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THE PENNEY-FUJIWARA PLANCHEREL FORMULA FOR ABELIAN SYMMETRIC SPACES AND COMPLETELY SOLVABLE HOMOGENEOUS SPACES

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A distribution theoretic version of the Plancherel formula for the decomposition of the quasi-regular representation of a Lie group G on $L^2(G/H)$ is presented. The formula is proven in two situations wherein the irreducible representations that occur in the decomposition are monomial. The intertwining operator that effects the decomposition is derived in terms of integral operators that arise from the distributions.

1. Introduction. We are concerned here with the quasi-regular representation $\tau = \mathrm{Ind}_H^G 1$ for G a connected Lie group and H a connected closed subgroup. The Orbit Method instructs us as to how to decompose such a representation into irreducibles. Indeed, if there is a "nice" orbital parameterization for the dual $\widehat{G}$, or at least for the part $\widehat{G}_H = \{\pi \in \widehat{G}: \pi$ is weakly contained in $\tau\}$ that supports the quasi-regular representation, then the Orbit Method [**11**] suggests the direct integral decomposition:

$$\tau = \int_{\mathfrak{h}^\perp/H}^{\oplus} \pi_\varphi \, d\dot{\varphi} = \int_{G\cdot\mathfrak{h}^\perp/G}^{\oplus} n_\varphi \pi_\varphi \, d\tilde{\varphi}\,, \tag{1.1}$$

where $\mathfrak{h}^\perp = \{\varphi \in \mathfrak{g}^*: \varphi(\mathfrak{h}) = 0\}$, $n_\varphi = \#[G\cdot\varphi \cap \mathfrak{h}^\perp]/H$, and $d\dot{\varphi}$, $d\tilde{\varphi}$ are push-forwards of Lebesgue measure. (Here $\mathfrak{g} = \mathrm{Lie}(G)$ and g^* is the real linear dual.) Such an orbital decomposition is valid if G is simply connected exponential solvable [**6**], [**13**], or G/H is abelian symmetric [**9**], [**11**], or Riemannian symmetric [**11**] (see [**11**] for other cases as well). The direct integral formula (1.1) is "soft" in that it provides the abstract representation theoretic decomposition of τ—i.e., it describes the spectrum, spectral multiplicity and spectral measure. But is is not "hard" in that it avoids the actual intertwining operator that effects the decomposition as well as the L^2 or L^1 convergence estimates usually necessary to derive the intertwining operator. Such analytic components are often needed in applications of the direct integral formula—e.g., to solvability of differential operators [**10**].

Penney [16] has shown how to formulate the direct integral decomposition more analytically in terms of C^∞ vectors and distributions. For nilpotent groups, and in the presence of finite multiplicity, Fujiwara [4] has cast Penney's theory into a concrete form by actually computing the analytic data. It is my goal to extend Fujiwara's work beyond the confines of nilpotent groups. In this paper I carry out that intention for the two cases: (i) G/H is abelian symmetric, and (ii) G is completely solvable and τ has finite multiplicity. More precisely, for these two cases, I shall compute the Penney distributions that correspond to the representations which appear in the spectrum, the actual Penney-Fujiwara Plancherel formula (PFPF), and the explicit intertwining operator that effects the direct integral decomposition (1.1).

The key feature that the two cases have in common is that the irreducible representations π which appear in the spectrum of the quasi-regular representation are monomial—that is, induced from a character $\pi = \mathrm{Ind}_B^G \chi$. The Hilbert space of such a representation can be realized in

$$\begin{aligned}\mathscr{H}_\chi &= L^2(G, B, \chi) \\ &= \left\{ f\colon G \to \mathbb{C},\ f(bg) = \chi(b)f(g),\ \int_{B\backslash G} |f(g)|^2 \, d\dot{g} < \infty \right\},\end{aligned}$$

where $d\dot{g}$ is a quasi-invariant measure on $B\backslash G$. Motivated by [5], [12], I assert that the Penney distributions are given on $\mathscr{H}_\chi$ by certain integrals over $(H \cap B)\backslash H$—see formulas (2.2) or (3.6). But the integrals are not evidently convergent for $f \in \mathscr{H}_\chi$ which are not compactly supported mod B. Nevertheless, we are able to prove that the integrals extend from compactly supported smooth functions (mod B) to (relatively invariant) distributions on the space $\mathscr{H}_\chi^\infty$ of C^∞ vectors. We then use them to obtain a distribution-theoretic version of the Plancherel formula for τ. The precise details are too complicated to state in these introductory remarks. The reader will find: the distributions in formulas (2.2) or (3.6); the Penney-Fujiwara Plancherel formula in Proposition 3.3 and Theorems 3.4, 4.1, 5.1; and the explicit intertwining operator in Proposition 3.2 and formulas (3.7) and (3.8).

Here is an outline of the paper. In §2 I describe in detail the integrals which figure in the definition of the relatively invariant distributions that correspond to the irreducibles in the spectrum of the quasi-regular representation. I derive the smoothness properties and compute the matrix coefficients (Theorem 2.1) for these distributions. I also compute the matrix coefficient of the canonical cyclical distribution for the representation τ (Prop. 2.2). All of this is done in the greatest possible generality—almost no assumptions are placed on G, H. In §3 I describe the general distribution theoretic formulation of the Plancherel formula, based on the Penney-Fujiwara model. I also describe the resulting intertwining operator (for the decomposition of the quasi-regular representation). I show (Proposition 3.3) that it is enough to work with positive definite test functions to guarantee convergence in the Plancherel formula. I then describe in detail the two categories of spaces considered in this paper, and I state the main result (Theorem 3.4). The categories are abelian symmetric spaces and finite multiplicity completely solvable homogeneous spaces. The main result describes the Penney distributions and the Penney-Fujiwara Plancherel formula for these simultaneously. The proof of the main result is carried out for abelian symmetric spaces in §4 (Theorem 4.1), and for finite multiplicity completely solvable homogeneous spaces in §5 (Theorem 5.1). The former proof is by direct computation, the latter by induction on $\dim G/H$.

It is worth noting that in **[12]** I have proven Theorem 3.4 for spaces which are both abelian symmetric and algebraic completely solvable. That was quite special, but the results there provide excellent inspiration for the theorems in this paper. Finally, I anticipate that the results of this paper might generalize to other categories of homogeneous spaces G/H. Three I have in mind are: more general exponential solvable spaces; the Grassmannian bundles considered by Strichartz in **[18]**; and G/H where G is real algebraic, $G = HU$ is a Levi decomposition with U unipotent and H reductive.

2. C^∞ vectors, direct integrals and monomial representations. Let G be a Lie group and H a closed subgroup. Fix a choice of right Haar measures dg, dh on G and H. We write Δ_G, Δ_H for the modular functions of G, H respectively (i.e., the derivative of right Haar measure with respect to left). We set $\Delta_{H,G} = \Delta_H/\Delta_G$ a positive character on H. If χ is a unitary character of H, the induced representation $\pi_\chi = \operatorname{Ind}_H^G \chi$ acts in the space

$$C_c^\infty(G, H, \chi) = \{f \in C^\infty(G) \colon f(hg) = \chi(h)f(g),\ h \in H,\ g \in G,\\ |f| \text{ compactly supported } \operatorname{mod} H\}$$

by the formula

$$\pi_\chi(g)f(x) = f(xg)[q(xg)/q(x)]^{1/2}. \tag{2.1}$$

Here q is a smooth function on G satisfying $q(e) = 1$, $q(hg) = \Delta_{H,G}(h)q(g)$. When G is exponential solvable, it is known that q must satisfy the formula

$$q(\exp X) = e^{\operatorname{tr}\, \operatorname{ad}_{\mathfrak{g}/\mathfrak{h}} X}, \qquad X \in \mathfrak{h}$$

(see [1, p. 96]). (Also if the groups are in doubt, I will write $q = q_{H,G}$.) The action (2.1) extends to a unitary representation. In fact, there is a quasi-invariant measure $d\dot{g}$ on $H\backslash G$ defined as follows. Any $f \in C_c(G, H)$ can be written

$$f(g) = \int_H F(hg)\, dh, \qquad F \in C_c(G);$$

then

$$\int_{H\backslash G} f(g) d\dot{g} \overset{(\text{def})}{=} \int_G F(g) q(g)\, dg.$$

The formula (2.1) defines the unitary action of G on $L^2(H\backslash G, d\dot{g}) = L^2(G, H, \chi)$. For all this see [8]. We note for future reference that the quasi-invariant measure $d\dot{g}$ is *relatively* invariant iff q extends to a continuous positive character on all of G, and in that case the modulus for the action is precisely q^{-1}. (Again, see [8].) Finally, we recall that—having fixed right Haar measures on G, H—the functions q and the quasi-invariant measures $d\dot{g}$ are in 1-1 correspondence, each uniquely specifies the other by the above procedure. Given one q, any other is determined by multiplication $q\rho$, $\rho \in C_1(G, H) = \{f \in C(G, H),\ f(e) = 1\}$. In what follows, given a Lie group G and a closed subgroup H, we assume that right Haar measures dg, dh and one $q = q_{H,G}$ have been chosen and fixed throughout.

Now suppose π is a unitary representation of G on a Hilbert space $\mathscr{H}_\pi$. We write $\mathscr{H}_\pi^\infty$ to denote the Fréchet space of C^∞ vectors of π. Its antidual space is denoted $\mathscr{H}_\pi^{-\infty}$. Each of $\mathscr{H}_\pi^\infty$, $\mathscr{H}_\pi^{-\infty}$ is acted upon by G, therefore also by $\mathscr{D}(G) = C_c^\infty(G)$. It is well known that

$$\pi(\mathscr{D}(G))\mathscr{H}_\pi^{-\infty} \subset \mathscr{H}_\pi^\infty.$$

Next suppose τ is a type I representation of G which is realized as a direct integral of irreducible unitary representations

$$\tau = \int_{\mathscr{S}}^{\oplus} \pi \, d\mu(\pi).$$

We recall some well-known facts due to Goodman [7] and Penney [**16**]. Namely we have

$$\mathscr{H}_\tau^\infty = \int_{\mathscr{S}}^{\oplus} \mathscr{H}_\pi^\infty \, d\mu(\pi), \qquad \mathscr{H}_\tau^{-\infty} = \int_{\mathscr{S}}^{\oplus} \mathscr{H}_\pi^{-\infty} \, d\mu(\pi).$$

(See [**16**, p. 180] for the precise formulation of these decompositions.) We now carry out a refinement of this scenario which will be applicable in the case that (almost) all of the π's are monomial.

Suppose that for μ-a.a π there exists a locally convex topological space $(\mathscr{H}_\pi)_c^\infty$ which lies inside $\mathscr{H}_\pi^\infty$, is dense there, and whose topology is finer than the relative topology. If we denote the antidual by $(\mathscr{H}_\pi)_c^{-\infty}$, then we have natural inclusions

$$(\mathscr{H}_\pi)_c^\infty \subset \mathscr{H}_\pi^\infty \subset \mathscr{H}_\pi^{-\infty} \subset (\mathscr{H}_\pi)_c^{-\infty},$$

and each embedding is continuous. Now in general, although $\pi(\mathscr{D}(G))\mathscr{H}_\pi^{-\infty} \subset \mathscr{H}_\pi^\infty$, we cannot expect $\pi(\mathscr{D}(G))(\mathscr{H}_\pi)_c^{-\infty} \subset (\mathscr{H}_\pi)_c^\infty$. However, for monomial representations and for certain distributions $\beta \in (\mathscr{H}_\pi)_c^{-\infty}$ (given by integrals with respect to a quasi-invariant measure), we shall show that $\pi(\mathscr{D}(G))(\beta) \in \mathscr{H}_\pi^{-\infty}$. This will be sufficient for our purposes. So suppose that (almost) all of the π's are monomial $\pi = \operatorname{Ind}_B^G \chi$, B a closed subgroup of G, χ a unitary character of B. We focus attention on *one* of these temporarily. We already saw how π is realized in $L^2(G, B, \chi)$. (This assumes right Haar measure db and $q_{B,G}$ have been chosen.) By [**17**] we know that $L^2(G, B, \chi)^\infty \subset C^\infty(G, B, \chi)$. It is also evident that $C_c^\infty(G, B, \chi) \subset L^2(G, B, \chi)^\infty$. The space $C_c^\infty(G, B, \chi)$ shall play the role of $(\mathscr{H}_\pi)_c^\infty$ in what follows.

Now we describe the natural distributions that arise in the Penney-Fujiwara Plancherel formula (PFPF). Let H be a closed subgroup of G for which $\chi|_{H\cap B} = 1$. We make two additional assumptions:

(I) BH is closed in G;

(II) $q_{H\cap B, H} q_{H\cap B, B} \equiv 1$ on $H \cap B$.

Because of (I) any $f \in C_c(G)$ satisfies $f|_{BH} \in C_c(BH)$. Hence $f \to f|_H$ projects $C_c^\infty(G, B, \chi)$ to $C_c^\infty(H, H \cap B)$. Fix a right Haar measure on $H \cap B$. Let $d\dot{h}$ denote a quasi-invariant measure on $H \cap B \backslash H$. Then the distributions $\beta \in (\mathscr{H}_\pi)_c^{-\infty}$ we are interested in are exactly

$$\beta : f \to \int_{H\cap B\backslash H} \overline{f} q_{B,G}^{1/2} q_{H\cap B,H}^{-1} q_{H,G}^{-1/2}\, d\dot{h}, \qquad f \in C_c^\infty(G, B, \chi). \tag{2.2}$$

THEOREM 2.1. (i) *We have the following identity of q functions on $H \cap B$*

$$q_{B,G}^{1/2} q_{H\cap B,H}^{-1} q_{H,G}^{-1/2} = q_{H\cap B,H}^{-1/2} q_{H\cap B,B}^{-1/2}.$$

Therefore by assumption (II), *β is well defined.*

(ii) *β is relatively invariant under the action of H with modulus $q_{H,G}^{-1/2} = (\Delta_{H,G})^{-1/2}$.*

(iii) $\pi(\mathscr{D}(G))\beta \subset C^\infty(G, B, \chi) \subset \mathscr{H}_\pi^{-\infty}$.

(iv) *In fact, for $\omega \in \mathscr{D}(G)$, this function is given by the formula*

$$\pi(\omega)\beta(g) = \int_{H\cap B\backslash B} \omega_H(bg)\overline{\chi(b)} q_{B,G}^{-1/2}(bg) q_{H,G}^{1/2}(bg) q_{H\cap B,B}^{-1}(b)\, d\dot{b},$$

where

$$\omega_H(g) = \Delta_G(g)^{-1} q_{H,G}^{-1/2}(g) \int_H \omega(g^{-1}h^{-1}) \Delta_G(h)^{-1} q_{H,G}^{-1/2}(h)\, dh.$$

(v) *The matrix coefficient of β is*

$$\begin{aligned}\langle \pi(\omega)\beta, \beta\rangle = \int_{H\cap B\backslash H} \int_{H\cap B\backslash B} & \omega_H(bh)\overline{\chi}(b) \\ & \cdot q_{B,G}^{-1/2}(b) q_{H,G}^{1/2}(h^{-1}bh) q_{H\cap B,B}^{-1}(b) q_{H\cap B,H}^{-1}(h)\, d\dot{b}\, d\dot{h}, \\ & \qquad \omega \in \mathscr{D}^+(G),\end{aligned}$$

where $\mathscr{D}^+(G)$ = positive linear combinations of functions of the form $\omega = \omega_1^* * \omega_1$, $\omega_1 \in \mathscr{D}(G)$. The matrix coefficient is a non-negative number, possibly equal to $+\infty$.

Proof. (i) This follows from the general fact that if $K \subset H \subset G$, all closed subgroups, then on K we have

$$q_{K,G} = \frac{\Delta_K}{\Delta_G} = \frac{\Delta_K}{\Delta_H}\frac{\Delta_H}{\Delta_G} = q_{K,H} q_{H,G}.$$

Therefore

$$q_{B,G}^{1/2}q_{H,G}^{-1/2} = q_{H\cap B,G}^{1/2}q_{H\cap B,B}^{-1/2}q_{H\cap B,G}^{-1/2}q_{H\cap B,H}^{1/2} = q_{H\cap B,B}^{-1/2}q_{H\cap B,H}^{1/2}.$$

Hence

$$q_{B,G}^{1/2}q_{H\cap B,H}^{-1}q_{H,G}^{1/2} = q_{H\cap B,B}^{-1/2}q_{H\cap B,H}^{-1/2}.$$

Incidentally another analogous fact which we shall use, and whose demonstration we leave to the reader, is the integral equation

$$\begin{aligned}(2.3)\quad \int_{K\backslash G} f(x)\,d\dot{x} = \int_{H\backslash G}\int_{K\backslash H} & f(hg)q_{K,G}(hg)q_{H,G}(hg)^{-1} \\ &\cdot q_{K,H}(h)^{-1}\,d\dot{h}\,d\dot{g}, \qquad f\in C_c(G,K).\end{aligned}$$

I also leave it to the reader to check that since $q_{H,G}$, $q_{B,G}$ have already been chosen, and since $q_{H\cap B,H}$ is paired with $d\dot{h}$, the integral (2.2) is independent of the choice of $d\dot{h}$. But of course it depends on the choice of the Haar measures on B and $H\cap B$.

$$\begin{aligned}\text{(ii)}\quad \langle\pi(h)\beta, f\rangle &= \langle\beta, \pi(h)^{-1}f\rangle = \langle\beta, f(\cdot h^{-1})[q_{B,G}(\cdot h^{-1})/q_{B,G}(\cdot)]^{1/2}\rangle \\ &= \int_{H\cap B\backslash H}\overline{f}(h_1h^{-1})\frac{q_{B,G}^{1/2}(h_1h^{-1})}{q_{B,G}^{1/2}(h_1)} \\ &\qquad\qquad \cdot q_{B,G}^{1/2}(h_1)q_{H\cap B,H}^{-1}(h_1)q_{H,G}^{-1/2}(h_1)\,d\dot{h}_1 \\ &= \int_{H\cap B\backslash H}\overline{f}(h_1h^{-1})q_{B,G}^{1/2}(h_1h^{-1})q_{H\cap B,H}^{-1}(h_1)q_{H,G}^{-1/2}(h_1)\,d\dot{h}_1 \\ &= \int_{H\cap B\backslash H}\overline{f}(h_1)q_{B,G}^{1/2}(h_1) \\ &\qquad\qquad \cdot q_{H\cap B,H}^{-1}(h_1h)q_{H,G}^{-1/2}(h_1h)\frac{q_{H\cap B,H}(h_1h)}{q_{H\cap B,H}(h_1)}\,d\dot{h}_1 \\ &= \int_{H\cap B\backslash H}\overline{f}(h_1)q_{B,G}^{1/2}(h_1)q_{H\cap B,H}^{-1}(h_1)q_{H,G}^{-1/2}(h_1h)\,d\dot{h}_1 \\ &= q_{H,G}^{-1/2}(h)\langle\beta, f\rangle.\end{aligned}$$

(iii) Let $\omega \in \mathscr{D}(G)$, $\psi \in \mathscr{H}_\pi^\infty$. As usual, the "adjoint" function is $\omega^*(g) = \overline{\omega}(g^{-1})\Delta_G(g)^{-1}$. Then

$$\begin{aligned}
\langle \pi(\omega)\beta, \psi\rangle &= \langle \beta, \pi(\omega^*)\psi\rangle = \int_G \omega(g^{-1})\Delta_G(g)^{-1}\langle \beta, \pi(g)\psi\rangle\, dg \\
&= \int_G \omega(g^{-1})\Delta_G(g)^{-1} \int_{H\cap B\backslash H} \overline{\psi}(hg)\left[\frac{q_{B,G}(hg)}{q_{B,G}(h)}\right]^{1/2} \\
&\qquad\qquad \cdot q_{B,G}^{1/2}(h) q_{H\cap B,H}^{-1}(h) q_{H,G}^{-1/2}(h)\, d\dot{h}\, dg \\
&= \int_G \omega(g^{-1})\Delta_G(g)^{-1} \int_{H\cap B\backslash H} \overline{\psi}(hg) q_{B,G}^{1/2}(hg) q_{H\cap B,H}^{-1}(h) \\
&\qquad\qquad \cdot q_{H,G}^{1/2}(h)\, d\dot{h}\, dg \\
&= \int_{H\backslash G}\int_H \omega(g^{-1}h_1^{-1})\Delta_G(h_1 g)^{-1} \\
&\qquad \cdot q_{H,G}^{-1}(h_1 g) \int_{H\cap B\backslash H} \overline{\psi}(hh_1 g) q_{B,G}^{1/2}(hh_1 g) \\
&\qquad\qquad \cdot q_{H\cap B,H}^{-1}(h) q_{H,G}^{-1/2}(h)\, d\dot{h}\, dh_1\, d\dot{g} \\
&= \int_{H\backslash G}\int_H \omega(g^{-1}h_1^{-1})\Delta_G(h_1 g)^{-1} q_{H,G}^{-1}(h_1 g) \\
&\quad \cdot \int_{H\cap B\backslash H} \overline{\psi}(hg) q_{B,G}^{1/2}(hg) q_{H\cap B,H}^{-1}(hh_1^{-1}) \cdot \\
&\qquad\qquad \cdot q_{H,G}^{-1/2}(hh_1^{-1}) \frac{q_{H\cap B,H}(hh_1^{-1})}{q_{H\cap B,H}(h)}\, d\dot{h}\, dh_1\, d\dot{g} \\
&= \int_{H\backslash G}\int_H \omega(g^{-1}h_1^{-1})\Delta_G(h_1 g)^{-1} q_{H,G}^{-1}(h, g) \\
&\quad \cdot \int_{H\cap B\backslash H} \overline{\psi}(hg) q_{B,G}^{1/2}(hg) q_{H\cap B,H}^{-1}(h) q_{H,G}^{-1/2}(hh_1^{-1})\, d\dot{h}\, dh_1\, d\dot{g} \\
&= \int_{H\backslash G}\int_H \omega_H(g)\overline{\psi}(hg) q_{B,G}^{1/2}(hg) q_{H\cap B,H}^{-1}(h) q_{H,G}^{-1/2}(hg)\, d\dot{h}\, d\dot{g},
\end{aligned}$$

where

$$\omega_H(g) = \Delta_G(g)^{-1} q_{H,G}^{-1/2}(g) \int_{H\cap B\backslash H} \omega(g^{-1}h_1^{-1})\Delta_G(h_1)^{-1} q_{H,G}^{-1/2}(h_1)\, dh_1 .$$

Note that ω_H is left H-invariant. We use that and equation (2.3) twice as we continue

$$\begin{aligned}
\langle \pi(\omega)\beta\,,\,\psi\rangle &= \int_{H\backslash G}\int_{H\cap B\backslash H} \omega_H(hg)\overline{\psi}(hg) \\
&\qquad\qquad \cdot q_{B,G}^{1/2}(hg)q_{H\cap B,H}^{-1}(h)q_{H,G}^{-1/2}(hg)\,dh\,d\dot{g} \\
&= \int_{H\cap B\backslash G} \omega_H(g)\overline{\psi}(g)q_{B,G}^{1/2}(g)q_{H,G}^{-1/2}(g)q_{H\cap B,G}^{-1}(g)q_{H,G}(g)\,d\dot{g} \\
&= \int_{H\cap B\backslash G} \omega_H(g)\overline{\psi}(g)q_{B,G}^{1/2}(g)q_{H\cap B,G}^{-1}(g)q_{H,G}^{1/2}(g)\,d\dot{g} \\
&= \int_{B\backslash G}\int_{H\cap B\backslash B} \omega_H(bg)\overline{\psi}(bg)q_{B,G}^{1/2}(bg)q_{H\cap B,G}^{-1}(bg)q_{H,G}^{1/2}(bg)\cdot \\
&\qquad\qquad \cdot q_{H\cap B,G}(bg)q_{B,G}^{-1}(bg)q_{H\cap B,B}^{-1}(b)\,d\dot{b}\,d\dot{g} \\
&= \int_{B\backslash G}\int_{H\cap B\backslash B} \omega_H(bg)\overline{\chi}(b)\overline{\psi}(g) \\
&\qquad\qquad \cdot q_{B,G}^{-1/2}(bg)q_{H\cap B,B}^{-1}(b)q_{H,G}^{1/2}(bg)\,d\dot{b}\,d\dot{g}\,.
\end{aligned}$$

Therefore the distribution—that is, the function—$\pi(\omega)\beta$ is given by

$$\pi(\omega)\beta(g) = \int_{H\cap B\backslash B} \omega_H(bg)\overline{\chi}(b)q_{B,G}^{-1/2}(bg)q_{H\cap B,B}^{-1}(b)q_{H,G}^{1/2}(bg)d\dot{b}\,.$$

Note the integrand is left $(H\cap B)$-invariant because: ω_H is left H-invariant, $\chi|_{H\cap B} = 1$, and assumption (II) and part (i) apply. We leave to the reader the verification that the integrand is compactly supported $\operatorname{mod} H\cap B$, and that the function $\pi(\omega)\beta$ transforms on the left under B by the character χ. It is obvious that $\pi(\omega)\beta$ is a C^∞ function. This completes the proof of (iii) and (iv).

(v) Suppose finally that $\omega\in\mathscr{D}(G)$ is of the form $\omega=\omega_1^* * \omega_1$, $\omega_1\in\mathscr{D}(G)$. Then, since $\pi(\omega)=\pi(\omega_1)^*\pi(\omega_1)$ and $\pi(\omega_1)\beta$ is a C^∞ function, it is evident that $\langle\pi(\omega)\beta\,,\,\beta\rangle$ is well defined by the expression

$$\langle\pi(\omega)\beta\,,\,\beta\rangle = \langle\pi(\omega_1)\beta\,,\,\pi(\omega_1)\beta\rangle = \int_{B\backslash G} |\pi(\omega_1)\beta(g)|^2\,d\dot{g}\,.$$

It is a non-negative number, possibly equal to $+\infty$. But, approximating $\pi(\omega_1)\beta$ by $C_c^\infty(G\,,B\,,\chi)$ functions if necessary, we can also write

$$\begin{aligned}\langle \pi(\omega)\beta\,,\,\beta\rangle &= \overline{\langle \beta\,,\,\pi(\omega)\beta\rangle}\\ &= \int_{H\cap B\backslash H}\int_{H\cap B\backslash B} \omega_H(bh)\overline{\chi}(b)q_{B,G}^{-1/2}(bh)q_{H\cap B,B}^{-1}(b)q_{H,G}^{1/2}(bh)\\ &\qquad\qquad \cdot q_{B,G}^{1/2}(h)q_{H\cap B,H}^{-1}(h)q_{H,G}^{-1/2}(h)\,d\dot b\,d\dot h\\ &= \int_{H\cap B\backslash H}\int_{H\cap B\backslash B} \omega_H(bh)\overline{\chi}(b)q_{B,G}^{-1/2}(b)q_{H,G}^{1/2}(h^{-1}bh)\\ &\qquad\qquad \cdot q_{H\cap B,B}^{-1}(b)q_{H\cap B,H}^{-1}(h)\,d\dot b\,d\dot h.\end{aligned}$$

Note. As in (i), the values of the function $\pi(\omega)\beta$ and the matrix coefficient $\langle \pi(\omega)\beta\,,\,\beta\rangle$ are independent of the choice of the quasi-invariant measures $d\dot b$, $d\dot h$ but they do depend on the original choices of Haar measure on G, H, B and $H\cap B$.

REMARK. It is important to observe that Theorem 2.1 is proven with no structural assumption on G. The only conditions are assumptions (I) and (II) and the equation $\chi|_{H\cap B} = 1$.

Now we shall compute the matrix coefficient for the canonical cyclic distribution in an arbitrary quasi-regular representation. So suppose G is any Lie group. $H \subset G$ any closed subgroup. Consider the quasi-regular representation $\tau = \mathrm{Ind}_H^G 1$ acting on $\mathscr{H}_\tau = L^2(G, H)$. The canonical cyclic distribution $\alpha_\tau \in \mathscr{H}_\tau^{-\infty}$ is given by

$$\alpha_\tau\colon f \to \overline{f(e)}.$$

That α_τ is cyclic means that $\alpha_\tau(\tau(g)f) = 0 \ \forall g \in G \Rightarrow f = 0$ (see **[16]**). It follows from **[17]** that $\alpha_\tau \in \mathscr{H}_\tau^{-\infty}$. Thus $\tau(\mathscr{D}(G))\alpha_\tau \in \mathscr{H}_\tau^{\infty}$, and $\langle \tau(\omega)\alpha_\tau\,,\,\alpha_\tau\rangle$ is well defined and finite for any $\omega \in \mathscr{D}(G)$.

PROPOSITION 2.2. (i) *α_τ is relatively invariant under the action of H with modulus $q_{H,G}^{-1/2}$.*

(ii) $\tau(\omega)\alpha_\tau(g) = \omega_H(g)$, *where ω_H is as in Theorem* 2.1, *namely*

$$\omega_H(g) = \Delta_G(g)^{-1}q_{H,G}^{-1/2}(g)\int_H \omega(g^{-1}h^{-1})\Delta_G(h)^{-1} \cdot q_{H,G}^{-1/2}(h)\,dh, \qquad \omega \in \mathscr{D}(G).$$

(iii) $\langle \tau(\omega)\alpha_\tau\,,\,\alpha_\tau\rangle = \omega_H(e) = \int_H \omega(h)q_{H,G}^{-1/2}(h)\,dh$.

Proof.

(i) $$\langle \tau(h)\alpha_\tau, f\rangle = \langle \alpha_\tau, \tau(h)^{-1} f\rangle = \left\langle \alpha, f(\cdot h^{-1}) \left[\frac{q_{H,G}(\cdot h^{-1})}{q_{H,G}(\cdot)}\right]^{1/2} \right\rangle$$
$$= \overline{f}(h^{-1}) q_{H,G}^{-1/2}(h) = \overline{f(e)} q_{H,G}^{-1/2}(h) = q_{H,G}^{-1/2}(h)\langle \alpha_\tau, f\rangle .$$

(ii) $$\langle \tau(\omega)\alpha_\tau, \psi\rangle = \langle \alpha_\tau, \tau(\omega^*)\psi\rangle$$
$$= \left\langle \alpha_\tau, \int_G \overline{\omega}(g^{-1})\Delta_G(g)^{-1}\tau(g)\psi\, dg \right\rangle$$
$$= \left\langle \alpha_\tau, \int_{H\backslash G}\int_H \overline{\omega}(g^{-1}h^{-1})\Delta_G(gh)^{-1} \cdot q_{H,G}^{-1}(hg)\tau(hg)\psi\, dh\, d\dot{g} \right\rangle$$
$$= \left\langle \alpha_\tau, \int_{H\backslash G}\int_H \overline{\omega}(g^{-1}h^{-1})\Delta_G(hg)^{-1} q_{H,G}^{-1}(hg)\psi(\cdot hg) \cdot \left[\frac{q_{H,G}(\cdot hg)}{q_{H,G}(\cdot)}\right]^{1/2} dh\, d\dot{g} \right\rangle$$
$$= \int_{H\backslash G}\int_H \omega(g^{-1}h^{-1})\Delta_G(hg)^{-1} \cdot q_{H,G}^{-1}(hg)\psi(hg) q_{H,G}^{1/2}(hg)\, dh\, d\dot{g}$$
$$= \int_{H\backslash G}\int_H \omega(g^{-1}h^{-1})\Delta_G(hg)^{-1} q_{H,G}^{-1/2}(hg)\psi(g)\, dh\, d\dot{g}.$$

Therefore,

$$\tau(\omega)\alpha_\tau(g) = \int_H \omega(g^{-1}h^{-1})\Delta_G(hg)^{-1} q_{H,G}^{-1/2}(hg)\, dh = \omega_H(g).$$

(iii) $$\langle \tau(\omega)\alpha_\tau, \alpha_\tau\rangle = \overline{\langle \alpha_\tau, \tau(\omega)\alpha_\tau\rangle} = \omega_H(e)$$
$$= \int_H \omega(h^{-1})\Delta_G(h)^{-1} q_{H,G}^{-1/2}(h)\, dh$$
$$= \int_H \omega(h^{-1})\Delta_G^{-1/2}(h)\Delta_H^{-1/2}(h)\, dh$$
$$= \int_H \omega(h^{-1}) q_{H,G}^{-1/2}(h^{-1})\Delta_H(h)^{-1}\, dh$$
$$= \int_H \omega(h) q_{H,G}^{-1/2}(h)\, dh .$$

3. The Penney-Fujiwara Plancherel formula. We continue with G a Lie group and $H \subset G$ a closed subgroup. We suppose we have a direct integral decomposition of the quasi-regular representation $\tau = \mathrm{Ind}_H^G 1$. That is, suppose τ is type I and

$$\tau = \int_{\mathscr{S}}^{\oplus} n_\tau(\pi)\pi \, d\mu_\tau(\pi),$$

where μ_τ is a Borel measure on $\widehat{G}$, $n_\tau(\pi)$ is a multiplicity function, and $\mathscr{S} \subset \widehat{G}$ is a minimal closed μ_τ-co-null subset. If $\pi \in \widehat{G}$, we shall write $(\mathscr{H}_\pi^{-\infty})^{H, q^{-1/2}}$ for the subspace of distribution vectors which transform under H by $q_{H,G}^{-1/2}$. A fundamental result of Penney is that $n_\tau(\pi) \leq \dim(\mathscr{H}_\pi^{-\infty})^{H, q^{-1/2}}$ **[16]**. Motivated by **[16]** and **[4]**, we ask the following

Questions 3.1. (i) When is $n_\tau(\pi) = \dim(\mathscr{H}_\pi^{-\infty})^{H, q^{-1/2}}$? In particular, when is it true that $(\mathscr{H}_\pi^{-\infty})^{H, q^{-1/2}} = \{0\}$ if π does not lie in the support of τ?

(ii) How does one produce $n_\tau(\pi)$ linearly independent $\alpha_\pi^1, \ldots, \alpha_\pi^{n_\tau(\pi)} \in (\mathscr{H}_\pi^{-\infty})^{H, q^{-1/2}}$ which satisfy

(iii) $\langle \tau(\omega)\alpha_\tau, \alpha_\tau \rangle = \int_{\mathscr{S}} \sum_{j=1}^{n_\tau(\pi)} \langle \pi(\omega)\alpha_\pi^j, \alpha_\pi^j \rangle \, d\mu_\tau(\pi)$, $\omega \in \mathscr{D}(G)$? (PFPF)

It is an observation of mine **[12]** that the map $\tau(\omega)\alpha \to \{\pi(\omega)\alpha_\pi^j\}$ must be an isometry that intertwines τ and the direct integral. But

(iv) is it surjective? i.e., is it an intertwining operator?

I have listed (ii) and (iii) as separate problems because the distributions α_π^j are usually (see §2) given by integrals which are not obviously (or actually) convergent. Thus the problem of making sense of the α_π^j is very different from that of actually proving the PFPF. Regarding item (i), this is a very subtle and difficult issue. Actually very little is known and much of that is negative. I shall not be concerned with it here. What I shall be concerned with is two categories of homogeneous spaces G/H for which one knows the direct integral decomposition of the quasi-regular representation—at least in the soft sense. In both of these situations, the multiplicity function is finite-valued a.e. In that case we have

PROPOSITION 3.2. *The answer to Question* 3.1 (iv) *is automatically yes in the presence of finite multiplicity.*

Proof. This follows instantly from the fact that a finite representation (i.e., a type I representation whose multiplicity is finite a.e.)

cannot be unitarily equivalent to a subrepresentation of itself—a simple consequence of **[15]**.

It follows that the answers to questions (ii) and (iii) provide the *hard* data for the decomposition of the quasi-regular representation $\tau = \operatorname{Ind}_H^G 1$. In fact, I shall write the PFPF slightly differently from (iii) above. Therein it is assumed (implicitly) that each distinct irreducible representation class π is realized on a single Hilbert space $\mathscr{H}_\pi$ and the $n_\tau(\pi)$ distributions α_π^j, $1 \le j \le n_\tau(\pi)$, are all defined on $\mathscr{H}_\pi^\infty$. For my purposes it will be more convenient to assume a direct integral decomposition

$$\tau = \int_{\mathscr{S}}^{\oplus} \pi \, d\mu(\pi) \tag{3.1}$$

where a.a. π are irreducible, but (finitely many) equivalencies (per π) are allowed. The PFPF then takes the simpler form

$$\langle \tau(\omega)\alpha_\tau, \alpha_\tau \rangle = \int_{\mathscr{S}} \langle \pi(\omega)\alpha_\pi, \alpha_\pi \rangle \, d\mu(\pi),$$

where each distribution α_π is realized on the C^∞ vectors of the Hilbert space associated to π. Different, but equivalent π, may be realized on different Hilbert spaces.

Now before moving on to the two categories of homogeneous spaces for which we derive the PFPF in this paper, I need to make two more important observations. First, if we apply Penney's results **[16]** to (3.1), we see there must exist distributions $\alpha_\pi \in \mathscr{H}_\pi^{-\infty}$ such that

$$\alpha_\tau = \int_{\mathscr{S}}^{\oplus} \alpha_\pi \, d\mu(\pi). \tag{3.2}$$

Then, for any $\omega \in \mathscr{D}(G)$, it is obvious that the direct integral decomposition of the C^∞ vector $\tau(\omega)\alpha_\tau$ is given by

$$\tau(\omega)\alpha_\tau = \int_{\mathscr{S}}^{\oplus} \pi(\omega)\alpha_\pi \, d\mu(\pi).$$

Consequently we obtain

$$\langle \tau(\omega)\alpha_\tau, \alpha_\tau \rangle = \int_{\mathscr{S}}^{\oplus} \langle \pi(\omega)\alpha_\pi, \alpha_\pi \rangle \, d\mu(\pi). \tag{3.3}$$

Conversely, given the PFPF in the form (3.3), it follows immediately from the cyclicity of α_τ that equation (3.2) is valid. That is the first observation.

The second is the following. Suppose we can only prove the PFPF with distributions $\alpha_\pi^c \in (\mathscr{H}_\pi)_c^{-\infty}$ which satisfy $\pi(\mathscr{D}(G))\alpha_\pi^c \subset \mathscr{H}_\pi^{-\infty}$, and for positive definite test functions:

$$\langle \tau(\omega)\alpha_\tau, \alpha_\tau\rangle = \int_{\mathscr{S}} \langle \pi(\omega)\alpha_\pi^c, \alpha_\pi^c\rangle \, d\mu(\pi), \qquad \omega \in \mathscr{D}^+(G). \tag{3.4}$$

(The integrand is defined by

$$\langle \pi(\omega)\alpha_\pi^c, \alpha_\pi^c\rangle = \begin{cases} \|\pi(\omega_1)\alpha_\pi^c\|^2 & \text{if } \omega = \omega_1^* * \omega_1, \\ & \text{and } \pi(\omega_1)\alpha_\pi^c \in \mathscr{H}_\pi \\ +\infty & \text{otherwise.)} \end{cases}$$

Implicit in the assumption is that for any fixed $\omega \in \mathscr{D}^+(G)$, the integrand is finite for a.a. π. Thus for any $\omega \in \mathscr{D}(G)$, we have

$$\langle \tau(\omega^* * \omega)\alpha_\tau, \alpha_\tau\rangle = \int_{\mathscr{S}} \langle \pi(\omega^* * \omega)\alpha_\pi^c, \alpha_\pi^c\rangle \, d\mu(\pi).$$

But, invoking the main result of **[3]**, we can conclude that

$$\langle \tau(\omega)\alpha_\tau, \alpha_\tau\rangle = \int_{\mathscr{S}} \langle \pi(\omega)\alpha_\pi^c, \alpha_\pi^c\rangle \, d\mu(\pi), \qquad \omega \in \mathscr{D}(G).$$

This says that

$$\alpha_\tau = \int_{\mathscr{S}}^{\oplus} \alpha_\pi^c \, d\mu(\pi),$$

which, together with the uniqueness of the decomposition (3.2) **[16]**, allows us to conclude

PROPOSITION 3.3. *The validity of equation* (3.4) *guarantees that for μ-a.a. $\pi \in \mathscr{S}$, the distributions α_π^c have unique extensions to $\alpha_\pi \in \mathscr{H}_\pi^{-\infty}$, and the PFPF*

$$\langle \tau(\omega)\alpha_\tau, \alpha_\tau\rangle = \int_{\mathscr{S}} \langle \pi(\omega)\alpha_\pi, \alpha_\pi\rangle \, d\mu(\pi), \qquad \omega \in \mathscr{D}(G)$$

obtains for all test functions.

Now we describe carefully the two categories of homogeneous spaces for which we can derive the PFPF.

(1) *Abelian symmetric spaces.* Let V be a real finite-dimensional vector space, H *any* connected Lie group acting on V by linear transformations. Form $G = VH$ the semidirect product. (Because we adopt the convention of putting group actions on the right—see **[8]**—we place the normal subgroup on the left.) Our homogeneous space

G/H has been dubbed **[9]** an *abelian symmetric space*. We assume that $\widehat{V}/H$ is countably separated. Then $\tau = \operatorname{Ind}_H^G 1$ is type I and

$$\tau = \int_{\widehat{V}/H}^{\oplus} \pi_\chi \, d\dot{\chi}, \tag{3.5}$$

where for $\chi \in \widehat{V}$, $\pi_\chi = \operatorname{Ind}_{VH_\chi}^G \chi \times 1$ and $d\dot{\chi}$ denotes a push-forward of Lebesgue measure on $\widehat{V}$. The decomposition can be reformulated orbitally. Obviously $\chi \in \widehat{V} \leftrightarrow \varphi \in \mathfrak{h}^\perp$. Also it is easy to check that

$$\dim G/VH_\chi = \dim VH_\chi/G_\varphi .$$

In fact, $B = VH_\chi$ is a real polarization for φ satisfying the Pukanszky condition. In addition, $\chi_\varphi = \chi \times 1$ is a character of B satisfying $d\chi_\varphi = i\varphi|_{\mathfrak{b}}$. Thus $\pi_\chi = \operatorname{Ind}_{VH_\chi}^G \chi \times 1 = \operatorname{Ind}_B^G \chi_\varphi = \pi_\varphi$. Hence it is reasonable to write (3.5) as

$$\tau = \int_{\mathfrak{h}^\perp/H}^{\oplus} \pi_\varphi \, d\dot{\varphi},$$

as in formula (1.1). Note here the multiplicity function is identically 1.

(2) *Finite multiplicity completely solvable homogeneous spaces* (*FMCS spaces*). Now take G simply connected completely solvable. H a closed connected subgroup. One has **[13]**

$$\tau = \operatorname{Ind}_H^G 1 = \int_{\mathfrak{h}^\perp/H}^{\oplus} \pi_\varphi \, d\dot{\varphi} = \int_{G\cdot\mathfrak{h}^\perp/G} n_\varphi \pi_\varphi \, d\tilde{\varphi}$$

(again, formula (1.1)), where $\pi_\varphi = \operatorname{Ind}_B^G \chi_\varphi$, B is a real polarization for $\varphi \in \mathfrak{h}^\perp$ satisfying Pukanszky, $\chi_\varphi \in \widehat{B}$ is a character with $\chi_\varphi(\exp X) = e^{i\varphi(X)}$. Consider the following condition:

(A) Generically on $\mathfrak{h}^\perp$, we have $\dim \mathfrak{g}\cdot\varphi = 2\dim \mathfrak{h}\cdot\varphi$.

If G is nilpotent, condition (A) is known **[2]**, **[11]** to be necessary and sufficient for the multiplicity $n_\varphi = \#[G\cdot\varphi \cap \mathfrak{h}^\perp]/H$ to be finite a.e. The corresponding statement is actually false **[11]** for general exponential solvable G. However, it is still true for G completely solvable—which I shall prove in a forthcoming publication. For convenience we refer to such G/H as FMS—finite multiplicity completely solvable homogeneous spaces. (In fact, I shall really only use condition (A) in the following, not finite multiplicity—see Remark 5.2 (1).)

Our main result of the paper is the next theorem. I have arranged the notation so that it makes sense for either of the above categories of homogeneous spaces G/H (abelian symmetric or FMCS).

THEOREM 3.4. *The following is true generically on* $\mathfrak{h}^\perp$. *For* $\varphi \in \mathfrak{h}^\perp$, *there exists a real polarization* $\mathfrak{b}$ *for* φ *satisfying Pukanszky such that when we realize* $\pi_\varphi = \operatorname{Ind}_B^G \chi_\varphi$ *in* $L^2(G, B, \chi_\varphi)$, *the conjugate-linear distributional functional*

$$\alpha_{\varphi,\mathfrak{b}}: f \to \int_{H\cap B\backslash H} \overline{f} q_{B,G}^{1/2} q_{H\cap B,H}^{-1} q_{H,G}^{-1/2}\, d\dot{h}, \qquad f \in C_c^\infty(G, B, \chi_\varphi) \tag{3.6}$$

is well defined and extends uniquely to a distribution in $(\mathscr{H}_{\pi_\varphi}^{-\infty})^{H, q_{H,G}^{-1/2}}$. *Moreover, if we fix a pseudo-image of Lebesgue measure* $d\dot{\varphi}$ *on* $\mathfrak{h}^\perp/H$, *then for a.a.* $\varphi \in \mathfrak{h}^\perp$, *the choice of the quasi-invariant measure* $d\dot{h}$ *is uniquely specified so that*

$$\langle \tau(\omega)\alpha_\tau, \alpha_\tau\rangle = \int_{\mathfrak{h}^\perp/H} \langle \pi_\varphi(\omega)\alpha_{\varphi,\mathfrak{b}}, \alpha_{\varphi,\mathfrak{b}}\rangle\, d\dot{\varphi}, \qquad \omega \in \mathscr{D}(G).$$

We shall prove Theorem 3.4 for abelian symmetric spaces in §4, for FMCS spaces in §5.

REMARKS 3.5. (1) In the latter case, Theorem 3.4 was proven by Fujiwara and Yamagami [5] in the very special instance that τ is a *finite* direct sum of irreducibles. But they present their formula in terms of the second of equations (1.1). Fujiwara does the same in [4] when working with nilpotent groups. Presenting the PFPF in that way requires singling out one H-orbit in $G \cdot \varphi \cap \mathfrak{h}^\perp$, singling out a polarization, and then relating the polarizations for the other H-orbits to the first. It is complicated and fairly messy—and unnecessary. Treating the different H-orbits in $G \cdot \varphi \cap \mathfrak{h}^\perp$ in a uniform manner as I do leads to a much neater formulation of the PFPF.

(2) Referring back to Question 3.1 (11), and to Proposition 3.2, we see that the intertwining operator for the decomposition of the quasi-regular representation is the unique unitary extension of the densely defined operator

$$\tau(\omega)\alpha \to \{\pi_\varphi(\omega)\alpha_{\varphi,\mathfrak{b}}\}_{\varphi\in\mathfrak{h}^\perp/H}, \qquad \omega \in \mathscr{D}(G). \tag{3.7}$$

We can use the computations in Theorem 2.1 and Proposition 2.2 to write it more explicitly. We exploit the fact that for $\omega \in \mathscr{D}(G)$, the function $\omega_H \in \mathscr{D}(G, H)$ and the map $\omega \to \omega_H$, $\mathscr{D}(G) \to \mathscr{D}(G, H)$ is surjective. Writing $\Omega = \omega_H$, we obtain the isometry

$$\begin{aligned} &\Omega(g) \to \{\Omega_\pi(g)\} \\ &\Omega_\pi(g) = \int_{H\cap B\backslash B} \Omega(bg)\overline{\chi}(b) q_{B,G}^{-1/2}(bg) q_{H,G}^{1/2}(bg) q_{H\cap B,B}^{-1}(b)\, d\dot{b}, \end{aligned} \tag{3.8}$$

which extends to $L^2(G, H)$ and is the intertwining operator for the direct integral decomposition of the quasi-regular representation $\tau = \operatorname{Ind}_H^G 1$.

4. Abelian symmetric spaces. Let $G = VH$ be a semidirect product of locally compact groups, V a normal real vector group, H a connected Lie group. We fix Haar measure dv on V and a right Haar measure dh on H. We let δ denote the modulus of the action of H on V

$$\delta(h)\int_V f(hvh^{-1})\,dv = \int_V f(v)\,dv, \qquad f \in C_c(V),\ h \in H.$$

Right Haar measure on G is then $dg = dv\,dh$ and the modular function of G is $\Delta_G(vh) = \delta(h)\Delta_H(g)$. Therefore $q = q_{H,G} = \Delta_H\Delta_G^{-1} = \delta^{-1}$. Since group actions are on the right, i.e., $v\cdot h = h^{-1}vh$, $(f\cdot h)(v) = f(v\cdot h^{-1})$, the above modular equality can be written

$$\delta(h)\int_V (f\cdot h)(v)\,dv = \int_V f(v)\,dv, \qquad f \in C_c(V),\ h \in H.$$

By duality we have

$$\delta(h)^{-1}\int_{\widehat{V}} (f\cdot h)(\chi)\,d\chi = \int_{\widehat{V}} f(\chi)\,d\chi, \qquad f \in C_c(\widehat{V}),\ h \in H,$$

where $d\chi$ is the Haar measure on $\widehat{V}$ dual to dv. Now since q extends to a character on all of G, $q(vh) = \delta(h)^{-1}$, the homogeneous space G/H has a relatively invariant measure with modulus $q^{-1} = \delta$. In fact, we have

$$(4.1)\quad \int_G f(g)q(g)\,dg = \int_V\int_H f(hv)\,dh\,dv = \int_{G/H}\int_H f(hg)\,dh\,d\dot{g}$$

and dv is said measure.

Next we disintegrate the Haar measure $d\chi$ under the action of H. We fix *once and for all* a choice of pseudo-image $d\dot{\chi}$. Since the modulus for the action of H on $\widehat{V}$ is $q = \delta^{-1}$, it follows that almost every orbit (of H) in $\widehat{V}$ has a relatively invariant measure with that modulus [**8**]. Henceforth we only consider such orbits, referring to them as generic. Moreover, those measures are *uniquely determined* by the choice of $d\dot{\chi}$ according to the formula

$$\int_{\widehat{V}} f(\chi)\,d\chi = \int_{\widehat{V}/H}\int_{\chi\cdot H} f(\chi\cdot h)\,d\mu_{\dot{\chi}}\,d\dot{\chi}, \qquad f \in C_c(\widehat{V})$$

(see [**8**, §2]). (Note we write $\dot{\chi} = \chi\cdot H \in \widehat{V}/H$.) Fix $\chi \in \widehat{V}$ generic. Then $H_\chi\backslash H$ has a relatively invariant measure of modulus $q = \delta^{-1}$.

This says Δ_{H_χ}/Δ_H extends from H_χ to H as a positive character, and in fact $\Delta_{H_\chi}(h) = \delta(h)\Delta_H(h)$ must hold **[8]**. Thus $q_{H_\chi, H} = \delta|_{H\chi}$. That is,

$$\int_H f(h)\delta(h)\,dh = \int_{H_\chi\backslash H}\int_{H_\chi} f(h_\chi h)\,dh_\chi\,d\mu_{\dot\chi}(\dot h)$$

where, since right Haar measure dh and the relatively invariant measure $d\mu_{\dot\chi}$ are already determined, the right Haar measure dh_χ is also uniquely determined. Now before proceeding we note that $q = \Delta_H/\Delta_G = \Delta_H/\Delta_{H_\chi}$ means that

$$\Delta_G|_{H_\chi} = \Delta_{H_\chi}. \tag{4.2}$$

This shows that $H_\chi\backslash G$ actually has an invariant measure. We shan't need that in the following, but we will make use of the equality (4.2) itself.

Now consider the quasi-regular representation $\tau = \operatorname{Ind}_H^G 1$. The goal is to prove Theorem 3.4. We begin by recalling Proposition 2.2: For $\omega \in \mathscr{D}(G)$, we have $\langle \tau(\omega)\alpha_\tau, \alpha_\tau\rangle = \omega_H(e)$, where

$$\omega_H(g) = \int_H \omega(g^{-1}h^{-1})\Delta_G(hg)^{-1}q_{H,G}^{-1/2}(hg)\,dh.$$

Next we study the spherical irreducible representations of G. These are

$$\pi_\chi = \operatorname{Ind}_{VH_\chi}^G \chi\times 1, \qquad \chi\in\widehat{V}.$$

We continue to assume χ is generic. The subgroup $B = VH_\chi$ plays the role of the polarization here. Clearly $H\cap B = H_\chi$. Next we check that the assumptions preceding Theorem 2.1 are satisfied. First, in this case, BH is not only closed in G, it equals G. Second, $(\chi\times 1)|_{H\cap B} = (\chi\times 1)|_{H_\chi} = 1$. And finally, we have $q_{H\cap B,H}q_{H\cap B,B} = 1$ on $H\cap B$. Indeed, we know $q_{H\cap B,B} = q_{H_\chi, VH_\chi} = \Delta_{H_\chi}/\Delta_{VH_\chi} = \delta^{-1}$. But since $q_{H\cap B,H} = q_{H_\chi,H} = \delta$, the assertion is evident. Moreover, the product of q functions $q_{B,G}^{1/2}q_{H\cap B,H}^{-1}q_{H,G}^{-1/2}$ is also identically one (on H). In fact, $q_{B,G}$ is a character on all of G since $q_{B,G} = \Delta_{VH_\chi}/\Delta_{VH} = \Delta_{H_\chi}/\Delta_H = \delta$. Combining with (4.2) we obtain

$$\begin{aligned} q_{B,G}^{1/2}q_{H\cap B,H}^{-1}q_{H,G}^{-1/2} &= (\Delta_{H_\chi}/\Delta_H)^{1/2}(\Delta_{\chi_\chi}/\Delta_H)^{-1}(\Delta_H/\Delta_G)^{-1/2} \\ &= (\Delta_{H_\chi}/\Delta_G)^{1/2} \equiv 1. \end{aligned}$$

Thus we are ready to apply Theorem 2.1. It says that if we consider the antidistribution

$$\alpha_\chi : f \to \int_{H_\chi \backslash H} \overline{f}\, d\dot{h}, \qquad f \in C_c^\infty(G, B, \chi),$$

then the matrix coefficient is given by

$$\langle \pi_\chi(\omega)\alpha_\chi, \alpha_\chi \rangle = \int_{H_\chi \backslash H} \int_{H_\chi \backslash B} \omega_H(bh)(b) q_{B,G}^{-1/2}(b) q_{H,G}^{1/2}(h^{-1}bh) \cdot q_{H\cap B,B}^{-1}(b) q_{H\cap B,H}^{-1}(h)\, d\dot{b}\, d\dot{h}, \qquad \omega \in \mathscr{D}^+(G).$$

Evaluating the q functions in the integrand (always keeping in mind that δ is a character on all of G), and using (4.1) (actually applied to B instead of G), we see that the matrix coefficient becomes

$$\langle \pi_\chi(\omega)\alpha_\chi, \alpha_\chi \rangle = \int_{H_\chi \backslash H} \int_V \omega_H(vh)\overline{\chi}(v)\delta^{-1}(h)\, dv\, d\dot{h}, \qquad \omega \in \mathscr{D}^+(G).$$

But ω_H is left H-invariant, so we can further evaluate

$$\begin{aligned} \langle \pi_\chi(\omega)\alpha_\chi, \alpha_\chi \rangle &= \int_{H_\chi \backslash H} \int_V \omega_H(h^{-1}vh)\overline{\chi}(v)\delta^{-1}(h)\, dv\, d\dot{h} \\ &= \int_{H_\chi \backslash H} \int_V \omega_H(v)\overline{\chi}(hvh^{-1})\, dv\, d\dot{h} \\ &= \int_{H_\chi \backslash H} \hat{\omega}_H(\overline{\chi \cdot h})\, d\dot{h}. \end{aligned}$$

The function $v \to \omega_H(v)$ belongs to $\mathscr{D}(V)$. Therefore, provided $\widehat{V}/H$ is countably separated, we can complete the computation of the PFPF

$$\begin{aligned} \int_{\widehat{V}/H} \langle \pi_\chi(\omega)\alpha_\chi, \alpha_\chi \rangle\, d\dot{\chi} &= \int_{\widehat{V}/H} \int_{H\chi \backslash H} \hat{\omega}_H(\overline{\chi} \cdot h)\, d\dot{h}\, d\dot{\chi} \\ &= \int_{\widehat{V}} \hat{\omega}_H(\overline{\chi})\, d\chi = \int_{\widehat{V}} \hat{\omega}_H(\chi)\, d\chi \\ &= \omega_H(e) = \langle \tau(\omega)\alpha_\tau, \alpha_\tau \rangle, \qquad \omega \in \mathscr{D}^+(G). \end{aligned}$$

An application of Proposition 3.3 yields at last

THEOREM 4.1. *Let $G = VH$ be a semidirect product, V a normal real vector group, H a connected Lie group, $\widehat{V}/H$ countably separated.*

Fix a pseudo-image $d\dot{\chi}$ in $\widehat{V}/H$ of Haar measure $d\chi$ on $\widehat{V}$. Then for almost all χ in $\widehat{V}$ there exists a unique relatively invariant measure $d\dot{h}$ on $H_\chi\backslash H$ such that the distribution

$$(4.3) \qquad \alpha_\chi : f \to \int_{H_\chi\backslash H} \overline{f}\, d\dot{h}, \qquad f \in C_c^\infty(H, H_\chi),$$

has a unique extension to $H_{\pi_\chi}^\infty$, $\pi_\chi = \mathrm{Ind}_{VH_\chi}^G \chi \times 1$, and so that

$$\langle \tau(\omega)\alpha_\tau, \alpha_\tau\rangle = \int_{\widehat{V}/H} \langle \pi_\chi(\omega)\alpha_\chi, \alpha_\chi\rangle\, d\dot{\chi}, \qquad \omega \in \mathscr{D}(G).$$

REMARK 4.2. Implicit in the statement of Theorem 4.1 is the realization of π_χ in $L^2(H, H_\chi)$. That is achieved by restricting the functions in $\mathscr{H}_{\pi_\chi} = L^2(G, B, \chi)$ to H. Of course we have

$$C_c^\infty(H, H_\chi) \subset L^2(H, H_\chi)^\infty \subset C^\infty(H, H_\chi).$$

The distribution integrals (4.3) converge for $f \in C_c^\infty(H, H_\chi)$ and the theory says that the distribution extends to the C^∞ vectors. It is tempting to speculate that the distribution integrals (4.3) actually converge absolutely for *any* $f \in L^2(H, H_\chi)^\infty$. I know of no example where that is not the case (see the typical example below). It was the case for the algebraic groups considered in **[12]**. In fact, one can generalize the argument of **[12]** to show that the integrals (4.3) converge on all of $L^2(H, H_\chi)^\infty$ whenever H is connected abelian. (In **[12]** we assumed the action of H was split.) For general H I do not know any technique to generalize those employed in **[12]**. No such techniques are apparent from **[5]**. Those authors use a knowledge of intertwining operators (corresponding to different polarizations) to show the distribution integrals extend. But no convergence on arbitrary C^∞ vectors is demonstrated. The technique of Proposition 3.3 seems more general to me. But the whole question of convergence of the integrals (4.3) for arbitrary C^∞ vectors is open (see also Remark 5.2 (2) and Conjecture 5.3).

EXAMPLE 4.3. Let $G = \mathbb{R}^2 \cdot \mathrm{SL}(2, \mathbb{R})$ with the natural action of $H = \mathrm{SL}(2, \mathbb{R})$ on $V = \mathbb{R}^2$. There is one spherical representation $\pi = \mathrm{Ind}_{VN}^G \chi \times 1$, $\chi(x, y) = e^{ix}$, $N = H_\chi = \{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) : b \in \mathbb{R}\}$. The Penney distribution is given by

$$(4.4) \qquad \alpha_\chi : f \to \int_{N\backslash H} \overline{f}\, d\mu, \qquad f \in C_c^\infty(H, N)$$

where $d\mu$ is the invariant measure on $N\backslash H$. In this case $\tau = \mathrm{Ind}_H^G 1 \cong \pi$ and the PFPF asserts, for $\omega \in \mathscr{D}(G)$, the equality of $\langle \tau(\omega)\alpha_\tau, \alpha_\tau\rangle$ and $\langle \pi(\omega)\alpha_\chi, \alpha_\chi\rangle$. In fact the computations in the proof of Theorem 4.1 show that both equal

$$\omega_H(e) = \int_H \omega(h)\, dh .$$

Now it is straightforward to check that for any $f \in \mathscr{H}_\pi^\infty = L^2(H, N)^\infty$: the requirement of $\pi(X)f \in L^2(H, N)$, $\forall X \in \mathfrak{U}(\mathfrak{h})$ yields $f \in C^\infty(H, N)$; and the condition $\pi(Y)f \in L^2(H, N)$, $\forall Y \in \mathfrak{n}$ gives sufficiently rapid decay at ∞ to guarantee convergence of (4.4). I leave the details to the reader.

5. Finite multiplicity completely solvable homogeneous spaces. We start this section with G simply connected exponential solvable. That means $\mathfrak{g}$ is solvable and has no purely imaginary eigenvalues. We assume familiarity with the Orbit Method: $\forall \varphi \in \mathfrak{g}^*$, there is a real polarization $\mathfrak{b}$ for φ satisfying Pukanszky; the representation $\pi_{\varphi, \mathfrak{b}} = \mathrm{Ind}_B^G \chi_\varphi$, $\chi_\varphi(\exp X) = e^{i\varphi(X)}$, is irreducible and its class π_φ is independent of $\mathfrak{b}$; the map $\varphi \to \pi_\varphi$, $\mathfrak{g}^* \to \widehat{G}$ factors to a bijection $\mathfrak{g}^*/G \to \widehat{G}$. Now suppose $H \subset G$ is a closed connected (and therefore simply connected) subgroup. We have already stated (in (1.1)) the Orbit Method formula for the decomposition of the quasi-regular representation $\tau = \mathrm{Ind}_H^G 1$, namely

$$\tau = \int_{\mathfrak{h}^\perp/H}^{\oplus} \pi_\varphi \, d\dot{\varphi} = \int_{G\cdot\mathfrak{h}^\perp/G}^{\oplus} n_\varphi \pi_\varphi \, d\tilde{\varphi} ,$$

where $n_\varphi = \#[G \cdot \varphi \cap \mathfrak{h}^\perp]/H$ and the measures are push-forwards of Lebesgue measure. For the reasons explained in Remark 3.5 (1), we work with the first version of the formula.

Now we assume that G is completely solvable and condition (A) is satisfied. That is, we assume G/H is a FMCS space as defined in §3. We shall use the results of **[13]**, **[14]** very heavily in this section. The main result of this section—the analog to Theorem 4.1 and our formulation of Theorem 3.4 for FMCS spaces—is the following.

THEOREM 5.1. *Let G/H be FMCS. Then for generic $\varphi \in \mathfrak{h}^\perp$, there exists a real polarization $\mathfrak{b}$ for φ satisfying Pukanszky such that:*

(i) $q_{H\cap B, H} q_{H\cap B, B} \equiv 1$ *on* $H \cap B$;

(ii) *BH is closed in G; and*

(iii) *the distribution*

$$\alpha_{\varphi,\mathfrak{b}}: f \to \int_{H\cap B\backslash H} \overline{f} q_{B,G}^{1/2} q_{H\cap B,H}^{-1} q_{H,G}^{-1/2}\, d\dot{h}, \qquad f \in C_c^\infty(G, B, \chi_\varphi),$$

is well defined and has a unique extension to $\mathscr{H}^\infty_{\pi_{\varphi,\mathfrak{b}}}$. Moreover,

(iv) *there exists a unique choice of pseudo-image $d\dot{\varphi}$ so that*

$$\langle \tau(\omega)\alpha_\tau, \alpha_\tau\rangle = \int_{\mathfrak{h}^\perp/H} \langle \tau_{\varphi,\mathfrak{b}}(\omega)\alpha_{\varphi,\mathfrak{b}}, \alpha_{\varphi,\mathfrak{b}}\rangle\, d\dot{\varphi}, \qquad \omega \in \mathscr{D}(G).$$

Notes. (1) In Theorem 3.4 we first fixed a pseudo-image $d\dot{\varphi}$ and then the theorem asserted the uniqueness of (almost all of) the quasi-invariant measures $d\dot{h}$. That sequence is preserved in Theorem 4.1, but in Theorem 5.1 we have reversed the procedure. We first choose Haar measures on B and $H \cap B$. The term $q_{H\cap B,H}^{-1}(h)\, d\dot{h}$ shows the integral defining $\alpha_{\varphi,\mathfrak{b}}$ does not depend on the choice of $d\dot{h}$. Thus the choices of Haar measure uniquely determine the pseudo-image $d\dot{\varphi}$. Of course the two procedures amount to the same thing (see e.g. [**8**]). If we had fixed the pseudo-image first, then we would have to adjust each choice of Haar measure corresponding to each $\varphi \in \mathfrak{h}^\perp$ so as to preserve the PFPF.

(2) We shall refer to a polarization obeying the properties of the theorem as "good." We shall also use standard notation:

$M(\varphi, \mathfrak{g})$ = subalgebras of $\mathfrak{g}$ which are maximal totally isotropic for the form $X, Y \to \varphi[X, Y]$,

$I(\varphi, \mathfrak{g}) = \{\mathfrak{b} \in M(\varphi, \mathfrak{g}) : \operatorname{Ind}_B^G \chi_\varphi$ is irreducible$\}$
$= \{\mathfrak{b} \in M(\varphi, \mathfrak{g}) : \mathfrak{b}$ satisfies Pukanszky, i.e., $B \cdot \varphi = \varphi + \mathfrak{b}^\perp\}$.

(3) In Theorem 4.1 we were able to check the assumptions I, II and verify the PFPF directly. Here we cannot do that. Our proof will be by induction on $\dim \mathfrak{g}/\mathfrak{h}$. We shall prove (i), (ii) and (iv) for $\omega \in \mathscr{D}^+(G)$ by induction. That (iii) holds and that (iv) holds for all $\omega \in \mathscr{D}(G)$ then follows from Proposition 3.3.

Proof of Theorem 5.1. We begin the induction argument with $\dim G/H = 1$. When G is nilpotent this forces H to be normal. But if G is only completely solvable, this is not so. We distinguish the two cases here, and in the induction step below.

$\dim G/H = 1$ and H normal in G. Then $G/H \cong \mathbb{R}$ and

$$\tau = \mathrm{Ind}_H^G 1 = \int_{\widehat{G/H}}^{\oplus} \chi \, d\chi .$$

Since $H \triangleleft G$, we have $\Delta_G|_H = \Delta_H$ and $q_{H,G} = 1$. In this case $B = G$ and properties (i), (ii) are trivially satisfied. Also, $\mathfrak{h}^\perp/H \leftrightarrow \widehat{G/H}$. For any $\omega \in \mathscr{D}^+(G)$, we know from Theorem 2.1 (v) that

$$\langle \pi_\chi(\omega)\alpha_\chi , \alpha_\chi \rangle = \int_{H\backslash G} \omega_H(g)\chi(g)\, d\dot{g} = \hat{\omega}_H(\chi).$$

Therefore

$$\int_{\mathfrak{h}^\perp/H} \langle \pi_\chi(\omega)\alpha_\chi , \alpha_\chi\rangle \, d\dot{\chi} = \int_{\widehat{G/H}} \hat{\omega}_H(\chi)\, d\chi = \omega_H(e) = \langle \tau(\omega)\alpha_\tau , \alpha_\tau\rangle .$$

$\dim G/H = 1$ and H not normal in G. Then (see **[13]**, Proposition 3.2] or Proposition 5.2 below) there is a canonical closed normal subgroup $G_0 \triangleleft G$, $G_0 \subset H \subset G$ such that $G/G_0 \cong ax+b$ group. This case reduces instantly to that of the representation of the $ax+b$ group induced by the identity on a nonnormal codimension 1 closed connected subgroup. In this case (a stronger result than) Theorem 5.1 is proven in **[12]**.

Now we pass to the induction step. Suppose $\dim G/H > 1$ and assume Theorem 5.1 is proven for all FMCS spaces of lower dimension. Since G is completely solvable we know that we can find a closed connected codimension 1 subgroup G_1 in between H and G, $H \subset G_1 \subset G$. G_1 may or may not be normal. First assume it is. To keep the notation in line with **[11]**, **[13]**, we write N for G_1, $H \subset N \triangleleft G$, $\dim G/N = 1$. We also adopt the terminology $\mathfrak{h}^\perp(\mathfrak{g}) = \{\varphi \in \mathfrak{g}^* : \varphi(h) = 0\}$ from **[11]**. Now by the induction hypothesis the theorem is true for $\mathrm{Ind}_H^N 1$. Also

$$\begin{aligned} \tau = \mathrm{Ind}_H^G 1 = \mathrm{Ind}_N^G \mathrm{Ind}_H^N 1 &= \mathrm{Ind}_N^G \int_{\mathfrak{h}^\perp(\mathfrak{n})/H}^{\oplus} \gamma_\theta \, d\dot{\theta} \\ &= \int_{\mathfrak{h}^\perp(\mathfrak{n})/H}^{\oplus} \mathrm{Ind}_N^G \gamma_\theta \, d\dot{\theta} \cong \int_{\mathfrak{h}^\perp(\mathfrak{g})/H}^{\oplus} \pi_\varphi \, d\dot{\varphi} . \end{aligned}$$

These equivalences are true by (resp.): induction in stages, the Orbit Method formula for the quasi-regular representation, commutation of direct integrals and induced representations, and the Orbit Method again. (Of course, the proof of the last equivalence is the main point of **[11]**.) Now the induced representations $\mathrm{Ind}_N^G \gamma_\theta$ have one generic pattern on $\mathfrak{h}^\perp(\mathfrak{n})$—either they are irreducible or a one-parameter direct integral of irreducibles. We further distinguish these cases.

(a) $\operatorname{Ind}_N^G \gamma_\theta = \pi_\varphi$ is irreducible, $\varphi \in \mathfrak{h}^\perp(\mathfrak{g})$, $\theta = \varphi|_\mathfrak{n} \in \mathfrak{h}^\perp(\mathfrak{n})$. According to the induction hypothesis, there is $\mathfrak{b} \in I(\theta, \mathfrak{n})$, a good polarization. We have

$$\mathfrak{g}_\varphi \subset \mathfrak{n}_\theta \subset \mathfrak{b} \subset \mathfrak{n} \subset \mathfrak{g}$$

and $\dim \mathfrak{g}/\mathfrak{n} = \dim \mathfrak{n}_\theta/\mathfrak{g}_\varphi = 1$ in this case [**11**, §3]. Therefore $\mathfrak{b} \in M(\varphi, \mathfrak{g})$. But since

$$\pi_\varphi = \operatorname{Ind}_N^G \gamma_\theta = \operatorname{Ind}_N^G \operatorname{Ind}_B^N \chi_\theta = \operatorname{Ind}_B^G \chi_\varphi$$

is irreducible, it is also true that $\mathfrak{b} \in I(\varphi, \mathfrak{g})$. Since BH is closed in N, it is closed in G. Also the product $q_{H\cap B, H} q_{H\cap B, B}$ doesn't notice whether it's computed in N or G. So properties (i) and (ii) are verified. It remains to check (iv), the PFPF. Recall the key formulas from Theorem 2.1

$$\langle \tau(\omega)\alpha_\tau, \alpha_\tau\rangle = \omega_H(e), \tag{5.1}$$

$$\omega_H(g) = \Delta_G(g)^{-1} q_{H,G}^{-1/2}(g) \int_H \omega(g^{-1}h^{-1}) \Delta_G(h)^{-1} q_{H,G}^{-1/2}(h)\, dh, \tag{5.2}$$

$$\begin{aligned}\langle \pi_\varphi(\omega)\alpha_{\varphi,\mathfrak{b}}, \alpha_{\varphi,\mathfrak{b}}\rangle &= \int_{H\cap B\backslash H} \int_{H\cap B\backslash B} \omega_H(bh) q_{B,G}^{-1/2}(b) q_{H,G}^{1/2}(h^{-1}bh) \\ &\qquad \cdot q_{H\cap B, B}^{-1}(b) q_{H\cap B, H}^{-1}(h)\, d\dot{b}\, d\dot{h}.\end{aligned} \tag{5.3}$$

Here and in the remainder we write $\omega_H = \omega_{H,G}$ if it is necessary to specify the supergroup. Now in situation (a), the map $\varphi \to \theta$ generates a bijection $\mathfrak{h}^\perp(\mathfrak{g})/H \to \mathfrak{h}^\perp(\mathfrak{n})/H$. (See [**11**, p. 446]. The proof uses condition (A). See also Remark 5.1 (1).) The PFPF for N/H will give that for G/H as soon as we observe

$$\Delta_G|_N = \Delta_N, \quad q_{H,G}|_N = q_{H,N}, \quad q_{B,G}|_N = q_{B,N} \tag{5.4}$$

(all because N is normal). Therefore we also have

$$\omega_{H,G}|_N = \omega_{H,N} \tag{5.5}$$

and property (iv) for N/H implies the same for G/H.

(b) $\operatorname{Ind}_N^G \gamma_\theta = \int_\mathbb{R}^\oplus \pi_{\varphi+t\beta}\, dt$, $\beta \in \mathfrak{n}^\perp(g)$, $\beta \neq 0$. In this case $\pi_{\varphi+t\beta}|_N = \gamma_\theta$ for every t. Also $\mathfrak{g}_\varphi = \mathfrak{g}_\theta$ and $\mathfrak{g} = \mathfrak{g}_\varphi + \mathfrak{n}$ [**11**]. By induction there is a good polarization $\mathfrak{b}_1 \in I(\theta, \mathfrak{n})$. It is no loss of generality to assume $\mathfrak{b}_1$ is G_θ-invariant (e.g., by building admissibility into the condition on $\mathfrak{b}$ if necessary). Set $\mathfrak{b} = \mathfrak{b}_\varphi + \mathfrak{b}_1$. It is simple to check $\mathfrak{b} \in M(\varphi, \mathfrak{g})$. Now $G = G_\varphi N$, so by the Subgroup Theorem we have

$$\operatorname{Ind}_B^G \chi_\varphi|_N = \operatorname{Ind}_{B\cap N}^N \chi_\varphi|_{B\cap N} = \operatorname{Ind}_{B_1}^N \chi_\theta = \gamma_\theta .$$

The restriction of a representation being irreducible implies the original is irreducible; hence $\mathfrak{b} \in I(\varphi, \mathfrak{g})$. Note that $BH = G_\varphi B_1 H$. We choose $X \in \mathfrak{g}_\varphi \backslash \mathfrak{n}$, $\beta(X) = 1$, so that

$$\mathbb{R} \times N \to G, \quad (t, n) \to \exp tXn$$

is a diffeomorphism. Since $B_1 H$ is closed in N, it is clear that BH is closed in G. This proves (ii). To prove (i) we observe

$$\mathfrak{h} \cap \mathfrak{b} = (\mathfrak{h} \cap \mathfrak{n}) \cap \mathfrak{b} = \mathfrak{h} \cap (\mathfrak{n} \cap \mathfrak{b}) = \mathfrak{h} \cap \mathfrak{b}_1 .$$

Also $b_1 \lhd \mathfrak{b}$. Thus

$$[\mathfrak{b}, \mathfrak{h} \cap \mathfrak{b}] \subset [\mathfrak{b}, \mathfrak{h} \cap \mathfrak{b}_1] \subset \mathfrak{b}_1 . \tag{5.6}$$

Now on $\mathfrak{h} \cap \mathfrak{b}$, we have

$$\operatorname{trad}_{\mathfrak{h}/\mathfrak{h}\cap\mathfrak{b}} + \operatorname{trad}_{\mathfrak{b}/\mathfrak{h}\cap\mathfrak{b}} = \operatorname{trad}_{\mathfrak{h}/\mathfrak{h}\cap\mathfrak{b}_1} + \operatorname{trad}_{\mathfrak{b}/\mathfrak{b}_1} + \operatorname{trad}_{\mathfrak{b}_1/\mathfrak{h}\cap\mathfrak{b}_1} .$$

But (5.6) says the middle term on the right vanishes. By the induction hypothesis, the remaining two add up to zero. Thus the left side vanishes and the q identity in (i) is established.

Finally we derive the PFPF in (iv). This argument is somewhat more subtle than in case (a). First we observe

$$B = B_1 \exp \mathbb{R}X, \quad B_1 \lhd B, \quad q_{B_1, B} = 1, \quad q_{B_1, G} = q_{B, G} .$$

Using (2.2) we obtain

$$\int_{H\cap B\backslash B} f(b)\, d\dot{b} = \int_{B_1\backslash B} \int_{H\cap B\backslash B_1} f(b_1 b_t) q_{H\cap B, B}(b_1, b_t) \cdot q_{H\cap B, B_1}^{-1}(b_1)\, d\dot{b}_1\, dt, \qquad b_t = \exp tX .$$

Therefore

$$\begin{aligned}
&\int_{H\cap B\backslash H} \int_{H\cap B\backslash B} \omega_{H,G}(bh) \overline{\chi}_\varphi(b) q_{B,G}^{-1/2}(b) q_{H,G}^{1/2}(h^{-1}bh) \cdot q_{H\cap B, B}^{-1}(b) q_{H\cap B, H}^{-1}(h)\, d\dot{b}\, d\dot{h} \\
&= \int_{H\cap B\backslash H} \int_{B_1\backslash B} \int_{H\cap B\backslash B_1} \omega_{H,G}(b_1 b_t h) \overline{\chi}_\varphi(b_1 b_t) \cdot q_{B_1, G}^{-1/2}(b_1 b_t) q_{H,G}^{1/2}(h^{-1} b_1 b_t h) \\
&\qquad \cdot q_{H\cap B, B_1}^{-1}(b_1) q_{H\cap B, H}^{-1}(h)\, d\dot{b}_1\, dt\, d\dot{h}.
\end{aligned}$$

Then using

$$\int_{\mathfrak{h}^\perp(\mathfrak{g})/H} f(\varphi)\,d\dot{\varphi} = \int_{\mathfrak{h}^\perp(\mathfrak{n})/H}\int_{\mathbb{R}} f(\theta + sX^*)\,ds\,d\dot{\theta}$$

we get

$$\begin{aligned}\int_{\mathfrak{h}^\perp/H} &\langle \pi_{\varphi,\mathfrak{b}}(\omega)\alpha_{\varphi,\mathfrak{b}},\, \alpha_{\varphi,\mathfrak{b}}\rangle\,d\dot{\varphi} \\ &= \int_{\mathfrak{h}^\perp(\mathfrak{n})/H}\int_{\mathbb{R}}\int_{H\cap B\backslash H}\int_{B_1\backslash B}\int_{H\cap B\backslash B_1} \omega_{H,G}(b_1 b_t h)\overline{\chi}_{\varphi+s\beta}(b_1 b_t) \\ &\qquad\qquad \cdot q_{B_1,G}^{-1/2}(b_1 b_t) q_{H,G}^{1/2}(h^{-1}b_1 b_t h) q_{H\cap B,B_1}^{-1}(b_1) \\ &\qquad\qquad \cdot q_{H\cap B,H}^{-1}(h)\,d\dot{b}_1\,dt\,d\dot{h}\,ds\,d\dot{\theta}\end{aligned}$$

which, by Fourier inversion on $B_1\backslash B$ and equations (5.4), (5.5), must equal

$$\begin{aligned}= \int_{\mathfrak{h}^\perp(\mathfrak{n})/H}\int_{H\cap B_1\backslash H}\int_{H\cap B_1\backslash B_1} &\omega_{H,N}(bh)\overline{\chi}_\theta(b_1) \\ &\cdot q_{B_1,N}^{-1/2}(b_1) q_{H,N}^{1/2}(h^{-1}b_1 h) \\ &\cdot q_{H\cap B_1,B_1}^{-1}(b_1) q_{H\cap B,H}^{-1}(h)\,d\dot{b}_1\,d\dot{h}\,d\dot{\theta}.\end{aligned}$$

An application of the induction hypothesis finishes the argument in case (b).

Now we drop the assumption of normality on the intermediate subgroup: $H \subset G_1 \subset G$, $\dim G/G_1 = 1$. We have the same sequence of equivalences

$$\begin{aligned}\tau = \operatorname{Ind}_H^G 1 = \operatorname{Ind}_{G_1}^G \operatorname{Ind}_H^{G_1} 1 &= \operatorname{Ind}_{G_1}^G \int_{\mathfrak{h}^\perp(\mathfrak{g}_1)/H}^{\oplus} \nu_\psi\,d\dot{\psi} \\ &= \int_{\mathfrak{h}^\perp(\mathfrak{g}_1)/H}^{\oplus} \operatorname{Ind}_{G_1}^G \nu_\psi\,d\dot{\psi} \\ &\cong \int_{\mathfrak{h}^\perp(\mathfrak{g})/H}^{\oplus} \pi_\psi\,d\dot{\psi},\end{aligned}$$

the last equivalence having been proven in **[13]**. The key difference is that the generic nature of the induced representation $\operatorname{Ind}_{G_1}^G \nu_\psi$ now has five possibilities instead of two. These are studied in great detail in **[13]**, **[14]**. We have to prove (i), (ii), (iv) for each of these five cases. Much of the argument is repetitive of the details in cases (a), (b). Therefore I shall only provide those details that involve something new. I will try as much as possible to keep to the notation and terminology of [13], [14]. Here is a summary of the main results we need.

PROPOSITION 5.2 [13]. *There exists a canonical subgroup* $G_0 \subset G_1$, *closed connected normal in* G *such that* $G/G_0 \cong ax+b$ *group. Let* $\psi \in \mathfrak{g}_1^*$, $\theta = \psi|_{\mathfrak{g}_0}$, $X \in \mathfrak{g}_1 \backslash \mathfrak{g}_0$, $Y \in \mathfrak{g} \backslash \mathfrak{g}_1$ *so that* $[X, Y] \equiv Y \operatorname{mod} \mathfrak{g}_0$. *Choose* α, $\beta \in \mathfrak{g}^*$ *which satisfy* $\alpha(Y) = \beta(X) = 1$, $\alpha(\mathfrak{g}_0) = \beta(\mathfrak{g}_0) = 0$, $\alpha(X) = \beta(Y) = 0$. *Let* $\varphi \in \mathfrak{g}^*$ *be defined by* $\varphi|_{\mathfrak{g}_1} = \psi$, $\varphi(Y) = 0$. *Then there are three possibilities*:

(c) $\operatorname{Ind}_{G_1}^G \nu_\psi = \pi_\varphi$ *is irreducible*;

(d) $\operatorname{Ind}_{G_1}^G \nu_\psi = \int_{\mathbb{R}}^{\oplus} \pi_{\varphi+s\alpha}\, ds$;

(e) $\operatorname{Ind}_{G_1}^G \nu_\psi = \pi^+ \oplus \pi^-$ *a sum of two irreducibles. In fact, there is a fixed* $s_0 \in \mathbb{R}$ *such that* $\varphi + s_1\alpha$, $\varphi + s_2\alpha$ *are in the same* G *orbit* $\Leftrightarrow$ s_1 *and* s_2 *lie on the same side of* s_0. *Then fixing* $s_1 < s_0 < s_2$, $\pi^+ = \pi_{\varphi+s_2\alpha}$, $\pi^- = \pi_{\varphi+s_1\alpha}$.

Note. In fact, there are *two* more cases. In one

$$\operatorname{Ind}_{G_1}^G \nu_\psi = \int^{\oplus} \pi_{\varphi+t\beta}\, dt,$$

and in the other, $\operatorname{Ind}_{G_1}^G \nu_\psi$ is irreducible (although $\mathfrak{g}_\theta + \mathfrak{g}_0$ can be either $\mathfrak{g}_1$ or $\mathfrak{g}_0$ and the two irreducible cases are structurally different). However, virtually the same arguments apply to these cases as in (c) and (d), resp., below, so I shall allow myself the luxury of ignoring them.

Now we continue with the proof of Theorem 5.1. The reader should keep (5.1)–(5.3) in mind.

(c) $\operatorname{Ind}_{G_1}^G \nu_\psi = \pi_\varphi$ is irreducible. By induction there is a good polarization $\mathfrak{h} \in I(\psi, \mathfrak{g}_1)$. One verifies that $\mathfrak{b} \in M(\varphi, \mathfrak{g})$ by using the fact (from [13]) that $\dim(\mathfrak{g}_1)_\psi/\mathfrak{g}_\varphi = 1$ in this case. That $\mathfrak{b}$ is in $I(\varphi, \mathfrak{g})$ is true because $\pi_\varphi = \operatorname{Ind}_{G_1}^G \nu_\psi = \operatorname{Ind}_{G_1}^G \operatorname{Ind}_{B^{G_1}} \chi_\psi = \operatorname{Ind}_B^G \chi_\varphi$ is irreducible. Since both B and H are in G_1, we have the closure of BH and the q identity $q_{H\cap B, H} q_{H\cap B, B} = 1$ on $H \cap B$ true by induction. Only property (iv) remains to be demonstrated. Looking back at case (a) we see the loss of normality invalidates formulas (5.4) and (5.5). We compensate as follows. Scanning the proof of Theorem

2.1 (i), we see that for $g \in G_1$

$$\begin{aligned}\omega_{H,G}(g) &= \int_H \omega((hg)^{-1})\Delta_G(hg)^{-1} q_{H,G}^{-1/2}(hg)\,dh \\ &= \int_H \omega((hg)^{-1})\Delta_{G_1}(hg)^{-1}\frac{\Delta_{G_1}(hg)}{\Delta_G(hg)} q_{H,G_1}^{-1/2}(hg) q_{G_1,G}^{-1/2}(hg)\,dh \\ &= \int_H \omega((hg)^{-1})\Delta_{G_1}(hg)^{-1} q_{H,G_1}^{-1/2}(hg) q_{G_1,G}^{-1/2}(hg)\,dh \\ &= (\omega q_{G_1,G}^{-1/2})_{H,G_1}(g),\end{aligned}$$

since $q_{G_1,G}$ is a homomorphism on G_1. Let us write $\omega_1 = \omega q_{G_1,G}^{-1/2} \in \mathscr{D}(G_1)$, noting that ω_1 may not be in $\mathscr{D}^+(G_1)$ even if $\omega \in \mathscr{D}^+(G)$.

Now the bijection between the H-orbit spaces $\mathfrak{h}^\perp(\mathfrak{g})/H$ and $\mathfrak{h}^\perp(\mathfrak{g}_1)/H$ is established in [**13**, §4] for case (c). We remark that the proof of the bijections requires condition (A). (See also Remark 5.2 (1) below.) It is obvious that

$$\omega_{H,G}(bh) = (\omega_1)_{H,G_1}(bh) \quad (\text{in particular, } \omega_{H,G}(e) = (\omega_1)_{H,G_1}(e)).$$

Once again, the fact that $q_{G_1,G}$ is a homomorphism and $bh \in G_1$ imply that all the q functions that appear in the integrand of (5.3) may have their G subscripts replaced by G_1's We therefore can apply the induction hypothesis: The PFPF is true for *all* $\omega_1 \in \mathscr{D}(G_1)$. We conclude the PFPF is true for $\omega \in \mathscr{D}^+(G)$, and so we are done in this case.

(d) $\operatorname{Ind}_{G_1}^G \nu_\psi = \int_{\mathbb{R}}^{\oplus} \pi_{\varphi+s\alpha}\,ds$. We know $\nu_\psi = \operatorname{Ind}_{G_0}^G \gamma_\theta$ in this case [**13**, Theorem 3.3 (iii)]. So if we start with a good polarization $\mathfrak{b}_1 \in I(\psi, \mathfrak{g}_1)$ (given by the induction assumption), then it must be that $\mathfrak{b}_1 \subset \mathfrak{g}_0$ (and in fact $\mathfrak{b}_1 \in I(\theta, \mathfrak{g}_0)$). If (as in [**13**]) we set $\mathfrak{g}_2 = \mathfrak{g}_0 + \mathbb{R}Y$, $\omega = \varphi|_{\mathfrak{g}_2}$ and $\mathfrak{b} = \mathfrak{b}_1 + (\mathfrak{g}_2)_\omega$, then it is easy to verify that $\mathfrak{b} \in I(\varphi + s\alpha, \mathfrak{g})$, $\forall s \in \mathbb{R}$. This uses $\mathfrak{g}_0 \triangleleft \mathfrak{g}_2$ and essentially the same analysis as in part (b). Continuing in that vein we verify that

$$\mathfrak{h} \cap \mathfrak{b} = \mathfrak{h} \cap \mathfrak{b}_1, \quad \mathfrak{b}_1 \triangleleft \mathfrak{b}$$

and (reasoning as in (b)) we obtain properties (i) and (ii) immediately. Even more, since everything is happening in G_2 which is normal in G, all of the q functions can have their G subscripts replaced by G_2, and then the proof of the PFPF proceeds exactly as in case (b).

(e) $\operatorname{Ind}_{G_1}^G \nu_\psi = \pi^+ \oplus \pi^-$. We start as usual with a good polarization $\mathfrak{b}_1 \in I(\psi, \mathfrak{g}_1)$. This case is interesting in that $\mathfrak{b}_1 \in M(\varphi, \mathfrak{g})$, but $\mathfrak{b}_1 \notin I(\varphi, \mathfrak{b})$. Set $\varphi^\pm = \varphi + s_2\alpha$, $\varphi + s_1\alpha$, respectively. Set $\mathfrak{b}_0 = \mathfrak{b}_1 \cap \mathfrak{g}_0 \in I(\theta, \mathfrak{g}_0)$. (I am omitting some details which can be checked as usual by using the results of **[13]**.) Defining $\mathfrak{b} = \mathfrak{b}_0 + (\mathfrak{g}_2)_\theta$, we can verify that $\mathfrak{b} \in I(\varphi^\pm, \mathfrak{g})$. We set $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0$ and we apply the induction hypothesis to the FMCS space G_0/H_0, which has dimension one less than that of G/H. It is almost obvious (as in (b)) that $\mathfrak{h} \cap \mathfrak{b} = \mathfrak{h} \cap \mathfrak{b}_0$ and

$$\begin{aligned}\operatorname{trad}_{\mathfrak{h}/\mathfrak{h}\cap\mathfrak{b}} + \operatorname{trad}_{\mathfrak{b}/\mathfrak{h}\cap\mathfrak{b}} = {} & \operatorname{trad}_{\mathfrak{h}/\mathfrak{h}_0} + \operatorname{trad}_{\mathfrak{h}_0/\mathfrak{h}_0\cap\mathfrak{b}_0} \\ & + \operatorname{trad}_{\mathfrak{b}_0/\mathfrak{h}_0\cap\mathfrak{b}_0} + \operatorname{trad}_{\mathfrak{b}/\mathfrak{b}_0}.\end{aligned}$$

The inside terms on the right add to zero by the induction hypothesis, and the outside terms add to zero because $\mathfrak{b}_0 \triangleleft \mathfrak{b}$ and $\mathfrak{h}_0 \triangleleft \mathfrak{h}$. The q identity follows. So does the closure because G is diffeomorphic to $G_0 \times \exp RX \times \exp \mathbb{R}Y$, X may be selected in $\mathfrak{h}\backslash\mathfrak{h}_0$ and $B = B_0(G_2)_\theta$. As usual the last detail is the PFPF. But in fact virtually the same argument as case (c) works here—because of the bijection between the H-orbits in $\mathfrak{h}^\perp(\mathfrak{g})$ and $\mathfrak{h}^\perp(\mathfrak{g}_1)$ in this case (see **[13]**). This completes case (e). It also concludes the proof of Theorem 5.1 and so the main result is established for FMCS homogeneous spaces.

REMARKS 5.2. (1) We only invoked condition (A) to obtain bijections of H-orbits in the proof of the PFPF. The actual assumption of finite multiplicity is never used. It seems likely that the main theorem is true for exponential solvable homogeneous spaces assuming only condition (A). That allows infinite multiplicity in the decomposition of the quasi-regular representation. But—as experts in the field have realized for a while—condition (A) forces a (perhaps infinite) *discrete* direct sum of equivalent irreducibles as opposed to a *continuous* direct integral of equivalent irreducibles—the latter of which occurs when condition (A) is violated.

(2) I remark that, as in abelian symmetric spaces, the question of convergence of the distribution integrals

$$(5.7) \qquad \alpha_{\varphi,\mathfrak{b}}\colon f \to \int_{H\cap B\backslash H} \bar{f} q_{B,G}^{1/2} q_{H\cap B,H}^{-1} q_{H,G}^{-1/2}\, d\dot{h}$$

is not completely settled. They converge for $f \in C_c^\infty(G, B, \chi)$ of course. For the special case of homogeneous spaces which are abelian symmetric and algebraic completely solvable, they are absolutely convergent for *all* $f \in L^2(G, B, \chi)^\infty$ **[12]**. I suspect that is always the case. It is an interesting question, so I shall state it as a

Conjecture 5.3. G/H a FMCS homogeneous space. Then generically on $\mathfrak{h}^\perp$, there exists a real polarization $\mathfrak{b} \in I(\varphi, \mathfrak{g})$ such that (5.7) is absolutely convergent for all $f \in \mathscr{H}_{\pi_{\varphi,\mathfrak{b}}}^\infty$.

We conclude with an example to illustrate that "generic" cannot be removed from the conjecture—indeed it cannot be removed from the main result of Theorem 3.4.

EXAMPLE 5.4. Let $\mathfrak{g}$ be spanned by generators T, X, Y, Z satisfying bracket relations $[T, X] = X$, $[T, Y] = -Y$, $[X, Y] = Z$. Set $H = \exp \mathbb{R}T$. $\tau = \mathrm{Ind}_H^G 1$ has uniform multiplicity 2 (see [**13**, Example 3(c)]). The affine space $\mathfrak{h}^{\perp}(\mathfrak{g})$ may be parameterized by $\varphi = \xi X^* + \eta Y^* + \zeta Z^*$. Then it is readily checked that for $\zeta = 0$, $\xi\eta \neq 0$ there are only good polarizations; for $\zeta(\xi^2 + \eta^2) \neq 0$, there are both good and bad polarizations; and for $\zeta \neq 0$, $\xi^2 + \eta^2 = 0$ there are only bad polarizations. The functionals satisfying $\xi\eta\zeta \neq 0$ are the generic ones.

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