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LINK HOMOTOPY IN $\mathbb{R}^3$ AND S^3

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We give the general homotopy classification of 2-component link maps in $\mathbb{R}^3$ and study 3-component link maps in S^3.

Introduction. For any sequence of integer numbers $p_1 \geq p_2 \geq \cdots \geq p_r \geq 0$ by an r-link map is meant a collection of continuous maps

$$f = \coprod_{1\leq j\leq r} f_j\colon \coprod_{1\leq j\leq r} S^{p_j} \to \mathbb{R}^3 \text{ or } S^3$$

with mutually disjoint images. A link homotopy is a homotopy through link maps.

In [M] J. Milnor studied the case $p_1 = \cdots = p_r = 1$ and classified links up to homotopy for $r = 2$ and $r = 3$. The classification in case $r > 3$ has recently been given by N. Habegger and S. Lin. Note that for $p_1 \leq 1$ the classifications in $\mathbb{R}^3$ and S^3 coincide. Moreover in this case all involved **0**-spheres can be omitted by transversality.

We write (p, q) and (p, q, r) instead of (p_1, p_2) and (p_1, p_2, p_3). Let $E(p, q)$, resp. $L(p, q, r)$, denote the set of link homotopy classes of link maps $S^p \amalg S^1 \to \mathbb{R}^3$, resp. $S^p \amalg S^q \amalg S^r \to S^3$.

The starting point is the following easy consequence of the sphere theorem (compare [**Ko1**]).

PROPOSITION. *If $q > 0$, and $p > 1$, then every link map $f\colon S^p \amalg S^q \to S^3$ is link homotopic to a trivial link map.*

Furthermore link maps $S^p \amalg S^0 \to S^3$ are easily seen to be classified by the homotopy *group* $\pi_p S^2$.

It is a remarkable fact that link homotopy in $\mathbb{R}^3$ contains a considerable amount of additional information. This is solely caused by the hole at $\infty \in S^3$ (compare [**K1, K2**]). On the other hand the strength of the sphere theorem implies that expectable phenomena are fully present, at least for $r = 2$.

There are two obvious constructions briefly described as follows: for $q < 3$ take the standard embedding $S^1 \subset S^3$ and map S^p into the complement which contains an embedded $S^{3-q-1} \vee S^2$ as deformation retract. This defines

$$e_*\colon [S^p, S^{3-q-1} \vee S^2] \to E(p, q),$$

[,] is the set of unbased homotopy classes. In the general situation we map one of the spheres onto the origin of $\mathbb{R}^3$ and wrap the second sphere into $S^2 \subset \mathbb{R}^3$. This defines

$$pt_*\colon \pi_p S^2 \vee \pi_q S^2 \to E(p, q).$$

Here, for two based sets M, N, i.e. sets with distinguished elements m_0, n_0, let $M \vee N$ denote $\{(m, n) \in M \times N \mid m = m_0 \text{ or } n = n_0\}$. If M, N are topological spaces, then $M \vee N$ is the usual wedge.

THEOREM 1. *The following assignments are* 1-1 *and onto*:

$$\begin{aligned} &e_*\colon [S^p, S^{3-q-1} \vee S^2] \to E(p, q), && \text{if } q \leq 1, \\ &pt_*\colon \pi_p S^2 \vee \pi_q S^2 \to E(p, q), && \text{if } q > 1. \end{aligned}$$

Note that the nontrivial elements of $[S^p, S^1 \vee S^2]$ are in 1-1 correspondence with sequences $(a_k)_{k\in\mathbb{N}}$, such that $a_1 \neq 0$, $a_k \in \pi_p S^2$ for $k \in \mathbb{N}$, almost all a_k trivial.

The techniques we develop to handle 2-link maps in $\mathbb{R}^3$ can easily be applied to 3-link maps in S^3. Define pt_* into $L(p, q, r)$ as above by mapping two spheres constantly. Let $j_*\colon E(p, q) \to L(p, q, 1)$ be defined by mapping the q-sphere onto $\infty \in S^3$ and identify $S^3 \backslash \infty \approx \mathbb{R}^3$. Define e_* into $L(p, 1, 1)$ by taking the unlinked disjoint union L of two unknotted circles and then mapping S^p into an embedded $S^2 \vee S^1 \vee S^1$, which is a deformation retract of $S^3 \backslash L$.

THEOREM 2. *The following assignments are* 1-1 *and onto*:

$$\text{(a)} \qquad \begin{aligned} &pt_*\colon \pi_p S^2 \vee \pi_q S^2 \vee \pi_r S^2 \to L(p, q, r), && \text{if } r > 1, \\ &j_*\colon E(p, 1) \vee E(q, 1) \to L(p, q, 1), && \text{if } q > 1. \end{aligned}$$

Moreover, the map

$$\text{(b)} \qquad j_* \vee e_*\colon E(1, 1) \vee [S^p, S^2 \vee S^1 \vee S^1] \to L(p, 1, 1)$$

is onto for $p > 1$.

In a future paper we will study r-link maps in $\mathbb{R}^3$ and S^3 for $r \geq 3$. For instance, if $p_r > 1$, the sphere theorem implies a funny general "periodicity" as follows: The natural map

$$\bigvee_{1 \leq i < j \leq r} L(p_1, \ldots, \hat{p}_i, \ldots, \hat{p}_j, \ldots, p_r, 0) \to L(p_1, \ldots, p_r)$$

is onto. Here ˆ means "omit the corresponding sphere."

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NOTATION. $\simeq$ means homotopic or homotopically equivalent, $\approx$ diffeomorphic. For each manifold M let $\mathrm{int}(M)$ denote the interior and ∂M denote the boundary. 1 is the identity map and [ ] is a homotopy or link homotopy class.

Proof of Theorem 1. The result is obvious for $q = 0$ and is known for $(p, q) = (1, 1)$. Assume $q > 1$, so that also $p > 1$. Recall the definition of a belt projection of a 2-component link map $g: S^p \amalg S^q \to S^3$. Just take a path $\gamma: I \to S^3$, such that $\gamma(0) \in g(S^p)$, $\gamma(1) \in g(S^q)$, $\gamma(0, 1) \cap g(S^p \amalg S^q) = \varnothing$, and define the belt projection of g to be the oriented stereographic projection from $\gamma(\frac{1}{2})$. This is well-defined up to link homotopy (compare **[Ko2]** or **[K1]**). So, if $f: S^p \amalg S^q \to \mathbb{R}^3$ maps each sphere into the unbounded component of the second sphere, then f is belt projection of a link map in S^3, thus trivial by the proposition. So we assume that f maps S^p into a bounded component of $\mathbb{R}^3 \backslash f(S^q)$, which is a component of the complement of $f(S^q)$ in S^3, thus aspherical **[P]**. Contract the map of S^p into a constant map on some point and deform the q-sphere into a surrounding 2-sphere. This proves $[f] \in pt_*(\pi_q S^2)$. It is proved in **[K1]** that pt_* injects.

As expected the only interesting case involves a circle S^1. A link map $f: S^p \amalg S^1 \to \mathbb{R}^3$ is called *proper*, if f is differentiable and embeds the circle. We may replace link homotopy of link maps by link homotopy of proper link maps. Let $f: S^p \amalg S^1 \to \mathbb{R}^3$ be proper, $K := f(S^1) \subset \mathbb{R}^3$.

To prove that e_* maps onto we have to unknot K by a link homotopy. Let T be a tubular neighborhood of K, such that $T \cap f(S^P) = \varnothing$. Choose an arc σ in $X := S^3 \backslash \mathrm{int}\, T$, which joins ∞ to a point on ∂T. Now deform X along this path to get a manifold $X' \subset \mathbb{R}^3 \backslash \mathrm{int}\, T$ diffeomorphic to X. Let S_∞ be a small sphere around ∞. We have the obvious embedding (see Figure 1) $e: X \vee S^2 \approx X' \vee S_\infty \to \mathbb{R}^3$ ($\approx$ means diffeomorphic outside the basepoints), such that $\mathbb{R}^3 \backslash K \simeq e(X \vee S^2) =: Y$. Thus we may assume that f maps S^P into Y. Let $p: \widetilde{X} \to X$ be the universal cover. The universal cover $\widetilde{Y}$ of Y can be described as follows (Figure 2): $p^{-1}(*) = \{*_j\}_{j \in \mathbb{Z}}$ is a countable set in $\widetilde{X}$. To each point $*_j$ we attach a separate 2-sphere S_j. Note that $\widetilde{X}$ is contractible. Let $r_t: \widetilde{X} \to \widetilde{X}$, $0 \le t \le 1$, be

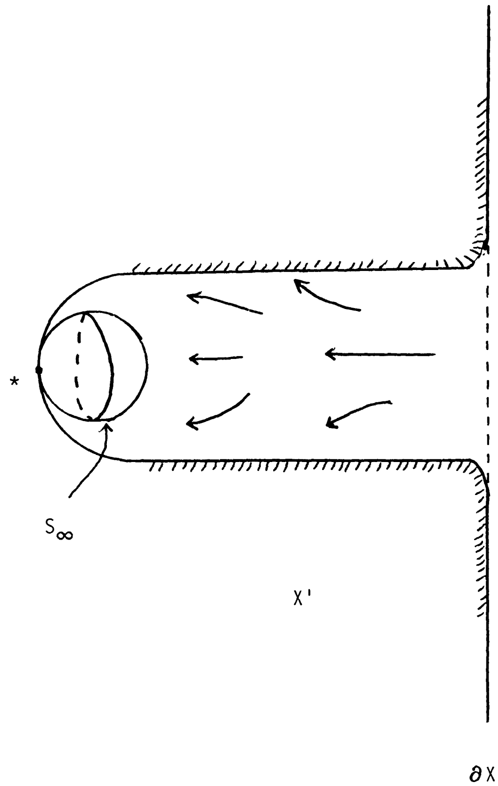

FIGURE 1

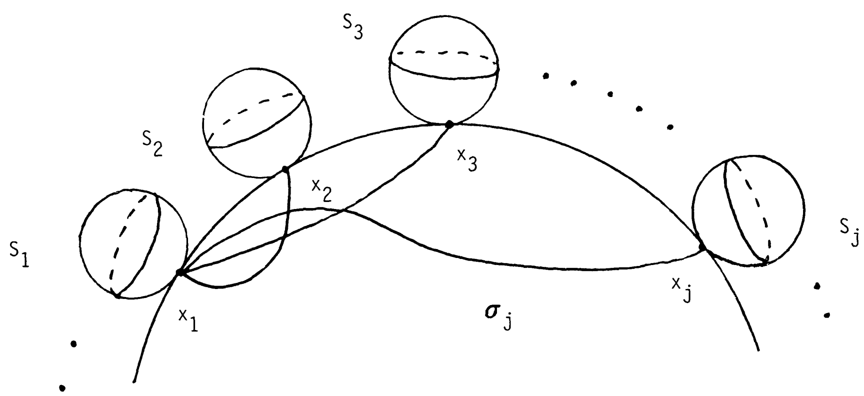

FIGURE 2

a contraction, $r_0 = 1$, $r_1(\widetilde{X}) = *_1$. For each $*_j \in p^{-1}(*)$ there is the path $\sigma_j : I \ni t \to r_t(*_j) \in \widetilde{X}$. Define $\tilde{r}_1 : \widetilde{Y} \to \bigvee_{j \in \mathbb{Z}}(S^2)_j$ as follows: $\tilde{r}_1 \mid \widetilde{X} = r_1$, r_1 maps S_j onto $(S^2)_j$ by a degree 1 map.

Similarly let $i\colon \bigvee_{j\in\mathbb{Z}}(S^2)_j \to \widetilde{Y}$ be the map which takes the upper hemispheres with degree 1 onto S_j. The restriction of i on the lower hemispheres maps the geodesic lines from the equator of S_j to the common basepoint onto the path σ_j. By homotopy extension it follows that $i\circ\tilde{r}_1 \simeq 1$. Lift f_1 to $\tilde{f}_1\colon S^p \to \widetilde{Y}$. Since S^p is compact, $\tilde{f}_1(S^p)\cap p^{-1}(*) = \{*_j\}_{j\in J}$, $J\subset\mathbb{Z}$ finite, and $\tilde{r}_1\circ\tilde{f}_1$ maps into $\bigvee_{j\in J}(S^2)_j$. Thus $(i\circ\tilde{r}_1)\circ\tilde{f}_1$ maps into $\bigcup_{j\in J}(S_j\cup\sigma_j(I))$. The projection of the homotopy $1\circ\tilde{f}_1 \simeq i\circ\tilde{r}_1\circ\tilde{f}_1$ is a homotopy of f_1 in $\mathbb{R}^3\backslash K$ to a map into the union of S_∞ and a finite collection of loops $p(\sigma_j(t))$ based in $*\in S_\infty\cap X'$. Now we can unknot K. This proves that e_* maps onto.

To prove injectivity of e_* we have to take advantage once more of the structure of knot complements. Recall that a knot $K\subset S^3$ comes naturally equipped with a Seifert map, i.e. a differentiable map $h = h(K)\colon X\to S^1$, which restricts to the meridional projection $\partial X\to S^1$ associated to a special framing. h is well defined up to homotopy **[Z]**. Recall that $h^{-1}(y)$ is a Seifert-surface of K for some regular value $y\in S^1$.

DEFINITION. A *based knot* is a pair (K,τ), such that $K\subset\mathbb{R}^3$ is an oriented differentiable knot and τ, the *basing*, is an arc in $X = S^3\backslash \operatorname{int} T$ for some tubular neighborhood $T\subset\mathbb{R}^3$; τ joins $\infty\in S^3$ to some point on ∂T. □

To each based knot we associate an unbased map $g = g(K,\tau)\colon Y := \mathbb{R}^3\backslash\operatorname{int}(T) \to S^1\vee S^2$ as follows: Use τ to construct $X'\vee S_\infty \approx X\vee S^2 \simeq Y$ as above. We can assume that $h(K)$ maps a closed tubular neighborhood N of τ onto $(-1)\in S^1$. Define $g(x) = h(x)$ for $x\in Y\backslash\operatorname{int}(N)$. Let $B_\infty\subset S^3$ denote the ball bounding S_∞. The cell $N' = N\backslash\operatorname{int}(B_\infty)$ can be collapsed onto $(\partial N')\backslash(N'\cap\partial X)$. Similarly we have the retraction $B_\infty\backslash\infty\to S_\infty$. This defines $g'\colon \operatorname{int}(N)\backslash\infty \to \partial X'\vee S_\infty$. We compose g' and $h\vee d$, where $d\colon S_\infty\to S^2$ is a diffeomorphism, to get $g\colon \operatorname{int}(N)\backslash\infty \to S^1\vee S^2$. It is easy to check that the unbased homotopy class of $g(K,\tau)$ does not depend on the choice of $h(K)$. Note that we may move τ in $S^3\backslash K$ fixing $\tau(0)$ and restricting $\tau(1)$ to ∂X without changing $[g(K,\tau)]\in[Y, S^1\vee S^2]$. Thus in case of an unknot $K = U$ the homotopy class of $g(K,\tau)\circ f_1$ does not depend on the choice of τ. This follows from the fact that any two arcs can be deformed into each other in $S^3\backslash K$ by a move as above and a homotopy fixing endpoints.

It is convenient to introduce the following

DEFINITION. A *based homotopy* of based knots (K_0, τ_0) and (K_1, τ_1) is a pair (F, τ) consisting of:

(i) $F : S^1 \times I \to \mathbb{R}^3$ is a homotopy, which restricts to K_0, resp. K_1, on $S^1 \times 0$, resp. $S^1 \times 1$.

(ii) $\tau : I \times I \to S^3$ is an isotopy of arcs and restricts to τ_0, resp. τ_1, on $I \times 0$, resp. $I \times 1$. Furthermore $\tau(0, t) = \infty$ for all $t \in I$ and $\tau(1, t)$ is a point on a meridional curve over some regular point of $F \mid S^1 \times t$. □

LEMMA 1. *Let* $\overline{F}: (S^p \amalg S^1) \times I \to \mathbb{R}^3$ *be a link homotopy between proper link maps and* $(\overline{F} \mid S^1, \tau)$ *be a based homotopy of knots. Then* $g(K_0, \tau_0) \circ (\overline{F} \mid S^p \times 0)$ *and* $g(K_1, \tau_1) \circ (\overline{F} \mid S^p \times 1)$ *are homotopic maps.*

Proof. The crucial point is already in **[M]**. The homomorphisms $H_1(S^3 \backslash K_0) \to \mathbb{Z}$ and $H_1(S^3 \backslash K_1) \to \mathbb{Z}$ corresponding to Seifert-maps for K_0 and K_1 extend to a map $H_1(S^3 \times I \backslash \overline{F}(S^1 \times I))$ onto $\mathbb{Z}$.[1] This can be proved by elementary obstruction theory and Poincaré duality. The resulting map $S^3 \times I \backslash \overline{F}(S^1 \times I) \to S^1$ and the basing τ can be used to construct $\mathbb{R}^3 \times I \backslash \overline{F}(S^1 \times I) \to S^1 \vee S^2$. Composition with the trace of $\overline{F} \mid S^p \times I$ yields the desired homotopy. □

LEMMA 2. *Let* $f = f_1 \amalg f_2 : S^p \amalg S^1 \to \mathbb{R}^3$ *be proper,* $K = f(S^1)$. *Then* $g(K, \tau) \circ f_1 \simeq g(K, \sigma) \circ f_1$ *for any two basings* σ, τ.

Proof. We know already that f can be homotoped into f', such that $f'(S^1)$ is the unknot U. A corresponding differentiable generic link homotopy can be split up into link homotopies which either restrict to isotopy on S^1 or involve a single crossing change of a knot. Since isotopies are ambient we get induced deformations of the basings σ, τ. If a crossing change is involved we may first move a given basing (at the corresponding stage of the homotopy) away from the singularity. This is possible because of transversality. Thus the link homotopy from f to f' induces based knot homotopies from (K, σ) to (U, σ') and (K, τ) to (U, τ'). By Lemma 1 we know $g(K, \tau) \circ f_1 \simeq g(U, \tau') \circ f_1'$ and $g(K, \sigma) \circ f_1 \simeq g(U, \sigma') \circ f_1'$. Now the assertion follows by a previous remark. □

[1] This observation is due to N. Habegger.

Lemmas 1 and 2 and the fact that the arguments in the proof of Lemma 2 can be applied to arbitrary link homotopies show that the assignment

$$\lambda\colon E(p, 1) \to [S^p, S^1 \vee S^2],$$
$$\lambda[f] = [g(K, \tau) \circ f_1], \quad K = f(S^1)$$

is well defined, i.e. independent of all involved choices (f is assumed proper!).

From the construction above follows immediately

LEMMA 3. *The composition*

$$[S^p, S^1 \vee S^2] \xrightarrow{e_*} E(p, 1) \xrightarrow{\lambda} [S^p, S^1 \vee S^2]$$

is given by the identity map. □

This proves the rest of Theorem 1. □

Proof of Theorem 2. If $r > 1$, thus $p, q, r > 1$, we consider a path σ in S^3 which meets the image of each component sphere. We assume $\sigma(0) \in f(S^p)$, $\sigma(t_0) \in f(S^q)$ and $\sigma[0, t_0] \cap f(S^r) = \emptyset$. Then, $f(S^p) \cup \sigma[0, t_0] \cup f(S^q) \subset S^3$ is a path connected subset of S^3. By **[Pa]** each component of the complement of this set is aspherical, so $f \mid S^r$ can be homotoped into a constant. Thus **[f]** is in the image of $j_*\colon E(p, q) \to L(p, q, r)$. But $pt_*\colon \pi_p S^2 \vee \pi_q S^2 \to E(p, q)$ is 1-1 and onto by Theorem 1. If we take into consideration all possibilities, clearly we have that $pt_*\colon \pi_p S^2 \vee \pi_q S^2 \vee \pi_r S^2 \to L(p, q, r)$ is onto. The map, which restricts each component to a map into the complement of the images of the basepoints of the other two components, is a two-sided inverse of pt_*.

Now assume $r = 1$ and $p, q > 1$. As above, a path σ which starts in $f(S^1)$ and meets each component sphere, has empty intersection with one of the remaining spheres for $t \leq t_0$. So we may assume that $f \mid S^q$ maps into a component of $S^3 \backslash (f(S^1) \cup \sigma[0, t_0] \cup f(S^p))$, which is aspherical by **[Pa]**. This proves that j_* is onto. Again, a two-sided inverse is obvious.

The proof of (b) is very similar to the proof of Theorem 1. If the link of the two circles does not split, then $[f]$ is in the image of j_*. Note that the complement of an unsplit link is aspherical by **[Pa]**, 27. Thus, we may assume that there is a 2-sphere S embedded in S^3, which separates two knots K_1, K_2. Choose a basepoint $x \in S$ and arcs σ_1, σ_2, which join points in tubular neighborhoods of the knots to

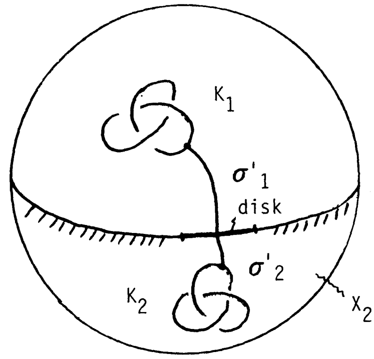

FIGURE 3

$* \in S$ and meet S only in their endpoints. Let X_1, resp. X_2, denote $S^3 \backslash K_1$, resp. $S^3 \backslash K_2$. Clearly, $S^3 \backslash (K_1 \cup K_2) \simeq X_1' \vee X_2' \vee S$, $X_i' \approx X_i$ for $i = 1, 2$. The covering space argument of Theorem 1 carries over first to deform $f \mid S^p$ and then unknot K_1 and K_2. Note that $X_1' \vee X_2'$ is homotopically equivalent to the complement of $K_1 \cup \sigma_1' \cup \sigma_2' \cup K_2$, when σ_1', σ_2' are canonical extensions of σ_1, σ_2 inside the tubular neighborhoods. This shows $[f] \in \mathrm{Im}(e_*)$ and completes the proof. □

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