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SINGULARITY OF THE RADIAL SUBALGEBRA OF $\mathscr{L}(F_N)$ AND THE PUKÁNSZKY INVARIANT

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Let $\mathscr{L}(F_N)$ be the von Neumann algebra of the free group with N generators $x_1, \dots, x_N$, $N \geq 2$ and let A be the abelian von Neumann subalgebra generated by $x_1 + x_1^{-1} + \cdots + x_N + x_N^{-1}$ acting as a left convolutor on $l^2(F_N)$. The radial algebra A appeared in the harmonic analysis of the free group as a maximal abelian subalgebra of $\mathscr{L}(F_N)$, the von Neumann algebra of the free group. The aim of this paper is to prove that A is singular (which means that there are no unitaries u in $\mathscr{L}(F_N)$ excepting those coming from A such that $u^*Au \subseteq A$). This is done by showing that the Pukánszky invariant of A is infinite, where the Pukánszky invariant of A is the type of the commutant of the algebra $\mathscr{A}$ in $B(l^2(F_N))$ generated by A and $x_1 + x_1^{-1} + \cdots + x_N + x_N^{-1}$ regarded also as a right convolutor on $l^2(F_N)$.

1. Introduction. Let M be a type II_1 factor with trace τ, $\tau(1) = 1$ and $A \subseteq M$ a maximal abelian von Neumann subalgebra (briefly a M.A.S.A.). Following J. Dixmier [**1**], let $N_M(A) = \{u \in M \mid u \text{ unitary}, uAu^* = A\}$ be the normalizer of A in M and $B = N_M(A)''$ the von Neumann subalgebra generated by $N_M(A)$ in M. According to the size of B in M, A is called singular if $B = A$ and A is called regular (or Cartan) if $B = M$. While examples of regular M.A.S.A.'s are readily available by the classical group measure space construction from a free action of a discrete group on a measure space, examples of singular M.A.S.A.'s are more difficult to obtain (see, e.g., [**1**], [**6**], [**9**], [**10**], [**5**]).

The aim of this paper is to show that in the von Neumann algebra $M = \mathscr{L}(F_N)$ of the free group with N generators $X_1, X_2, \dots, X_N$, the radial algebra (i.e. the abelian von Neumann subalgebra generated by $X_1 + \cdots + X_N + X_1^{-1} + \cdots + X_N^{-1}$) is singular. This algebra has been studied intensively in [**2**], [**3**], [**7**] because of its connections with the problem of computing spectra of convolutors and with representation theory of F_N. In particular in [**7**] it is shown that the radial algebra is a M.A.S.A. in $\mathscr{L}(F_N)$.

To prove our result we need in fact to prove more than the singularity of A. In order to express this we recall some definitions.

Let $||X||_2 = \tau(X^*X)^{1/2}$ be the Hilbert norm given by τ on M, let $L^2(M, \tau)$ be the completion of M with respect to this norm so that M acts (in the standard way) on $L^2(M, \tau)$. Let also J: $L^2(M, \tau) \mapsto L^2(M, \tau)$ be the canonical conjugation (given by $Jx = x^*$ for x in M) and let $\mathscr{A} = (A \vee JAJ)''$ be the (abelian) von Neumann subalgebra generated in $B(L^2(M, \tau))$ by A and JAJ. Since any automorphism of M is unitarily implemented on $L^2(M, \tau)$, it follows that the type of the algebra $\mathscr{A}'$ is an invariant for A. This invariant was considered by Ambrose-Singer and also by Pukánszky in **[6]**. Moreover the latter showed that in the hyperfinite factor there are singular M.A.S.A.'s, A_n such that the corresponding $\mathscr{A}_n$'s are of the homogeneous type I_n on $I_{B(L^2(M,\tau))} - p_1$ (where p_1 is the cyclic projection onto $\overline{A}^{|| \,||_2} \subseteq L^2(M, \tau))$.

The link between this invariant and the classification of M.A.S.A.'s recalled at the beginning is given by a result of S. Popa (**[4]**); $\mathscr{A}$ is maximal abelian whenever A is a Cartan M.A.S.A. and (consequently) if $\mathscr{A}$ is of the homogeneous type I_n $(n \geq 2)$ on $I_{B(L^2(M,\tau))} - p_1$, then A is singular.

We prove that for the radial algebra this invariant is infinite (for each N) and therefore A is singular.

In an earlier version of this paper the proofs were very complicated. I am greatly indebted to Florin Boca who simplified the proofs by noticing the relations in Lemma 1. Also the actual proof of Lemma 5 is due to him.

2. Singularity of the radial algebra. Let $N \geq 2$ be an integer and F_N be the free group with N generators $X_1, X_2, \dots, X_N$. Let $M = \mathscr{L}(F_N)$ be the associated von Neuman algebra (which is the weakly closed subalgebra of $B(l^2(F_N))$ generated by the left convolution operators on $l^2(F_N))$. It is well known that $\mathscr{L}(F_N)$ is a type II_1 factor that acts standardly on $l^2(F_N)$, (see **[8]**) and that with this identification the norm $|| \,||_\tau$ coincides with the usual norm $|| \,||_2$ on $l^2(F_N)$.

By $M_0 = \mathbb{C}[F_N]$ we denote the group ring of F_N over $\mathbb{C}$, viewed as a subalgebra of M (thus M_0 is the linear space of all finite sums $\sum_{w \in F_n} \lambda_w \cdot w$ (where λ_w are complex numbers) endowed with the usual product structure). Since the void word $\varnothing$ (the unity of F_N) coincides with the unity of $\mathscr{L}(F_N)$, we shall simply write 1 instead of $\lambda_\varnothing$. By means of the identification of M with a subspace of $L^2(M, \tau)$, M_0 corresponds to the subspace of finite support sequences in $l^2(F_N)$.

Recall that the canonical length function $|\cdot|$ on F_N is defined by $|w| = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_p|$ if $w \in F_N$ has the reduced form $X_{i_1}^{\alpha_1} \cdots X_{i_p}^{\alpha_p}$ $(i_j \neq i_{j+1})$.

For any integer $n \geq 0$ we denote by $M_0^n \subseteq M_0$ the linear span of all words in F_N of length n and by q_n we denote the projection of $l^2(F_N)$ onto M_0^n; χ_n will be the vector in M_0^n defined by

$$\chi_n = \sum_{|w|=n} w.$$

In [2], [3], [7] it is proved that

$$A = \overline{\mathrm{Sp}(\{\chi_n | n \geq 0\})^w} \subseteq M$$

(the closure being taken with respect to the weak topology on M) is an abelian von Neumann subalgebra of $\mathscr{L}(F_N)$ (called the radial algebra); this in fact a consequence of the following recurrence relations:

$$\begin{aligned} &\chi_1 \cdot \chi_1 = \chi_2 + 2N, \\ &\chi_1 \cdot \chi_n = \chi_n \chi_1 = \chi_{n+1} + (2N-1)\chi_{n-1}, \qquad n \geq 1. \end{aligned} \tag{1}$$

Moreover, by [7], A is a M.A.S.A. in M.

Let as before $\mathscr{A} = (A \vee JAJ)''$ be the (abelian) von Neumann subalgebra of $B(l^2(F_N))$ generated by A and JAJ and for any ξ in $l^2(F_N)$ let p_ξ denote the cyclic projection of $l^2(F_N)$ onto $\overline{\mathrm{Sp}\mathscr{A}\xi}^{\|\,\|_2} = \overline{\mathrm{Sp}\, A\xi A}^{\|\,\|_2}$. Our aim is to show that $\mathscr{A}'$ is of the homogeneous type I_∞ on $I - p_1$ (where $1 \in M \subseteq L^2(M, \tau)$) and to do this we will construct an infinite family of vectors $\{\xi_n\}_n$ in $l^2(F_N)$ such that their corresponding cyclic projections $\{p_{\xi_n}\} \in \mathscr{A}'$ are mutually orthogonal and of central support $I - p_1$ in $\mathscr{A}'$.

In proving this we will need also to consider for each integer $n \geq 0$ the projections P_n onto $\overline{\mathrm{Sp}\{AwA \mid w \in F_N,\ |w| \leq n\}}^{\|\,\|_2}$ and the linear subspaces $S_n \subseteq M_0^n$ spanned by $\{q_n(\chi_1 w),\ q_n(w\chi_1),\ w \in F_N,\ |w| \leq n-1\}$ (so that $S_0 = (0)$). We will later identify S_n with the range of the projection $P_{n-1} \wedge q_n$. Note that with the notation before $p_1 = P_0$.

To describe the structure of the cyclic projections associated with an arbitrary vector ξ in M_0^l, $l \geq 1$, we will also use the following notation: for all r, s integers let

$$\xi_{r,s} = q_{r+s+1}(\chi_r \xi \chi_s) \quad \text{if } r, s \geq 0$$

and if r or s are strictly negative, let $\xi_{r,s} = 0$.

Our first purpose is to show that for every γ in $M_0^l \ominus S_l$, $l \geq 1$ the projection p_γ commutes with all q_n, $n \geq 0$ and to describe the range of $p_\gamma \wedge q_n$. To do this we need first the following lemma which is analogous to the recurrence relations (1).

LEMMA 1. (a) *If γ is in M_0^l, $l \geq 1$, then*

$$\chi_1 \gamma_{r,s} = \gamma_{r+1,s} + (2N-1)\gamma_{r-1,s} \quad \text{for } r \geq 1,\ s \geq 0,$$
$$\gamma_{r,s}\chi_1 = \gamma_{r,s+1} + (2N-1)\gamma_{r,s-1} \quad \text{for } r \geq 0,\ s \geq 1.$$

(b) *If γ is in $M_0^l \ominus S_l$ for some $l \geq 2$ then the relations in* (a) *are also true for $r, s \geq 0$, i.e. we have*

$$\chi_1 \gamma_{0,s} = \gamma_{1,s};\ \gamma_{s,0}\chi_1 = \gamma_{s,1}, \qquad s \geq 0.$$

(c) *If $\gamma = \sum_{|X|=1} c_X \cdot X$ belongs to $M_0^1 \ominus S_1$ and $\varepsilon \in \{\pm 1\}$ is such that $c_{(X^{-1})} = \varepsilon c_X$ for all X in F_N with $|X| = 1$ then*

$$\chi_1 \gamma_{0,s} = \gamma_{1,s} - \varepsilon \gamma_{0,s-1}, \qquad s \geq 0,$$
$$\gamma_{s,0}\chi_1 = \gamma_{s,1} - \varepsilon \gamma_{s-1,0}, \qquad s \geq 0.$$

(d) *Finally if γ belongs to $M_0^l \ominus S_l$, $l \geq 1$, then*

$$\gamma \chi_s = \gamma_{0,s} - \gamma_{0,s-2}, \qquad s \geq 0.$$

Proof. (a) is proved exactly as the relations (1) are proved in **[2]** and (b), (c) are proved by similar arguments. For example, if γ is as in the statement of (c) and $s \geq 1$ an integer, then

$$\begin{aligned}\chi_1 \gamma_{0,s} &= \gamma_{1,s} + \sum_{|w|=s} \left(\sum_{X \neq (f(w))^{-1}} c_X \right) \cdot w \\ &= \gamma_{1,s} - \sum_{|w|=s} (c_{(f(w))^{-1}}) \cdot w = \gamma_{1,s} - \varepsilon \sum_{|w|=s} (c_{f(w)}) w \\ &= \gamma_{1,s} - \varepsilon \gamma_{0,s-1},\end{aligned}$$

where by $f(w)$ we denoted the first letter of w and where we used the equality $\sum_{|X|=1} c_X = 0$ which follows from the fact that γ being orthogonal to S_1 it is also orthogonal on χ_1.

Finally (d) is proved by induction from the preceding relations.

The next lemma shows that for γ in $M_0^l \ominus S_l$, p_γ commutes with q_n and range $p_\gamma \wedge q_n = \mathrm{Sp}\{\gamma_{r,s;r+s=n-1}\}$ for all $n \geq 1$.

LEMMA 2. *Suppose* γ *is in* $M_l^0 \ominus S_l$, $l \geq 2$, ε *in* $\{\pm 1\}$ *and* β *in* $M_0^1 \ominus S_1$, $\beta = \sum c_X$ *such that* $c_X = \varepsilon c_{(X^{-1})}$ *for* $|X| = 1$. *Then for all* $n, m \geq 0$

(a) $\chi_n \gamma \chi_m = \gamma_{n,m} - (\gamma_{n,m-2} + \gamma_{n-2,m}) + \gamma_{n-2,m-2}$,

(b)

$$\chi_n \beta \chi_m = \beta_{n,m} - (\beta_{n-2,m} + \beta_{n,m-2} + \varepsilon \beta_{n-1,m-1}) + \sum_{k \geq 2} (-\varepsilon)^k (\varepsilon \beta_{n-k-1,m-k+1} + \varepsilon \beta_{n-k+1,m-k-1} + 2\beta_{n-k,m-k}),$$

(c) $\gamma_{n,m} = \sum_{r \leq n, s \leq m} \chi_r \gamma \chi_s$ *where* (r, s) *runs over all possible values such that* r, s *have the same parity as* n, m *respectively.*

(d) $\beta_{n,m} = \sum_{r \leq n, s \leq m} (\varepsilon)^{n-r} \chi_r \beta \chi_s$ *where* (r, s) *runs over all possible values such that* $r - s$ *has the same parity as* $n - m$.

In particular $\mathrm{Sp}((\chi_n \gamma \chi_m)_{n,m \geq 0}) = \mathrm{Sp}((\gamma_{n,m})_{n,m \geq 0})$ *and similarly for* β.

Proof. We will prove only (b), (d) since (a), (c) can be proved in a similar (but easier) way.

To prove (b) note that the case $n = 0$ follows straight from the preceding lemma (point (d)) and hence we can proceed with the proof by induction according to n. Assume that we have already carried the induction up to n and hence we have to compute $\chi_{n+1} \beta \chi_m$. For the sake of simplicity we assume $n \geq 2$ and by the use of the second relation from (1) (when $n = 1$ we have to use the first) we get for any $m \geq 0$

$$\begin{aligned} \chi_{n+1} \beta \chi_m &= (\chi_1 \cdot \chi_n - (2N - 1) \chi_{n-1}) \beta \chi_m \\ &= \chi_1 (\chi_n \beta \chi_m) - (2N - 1) \chi_{n-1} \beta \chi_m. \end{aligned}$$

In order to describe the proof we introduce the following linear operators L and M on $\mathrm{Sp}(\beta_{n,m})$ defined by $L(\beta_{n,m}) = \beta_{n+1,m}$ and

$$M(\beta_{n,m}) = \begin{cases} (2N - 1) \beta_{n-1,m} & \text{if } n \geq 1, \\ (-\varepsilon) \beta_{0,m-1} & \text{if } n = 0, \end{cases}$$

so that by Lemma 1.c

$$\chi_{n+1} \beta \chi_m = (L + M)(\chi_n \beta \chi_m) - (2N - 1) \chi_{n-1} \beta \chi_m.$$

One easily observes that $L(\chi_n \beta \chi_m)$ contains all the required terms in the expansion of $\chi_{n+1} \beta \chi_m$ except those of the form $\beta_{0,r}$. But on the other hand $M(\chi_n \beta \chi_m)$ gives $(2N - 1) \chi_{n-1} \beta \chi_m$ (by the induction hypothesis) plus the above-mentioned missing terms. This completes the proof of (b). Similarly (d) is proved by using

$$\beta_{n+1,m} = \chi_1\beta_{n,m} - (2N-1)\beta_{n-1,m}, \qquad n \geq 1.$$

The next lemma shows in particular that whenever γ, γ' are orthogonal vectors in $M_0^l \ominus S_l$ the associated cyclic projections are orthogonal.

LEMMA 3. *Let* γ, γ' *be vectors in* $M_0^l \ominus S_l$, $l \geq 2$. *Corresponding to* ε, $\varepsilon' \in \{\pm 1\}$ *as in the statement of the preceding lemma, let* β, β' *be in* $M_0^1 \ominus S_1$. *Then for any* n, n', m, $m' \geq 0$

(a) $\langle \gamma_{n,m}, \gamma_{n',m'}\rangle_2 = \delta_{n,n'}\delta_{m,m'}(2N-1)^{n+m}\langle\gamma, \gamma'\rangle_2$.

(b)

$$\langle \beta_{n,m}, \beta_{n',m'}\rangle_2 = \delta_{\varepsilon,\varepsilon'}\delta_{n+m,n'+m'}(2N-1)^{n+m} \cdot(-\varepsilon(2N-1))^{-|n-n'|}\langle\beta, \beta'\rangle_2,$$

(*where by* $\langle\cdot,\cdot\rangle_2$ *we denote the scalar product on* $l^2(F_N)$ *and* δ_{ij} *is the Kronecker symbol:* δ_{ij} *is nonzero only if* $i = j$ *and in this case* $\delta_{ii} = 1$).

Proof. It is obvious that both sides in (a) and (b) vanish if $n+m \neq n'+m'$. Hence we will assume that $n+m = n'+m' = k$. Let first ξ, ξ' be elements in $M_0^l \ominus S_l$, $l \geq 1$; using (1) and the fact that $q_l(\chi_n \cdot \xi_{n',m'} \cdot \chi_{m-2}) = 0$ we deduce

$$q_l(\chi_n\xi_{n',m'}\chi_m) = q_l(\chi_n \cdot \xi_{n',m'} \cdot \chi_1 \cdot \chi_{m-1})$$

whenever $m \geq 1$ (and a similar relation to the left of ξ). Using this and Lemma 1 (and since $q_l(\chi_n\xi_{n',m'+1}\chi_{m-1}) = 0$) we obtain

$$q_l(\chi_n\xi_{n',m'}\chi_m) = (2N-1)q_l(\chi_n\xi_{n',m'-1}\chi_{m-1}), \quad \text{for } m, m' \geq 1,$$

$$q_l(\xi_{s,0}\cdot\chi_s) = \begin{cases} 0 \quad \text{if } \xi = \gamma \quad \text{for } s \geq 1, \\ (-\varepsilon)q_l(\xi_{s-1,0}\chi_{s-1}) \quad \text{if } \xi = \beta, \quad \text{for } s \geq 1. \end{cases}$$

(and similar relations to the left of ξ). Assuming $n \geq n'$ (so $m \leq m'$) we obtain by induction

$$q_l(\chi_n\gamma_{n',m'}\chi_m) = \delta_{n,n'}(2N-1)^{n+m'}\gamma,$$
$$q_l(\chi_n\beta_{n',m'}\chi_m) = (2N-1)^{n+m'}(-\varepsilon)^{|n-n'|}\beta.$$

The proof is now accomplished by noticing that

$$\langle\xi'_{n',m'}, \xi_{n,m}\rangle_2 = \langle\xi'_{n',m'}, \chi_n\xi\chi_m\rangle_2$$
$$= \langle\chi_n\xi'_{n',m'}\chi_m, \xi\rangle_2 = \langle q_l(\chi_n\xi'_{n',m'}\chi_m), \xi\rangle_2.$$

We can now show that the space S_l coincides with the range of the projection $P_{l-1} \wedge q_l$ and deduce that P_{l-1} and q_l are commuting projections. As a corollary we will obtain that the cyclic projections corresponding to two orthogonal vectors in $\mathrm{Sp}(M_0^l \ominus S_l \mid l \geq 0)$ are orthogonal:

LEMMA 4. (a) *The projections P_{l-1} and q_l commute and S_l is the range of $P_{l-1}q_l = P_{l-1} \wedge q_l$ (so that $M_0^l \ominus S_l$ is the range of $q_l \ominus P_{l-1}$) for all $l \geq 0$.*

(b) *If ξ_i belongs to $\mathrm{Sp}(M_0^l \ominus S_l \mid l \geq 0)$, $i = 1, 2$, and $\langle \xi_1, \xi_2 \rangle_2 = 0$ then p_{ξ_1} and p_{ξ_2} are orthogonal.*

(c) *If $p = 2N - 1$, $\alpha_l = \dim(q_l \ominus P_{l-1})$, $l \geq 0$, then $\alpha_0 = 1$, $\alpha_1 = p$, $\alpha_2 = p^2 - p - 1$ and $\alpha_l = p^{l-3}(p-1)^2(p+1)$ for $l \geq 3$, In particular $q_l \ominus P_{l-1}$ is nonnull for all l.*

Proof. (b) is an easy consequence of (a) and of the preceding lemma. (a) is obvious for $l = 0$ since $P_0 = 0$ and in general it will be proved by induction according to l. Assuming that we have carried the induction up to l, we prove (a) for $(l+1)$ instead of l. First we prove that for each w in F_N, $|w| \leq l$,

$$q_{l+1}(\chi_p w \chi_q) \in S_{l+1} \quad \text{for all } p, q \geq 0. \tag{3}$$

By the induction hypothesis it is sufficient to show that for any $k \leq l$ and any γ in $M_0^k \ominus S_k$,

$$q_{l+1}(\chi_p \gamma \chi_q) \text{ belongs } S_{l+1} \quad \text{for all } p, q \geq 0.$$

This means (by Lemma 2) that we have to show that $\{\gamma_{p,q}\}_{-p+q=l+1-k}$ is contained in S_{l+1}. But this follows from the fact that whenever $p + q = l - k$, $q_{l+1}(\chi_1 \gamma_{p,q}) = \gamma_{p+1,q}$ and $q_{l+1}(\gamma_{p,q}\chi_1) = \gamma_{p,q+1}$ (by Lemma 1). Hence (3) is true and it follows that

$$\operatorname{range} q_{l+1}P_l \subseteq S_{l+1} \subseteq \operatorname{range} P_l \wedge q_{l+1}$$

(the second inclusion being obvious). Hence P_l commutes with q_{l+1} and S_{l+1} equals the range of $q_{l+1}P_l$.

To prove (c) note that by Lemma 2 for any γ in $M_0^l \ominus S_l$, $l \geq 1$, p_γ commutes with all q_n and $\dim(p_\gamma \wedge q_n) = n - l + 1$ for $n \geq l$ (and zero for $n < l$). Hence the following formula holds for $l \geq 1$:

$$\alpha_l = p^{l-1}(p+1) - (1 + l\alpha_1 + (l-1)\alpha_2 + \cdots + 2\alpha_{l-1})$$

and by an elementary induction argument we get (c).

The following computational lemma will be used only in the next lemma. Probably it is known but for the sake of completeness we include its proof here.

LEMMA 5. *Let a be a real number with $|a| < 1$. There are strictly positive numbers B, C depending on a such that for any integer $k \geq 0$ and any $\lambda_0, \dots, \lambda_k$ in $\mathbb{C}$*

$$B\Big(\sum |\lambda_i|^2\Big) \leq \sum_{i,j} \lambda_i \bar{\lambda}_j a^{|i-j|} \leq C\Big(\sum_i |\lambda_i|^2\Big).$$

Proof. Let A be the linear operator on $\mathbb{C}^{k+1}$ given by the matrix $a_{ij} = a^{|i-j|}$, $i, j = 0, 1, \dots, k$, and let N_0 be the nilpotent operator given by $n_{ij} = \delta_{i,j+1}$ and D the diagonal operator with only nonnull entries $d_{00} = d_{kk} = 1$. Then

$$A = I + a(N_0 + N_0^*) + \cdots + a^k(N_0^k + (N_0^*)^k)$$

and an elementary computation shows that A is invertible and

$$A^{-1} = (1-a^2)^{-1}((1+a^2)I - a(N_0 + N_0^*) - a^2 D).$$

Hence

$$\begin{aligned} ||A|| &\leq (1+|a|)(1-|a|)^{-1} = C, \\ ||A^{-1}|| &\leq (1-a^2)^{-1}(1+2|a|+2a^2) = B^{-1} \end{aligned}$$

and the lemma follows now from the inequality

$$||A^{-1}||^{-1}||\xi||^2 \leq \langle A\xi, \xi\rangle \leq ||A||\,||\xi||^2, \qquad \xi \in \mathbb{C}^{k+1}$$

which holds true since A is positive definite.

Finally start with nonnull vectors β in $M_0^1 \ominus S_1$ and γ in $M_0^l \ominus S_l$, $l \geq 2$. To prove that the cyclic projections p_β and p_γ have the same central support in $\mathscr{A}'$ it is clearly sufficient to show that the linear mapping T_0 defined by the requirement $T_0(\chi_n \beta \chi_m) = \chi_n \gamma \chi_m$ is well defined and extends to a bounded invertible operator from $\overline{\mathrm{Sp}(\chi_n \beta \chi_m)}^{\|\ \|_2}$ onto $\overline{\mathrm{Sp}(\chi_n \gamma \chi_m)}^{\|\ \|_2}$. This is done in the following lemma.

LEMMA 6. *Let β and γ be as in the statement of Lemma 2. Then the linear mapping T_0 on* $\mathrm{Sp}\{\chi_n \beta \chi_m\}$ *into* $\mathrm{Sp}\{\chi_n \gamma \chi_m\}$ *defined by $T_0(\chi_n \beta \chi_m) = \chi_n \gamma \chi_m$ is well defined and extends to a bounded linear* *operator from* $\overline{\mathrm{Sp}(\chi_n\beta\chi_m)}^{\|\ \|_2}$ *onto* $\overline{\mathrm{Sp}(\chi_n\gamma\chi_m)}^{\|\ \|_2}$. *In particular* p_β *and* p_γ *have the same central support in* $\mathscr{A}'$.

Proof. Let T_0': $\mathrm{Sp}(\chi_n\beta\chi_m) \mapsto \mathrm{Sp}(\chi_n\gamma\chi_m)$ be the linear map defined by requiring $T_0'\beta_{n,m} = \gamma_{n,m}$, and let S: $\mathrm{Sp}\{\chi_n\gamma\chi_m\} \to \{\mathrm{Sp}\,\chi_n\gamma\chi_m\}$ be defined by $S\gamma_{n,m} = \gamma_{n-1,m-1}$. By the preceding lemma and by Lemma 3, T_0' extends to a bounded invertible operator (also denoted by T_0') from $\overline{\mathrm{Sp}(\chi_n\beta\chi_m)}^{\|\ \|_2}$ into $\overline{\mathrm{Sp}\{\chi_n\gamma\chi_m\}}^{\|\ \|_2}$. Also by Lemma 3, we have $\|S\| \leq (2N-1)^{-2}$ so that $I_{\overline{\mathrm{Sp}\,\chi_n\gamma\chi_m}^{\|\ \|_2}} + S$ is also bounded and invertible. Hence the linear map (which is the composite map $T_0'(I+S)$)

$$\beta_{n,m} \mapsto \gamma_{n,m} + \varepsilon\gamma_{n-1,m-1}$$

extends to a bounded invertible operator from $\overline{\mathrm{Sp}\{\chi_n\beta\chi_m\}}^{\|\ \|_2}$ into $\overline{\mathrm{Sp}\{\chi_n\gamma\chi_m\}}^{\|\ \|_2}$ which by Lemma 2(b) and (d) coincides with T_0. This completes the proof of the lemma.

We can now state and prove our main result.

THEOREM 7. *Let* $A = \{X_1 + \cdots + X_N + X_1^{-1} + \cdots + X_N^{-1}\}''$ *be the radial algebra in* $\mathscr{L}(F_N)$. *Let* $\mathscr{A} = (A \vee JAJ)''$ *be the* (*abelian*) *von Neumann algebra generated in* $B(l^2(F_N))$ *by* A *and* JAJ. *Then* $\mathscr{A}' \subseteq B(l^2(F_N))$ *is of the homogeneous type* I_∞ *on* $I_{B(l^2(F_N))} - p_1$ (*where* p_1 *is the cyclic projection from* $l^2(F_N)$ *onto* $\overline{A}^{\|\ \|_2}$). *In particular* (*by* **[4]**) A *is singular.*

Proof. For each $l \geq 0$ take a basis $\{\xi_{i,l}\}_i$ of $M_0^l \ominus S_l$ with $\xi_{0,0} = 1 \in M_0^0$, and when $l = 1$ we have to choose the vectors $\xi_{i,1}$ to be like in Lemma 2 for some $\varepsilon = \pm 1$ (which is always possible). By Lemma 6 $\{p_{\xi_{i,l}}\}_{i,l}$ have all the same central support in $\mathscr{A}'$. By Lemma 4, $\sum_{l\geq 1} p_{\xi_{i,l}} = I - p_1$ and hence, since $p_1 \in \mathscr{A}$ (by Lemma 3.1 in **[4]**), it follows that the central support of $p_{\xi_{i,l}}$ in $\mathscr{A}'$ is $I - p_1$ for each $l \geq 1$. Since by Lemma 4 the family $\{p_{\xi_{i,l}}\}_{l\geq 1}$ is infinite, the theorem is proved.

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