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MULTIPLE HARMONIC SERIES

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We consider several identities involving the multiple harmonic series

$$\sum_{n_1>n_2>\cdots>n_k\geq 1} \frac{1}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}},$$

which converge when the exponents i_j are at least 1 and $i_1 > 1$. There is a simple relation of these series with products of Riemann zeta functions (the case $k = 1$) when all the i_j exceed 1. There are also two plausible identities concerning these series for integer exponents, which we call the sum and duality conjectures. Both generalize identities first proved by Euler. We give a partial proof of the duality conjecture, which coincides with the sum conjecture in one family of cases. We also prove all cases of the sum and duality conjectures when the sum of the exponents is at most 6.

1. Introduction. The problem of computing the doubly infinite series

$$\sum_{n_1\geq n_2\geq 1} \frac{1}{n_1^a n_2^b}, \tag{1}$$

which converges when $a > 1$ and $b \geq 1$, was discussed by Euler and Goldbach in their correspondence of 1742–3 [**3**]. Euler evaluated several special cases of (1) in terms of the Riemann zeta function

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}.$$

Later, in a paper of 1775 [**2**], Euler found a general formula for (1) in terms of the zeta function when a and b are positive integers whose sum is odd. The simplest such result is

$$\sum_{n_1\geq n_2\geq 1} \frac{1}{n_1^2 n_2} = 2\zeta(3), \tag{2}$$

which has been rediscovered many times since (see [**1**, p. 252] and the references cited there).

We shall consider multiple series of the form

$$S(i_1, i_2, \ldots, i_k) = \sum_{n_1\geq n_2\geq\cdots\geq n_k\geq 1} \frac{1}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}}$$

and

$$A(i_1, i_2, \ldots, i_k) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}}$$

(so (1) is $S(a, b)$). With this notation, $S(i) = A(i) = \zeta(i)$. The relation between the S's and A's should be clear: for example,

$$S(i_1, i_2) = A(i_1, i_2) + A(i_1 + i_2)$$

and

$$\begin{aligned} S(i_1, i_2, i_3) = A(i_1, i_2, i_3) &+ A(i_1 + i_2, i_3) \\ &+ A(i_1, i_2 + i_3) + A(i_1 + i_2 + i_3). \end{aligned}$$

Note that (2) implies $A(2, 1) = \zeta(3)$.

It is immediate from the definitions that

$$S(i_1, i_2) + S(i_2, i_1) = \zeta(i_1)\zeta(i_2) + \zeta(i_1 + i_2)$$

and

$$A(i_1, i_2) + A(i_2, i_1) = \zeta(i_1)\zeta(i_2) - \zeta(i_1 + i_2)$$

whenever $i_1, i_2 > 1$. More generally, if $i_1, i_2, \ldots, i_k > 1$ the sums

$$\sum_{\sigma \in \Sigma_k} S(i_{\sigma(1)}, \ldots, i_{\sigma(k)}) \quad \text{and} \quad \sum_{\sigma \in \Sigma_k} A(i_{\sigma(1)}, \ldots, i_{\sigma(k)})$$

(Σ_k is the symmetric group of degree k) can be expressed in terms of the zeta function. We state and prove such formulas in §2.

There are also two interesting general conjectures about the quantities $A(i_1, \ldots, i_k)$ for positive integer exponents $i_1, \ldots, i_k$, which we call the sum and duality conjectures. Both generalize the identity $A(2, 1) = A(3)$. We state them in §3, and give a partial proof of the duality conjecture in §4. Further evidence for the two conjectures is discussed in §5.

2. Symmetric sums in terms of the zeta function. To state our results we shall require some notation. For a partition $\Pi = \{P_1, P_2, \ldots, P_l\}$ of the set $\{1, 2, \ldots, k\}$, let

$$c(\Pi) = (\operatorname{card} P_1 - 1)!(\operatorname{card} P_2 - 1)! \cdots (\operatorname{card} P_l - 1)!.$$

Also, given such a Π and a k-tuple $\mathbf{i} = \{i_1, \ldots, i_k\}$ of exponents, define

$$\zeta(\mathbf{i}, \Pi) = \prod_{s=1}^{l} \zeta\left(\sum_{j \in P_s} i_j\right).$$

THEOREM 2.1. *For any real* $i_1, \ldots, i_k > 1$,

$$(1) \qquad \sum_{\sigma\in\Sigma_k} S(i_{\sigma(1)}, \ldots, i_{\sigma(k)}) = \sum_{\text{partitions } \Pi \text{ of } \{1,\ldots,k\}} c(\Pi)\zeta(\mathbf{i}, \Pi).$$

Proof. Assume that the i_j are all distinct. (There is no loss of generality, since we can take limits.) The left-hand side of (1) can be written

$$\sum_{\sigma} \sum_{n_1\geq\cdots\geq n_k\geq 1} \frac{1}{n_{\sigma(1)}^{i_1} n_{\sigma(2)}^{i_2} \cdots n_{\sigma(k)}^{i_k}}.$$

Now think of the symmetric group Σ_k as acting on k-tuples $(n_1, \ldots, n_k)$ of positive integers. A given k-tuple $\mathbf{n} = (n_1, \ldots, n_k)$ has an isotropy group $\Sigma_k(\mathbf{n})$ and an associated partition Λ of $\{1, 2, \ldots, k\}$: Λ is the set of equivalence classes of the relation given by $i \sim j$ iff $n_i = n_j$, and $\Sigma_k(\mathbf{n}) = \{\sigma \in \Sigma_k \mid \sigma(i) \sim i \ \forall i\}$. Now the term

$$(2) \qquad \frac{1}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}}$$

occurs on the left-hand side of (1) exactly $\operatorname{card}\Sigma_k(\mathbf{n})$ times. It occurs on the right-hand side in those terms corresponding to partitions Π that are refinements of Λ: letting $\succeq$ denote refinement, (2) occurs

$$\sum_{\Pi\succeq\Lambda} c(\Pi)$$

times. Thus, the conclusion will follow if

$$\operatorname{card}\Sigma_k(\mathbf{n}) = \sum_{\Pi\succeq\Lambda} c(\Pi)$$

for any k-tuple $\mathbf{n}$ and associated partition Λ. To see this, note that $c(\Pi)$ counts the permutations having cycle-type specified by Π: since any element of $\Sigma_k(\mathbf{n})$ has a unique cycle-type specified by a partition that refines Λ, the result follows.

For $k = 3$, the theorem says

$$\begin{aligned}\sum_{\sigma\in\Sigma_3} & S(i_{\sigma(1)}, i_{\sigma(2)}, i_{\sigma(3)}) \\ &= \zeta(i_1)\zeta(i_2)\zeta(i_3) + \zeta(i_1+i_2)\zeta(i_3) + \zeta(i_1)\zeta(i_2+i_3) \\ &\quad + \zeta(i_1+i_3)\zeta(i_2) + 2\zeta(i_1+i_2+i_3)\end{aligned}$$

for $i_1, i_2, i_3 > 1$. This is the main result of [7].

To state the analog of 2.1 for the A's, we require one more bit of notation. For a partition $\Pi = \{P_1, \dots, P_l\}$ of $\{1, 2, \dots, k\}$, let $\tilde{c}(\Pi) = (-1)^{k-l}c(\Pi)$.

THEOREM 2.2. *For any real* $i_1, \dots, i_k > 1$,

$$\sum_{\sigma\in\Sigma_k} A(i_{\sigma(1)}, \dots, i_{\sigma(k)}) = \sum_{\text{partitions } \Pi \text{ of } \{1,\dots,k\}} \tilde{c}(\Pi)\zeta(\mathbf{i}, \Pi).$$

Proof. We follow the same line of argument as in the preceding proof. The left-hand side is now

$$\sum_{\sigma} \sum_{n_1>n_2>\cdots>n_k\geq 1} \frac{1}{n_{\sigma(1)}^{i_1} n_{\sigma(2)}^{i_2} \cdots n_{\sigma(k)}^{i_k}},$$

and a term (2) occurs on the left-hand side once if all the n_i are distinct, and not at all otherwise. Thus, it suffices to show

$$\sum_{\Pi\succeq\Lambda} \tilde{c}(\Pi) = \begin{cases} 1, & \text{if } \operatorname{card}\Lambda = k, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

To prove this, note first that the sign of $\tilde{c}(\Pi)$ is positive if the permutations of cycle-type Π are even, and negative if they are odd: thus, the left-hand side of (3) is the signed sum of the number of even and odd permutations in the isotropy group $\Sigma_k(\mathbf{n})$. But such an isotropy group has equal numbers of even and odd permutations unless it is trivial, i.e. unless the associated partition Λ is $\{\{1\}, \{2\}, \dots, \{k\}\}$.

When all the exponents are 2 we have the following result, which proves a conjecture of C. Moen.

COROLLARY 2.3. *For* $k \geq 1$,

$$A(\underbrace{2, 2, \dots, 2}_{k}) = \frac{\pi^{2k}}{(2k+1)!}.$$

Proof. Applying Theorem 2.2, we obtain

$$A(\underbrace{2, \dots, 2}_{k}) = \frac{1}{k!}\sum_{l=1}^{k} \sum_{\text{partitions } \{P_1,\dots,P_l\} \text{ of } \{1,\dots,k\}} (-1)^{k-l} \times \prod_{s=1}^{l} (\operatorname{card} P_s - 1)!\,\zeta(2\operatorname{card} P_s).$$

Using the well-known formula for values of the zeta function at even integers in terms of Bernoulli numbers (see, e.g., [4]), and writing p_s for card P_s, we can express the right-hand side as

$$\frac{1}{k!}\sum_{l=1}^{k}\sum_{\text{partitions } \{P_1,\dots,P_l\} \text{ of } \{1,\dots,k\}}(-1)^{k-1}$$
$$\times\prod_{s=1}^{l}\frac{(p_s-1)!2^{2p_s-1}(-1)^{p_s+1}B_{2p_s}\pi^{2p_s}}{(2p_s)!}$$
$$=\frac{1}{k!}\sum_{l=1}^{k}\sum_{\text{partitions } \{P_1,\dots,P_l\} \text{ of } \{1,\dots,k\}}2^{2k-l}\pi^{2k}\prod_{s=1}^{l}\frac{(p_s-1)!B_{2p_s}}{(2p_s)!},$$

since the sum of the p_s is k. Now for a given (unordered) sequence $p_1,\dots,p_l$, the number of partitions $\{P_1,\dots,P_l\}$ of $\{1,\dots,k\}$ with $\operatorname{card} P_s=p_s$ for $1\le s\le l$ is

$$\frac{1}{m_1!m_2!\cdots m_k!}\frac{k!}{p_1!p_2!\cdots p_l!},$$

where $m_i=\operatorname{card}\{s\mid p_s=i\}$. Thus, we can rewrite the sum as

$$\frac{(2\pi)^{2k}}{k!}\sum_{l=1}^{k}\frac{1}{2^l}\sum_{p_1+\cdots+p_l=k}\frac{k!}{m_1!\cdots m_k!}\prod_{s=1}^{l}\frac{(p_s-1)!B_{2p_s}}{p_s!(2p_s)!}$$
$$=(2\pi)^{2k}\sum_{l=1}^{k}\sum_{p_1+\cdots+p_l=k}\frac{1}{m_1!\cdots m_k!}\prod_{s=1}^{l}\frac{B_{2p_s}}{2p_s(2p_s)!}$$
$$=(2\pi)^{2k}\sum_{m_1+2m_2+\cdots+km_k=k}\frac{1}{m_1!\cdots m_k!}\left(\frac{B_2}{2\,2!}\right)^{m_1}\cdots\left(\frac{B_{2k}}{2k(2k)!}\right)^{m_k}.$$

It is then enough to prove the following proposition.

PROPOSITION 2.4. *If B_n denotes the nth Bernoulli number, then for $k\ge 1$*

$$\sum_{m_1+2m_2+\cdots+km_k=k}\frac{1}{m_1!\cdots m_k!}\left(\frac{B_2}{2\,2!}\right)^{m_1}\cdots\left(\frac{B_{2k}}{2k(2k)!}\right)^{m_k}=\frac{1}{2^{2k}(2k+1)!}.$$

Proof. Define

$$\beta(x)=\sum_{k=1}^{\infty}\frac{B_{2k}x^{2k}}{2k(2k)!}.$$

If we also let

$$f(x) = \log\left(\frac{\sinh x/2}{x/2}\right) = \log\left(\frac{e^x - 1}{x}\right) - \frac{x}{2},$$

then $\beta(x) = f(x)$ since $\beta(0) = f(0) = 0$ and

$$\beta'(x) = \sum_{k=1}^{\infty} \frac{B_{2k}x^{2k-1}}{(2k)!} = \frac{1}{x}\sum_{n=2}^{\infty}\frac{B_n x^n}{n!} = \frac{1}{x}\left(\frac{x}{e^x-1} - 1 + \frac{x}{2}\right)$$
$$= \frac{1}{e^x-1} - \frac{1}{x} + \frac{1}{2} = \frac{e^x}{e^x-1} - \frac{1}{x} - \frac{1}{2} = f'(x).$$

(Here we have used the generating function for the Bernoulli numbers.) Thus

$$\sum_{k=0}^{\infty} \frac{x^{2k}}{2^{2k}(2k+1)!} = \frac{\sinh x/2}{x/2} = e^{\beta(x)} \tag{4}$$
$$= \sum_{l=0}^{\infty}\frac{1}{l!}\left(\frac{B_2x^2}{2\,2!} + \frac{B_4x^4}{4\,4!} + \dots\right)^l .$$

Using the multinomial theorem, expand out the right-hand side to obtain

$$\sum_{l=0}^{\infty}\frac{1}{l!}\sum_{m_1+m_2+\cdots+m_k=l}\binom{l}{m_1 m_2\cdots m_k}\left(\frac{B_2x^2}{2\,2!}\right)^{m_1}\cdots\left(\frac{B_{2k}x^{2k}}{2k(2k)!}\right)^{m_k}$$
$$= \sum_{k=0}^{\infty} x^{2k}\sum_{m_1+2m_2+\cdots+km_k=k}\frac{1}{m_1!m_2!\cdots m_k!}$$
$$\times\left(\frac{B_2}{2\,2!}\right)^{m_1}\cdots\left(\frac{B_{2k}}{2k(2k)!}\right)^{m_k},$$

and the conclusion follows by equating coefficients of x^{2k} in (4).

3. The sum and duality conjectures. We first state the sum conjecture, which is due to C. Moen [5].

Sum conjecture. For positive integers $k < n$,

$$\sum_{\substack{i_1+\cdots+i_k=n \\ i_1>1}} A(i_1, \dots, i_k) = \zeta(n),$$

where the sum is extended over k-tuples $i_1, \dots, i_k$ of positive integers with $i_1 > 1$.

Three remarks concerning this conjecture are in order. First, it implies

$$\sum_{\substack{i_1+\cdots+i_k=n \\ i_1>1}} S(i_1, \ldots, i_k) = \binom{n-1}{k-1} \zeta(n), \tag{1}$$

as we prove below. Second, in the case $k = 2$ it says that

$$A(n-1, 1) + A(n-2, 2) + \cdots + A(2, n-2) = \zeta(n),$$

or, using the relation between the A's and S's and Theorem 2.1,

$$2S(n-1, 1) = (n+1)\zeta(n) - \sum_{k=2}^{n-2} \zeta(k)\zeta(n-k).$$

This was proved in Euler's paper [2] and has been rediscovered several times, in particular by Williams [8]. Finally, C. Moen [5] has proved the sum conjecture for $k = 3$ by lengthy but elementary arguments.

For the duality conjecture, we first define an involution τ on the set $\mathfrak{S}$ of finite sequences of positive integers whose first element is greater than 1. Let $\mathfrak{I}$ be the set of strictly increasing finite sequences of positive integers, and let $\Sigma\colon \mathfrak{S} \to \mathfrak{I}$ be the function that sends a sequence in $\mathfrak{S}$ to its sequence of partial sums. If $\mathfrak{I}_n$ is the set of sequences in $\mathfrak{I}$ whose last element is at most n, we have two commuting involutions R_n and C_n on $\mathfrak{I}_n$ defined by

$$R_n(a_1, a_2, \ldots, a_l) = (n+1-a_l, n+1-a_{l-1}, \ldots, n+1-a_1)$$

and

$$C_n(a_1, \ldots, a_l) = \text{complement of } \{a_1, \ldots, a_l\} \text{ in } \{1, 2, \ldots, n\} \text{ arranged in increasing order.}$$

Then our definition of τ is

$$\tau(I) = \Sigma^{-1} R_n C_n \Sigma(I) = \Sigma^{-1} C_n R_n \Sigma(I)$$

for $I = (i_1, i_2, \ldots, i_k) \in \mathfrak{S}$ with $i_1 + \cdots + i_k = n$. (The reader may verify that $\tau(I)$ is actually in $\mathfrak{S}$, has length $n-k$, and its elements have sum n.) For example,

$$\begin{aligned} \tau(3, 4, 1) &= \Sigma^{-1} C_8 R_8(3, 7, 8) \\ &= \Sigma^{-1}(3, 4, 5, 7, 8) = (3, 1, 1, 2, 1). \end{aligned}$$

We shall say the sequences $(i_1, \ldots, i_k)$ and $\tau(i_1, \ldots, i_k)$ are dual to each other, and refer to a sequence fixed by τ as self-dual.

Now we can state our second conjecture.

Duality conjecture. If $(h_1, \ldots, h_{n-k})$ is dual to $(i_1, \ldots, i_k)$, then $A(h_1, \ldots, h_{n-k}) = A(i_1, \ldots, i_k)$.

We include some remarks on how τ may be more easily computed. The set $\mathfrak{S}$ is a semigroup under the operation given by concatenation, and the indecomposables are evidently sequences of the form $(h+1, 1, \ldots, 1)$ with $h \geq 1$. It is easily computed that

$$\tau(h+1, \underbrace{1, \ldots, 1}_{k-1}) = (k+1, \underbrace{1, \ldots, 1}_{h-1}). \tag{2}$$

(In particular, the duality conjecture implies

$$A(2, \underbrace{1, \ldots, 1}_{n-2}) = A(n) = \zeta(n)$$

for integer $n \geq 2$. Note that this is also the sum conjecture for the case $k = n-1$. This case follows from Theorem 4.4 below.) Together with the following proposition, (2) gives an effective method for computing $\tau(I)$ for any $I \in \mathfrak{S}$.

PROPOSITION 3.1. *For $I_1, I_2 \in \mathfrak{S}$, $\tau(I_1 I_2) = \tau(I_2)\tau(I_1)$.*

Proof. By induction on the number of indecomposables in I_2 we can reduce to the case where I_2 is indecomposable. So let

$$I_1 = (i_1, \ldots, i_k), \quad I_2 = (h+1, \underbrace{1, \ldots, 1}_{l-1}),$$
$$n = i_1 + \cdots + i_k, \quad m = n+h+l.$$

Then

$$\begin{aligned}
\tau(I_1 I_2) &= \Sigma^{-1} C_m R_m \Sigma(i_1, \ldots, i_k, h+1, \underbrace{1, \ldots, 1}_{l-1}) \\
&= \Sigma^{-1} C_m R_m (i_1, i_1 + i_2, \ldots, i_1 + \cdots + i_k, n+h+1, \\
&\qquad\qquad n+h+2, \ldots, n+h+l) \\
&= \Sigma^{-1} C_m (1, 2, \ldots, l, l+h+1, l+h+i_k+1, \ldots, \\
&\qquad\qquad l+h+i_2+\cdots+i_k+1) \\
&= \Sigma^{-1}(l+1, \ldots, l+h, l+h+c_1, \ldots, l+h+c_{n-k}) \\
&= (l+1, 1, \ldots, 1, c_1, c_2 - c_1, \ldots, c_{n-k} - c_{n-k-1}),
\end{aligned}$$

where $C_n R_n \Sigma(I_1) = (c_1, c_2, \dots, c_{n-k})$, from which the conclusion follows.

We close this section by proving that the sum conjecture implies (1). We first note that $\mathfrak{S}$ has a partial order given by refinement, e.g. $(2, 1, 2)$ and $(3, 1, 1)$ both refine $(3, 2)$. Further, $S(i_1, \dots, i_k)$ is the sum of those $A(j_1, \dots, j_l)$ for which $(i_1, \dots, i_k)$ is a refinement of $(j_1, \dots, j_l)$. Thus the sum

$$\sum_{\substack{i_1+\cdots+i_k=n \\ i_1>1}} S(i_1, \dots, i_k)$$

can be written as a sum of terms $A(j_1, \dots, j_l)$, each of which appears with multiplicity

$$\begin{aligned} &\operatorname{card}\{(i_1, \dots, i_k) \mid i_1 > 1 \text{ and } (i_1, \dots, i_k) \text{ refines } (j_1, \dots, j_l)\} \\ &\quad = \binom{n-l-1}{k-l}. \end{aligned}$$

The equality can be seen combinatorially: think of an ordered sum $i_1 + \cdots + i_k = n$ that refines $j_1 + \cdots + j_l = n$ as defined by choosing $k - l$ division points (in addition to those defining the first sum) out of $n - l - 1$ possibilities. Thus

$$\sum_{\substack{i_1+\cdots+i_k=n \\ i_1>1}} S(i_1, \dots, i_k) = \sum_{l=1}^{k} \binom{n-l-1}{k-l} \sum_{\substack{j_1+\cdots+j_l=n \\ j_1>1}} A(j_1, \dots, j_l).$$

Assuming the sum conjecture, the latter sum is

$$\sum_{l=1}^{k} \binom{n-l-1}{k-l} \zeta(n) = \binom{n-1}{k-1} \zeta(n).$$

4. Partial proof of the duality conjecture. We shall prove the duality conjecture for sequences $(i_1, 1, \dots, 1)$ with $i_1 > 1$. We use the following theorem of L. J. Mordell [**6**].

THEOREM 4.1 (Mordell). *For positive integer k and any $a > -k$,*

$$\sum_{n_1, \dots, n_k \geq 1} \frac{1}{n_1 n_2 \cdots n_k (n_1 + \cdots + n_k + a)} = k! \sum_{i=0}^{\infty} \frac{(-1)^i}{i!(i+1)^{k+1}} \binom{a-1}{i}.$$

From this we deduce the following result.

COROLLARY 4.2. *For integer* $h \geq 1$,

$$\sum_{n_1,\dots,n_k \geq 1} \frac{1}{n_1 n_2 \cdots n_k (n_1 + \cdots + n_k)^h} = k! A(k+1, \underbrace{1, \dots, 1}_{h-1}).$$

Proof. Differentiate Mordell's formula p times with respect to a to get

$$\sum_{n_1,\dots,n_k \geq 1} \frac{1}{n_1 n_2 \cdots n_k (n_1 + \cdots + n_k + a)^{p+1}}$$
$$= k! \sum_{i=p}^{\infty} \frac{(-1)^{i-p}}{i!(i+1)^{k+1}} \sum_{1 \leq l_1 < \cdots < l_p \leq i} \frac{1}{(a-l_1)\cdots(a-l_p)} \binom{a-1}{i}.$$

Now set $a = 0$ to obtain

$$\sum_{n_1,\dots,n_k \geq 1} \frac{1}{n_1 n_2 \dots n_k (n_1 + \cdots + n_k)^{p+1}}$$
$$= k! \sum_{i=p}^{\infty} \frac{(-1)^{i-p}}{i!(i+1)^{k+1}} \sum_{1 \leq l_1 < \cdots < l_p \leq i} \frac{(-1)^{p+i} i!}{l_1 \cdots l_p}$$
$$= k! \sum_{1 \leq l_1 < \cdots < l_p < j} \frac{1}{l_1 \cdots l_p j^{k+1}},$$

from which the conclusion follows by setting $h = p + 1$.

On the other hand, we can use the following rearrangement lemma for multiple series to rewrite the left-hand side of 4.2 another way.

LEMMA 4.3. *Let* f *be a symmetric function in* k *variables. Then*

$$\sum_{n_1,\dots,n_k \geq 1} \frac{k! f(n_1, \dots, n_k)}{n_1(n_1+n_2)\cdots(n_1+\cdots+n_k)} = \sum_{n_1,\dots,n_k \geq 1} \frac{f(n_1,\dots,n_k)}{n_1 n_2 \cdots n_k},$$

provided the sums converge.

Proof. We proceed by induction on k. The result is immediate for

$k = 1$. Now assume it for k and consider

$$\sum_{n_1,\dots,n_{k+1}\ge 1} \frac{(k+1)!f(n_1,\dots,n_{k+1})}{n_1(n_1+n_2)\cdots(n_1+\cdots+n_{k+1})} \tag{1}$$

$$= \sum_{n_1,\dots,n_k\ge 1} \frac{k!}{n_1(n_1+n_2)\cdots(n_1+\cdots+n_k)} \times \sum_{n_{k+1}=1}^{\infty} \frac{(k+1)f(n_1,\dots,n_{k+1})}{n_1+\cdots+n_{k+1}},$$

where f is symmetric in $k+1$ variables. Since the function

$$\sum_{n_{k+1}=1}^{\infty} \frac{(k+1)f(n_1,\dots,n_{k+1})}{n_1+\cdots+n_{k+1}}$$

is symmetric in $n_1, \dots, n_k$, we can apply the induction hypothesis to transform the right-hand side of (1) into

$$\sum_{n_1,\dots,n_k\ge 1} \frac{k+1}{n_1 n_2\cdots n_k} \sum_{n_{k+1}=1}^{\infty} \frac{f(n_1,\dots,n_{k+1})}{n_1+\cdots+n_{k+1}}.$$

Let S be this sum divided by $k+1$. Then

$$\sum_{n_1,\dots,n_{k+1}\ge 1} \frac{f(n_1,\dots,n_{k+1})}{n_1 n_2\cdots n_{k+1}} - S$$

$$= \sum_{n_1,\dots,n_k\ge 1} \frac{1}{n_1\cdots n_k} \times \sum_{n_{k+1}=1}^{\infty} f(n_1,\dots,n_{k+1}) \left[\frac{1}{n_{k+1}} - \frac{1}{n_1+\cdots+n_{k+1}}\right]$$

$$= \sum_{n_1,\dots,n_k\ge 1} \frac{1}{n_1\cdots n_k} \sum_{n_{k+1}=1}^{\infty} \frac{f(n_1,\dots,n_{k+1})(n_1+\cdots+n_k)}{n_{k+1}(n_1+\cdots+n_{k+1})}$$

$$= \sum_{j=1}^{k} \sum_{n_1,\dots,n_{k+1}\ge 1} \frac{f(n_1,\dots,n_{k+1})n_j}{n_1\cdots n_{k+1}(n_1+\cdots+n_{k+1})}.$$

By permuting the variables, we see that each of the k terms in the latter summation is just S. Thus, the sum is kS and we have

$$\sum_{n_1,\dots,n_{k+1}\geq 1} \frac{f(n_1,\dots,n_{k+1})}{n_1 n_2 \cdots n_{k+1}} = (k+1)S$$
$$= \sum_{n_1,\dots,n_{k+1}\geq 1} \frac{(k+1)!f(n_1,\dots,n_{k+1})}{n_1(n_1+n_2)\cdots(n_1+\cdots+n_{k+1})}.$$

Applying the lemma with $f(n_1,\dots,n_k) = (n_1+\cdots+n_k)^{-h}$ gives

$$A(h+1,\underbrace{1,\dots,1}_{k-1}) = \sum_{n_1,\dots,n_k\geq 1} \frac{1}{n_1(n_1+n_2)\cdots(n_1+\cdots+n_k)^{h+1}}$$
$$= \frac{1}{k!}\sum_{n_1,\dots,n_k\geq 1} \frac{1}{n_1\cdots n_k(n_1+\cdots+n_k)^h}.$$

Putting this together with Corollary 4.2, we have

THEOREM 4.4. *For integers $h, k \geq 1$,*

$$A(h+1,\underbrace{1,\dots,1}_{k-1}) = A(k+1,\underbrace{1,\dots,1}_{h-1}).$$

From the remarks in §3, Theorem 4.4 is just the duality conjecture for indecomposable sequences of $\mathfrak{S}$.

5. Evidence for the conjectures. For the computations of this section, the following result will be useful.

THEOREM 5.1. *Let $i_1, i_2, \dots, i_k$ be any sequence of positive integers with $i_1 > 1$. Then*

$$\sum_{l=1}^{k} A(i_1,\dots,i_l+1,\dots,i_k)$$
$$= \sum_{\substack{1\leq l\leq k\\ i_l\geq 2}} \sum_{j=0}^{i_l-2} A(i_1,\dots,i_{l-1},i_l-j,j+1,i_{l+1},\dots,i_k).$$

Proof. By multiplying series, we have

$$\begin{aligned}(1)\quad & A(i_1+1,i_2,\dots,i_k)+A(i_1,1,i_2,\dots,i_k)\\ &+A(i_1,i_2+1,\dots,i_k)+\cdots+A(i_1,i_2,\dots,i_k,1)\end{aligned}$$
$$= \sum_{n_1,\dots,n_k\geq 1} \frac{1}{s_1^{i_k}s_2^{i_{k-1}}\cdots s_k^{i_1}} \sum_{j=1}^{s_k}\frac{1}{j},$$

where we write s_r for $n_1 + \cdots + n_r$. But the right-hand side of (1) can be written

$$(2) \qquad \sum_{n_1, \ldots, n_k \geq 1} \frac{1}{s_1^{i_k} s_2^{i_{k-1}} \cdots s_k^{i_1}} \sum_{n_{k+1} \geq 1} \left[\frac{1}{n_{k+1}} - \frac{1}{s_{k+1}} \right],$$

and by a standard partial-fractions identity we have

$$\frac{1}{n_{k+1} s_{k+1}^{i_1}} = \frac{1}{n_{k+1} s_k^{i_1}} - \sum_{j=0}^{i_1-1} \frac{1}{s_k^{j+1} s_{k+1}^{i_1-j}}$$

or

$$\frac{1}{s_k^{i_1}} \left[\frac{1}{n_{k+1}} - \frac{1}{s_{k+1}} \right] = \frac{1}{n_{k+1} s_{k+1}^{i_1}} + \sum_{j=0}^{i_1-2} \frac{1}{s_k^{j+1} s_{k+1}^{i_1-j}}.$$

Thus, (2) can be rewritten as

$$\sum_{n_1, \ldots, n_{k+1} \geq 1} \frac{1}{s_1^{i_k} s_2^{i_{k-1}} \cdots s_{k-1}^{i_2} n_{k+1} s_{k+1}^{i_1}} + \sum_{j=0}^{i_1-2} \sum_{n_1, \ldots, n_{k+1} \geq 1} \frac{1}{s_1^{i_k} s_2^{i_{k-1}} \cdots s_{k-1}^{i_2} s_k^{j+1} s_{k+1}^{i_1-j}},$$

or, since the first sum is unchanged by permuting n_k and n_{k+1},

$$\sum_{n_1, \ldots, n_{k+1} \geq 1} \frac{1}{s_1^{i_k} s_2^{i_{k-1}} \cdots s_{k-1}^{i_2} n_k s_{k+1}^{i_1}} + \sum_{j=0}^{i_1-2} A(i_1 - j, j+1, i_2, \ldots, i_k).$$

Now we use the partial-fractions expansion

$$\frac{1}{n_k s_k^{i_2}} = \frac{1}{n_k s_{k-1}^{i_2}} - \sum_{j=0}^{i_2-1} \frac{1}{s_{k-1}^{j+1} s_k^{i_2-j}}$$

to obtain

$$\sum_{n_1, \ldots, n_{k+1} \geq 1} \frac{1}{s_1^{i_k} s_2^{i_{k-1}} \cdots s_{k-1}^{i_2} n_k s_{k+1}^{i_1}} = \sum_{n_1, \ldots, n_{k+1} \geq 1} \frac{1}{s_1^{i_k} \cdots s_{k-2}^{i_3} n_{k-1} s_k^{i_2} s_{k+1}^{i_1}} + \sum_{j=0}^{i_2-1} A(i_1, i_2 - j, j+1, i_3, \ldots, i_k).$$

Continuing in this way, we conclude that (2) equals

$$\begin{aligned}&\sum_{j=0}^{i_1-2} A(i_1 - j, j+1, i_2, \dots, i_k)\\ &\quad + \sum_{j=0}^{i_2-1} A(i_1, i_2 - j, j+1, i_3, \dots, i_k)\\ &\quad + \dots + \sum_{j=0}^{i_k-1} A(i_1, \dots, i_{k-1}, i_k - j, j+1)\\ &\quad + \sum_{n_1, \dots, n_{k+1} \geq 1} \frac{1}{n_1 s_2^{i_k} s_3^{i_{k-1}} \cdots s_{k+1}^{i_1}},\end{aligned}$$

and since the last sum is $A(i_1, \dots, i_k, 1)$ the conclusion follows by substitution for (2) on the right-hand side of (1) and appropriate cancellation.

Note that by taking $k = 1$ in Theorem 5.1 we get

$$A(i_1 + 1) = A(i_1, 1) + A(i_1 - 1, 2) + \dots + A(2, i_1 - 1),$$

which is just the sum conjecture for two arguments. Recall that the sum conjecture for $n-1$ arguments (where n is the sum of the arguments) follows from Theorem 4.4: using this together with Theorem 5.1 applied to the sequence

$$(2, \underbrace{1, \dots, 1}_{n-3}),$$

we get the sum conjecture for $n-2$ arguments.

Now we consider relations among the quantities $A(i_1, \dots, i_k)$ with $n = i_1 + \dots + i_k \leq 6$. For $n = 3$ we have $A(2, 1) = A(3)$ by Theorem 4.4. For $n = 4$ we have $A(2, 2, 1) = A(4)$ and $A(3, 1) + A(2, 2) = A(4)$ from Theorems 4.4 and 5.1; since the sequences $(3, 1)$ and $(2, 2)$ are both self-dual, all instances of both conjectures hold in this case. For $n = 5$, Theorems 4.4 and 5.1 establish the sum conjecture for all values of k. Also, 5.1 applied to the sequence $(3, 1)$ gives

$$A(4, 1) + A(3, 2) = A(3, 1, 1) + A(2, 2, 1). \tag{3}$$

But the sum conjecture implies

$$A(4, 1) + A(3, 2) + A(2, 3) = A(3, 1, 1) + A(2, 2, 1) + A(2, 1, 2),$$

so $A(2, 3) = A(2, 1, 2)$ (an instance of the duality conjecture). Now $A(3, 1, 1) = A(4, 1)$ by Theorem 4.4, so $A(2, 2, 1) = A(3, 2)$ from (3) and all instances of the duality conjecture hold in this case.

For $n = 6$, we get the sum conjecture immediately for $k = 2, 4, 5$. Theorem 5.1 applied to the sequences $(4, 1)$, $(3, 2)$, and $(2, 3)$ gives respectively

$$A(5, 1) + A(4, 2) = A(4, 1, 1) + A(3, 2, 1) + A(2, 3, 1) \tag{4}$$

$$A(4, 2) + A(3, 3) = A(3, 1, 2) + A(2, 2, 2) + A(3, 2, 1) \tag{5}$$

$$A(3, 3) + A(2, 4) = A(2, 1, 3) + A(2, 3, 1) + A(2, 2, 2). \tag{6}$$

On the other hand, 5.1 applied to the sequences $(3, 1, 1)$, $(2, 2, 1)$, and $(2, 1, 2)$ give

$$\begin{aligned} &A(4, 1, 1) + A(3, 2, 1) + A(3, 1, 2) \\ &\quad = A(3, 1, 1, 1) + A(2, 2, 1, 1) \end{aligned} \tag{7}$$

$$\begin{aligned} &A(3, 2, 1) + A(2, 3, 1) + A(2, 2, 2) \\ &\quad = A(2, 1, 2, 1) + A(2, 2, 1, 1) \end{aligned} \tag{8}$$

$$\begin{aligned} &A(3, 1, 2) + A(2, 2, 2) + A(2, 1, 3) \\ &\quad = A(2, 1, 1, 2) + A(2, 1, 2, 1) \end{aligned} \tag{9}$$

respectively. Since we know the sum conjecture holds for $k = 2$ and $k = 4$ in this case, the sum of the left-hand sides of (4) and (6) is $\zeta(6)$, as is the sum of the right-hand sides of (7) and (9). Thus

$$\begin{aligned} &A(4, 1, 1) + A(3, 2, 1) + 2A(2, 3, 1) + A(2, 1, 3) + A(2, 2, 2) \\ &\quad = A(4, 1, 1) + A(3, 2, 1) + 2A(3, 1, 2) + A(2, 2, 2) + A(2, 1, 3), \end{aligned}$$

or $A(2, 3, 1) = A(3, 1, 2)$, an instance of the duality conjecture. (Note that all other sequences of length 3 are self-dual.) Using this fact, we can add equations (4) and (6) to get the sum conjecture for $k = 3$. Also, we can conclude from (4) and (7) that

$$A(5, 1) + A(4, 2) = A(3, 1, 1, 1) + A(2, 2, 1, 1).$$

But $A(5, 1) = A(3, 1, 1, 1)$ from Theorem 4.4, so this means $A(4, 2) = A(2, 2, 1, 1)$. Now we can use (5) and (8) to conclude similarly that $A(3, 3) = A(2, 1, 2, 1)$, and finally (6) and (9) to get $A(2, 4) = A(2, 1, 1, 2)$. Thus all instances of both conjectures are true when $n = 6$.

References

[1] Bruce C. Berndt, *Ramanujan's Notebooks, Part* I, Springer-Verlag, New York, 1985.

[2] L. Euler, *Meditationes circa singulare serierum genus*, Novi Comm. Acad. Sci. Petropol., **20** (1775), 140–186. Reprinted in "Opera Omnia", ser. I, vol. 15, B. G. Teubner, Berlin, 1927, pp. 217–267.

[3] P. H. Fuss (ed.), *Correspondence Mathématique et Physique de quelques célèbres Géomètres* (Tome 1), St. Petersburg, 1843.

[4] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, Springer-Verlag, New York, 1982. Chap. 15.

[5] C. Moen, *Sums of multiple series*, preprint.

[6] L. J. Mordell, *On the evaluation of some multiple series*, J. London Math. Soc., **33** (1958), 368–371.

[7] R. Sita Ramachandra Rao and M. V. Subbarao, *Transformation formulae for multiple series*, Pacific J. Math., **113** (1984), 471–479.

[8] G. T. Williams, *A new method of evaluating* $\zeta(2n)$, Amer. Math. Monthly, **60** (1953), 19–25.

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