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ACTIONS OF FINITE GROUPS ON KNOT COMPLEMENTS

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We examine the symmetry of the complement of a non-trivial knot K in S^3 and obtain a classification of the actions of finite groups on the complement of a non-trivial knot in the case where the actions extend to non-fixed point free actions on the three sphere. We prove the result by showing first an extension theorem which says that any action of finite group on a non-trivial knot complement extends to an action on the three sphere and then by applying the solution of the Smith conjecture.

Let $N(K)$ be a regular neighborhood of K; m, l be a meridian and a preferred longitude of K in $\partial N(K)$ respectively; $[m]$, $[l]$ be the homology classes in $H_1(\partial N(K), Z)$ represented by m, l respectively. A knot is called a hyperbolic knot if $S^3 - K$ has a hyperbolic structure. See $[R]$, or $[B, Z]$ for the standard terminology that we use. The main results of this note are the following. Theorem 1 also follows from the recent result of Gordon and Luecke **[G, L]**. Since the proof is simple, it is included here for completeness.

THEOREM 1. *If K is a hyperbolic knot, then any self-diffeomorphism of the knot complement $S^3 - \text{int}(N(K))$ extends to a self-diffeomorphism of S^3.*

Satellite knots have property P by Gabai's work, and torus knots are also known to have property P. One obtains the following theorem.

COROLLARY 1. *Any self-diffeomorphism of a non-trivial knot complement $S^3 - N(K)$ extends to a self-diffeomorphism of S^3.*

THEOREM 2. *If G is a finite group acting smoothly on the complement $S^3 - \text{int}(N(K))$ of a non-trivial knot K, then the group G is a cyclic or a dihedral group, and the G-action extends to a G-action on S^3. In particular, if K is a hyperbolic knot, then $\text{Out}(\pi_1(S^3 - K))$ (or what is the same $\text{Isom}(S^3 - K)$) is a cyclic or a dihedral group.*

With the help of a computer, Riley **[Ri]** has calculated the

$\mathrm{Out}(\pi_1(S^3 - K))$ for the following hyperbolic knots, 5_2, 6_3, 7_7, 8_{21}, 9_{35}, 9_{43}, and 9_{48}, the corresponding groups are: D_2, D_4, D_4, D_2, D_6, Z_2, and D_6. The theorem explains the general fact behind Riley's work. Combining with the work of Culler, Gordon, Luecke, Shalen (see **[CGLS]**), Bleiler and Scharlemann **[B, S]** on the property P of non-trivial knots invariant under non-trivial periodic automorphisms of S^3, we have the following.

COROLLARY 2. *If there exists a finite group acting smoothly non-trivially on a knot complement in S^3, then the knot has property* P. *In particular, if K is a hyperbolic knot with non-trivial* $\mathrm{Out}(\pi_1(S^3 - K))$, *then K has property* P.

If the group G in Theorem 2 is cyclic, the G-action on the knot complement can be described more explicitly. Before stating the corollary, let us make the following conventions. A $2\pi/n$-rotation of S^3 is a Z_n-action which is conjugate to the orientation preserving Z_n-action generated by A where A sends a point (x, z) in $S^3 = R^1 \times C \cup \{\text{infinity}\}$ to $(x, e^{2\pi i/n} z)$ and infinity to infinity. The circle $\{(x, z) | z = 0\} \cup \{\text{infinity}\}$ is called the axis of the rotation. A twisted $2\pi/n$-rotation of S^3 is an action conjugate to the non-orientation preserving Z_n-action generated by α, where α is described as follows. Represent S^3 as $(R^1 \times C) \cup \{\text{infinity}\}$, α is the automorphism sending (x, z) to $(-x, -e^{2\pi i/n} z)$, and infinity to infinity. The circle $\{(x, z) | z = 0\} \cup \{\text{infinity}\}$ is called the axis of the twisted rotation. A reflection of S^3 through two points is an action conjugate to the orientation reversing involution of S^3 generated by β, where β is the automorphism of S^3 considered as $R^3 \cup \{\text{infinity}\}$ sending x to $-x$, for x in R^3, and infinity to infinity.

COROLLARY 3. *The smooth action of a cyclic group Z_n on a non-trivial knot complement $S^3 - \mathrm{int}(N(K))$ are classified as follows.*

(I) *The action preserves the orientation. There are two cases.*

(a) *The action on $S^3 - \mathrm{int}(N(K))$ is free. Then the action is induced by a fixed point free Z_n-action on S^3. K is invariant under the action.*

(b) *The action is not free. Then the Z_n-action is induced by a $2\pi/n$-rotation of S^3 about a trivial knot L. K is invariant under the rotation. K is disjoint from L, or K intersects L transversely in two points. If the latter happens, $n = 2$.*

(II) *The Z_n-action on $S^3 - \mathrm{int}(N(K))$ does not preserve the orientation. Then the Z_n-action has fixed points in S^3, and is of even order.*

There are four kinds:

(c) $n = 2$. *Then the action is induced by a reflection* R *of* S^3 *through two points, or is induced by a reflection* R' *of* S^3 *with respect to a two-sphere, which is the same as a twisted* π*-rotation of* S^3. K *is invariant under the involution. There are three types of* Z_2*-actions on* $S^3 - \operatorname{int}(N(K))$.

$(c)_1$ K *is disjoint from the two fixed points of the reflection* R. *In this case the* Z_2*-action on* $S^3 - \operatorname{int}(N(K))$ *has two fixed points.*

$(c)_2$ K *contains the two fixed points of* R. *In this case, the* Z_2*-action is a free action on* $S^3 - \operatorname{int}(N(K))$.

$(c)_3$ K *intersects the 2-sphere fixed points of* R' *transversely in two points. In this case,* K *is of the form* $K = L\#(-L)$ *for some knot* L.

(d) $n \geq 4$. *Then the action is induced by a twisted* $2\pi/n$*-rotation of* S^3 *about an axis* L. K *is invariant, and is disjoint from* L.

We state the following as a corollary for convenience.

COROLLARY 4. *If a cyclic group* Z_n *generated by* g *acts smoothly on a non-trivial knot complement* $S^3 - \operatorname{int}(N(K))$ *such that* $g_*([l]) = -[l]$ *in* $H_1(\partial N(K), Z)$, *then* g *is an involution.*

Combining Corollaries 3 and 4, smooth action of dihedral groups on a knot complement can also be classified. We omit it here.

Recall that a knot K is invertible if K is oriented equivalent to $-K$, the inverted knot of K; K is amphicheiral if K is equivalent to its mirror-image K^*.

COROLLARY 5. *If* K *is a hyperbolic knot in* S^3, *then the following holds.*

(a) K *is invertible if and only if* K *is invariant under a* π*-rotation in* S^3 *about an axis* L *such that* L *intersects* K *transversely in two points.*

(b) K *is amphicheiral if and only if* K *is invariant under a twisted* $2\pi/n$*-rotation of* S^3 *about an axis missing* K, *for* $n \geq 4$, *or* K *is invariant under a reflection of* S^3 *through two points missing* K.

(c) *If* K *is both invertible and amphicheiral, then* K *is invariant under a reflection of* S^3 *through two points contained in* K.

In §1, we prove Theorem 1. In §2, we prove Theorem 2, and its corollaries. In the appendix, we prove the following proposition concerning smooth non-orientation preserving cyclic group actions on S^3.

PROPOSITION. *Any smooth non-orientation preserving cyclic group action on the* 3*-sphere is conjugate to a twisted rotation or a reflection of the sphere through two points.*

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1. Proof of Theorem 1. Let K be a hyperbolic knot in S^3 with $S^3 - K$ having a hyperbolic metric; $N(K)$ be a regular neighborhood of K such that $\partial N(K)$ is a flat torus in $S^3 - K$ with respect to the hyperbolic metric; m, l be a meridian and a preferred longitude of K respectively, m, l lie in $\partial N(K)$ and be realized as geodesics. m, l will also be used to denote the elements in $\pi_1(S^3 - \mathrm{int}(N(K)))$ represented by them. Let $[m]$, $[l]$ be the homology classes in $H_1(\partial N(K), Z)$ represented by m, l respectively. Let h be a self-diffeomorphism of $S^3 - \mathrm{int}(N(K))$. Our goal is to prove that $h^*([m])$ is $\pm[m]$ in $H_1(\partial N(K), Z)$. Since if this condition is satisfied,

$$h|_{\partial N(K)}: \partial N(K) \to \partial N(K)$$

extends to be a self-diffeomorphism of $N(K)$ which in turn gives an extension of h to S^3 by gluing. By Mostow Rigidity, one can assume that h is a hyperbolic isometry. $h_*([l]) = \varepsilon_1[l]$ with ε_1 being ± 1 in $H_1(\partial N(K), Z)$, because $\pm[l]$ are the only primitive homology classes in $H_1(\partial N(K), Z)$ which vanish in $H_1(S^3 - \mathrm{int}(N(K)), Z)$ under the inclusion homomorphism. h_* is an automorphism of $H_1(\partial N(K), Z)$; hence $h_*[m] = \varepsilon[m] + a[l]$, where $\varepsilon_2 = \pm 1$, and a is in Z. Our goal is to show $a = 0$. If $\varepsilon_1 = \varepsilon_2$, i.e., h is orientation preserving, the result is trivial because on one hand h, being an isometry of a hyperbolic manifold of finite volume, is of finite order (i.e., composition of h finite times is the identity map; see **[M, B]**, or **[Th]**), on the other hand the matrix $\left[\begin{smallmatrix} \varepsilon_1 & a \\ 0 & \varepsilon_2 \end{smallmatrix}\right]$ has infinite order if a is non-zero. Therefore, we need only to consider the case where $\varepsilon_1 = -\varepsilon_2$. Suppose conversely $a \neq 0$. Then by Culler, Gordon, Luecke, Shalen **[CGLS]**, one has that $a = \pm 1$, and that K does not have property P. Since the matrix $\left[\begin{smallmatrix} \varepsilon_1 & a \\ 0 & \varepsilon_2 \end{smallmatrix}\right]$ is of order two, $h_* h_* = \mathrm{id}$ in $H_1(\partial N(K), Z)$. Consider the orientation preserving isometry $g = h \circ h$. g is of finite order; hence it generates a finite cyclic group G acting isometrically on the flat torus $\partial N(K)$. Because $g_*([m]) = [m]$ and $g_*[l] = [l]$ in $H_1(\partial N(K), Z)$, G preserves the foliations $\partial N(K)$ by geodesic meridians and by geodesic longitudes. The following lemma shows that the G-action on $\partial N(K)$ can be extended to a G-action on $N(K)$.

LEMMA 1. *If G acts isometrically on a flat boundary ∂N of a solid torus N and $g_*[m] = \pm[m]$, $g_*[l] = \pm[l]$ in $H_1(\partial N, Z)$ where g is a generator of G, m, l are a meridian and a longitude of ∂N respectively, then the G-action can be extended to an action on N. Moreover the extended G-action on the core of N preserves a flat Riemannian metric on it.*

Proof. Parametrize ∂N by (u, v), where u, v are the unit complex numbers such that $S^1 \times \{v\}$ and $\{u\} \times S^1$ correspond to the geodesic meridian m and the geodesic longitude l in ∂N. Since the action on the homology group $H_1(\partial N, Z)$ satisfies the conditions above, the G-action on ∂N corresponds now to a G-action on $S^1 \times S^1$ preserving the standard product metric and the product structure. Extending the G-action on ∂N to N is the same as extending the G-action on $S^1 \times S^1$ to $D^2 \times S^1$. The extension of the latter is trivial. To see this, for $g \in G$, we have,

$$g(u, v) = (\phi(u, g), \psi(v, g))$$

where $u, v \in S^1$, $\phi(u, g) = \alpha u$, or $\alpha\bar{u}$, and $\psi(v, g) = \beta v$ or $\beta\bar{v}$, for some roots of unity α and β. The extension of the G-action to $D^2 \times S^1$ is given by the same formula with u in $D^2 = \{z \in C \mid |z| \leq 1\}$. The extended G-action still preserves the product metric and acts on the core $\{0\} \times S^1$ isometrically with respect to the flat metric induced from $D^2 \times S^1$.

We have now a cyclic group G which acts on S^3 preserving K. If G is non-trivial, then K has property P by Corollary 7 of Culler, Gordon, Luecke, Shalen **[CGLS]** which contradicts $a \neq 0$. Therefore $h \circ h = \text{id}$ in $S^3 - \text{int}(N(K))$. It is easy to check, using $a = \pm 1$, $h_*([m]) = -\varepsilon_1[m] + a[l]$ and $h_*([l]) = \varepsilon_1[l]$, that

$$h_*(-2\varepsilon_1 a[m] + [l]) = -\varepsilon_1(-2\varepsilon_1 a[m] + [l]).$$

Note that $[l]$, and $-2\varepsilon_1 a[m] + [l]$ are primitive elements, and are the (± 1)-eigenvectors of h_* in $H_1(\partial N(K), Z)$. The algebraic intersection number of $[l]$ and $-2\varepsilon_1 a[m] + [l]$ is ± 2. The following lemma shows that h has fixed points in $\partial N(K)$.

LEMMA 2. *Suppose h is an orientation reversing fixed point free involution of a torus T^2, then the (± 1)-eigenspaces of h_* are generated by two primitive classes with ± 1 as their algebraic intersection number.*

Proof. Since any orientation reversing fixed point free involution of T^2 has the quotient space homeomorphic to the Klein bottle, and since the Klein bottle has only one orientable two-fold cover up to covering equivalence, any two orientation reversing fixed point free involutions on T^2 are conjugate. Because the hypothesis and the conclusion of the lemma are invariant under conjugation, the lemma follows by checking a concrete example. Take T^2 to be $S^1 \times S^1$ parametrized by (u, v), where $u, v \in S^1$, the unit circle in the complex plane. Let $h: T^2 \to T^2$ be the automorphism sending (u, v) to $(\overline{u}, -v)$. h generates a fixed point free orientation reversing involution of T^2. The 1-eigenspace of h_* is generated by the homology class of the curve $\{1\} \times S^1$, and the (-1)-eigenspace of h_* is generated by the homology class of the curve $S^1 \times \{1\}$. Hence the algebraic intersection number of the primitive generators of (± 1)-eigenspaces is ± 1.

By the lemma, h has fixed points in $\partial N(K)$. However, h is an orientation reversing involution, $\mathrm{Fix}(h|_{\partial N(K)})$ is a 1-dimensional submanifold. This implies that $\mathrm{Fix}(h)$ contains a 2-manifold, say F. We claim that this is impossible. By Smith theory (see [**B**], Theorem 5.1), for the Z_2-action generated by h on the 1-dimensional Z_2-homology sphere $S^3 - \mathrm{int}(N(K))$, the fixed point set $\mathrm{Fix}(h)$ is a Z_2-homology sphere of dimension at most one. Hence $\mathrm{Fix}(h)$ $(= F)$ is an annulus or a Möbius band.

Case 1. F is an annulus. Since $S^3 - K$ has a hyperbolic structure, $S^3 = \mathrm{int}(N(K))$ is annulus free. Hence F is parallel to an annulus in $\partial N(K)$. In particular, F is separating. The two components of the complement of F in $S^3 - \mathrm{int}(N(K))$ are interchanged by h and hence are homeomorphic. Therefore both of them are solid tori. This implies that $S^3 - \mathrm{int}(N(K))$ is the union of two solid tori along an annulus in their boundaries which contradicts the existence of the hyperbolic structure of finite volume in $S^3 - K$.

Case 2. F is a Möbius band. ∂F is now a simple closed curve in $\partial N(K)$ fixed by h, and hence $[\partial F]$ is in the 1-eigenspace of h_* which is generated by $[l]$, or by $2a[m] + [l]$ according to $\varepsilon_1 = 1$, or -1. Thus ∂F and K bound an annulus A in $N(K)$. The Möbius band $F \cup_\partial A$ in S^3 has K as its boundary. Let L be the core of the Möbius band. If L is non-trivial, K is the cable knot of L. This contradicts that K is a hyperbolic knot. If L is the trivial knot, then K is the $(2, n)$-torus which is again absurd.

This completes the proof of Theorem 1.

Since any non-trivial knot with property P has the property that any self-diffeomorphism of the knot complement preserves the meridian, and since the only non-trivial knots which are not known to have property P are some hyperbolic knots by the work of Gabai and others, Corollary 2 follows from Theorem 1.

2. Proof of Theorem 2. We shall still use the same notations introduced in §1. Hence K is a non-trivial knot in S^3; $N(K)$ is a regular neighborhood of K; m, l are a meridian and a preferred longitude of K respectively. m, l lie in $\partial N(K)$. Our first observation is that there exists a flat metric on $\partial N(K)$ such that G acts on $\partial N(K)$ isometrically. This follows from the Geometrization Theorem that any action of a finite group G on a 2-manifold is equivalent to a geometric group action (see [**E**]). Fix the metric on $\partial N(K)$, and realize m, l by geodesics in $\partial N(K)$. Theorem 1 shows that the G-action on $\partial N(K)$ preserves the geodesic meridians and geodesic longitudes in $\partial N(K)$. By Lemma 1, the G-action on $\partial N(K)$ extends to a G-action on $N(K)$ such that the extended G-action preserves a flat metric on K. Hence the G-action on $S^3 - \mathrm{int}(N(K))$ extends to a G-action on S^3 which preserves K and acts on K preserving a flat metric d. The restriction of the G-action to K gives a representation:

$$\sigma: G \to \mathrm{Isom}(K, d).$$

The solution of the Smith Conjecture shows that σ is a monomorphism. To see this, let $h \in \ker(\sigma)$, and H be the cyclic group by h. Then H acts on S^3 with fixed point set containing K, and H preserves each geodesic meridian in $\partial N(K)$. Moreover, $h_*([l]) = [l]$ in $H_1(\partial N(K), Z)$. There are now two cases that might happen.

Case 1. $h_*([m]) = [m]$. h is now an orientation preserving homeomorphism because $h_*([l]) = [l]$ and $h_*([m]) = [m]$ imply that h is an orientation preserving homeomorphism in $H_1(\partial N(K), Z)$. Therefore the H-action on a geodesic meridian m is a rotation. Suppose $h \neq \mathrm{id}$; then H acts non-trivially on m. Therefore K is the only fixed point set of h in $N(K)$. By Smith theory, $\mathrm{Fix}(h) = K$, which then contradicts the solution of the Smith Conjecture.

Case 2. $h_*([m]) = -[m]$. h is now an orientation reversing homeomorphism. Since $h \circ h \in \ker(\sigma)$, and $h_* h_*([m]) = [m]$, one has $h \circ h = \mathrm{id}$ by the solution of Case 1. Hence h is an orientation reversing involution of S^3 with fixed point set containing K. Because the $\mathrm{Fix}(h)$ is a submanifold of odd codimension and contains K, $\mathrm{Fix}(h)$ contains a 2-manifold. By Smith Theory, the $\mathrm{Fix}(h)$ is a Z_2-homology sphere. Hence $\mathrm{Fix}(h)$ is a 2-sphere and contains K. This implies that K is a trivial knot which is absurd.

Therefore G is a subgroup of $\mathrm{Isom}(K, d)$. It is well known that a finite subgroup of $\mathrm{Isom}(K, d)$ is a cyclic or a dihedral group. In case K is a hyperbolic knot, $\mathrm{Out}(\pi_1(S^3 - K))$ acts isometrically on $S^3 - \mathrm{int}(N(K))$ where $\partial N(K)$ is a flat torus in $S^3 = K$ (see **[M, B]**, or **[Th]**). Hence $\mathrm{Out}(\pi_1(S^3 - K))$ (or the same $\mathrm{Isom}(S^3 - K)$) is a cyclic or a dihedral group.

Proof of Corollary 3. By Theorem 2 and its proof, the Z_n-action extends to a Z_n-action on S^3 such that K is invariant and K intersects the fixed point set of a nontrivial element f in Z_n if and only if $\mathrm{Fix}(\sigma(f)) \cap K \neq \varnothing$. But $\mathrm{Fix}(\sigma(f)) \cap K \neq \varnothing$ if and only if $\sigma(f)$ is a reflection on K which in turn is the same as $f_*([l]) = -[l]$ in $H_1(\partial N(K), Z)$. Moreover, in this case, K intersects $\mathrm{Fix}(f)$ transversely in two points. The classification is now reduced to the classification of smooth cyclic group actions on S^3.

(I) The Z_n-action preserves the orientation.

If the Z_n-action on S^3 is fixed point free, we have (a). Otherwise, by Smith theory, the fixed point set is a knot, say L. The solution of the Smith Conjecture shows that L is a trivial knot, and the Z_n-action is a $2\pi/n$-rotation about L. Let g be a generator of the Z_n-action. If L intersects K, then by the remark above, we have $g_*([l]) = [l]$, and $\sigma(g)$ is a reflection in K. Hence $\mathrm{Fix}(g \circ g)$ contains K. However gg is orientation preserving. Therefore the solution of the Smith Conjecture implies that $g \circ g$ is the identity, i.e., $n = 2$. This proves (b).

(II) The Z_n-action does not preserve the orientation.

Let g still be the generator of the Z_n-action on S^3. Since g reverses the orientation, g has fixed points in S^3, n is even, and $\mathrm{Fix}(g)$ is a submanifold of odd codimension in S^3.

(c) $n = 2$.

By Smith theory, $\mathrm{Fix}(g)$ is a Z_2-homology sphere. Hence $\mathrm{Fix}(g)$ is the two points set or the 2-sphere. If $\mathrm{Fix}(g)$ is the two points set, by Livesay's theorem [L], the Z_2-action is a reflection of S^3 through two points; if $\mathrm{Fix}(g)$ is a 2-sphere, then the action is a reflection of S^3 with respect to a 2-sphere by Schonflies theorem. Now the Z_2-action is classified as follows. If $g_*([l]) = [l]$, then $\mathrm{Fix}(g) \cap K = \varnothing$. In this case $\mathrm{Fix}(G)$ cannot be a 2-sphere. To see this, $\mathrm{Fix}(g) \cap K = \varnothing$. In this case $\mathrm{Fix}(G)$ cannot be a 2-sphere. To see this, $\mathrm{Fix}(g) \cap K = \varnothing$ implies the fixed point set of g in S^3 is actually in $S^3 - \mathrm{int}(N(K))$. By Smith theory, for the g involution on the one-dimensional homology sphere $S^3 - \mathrm{int}(N(K))$, $\mathrm{Fix}(g|_{S^3-\mathrm{int}(N(K))})$ is a Z_2-homology sphere of dimension at most one. Hence $\mathrm{Fix}(g)$ are two points. This gives $(\mathrm{c})_1$. If $g_*([l]) = -[l]$, then $\sigma(g)$ is a reflection in K, and K intersects $\mathrm{Fix}(g)$ transversely in two points. $(\mathrm{c})_2$, $(\mathrm{c})_3$ follow from the above mentioned classification of the orientation reversing involutions of S^3.

(d) $n \geq 4$.

The result is a consequence of the following proposition which will be proven in the appendix.

PROPOSITION. *Any smooth cyclic group action on S^3 which does not preserve the orientation is conjugate to a twisted rotation of S^3, or to a reflection of S^3 through two points.*

Applying the proposition, we need only to check that K is disjoint from the axis of the twisted rotation g. However the axis of g is $\mathrm{Fix}(g \circ g)$. $\mathrm{Fix}(g \circ g)$ does not intersect K follows now from $g_* g_*([l]) = [l]$, and $g \circ g \neq \mathrm{id}$. This completes the proof of (d).

Corollary 4 is actually proven in the proof of Corollary 3.

Proof of Corollary 5. (a) By Proposition 3.19 of [**B, Z**], K is invertible if and only if there is an automorphism

$$\phi: \pi_1(S^3 - \mathrm{int}(N(K))) \to \pi_1(S^3 - \mathrm{int}(N(K)))$$

such that $\phi(m) = m^{-1}$ and $\phi(l) = l^{-1}$. Since K is a hyperbolic knot, Mostow Rigidity Theorem shows that ϕ can be realized by a hyperbolic isometry $h: S^3 - \mathrm{int}(N(K)) \to S^3 - \mathrm{int}(N(K))$ such that $h_*([m]) = -[m]$, and $h_*([l]) = -[l]$ in $H_1(\partial N(K), Z)$. Here we have assumed that $\partial N(K)$ is a flat torus in $S^3 - K$. The condition $h_*([l]) = -[l]$ implies that h is an involution by Corollary 4. Because $h_*([m]) = -[m]$, h is orientation preserving. Hence by Corollary 3, the Z_2-action generated by the extension of h on S^3 is induced by a π-rotation of S^3 about an axis L. $H_*([l]) = -[l]$ implies that

L intersects K transversely in two points. Therefore K is invariant under a π-rotation about an axis intersecting K at two points. The inverse implication is trivial.

(b) By Proposition 3.19 of **[B, Z]**, K is amphicheiral if and only if there is an automorphism

$$\phi: \pi_1(S^3 - \mathrm{int}(N(K))) \to \pi_1(S^3 - \mathrm{int}(N(K)))$$

such that $\phi(m) = m^{-1}$ and $\phi(l) = l$. Realize ϕ by an isometry $h: S^3 - \mathrm{int}(N(K)) \to S^3 - \mathrm{int}(N(K))$. h is orientation reversing since $h_*([m]) = -[m]$, and $h_*([l]) = [l]$ in $H_1(\partial N(K), Z)$. h generates a smooth cyclic group action on $S^3 - \mathrm{int}(N(K))$ which does not preserve the orientation. Hence by Corollary 3, h is induced by a twisted rotation of S^3 about an axis L missing K if the order of h is at least four. If the order of h is two, the h involution is the case $(\mathrm{c})_1$ in Corollary 3 because $h_*([l]) = [l]$. Therefore, in this case K is invariant under a reflection of S^3 through two points missing K. Then the condition is clearly sufficient.

(c) If the knot is both invertible and amphicheiral, then there exists an automorphism

$$\phi: \pi_1(S^3 - \mathrm{int}(N(K))) \to \pi_1(S^3 - \mathrm{int}(N(K)))$$

such that $\phi(m) = m$, and $\phi(l) = l^{-1}$. ϕ is the composition of the two automorphisms coming from (a) and (b). Realize ϕ by an orientation reversing hyperbolic isometry h such that $h_*([m]) = [m]$, and $h_*([l]) = -[l]$ in $H_1(\partial N(K), Z)$. By Corollary 4, $h_*([l]) = -[l]$ and $h_*([m]) = [m]$ imply h is an orientation reversing involution of $S^3 - \mathrm{int}(N(K)) \to S^3 - N(K)$. By Corollary 3, h is the case $(\mathrm{c})_2$ or the case $(\mathrm{c})_3$. Case $(\mathrm{c})_3$ cannot happen since K is a prime knot. Hence K is invariant under the reflection of S^3 through two points contained in K.

Appendix. We prove the following proposition concerning smooth cyclic group action on the 3-sphere which does not preserve the orientation.

PROPOSITION. *Any smooth non-orientation preserving cyclic group action on S^3 is conjugate to a twisted rotation of S^3, or to a reflection of S^3 through two points.*

Proof. Let g be a generator of the Z_n-action. n has to be even. g is orientation reversing, and hence has fixed points in S^3. If $n = 2$, we have shown in the proof of Corollary 3 (c) that the result holds. Assume $n \geq 4$ from now on. Let $h = g \circ g$. h is an orientation preserving automorphism of order m, and has fixed points. The solution of the Smith Conjecture shows that the $\mathrm{Fix}(h)$ is a trivial knot, say L. Now L is invariant under g. g acts on L with fixed point and is of order two in L. Hence the action of g on L is a reflection by the classification of Z_2-action on the circle. Take a Z_n-equivariant regular neighborhood $N(L)$ of L in S^3 (see [**B**]). By the choice of the regular neighborhood, one knows that the action of Z_n on $N(L)$ is standard. Therefore by choosing the generator g of the Z_n-action appropriately, we can assume that the restriction of g on $N(L) = D^2 \times S^1$ is conjugate to α, where

$$\alpha: D^2 \times S^1 \to D^2 \times S^1$$

sends (z, w) to $(e^{2\pi i/n} z, \overline{w})$, with z in $D^2 = \{z \in C \mid |z| \leq 1\}$ and w in $S^1 = \{z \in C \mid |z| = 1\}$. Note that α generates an orientation reversing Z_n-action on $D^2 \times S^1$ with two fixed points in $\{0\} \times S^1$. Since L is the trivial knot, $S^3 - \mathrm{int}(N(L))$ is a solid torus. Let ϕ: $S^3 = (S^3 - \mathrm{int}(N(L))) \cup N(L) \to \overline{S}^3 = (S^1 \times D^2) \cup_{\mathrm{id}} (D^2 \times S^1)$ be a diffeomorphism taking $N(L)$ to $D^2 \times S^1$ such that $\phi g|_{N(L)} \phi^{-1} = \alpha$. Now extend α to be a self-diffeomorphism $\overline{\alpha}$ of $\overline{S}^3$ by sending $(z, w) S^1 \times D^2$ to $(e^{2\pi i/n} z, \overline{w})$ with $z \in S^1$ and $w \in D^2$. Then $\overline{\alpha}$ generates a twisted $2\pi/n$-rotation of $\overline{S}^3$. Our goal is to show that $\phi g \phi^{-1}$ is conjugate to $\overline{\alpha}$ in $\overline{S}^3$. This is consequence of the following claim.

Claim. $g' = \phi g \phi^{-1}|_{S^1 \times D^2}$ is conjugate to $\beta = \overline{\alpha}|_{S^1 \times D^2}$ by a piecewise smooth diffeomorphism ψ such that ψ is the identity map on $\partial(S^1 \times D^2)$.

Let us assume the claim and finish the proof. By gluing ψ with $\mathrm{Id}|_{D^2 \times S^1}$ along the boundaries, we obtain a piecewise smooth self-diffeomorphism of $\overline{S}^3$ which conjugates $\phi g \phi^{-1}$ to $\overline{\alpha}$. Therefore $\phi g \phi^{-1}$ is smoothly conjugate to $\overline{\alpha}$ by the work of Moise.

Proof of the Claim. By the choice of ϕ, g' is the same as β on $\partial(S^1 \times D^2)$. Using the equivariant Dehn's lemma, we can find n copies of disjoint properly embedded disks $D_1, D_2, \ldots, D_n$ with $\partial D_j = e^{e\pi j i/n} \times \partial D^2$ in $S^1 \times D^2$, such that $g'(D_j) = D_{j+1}$ for $j =$

$1, 2, \ldots, n$, where $D_1 = D_{n+1}$. g': $D_j \to D_{j+1}$ is a diffeomorphism for each j. These disks cut $S^1 \times D^2$ into n components, say $B_1, B_2, \ldots, B_n$ with $D_j \cup D_{j+1} \subset \partial B_j$, and each of B_j is a 3-ball by Schonflies' theorem. Let $D'_j = e^{2\pi i j/n} \times D^2$ (where $D'_{n+1} = D'_1$); $B'_j = \{e^{2\pi i t/n} | j \leq t \leq j+1\} \times D^2$; and $E_j = \partial B'_j - (D_j \cup D_{j+1})$, the annulus, for each $i = 1, 2, \ldots, n$. The construction of ψ is now as follows. Let A_1: $D_1 \to D'_1$ be a diffeomorphism which is the identity on ∂D_1. Define A_2: $D_2 \to D'_2$ to be $\beta|_{D'_1} A_1 g'^{-1}|_{D_2}$. It is still a diffeomorphism which fixes ∂D_2 pointwise. Since $\partial B_1 = D_1 \cup E_1 \cup D_2$ and $\partial B'_1 = D'_1 \cup E_1 \cup D'_2$, glue A_1, A_2 and $\mathrm{id}|_{E_1}$ along the boundaries, one obtains a piecewise smooth diffeomorphism from $\partial B_1 \to \partial B'_1$ which is the identity on E_1. Extend it to be a piecewise smooth diffeomorphism from B_1 to B'_1 by Alexander's lemma, and call it ψ_1. Now ψ_j: $B_j \to B'_j$ is defined to be

$$\beta_j|_{B'_1} \psi_1 g'^{-j}|_{B_j}$$

for $j = 2, 3, \ldots, n$. All these piecewise smooth diffeomorphisms match on the D_j's. Gluing them together along the D_j's, we obtain a piecewise diffeomorphism $\psi: S^1 \times D^2 \to S^1 \times D^2$. Then $\psi|_{\partial(S^1 \times D^2)} =$ id and $\beta = \psi^{-1} \beta \psi$.

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