any alternating diagram; therefore we can immediately discount links whose reduced Montesinos presentations are non-alternating.

Now suppose L is an alternating, elementary link. Either L is 2-bridged and admits a standard 4-string plait diagram, or L admits an alternating, reduced Montesinos diagram with three substituent rational tangles, of the type illustrated in Fig. 6(iii). Indeed, if D is any alternating, *algebraic* diagram of L, i.e. an alternating diagram whose Conway basic polyhedron is 1^* [C], then from the absence of Conway spheres in (S^3, L) it is easily seen that D must be either a 4-string plait diagram or a reduced Montesinos diagram. However, we must still consider the possibility that L might admit an

alternating diagram whose Conway basic polyhedron is other than 1*. This possibility is duly excluded by the following theorem.

THEOREM 4.1. *If a link L admits an alternating diagram which is algebraic, then every alternating diagram of L is algebraic.*

A longer proof of a slightly more general theorem is set forth in [T2]; there it is proved that the basic polyhedra 1*, 6*, 8* are all "characteristic" for reduced alternating diagrams.

As in [T2], we shall use the integer invariant $\kappa(D)$ of regular isotopy classes of alternating diagrams D with at least two crossings, determined by the Kauffman two-variable polynomial $\Lambda_D(a, z)$. We recall the following facts about κ:

(i) $\kappa(D) \geq 0$, and $\kappa(D) > 0$ if and only if D is reduced and prime;

(ii) $\kappa(D)$ depends only on the basic polyhedron of D;

(iii) if D is algebraic, $\kappa(D) = 1$.

Therefore, to prove Theorem 4.1, it will be sufficient to show that $\kappa(D) > 1$ if D is a reduced alternating diagram whose underlying projection is a basic polyhedron other than 1*.

Now let $G = G(D)$ be the connected, planar graph obtained from a black-and-white shading of D by associating, in the usual fashion, a vertex of G to each white region say of D, and an edge of G to each crossing of D. It will be convenient to discuss the invariant κ applied to $G(D)$, rather than to D. We shall use the standard notation G_e', G_e'' for the graphs obtained from G by respectively deleting an edge e of G, contracting the edge e. We shall say that an edge of G *links* its endpoints.

DEFINITION 4.2. A connected graph G is *separable* if either of the following holds:

(i) G has more than one edge, and at least one edge of G is a loop;

(ii) G may be disconnected by the removal of a single vertex, called a *cutvertex* of G, together with its incident edges.

DEFINITION 4.3. A subgraph H of a separable graph G is a *cut component of G with respect to v* if it is a component of the graph obtained by deleting from G some cutvertex v of G together with its incident edges.

The nonnegative integer invariant κ is characterized by the following properties:

I $\kappa(G) = 0$ if and only if G is separable;

II if G is the graph ⬭, $\kappa(G) = 1$;

III if G has at least three edges, and e is an edge of G which is neither an isthmus nor a loop, then

$$\kappa(G) = \kappa(G'_e) + \kappa(G''_e).$$

Note 4.4. The above characterization is somewhat redundant, as the implication $(\kappa(G) = 0 \Rightarrow G$ is separable) follows from properties II, III. We leave this as an exercise to the interested reader.

Now, if the underlying projection of a diagram D is a non-trivial basic polyhedron, then that diagram has no two-sided region. Therefore the planar graph $G(D)$ contains no vertex of valency 2, nor any pair of vertices linked by more than one edge (Fig. 7).
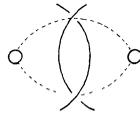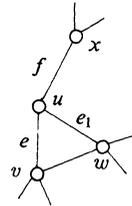

FIGURE 7   FIGURE 8

The proof of the Theorem is now reduced to the following.

PROPOSITION 4.5. *Let G be a connected graph with at least two edges, such that $\kappa(G) = 1$. Then G has a vertex of valency 2 or a pair of vertices linked by more than one edge.*

Proof. The conclusion certainly holds for the graph ⬭. We proceed by induction on the number of edges of G. Suppose the statement holds for all graphs with fewer than n edges; let G be a graph with $n > 2$ edges, such that $\kappa(G) = 1$ and G fails to satisfy the conclusion of the Proposition. Since G is non-separable by Property I above, G contains no isthmus or loop. If e is any edge of G, by Property III and the non-negativity of κ, precisely one of G'_e, G''_e is separable. Let us choose e so that this separable graph (G'_e or G''_e) has a cut component H say with smallest possible number of edges. We shall assume henceforth that this separable graph is G'_e; the case where the separable graph is G''_e is dealt with by means of

a dual argument. Therefore we assume that G'_e is separable with a cut component, H say, with minimal number of edges for all choices of G'_e, G''_e, and that G''_e is non-separable. Since $\kappa(G''_e) = 1$, by the inductive hypothesis G''_e has a vertex of valency 2 or a pair of vertices linked by more than one edge. However, by hypothesis G has no vertex of valency 2, and contraction of e cannot decrease vertex valencies; therefore G''_e must have a pair of vertices linked by more than one edge. It follows that e must belong to a cycle of length 3 in G, whose vertices are u, v, w say, where u, v are incident to e and w is the cutvertex of G'_e giving rise to the minimal cut component H of G'_e (Fig. 8). As G is non-separable, the graph K, obtained by deleting from G the vertex w together with its incident edges, is connected. In this graph K, e is an isthmus, so $K - e$ has two components; these components are precisely the cut components of G'_e with respect to w. Therefore the minimal cut component H of G'_e must contain precisely one of the vertices u, v; let this vertex be u. Let e_1 be the edge of G joining u, w. Note that H must contain at least one other edge f say, incident to u and not incident to w, as u cannot have valency 2 in G and u, w cannot be linked in G by more than one edge. Let x be the other end of f (Fig. 8). G''_{e_1} is separable, with cutvertex the single vertex, w_1 say, obtained from the identification of u, w under the contraction of e_1. But the cut component of G''_{e_1}, corresponding to w_1 and containing the vertex x, has fewer edges than H, contradicting the minimality of H. □
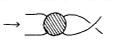
5. Epilogue. An unexpected consequence of the method of proof of Theorem 3.1 is the following.

PROPOSITION 5.1. *Let D_1, D_2 be alternating link diagrams such that there exists a homeomorphism $h: (S^3, l(D_1)) \rightarrow (S^3, l(D_2))$, and let F be a Conway sphere for the link $l(D_1)$ which is visible for D_1. Then $h(F)$ is visible for D_2.*

Proof. D_1, D_2 have the same number of crossings by Theorem 1 of [T1]; let their common crossing-number be n. Let F be as in the statement of the proposition. Then we may assume that the intersection of F with the projection plane is a circle which divides the link diagram D_1 into two tangle diagrams U_1 and V_1 say. Let us suppose that D_2 and $h(F)$ conform to Fig. 3(ii). Then, as we have already seen in §3, we may transform the diagram D_2 to a non-

alternating diagram D_2' with $(n+2)$ crossings, in which the image of $h(F)$ under this isotopy has become visible (Fig. 3(iii)). Let h' be the modified homeomorphism of S^3 which carries $l(D_1)$ to $l(D_2)$. Then the intersection of $F' = h'(F)$ with the projection 2-sphere is a circle which divides the diagram D_2' into two tangle diagrams U_2, V_2 such that $h'(U_1) = U_2$ and $h'(V_1) = V_2$. Since D_1 has fewer crossings than D_2', it follows that at least one of the tangle diagrams U_2, V_2 does not have minimal crossing-number for the tangle it represents. But, as already mentioned in the proof of Corollary 3.3, doubling either of these tangle diagrams yields an adequate diagram, thereby forcing a contradiction. □

Thus there are two discernible types of Conway sphere for alternating links: the type of sphere which, informally speaking, is "always visible", and the type which is "always hidden". Bonahon states that he has a purely geometric proof (unpublished) of Proposition 5.1.

Proposition 5.1 would follow from the celebrated *Tait flyping conjecture*, which proclaims that, given any two alternating diagrams of a link, one can transform one diagram to the other by means of a sequence of *flypes* . The Tait conjecture is unsolved in general, but in light of the results of [**B-S**] and the discussion of the previous section, we may make the following assertion.

PROPOSITION 5.2. *Let L be a link admitting an alternating, algebraic diagram. Then any two alternating diagrams of L are related via a sequence of flypes.*

Sketch proof. From Theorem 4.1, every alternating diagram of L is algebraic. But in [**B-S**] it is proved that any two algebraic diagrams of a link must be related via certain moves; in particular, it follows from their analysis that any two alternating, algebraic diagrams of a link are related via a sequence of flypes. □

One may deduce in particular from Proposition 5.2 that every alternating diagram representing a 2-bridged link is "standard". There does not appear to be any purely geometric proof of this available at present.

Proposition 5.2 does not cater for those alternating diagrams which represent links which are algebraic according to Definition 2.7, yet which do not have basic polyhedron 1^* (for example the 6-crossing diagram of the Borromean rings). From Proposition 5.1, if a link
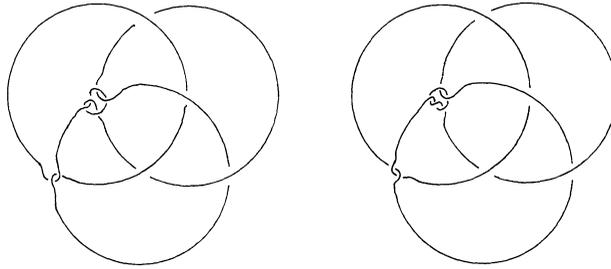
FIGURE 9

admits such an alternating diagram, every alternating diagram of the link must be of that form. Such a diagram can always be transformed to a non-alternating diagram which does have basic polyhedron 1*, by repeated application of the move which transforms Fig. 3(ii) to Fig. 3(iii), at which stage the machinery of [B-S] can be applied. It should be possible, by means of this approach, to show that any two alternating diagrams of an algebraic link are related via a sequence of flypes, whether or not the diagrams themselves are algebraic; however, we shall not pursue this matter here.

We close by remarking that the flyping conjecture does not hold for the more general class of adequate diagrams [T3]; Fig. 9 illustrates two adequate 11-crossing diagrams of the famous Kinoshita-Terasaka knot with trivial Alexander polynomial, each conforming to the pattern of Fig. 3(ii). These diagrams are not related via a sequence of flypes, and indeed at first sight it is not obvious that they represent the same knot type. However, their equivalence becomes fairly easy to see once they have been transformed to the pattern of Fig. 3(iii).

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