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# KLOOSTERMAN INTEGRALS FOR SKEW SYMMETRIC MATRICES

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**If $G$ is a reductive group quasi-split over a number field $F$ and $K$ the kernel of the trace formula, one can integrate $K$ in the two variables against a generic character of a maximal unipotent subgroup $N$ to obtain the Kuznietsov trace formula. If $H$ is the fixator of an involution of $G$, one can also integrate $K$ in one variable over $H$ and in the other variable against a generic character of $N$: one obtains then a "relative" version of the Kuznietsov trace formula. We propose as a conjecture that the relative Kuznietsov trace formula can be "matched" with the Kuznietsov trace formula for another group $G'$. A consequence of this formula would be the characterization of the automorphic representations of $G$ which admit an element whose integral over $H$ is non-zero: they should be functorial image of representations of $G'$. In this article, we study the case where $H$ is the symplectic group inside the linear group; we prove the "fundamental lemma" for the situation at hand and outline the identity of the trace formulas. This case is elementary and should serve as a model for the general case.**

**1. Introduction.** Let $F$ be a local field. We will denote by $G'$ the group $\mathrm{GL}(m)$, regarded as an algebraic group over $F$, by $N'$ the group of upper triangular matrices with unit diagonal, by $A'$ the group of diagonal matrices and by $W'$ the Weyl group of $A'$ identified to the group of permutation matrices. Let $\psi$ be a non-trivial additive character of $F$ and $\theta'$ the character of $N'$ defined by:

$$\theta'(n) = \psi\left(\sum_{1\le i\le m-1} n_{i,i+1}\right). \tag{1}$$

The group $N'$ operates on $G'$ by

$$g' \mapsto {}^t n'_1 g' n'_2. \tag{2}$$

We say that the orbit of $g'$ under $N' \times N'$ on $G'$ is relevant (with respect to $\theta'$) if $\theta' \otimes \theta'$ is trivial on the fixator of $g'$ in $N' \times N'$. If $f'$ is a smooth function of compact support of $G'$, we define for every relevant element $g'$ an orbital integral:

$$I(g', f') = \int f'({}^t n'_1 g' n'_2)\theta'(n'_1 n'_2)\, dn'_1\, dn'_2. \tag{3}$$

The integral is taken over the quotient of $N' \times N'$ by the fixator of $g'$. At the moment, we do not specify the choice of the Haar measures. This type of integral is a "Kloosterman integral," the local analogue of a Kloosterman sum (see [G] and the references therein).

Now let $S$ or $S_F$ be the space of skew symmetric matrices of size $2m \times 2m$ and rank $2m$. The group $G = \mathrm{GL}(2m, F)$ operates on $S$ by:

$$s \mapsto {}^t gsg \,. \tag{4}$$

We denote by $N$ the group of upper triangular matrices with unit diagonal, by $A$ the group of diagonal matrices and by $W$ the group of permutation matrices in $G$. In particular, the group $N$ operates on $S$ and its orbits are easily described: each orbit has exactly one representative in the normalizer of $A$. More precisely, every orbit has a representative of the form $wa$ where $w \in W$, $a \in A$, $w^2 = 1$ and $waw = -a$. We let $\theta$ be the character of $N$ defined by:

$$\theta(n) = \theta'(n_1)\theta'(n_2) \tag{5}$$

if

$$n = \begin{pmatrix} n_1 & u \\ 0 & n_2 \end{pmatrix} . \tag{6}$$

In particular, $\theta$ is a degenerate character. We define again the notion of a relevant orbit (with respect to the character $\theta$). If $\Phi$ is a smooth function of compact support on $S$ we will denote by $I(s, \Phi)$ its orbital integrals:

$$I(s, \Phi) = \int \Phi({}^t nsn)\theta(n)\, dn ;$$

the integral is over $N$ divided by the fixator $N_s$ of $s$; by definition, the character $\theta$ is trivial in $N_s$, so that the integral makes sense. We think of this integral as a relative Kloosterman integral. Remarkably, the two kinds of Kloosterman integrals lead to the same kind of functions.

Let $w_0$ be the following matrix:

$$w_0 = \begin{pmatrix} 0 & 1_m \\ 1_m & 0 \end{pmatrix} . \tag{7}$$

We will first show that every relevant orbit has a representative of the form

$$s = w_0 \begin{pmatrix} g' & 0 \\ 0 & -{}^t g' \end{pmatrix} , \tag{8}$$

where $g'$ is in $G'$ and the orbit of $g'$ under $N' \times N'$ is relevant for $\theta'$; furthermore, the orbit of $g'$ determines the orbit of $s$ under $N$. Next, we will prove that given $\Phi$ there is $f'$ such that the orbital integrals of $\Phi$ and $f$ match in the sense that:

$$I(s, \Phi) = I(g', f')|\det g'|^{(m-1)/2},$$

if $s$ and $g'$ are as above. Although $f'$ is not unique, there is a canonical choice for $f'$ in terms of $\Phi$: we will use the notation $\Phi \mapsto f'$ for this canonical choice.

Now let $\varepsilon$ be the following skew matrix:

$$\varepsilon = \begin{pmatrix} 0 & 1_m \\ -1_m & 0 \end{pmatrix} \tag{9}$$

and let $H$ be the fixator of $\varepsilon$ in $G$. For any smooth function of compact support $f$ on $G$, the function $\Phi_f$ on $S$ defined by:

$$\Phi_f({}^t g \varepsilon g) = \int_H f(hg)\, dh$$

is smooth of compact support. Moreover, if $g$ is relevant, we have:

$$\iint f(hgn)\, dh \theta(n)\, dn = I({}^t g \varepsilon g, \Phi_f),$$

the integral on the left is for $h \in H$ and $n$ in the quotient of $N$ by the fixator of ${}^t g \varepsilon g$ in $N$, that is, the group

$$N \cap g^{-1} H g.$$

Consider now the function $f'$ associated to $\Phi_f$. We will write

$$f \mapsto f'.$$

If furthermore $f$ is a Hecke function then $f'$ is also a Hecke function; we will show the map $f \mapsto f'$ is an homomorphism of Hecke algebras. There is a dual correspondence on the set of unramified representations that we now describe: let $\pi'$ be an unramified representation of $G'$; consider then the parabolic subgroup of $P = MU$ of type $(m, m)$ in $G$ and the representation $\sigma$ of $G$ induced by the representation

$$\begin{pmatrix} g_1 & u \\ 0 & g_2 \end{pmatrix} \mapsto \pi'(g_1)|\det g_1|^{1/2} \otimes \pi'(g_2)|\det g_2|^{-1/2}. \tag{10}$$

Let $\pi$ be its unique unramified constituent. We will show that the Satake transforms of these functions (see (19) below) are related by:

$$\hat{f}(\pi) = \hat{f}'(\pi').$$

Once these results are established, we can write down a global trace formula: let $F$ be a number field, and $f$ a smooth function of compact support on $G(F_{\mathbb{A}})$; we will assume that $f$ is a product of functions $f_v$; for each place $v$, we will denote by $\Phi_v$ the corresponding function on $S_v$ and by $f'_v$ the corresponding function on $G'_v$: thus $f_v \mapsto f'_v$. We will denote by $f'$ the product of the functions $f'_v$. Now we can form the geometric kernels $K$ and $K'$ attached to $f$ and $f'$ respectively:

$$K(x, y) = \sum_{\xi \in G(F)} f(x^{-1}\xi y), \quad K'(x, y) = \sum_{\xi' \in G'(F)} f'(x^{-1}\xi' y).$$

Then we have the following identity

$$\iint K(h, n)\, dh \theta(n)\, dn = \iint K'(n_1, n_2)\theta'(n_1)^{-1}\, dn_1 \theta'(n_2)\, dn_2; \tag{11}$$

here

$$h \in H(F)\backslash H(F_{\mathbb{A}}), \quad n \in N(F)\backslash N(F_{\mathbb{A}}), \quad n_i \in N'(F)\backslash N'(F_{\mathbb{A}}).$$

The right-hand side is the Kuznietsov trace formula and the left-hand side the relative Kuznietsov trace formula.

Indeed, the left-hand side can be written as:

$$\sum_{\xi} \int f(h\xi n)\theta(n)\, dn$$

where $h$ is in $H(F_{\mathbb{A}})$, $\xi$ in $H(F)\backslash G(F)$, $n$ in $N(F)\backslash N(F_{\mathbb{A}})$. Let $\Phi$ be the product of the functions $\Phi_v$. Then this expression can be expressed in terms of $\Phi$ as follows:

$$\sum_{\sigma} \int \Phi({}^t n \sigma n)\theta(n)\, dn.$$

Here $\sigma$ is $S_F$ and $n$ is as before. We can also rewrite this integral with $\sigma$ in a set of representatives for the orbits of $N(F)$ on $S_F$ and $n$ in $N_\sigma(F)\backslash N(F_{\mathbb{A}})$. For each $\sigma$ the integral factors through an integral

$$\int_{N_\sigma(F)\backslash N_\sigma(F_{\mathbb{A}})} \theta(n_\sigma)\, dn_\sigma$$

and is thus zero unless $\theta$ is trivial on $N_\sigma$, that is, $\sigma$ is relevant. Thus we can rewrite our expression as

$$\sum_{\sigma} \int \Phi({}^t n \sigma n)\theta(n)\, dn,$$

where now $\sigma$ is in a set of representatives for the relevant orbits of $N(F)$ and $n$ in

$$N_{\sigma}(F_{\mathbb{A}})\backslash N(F_{\mathbb{A}}).$$

This can be written also as a sum of global orbital integrals:

$$\sum_{\sigma} I(\sigma, \Phi),$$

each of which is a product of local ones. Next, we choose for $\sigma$ elements of the form:

$$\sigma = w_0 \begin{pmatrix} \tau & 0 \\ 0 & -{}^t\tau \end{pmatrix},$$

where $\tau$ is in a set of representatives for the relevant orbits of $N'(F) \times N'(F)$. Then our expression can be written also

$$\sum_{\tau} I(\tau, f'),$$

since $|\det \tau| = 1$. In turn, this is equal to the right-hand side of (11) and the identity (11) is proved.

It would be more difficult to prove a spectral version of this identity where $K$ and $K'$ are replaced by the discrete spectrum kernels. Dual to this formula, would be a functorial map from the automorphic cuspidal representations of $G'$ to the automorphic representations of $G$. The image would consist of automorphic representations of $G$ which are distinguished with respect to $H$, that is, which admit a vector $\phi$ such that the integral is non-zero:

$$\int_{H(F)\backslash H(F_{\mathbb{A}})} \phi(h)\, dh$$

is non-zero. Moreover, if $\pi'$ corresponds to $\pi$, then, for almost all $v$, the representation $\pi_v$ and $\pi'_v$ are unramified and $\pi_v$ is the unramified component of the induced representation $\sigma_v$ (see (10)) determined by $\pi'_v$. In fact, the image consists of residues of Eisenstein series and it is more convenient to study directly the above integral as in **[J-R]**.

Our motivation for studying this question is that we expect similar results for any symmetric space. For instance in **[J-Y]**, we begun to study the case where $S$ is the space of Hermitian matrices with respect to a quadratic extension $E$ of $F$: in this case, $G = \mathrm{GL}(n, E)$, the character $\theta$ is generic and $G' = \mathrm{GL}(n, F)$. The correspondence $\pi' \mapsto \pi$ is the quadratic base change. The case at hand is much simpler, in fact elementary, and should serve as a model for the general case.

**2. Relevant orbits in $G'$.** We first recall the classification of relevant orbits for the action of $N' \times N'$ on $G'$ (see [G]). Let $\Delta'$ be the set of roots which are simple with respect to $A'N'$. Let $W'$ be the Weyl group of $A'$. We identify $W'$ to the group of permutation matrices. For each root $\alpha$, we denote by $X_\alpha$ the corresponding root vector. Thus if $\alpha(a) = a_i/a_j$, $X_\alpha$ is the matrix with 1 in the $i$th row and $j$ column entry and 0 in all other entries.

Suppose $wa$ is relevant. Let $\Theta'$ be the set of simple roots $\alpha$ such that $w\alpha < 0$ and let $\Theta''$ be the set of positive roots of the form $-w(\alpha)$ with $\alpha \in \Theta'$. Then $\Theta''$ is contained in $\Delta'$. Indeed, if $\beta = -w(\alpha)$ is not simple, we consider a pair $(n_1, n_2)$ with

$$n_1 = 1 + yX_\beta, \quad n_2 = 1 + xX_\alpha.$$

The relation

$${}^t n_1 w a n_2 = wa$$

is equivalent to

$$y + \alpha(a)x = 0.$$

Choose $x$ such that $\psi(x) \neq 0$ and then determine $y$ by the above equation. The pair $(n_1, n_2)$ is then in the fixator of $wa$ and yet $\theta'(n_1)\theta'(n_2) = \psi(x) \neq 1$, a contradiction.

Thus $\Theta''$ is contained in $\Delta'$. Next, exchanging the roles of $w$ and $w^{-1}$ we see that $\Theta''$ is the set of simple roots $\alpha$ such that $w^{-1}(\alpha) < 0$. We now appeal to a simple lemma that we will use again.

LEMMA 1. *Let $w \in W'$ and $\Theta'$ (resp. $\Theta''$) the set of simple roots $\alpha$ such that $w\alpha < 0$ (resp. $w^{-1}\alpha < 0$). Suppose $w(\Theta') = -\Theta''$. Then, in fact $\Theta' = \Theta''$ and $w^2 = 1$. Furthermore, let $P' = M'U'$ be the standard parabolic subgroup determined by $\Theta'$. Then $w$ is in $M'$ and is the longest element of $W \cap M'$.*

Here standard means that $P'$ contains $A'N'$; the Levi-factor $M'$ is the one containing $A'$; thus $M'$ is determined by the condition that the roots $\alpha$ in $\Theta'$ be "contained" in $M'$ (i.e. the root group $N_\alpha$ is contained in $M'$). For the proof, we introduce also the parabolic subgroup $P'' = M''U''$ determined by $\Theta''$. Clearly, $wM'w^{-1} = M''$. Next we claim any root $\gamma$ in $U'$ is transformed into a positive root in $U''$ by $w$. Indeed, $\beta = w\gamma$ cannot be in $M''$. Otherwise, applying $w^{-1}$ to $\beta$ we would find that $\gamma$ is in $M'$. Now suppose $\beta$ is in $\overline{U}''$. Then

$$\beta = -\sum_{\alpha \in \Delta} n_\alpha \alpha$$

with $n_\alpha \geq 0$ for all $\alpha$ and $n_\alpha > 0$ for at least one $\alpha \notin \Theta''$. On the other hand,

$$\gamma = \sum_{\alpha\in\Delta} m_\alpha \alpha ,$$

with $m_\alpha \geq 0$ for all $\alpha$ and $m_\alpha > 0$ for at least one $\alpha \notin \Theta'$. Applying $w$ to this relation we get

$$\beta = w\gamma = -\sum_{\alpha\in\Theta'} m_\alpha(-w\alpha) + \sum_{\alpha\notin\Theta'} m_\alpha w\alpha .$$

Since $\alpha \mapsto -w\alpha$ is a bijection of $\Theta'$ onto $\Theta''$, we get that

$$\sum_{\alpha\notin\Theta'} m_\alpha w\alpha = \sum_{\alpha\in\Theta''} n'_\alpha \alpha - \sum_{\alpha\notin\Theta''} n_\alpha \alpha ,$$

for suitable $n'_\alpha$. The left-hand side is a non-zero sum of positive roots, hence a linear combination of simple roots with positive coefficients, one of which is strictly positive. However, this is not the case for the right-hand side and we get a contradiction. We conclude that $w\gamma$ is in $U''$ for $\gamma$ in $U'$. Hence $w$ transforms $P'$ into $P''$. Since both are standard we must have $P' = P''$ and $\Theta' = \Theta''$. Thus $w$ is in $M'$ and $w\Theta' = -\Theta'$. In other words, $w$ is the longest element of $W \cap M'$. This concludes the proof of the lemma.

The lemma being proved, we apply it to our relevant element $wa$. Thus $w$ is the longest element in $W \cap M'$, where $P' = M'U'$ is determined by $\Theta'$. It remains to see that $a$ is in the center of $M'$. Let $\alpha$ be a root in $\Theta'$ and set $\beta = -w\alpha$. Consider elements $(n_1, n_2)$ with

$$n_1 = 1 + yX_\beta , \quad n_2 = 1 + xX_\alpha .$$

The relation

$${}^t n_1 w a n_2 = wa$$

is again equivalent to

$$y + \alpha(a)x = 0 ,$$

but now

$$\theta'(n_1 n_2) = \psi(x + y) .$$

We see that

$$y + \alpha(a)x = 0 ,$$

must imply $\psi(x + y) = 1$. This means that $\alpha(a) = 1$. Thus $a$ is indeed in the center of $M'$.

In conclusion, we have proved the following result:

PROPOSITION 1. *Let $M'$ be a standard parabolic subgroup of $G'$, $w$ the longest element in $W \cap M'$ and $a$ in the center of $M'$. Then $wa$ is relevant. All relevant elements have this form.*

**3. Relevant orbits in $S$.** We first describe all orbits of $N$ on $S$:

LEMMA 2 ([S]). *Every $s \in S$ can be written in the form*

$$s = {}^t nwan$$

*with $n \in N$, $a \in A$, $w \in W$, $w^2 = 1$ and $waw = -a$.*

Indeed, every element can be written in the form

$$s = {}^t n_1 wan_2$$

with $n_i \in N$. Since ${}^t s = -s$ and $w$ and $a$ are uniquely determined, we have ${}^t(wa) = -wa$. Since ${}^t w = w^{-1}$ we get the condition of the lemma for $wa$. Next, we let $N^+$ (resp. $N^-$) be the group generated by the root groups $N_\alpha$ with $\alpha > 0$ and $w\alpha > 0$ (resp. $w\alpha < 0$). We have $N = N^+N^- = N^-N^+$. Let us write

$$n_1 = n_1^- n_1^+, \quad n_2 = n_2^+ n_2^-.$$

Replacing $s$ by ${}^t(n_2^-)^{-1}s(n_2^-)^{-1}$, we may assume $n_2 \in N^+$. We have then

$$s = -{}^t s = {}^t n_2 wan_1^- n_1^+$$

and so $n_1^+ = n_2$. Replacing $s$ by an element of the same orbit, we see we may assume $n_2 = 1$, $n_1^+ = 1$. In other words:

$$s = {}^t n_1 wa$$

with $n_1 \in N^-$. Writing once more that $s$ is skew symmetric we find:

$${}^t n_1 = wan_1 a^{-1} w^{-1}.$$

For $n \in N^-$, let us write

$$\xi(n) = wa^{-1}({}^t n^{-1})aw^{-1}.$$

Since $w^2 = 1$, $\xi(n)$ is again in $N^-$ and $\xi^2 = 1$. Thus $\xi$ is an automorphism of $N^-$ of order 2. Now the above condition reads $\xi(n_1) = n_1^{-1}$. Since $N$ is nilpotent there is $n \in N^-$ such that

$$n_1 = \xi(n)^{-1}n.$$

This condition reads

$$s = {}^t nwan$$

and we are done.

We first consider the case of a generic character. Thus we temporarily define $\theta$ by formula (1) with $m$ replaced by $2m$. Then:

**Proposition 2.** *Suppose $\theta$ is a generic character of $N$. Then there are no relevant orbits for $\theta$ in $S$.*

We first observe the following: if $\alpha$ is a simple root and $w\alpha$ is negative, then $w\alpha = -\beta$ with $\beta \in \Delta$. Indeed, suppose $\beta$ is not a simple root. We choose root vectors $X_\gamma$ in the usual way but agree that $X_{\alpha+\beta} = 0$ if $\alpha + \beta$ is not a root. Define

$$n = 1 + xX_\alpha + yX_\beta + zX_{\alpha+\beta}. \tag{12}$$

Then

$$\begin{aligned} {}^t nwana^{-1}w^{-1} = 1 &+ (x + y\beta(a))X_{-\alpha} + (y + x\alpha(a))X_{-\beta} \\ &+ (z(1 + \alpha(a)\beta(a)) + u)X_{-\alpha-\beta}, \end{aligned}$$

where $u$ depends only on $(x, y)$ but not on $z$. We have

$$\theta(n) = \psi(x).$$

On the other hand,

$$\beta(a) = (w\alpha(a))^{-1} = \alpha(waw^{-1})^{-1} = \alpha(-a)^{-1} = \alpha(a)^{-1}.$$

Thus, if

$$\psi(x) \neq 1, \quad y = -\alpha(a)x, \quad z = -\tfrac{u}{2}$$

then $n$ is in the fixator of $wa$ and yet $\theta(n) \neq 1$, a contradiction. Thus $\beta$ is in $\Delta$. Let $\Theta$ be the set of simple roots $\alpha$ such that $w(\alpha) < 0$. Then $w(\Theta) = -\Theta'$, with $\Theta'$ contained in $\Delta$. Since $w^2 = 1$, we have $\Theta' \subseteq \Theta$ and thus $\Theta' = \Theta$. Let $P = MU$ be the parabolic subgroup determined by $\Theta$. Then, by Lemma 1, $w$ is the longest element in $W \cap M$.

Finally, we consider again a root $\alpha$ in $\Theta$ and the root $\beta = -w\alpha$. Consider $n$ as in (12), then

$$\theta(n) = \psi(x + y).$$

Thus $\psi(x + y) = 1$ if $y = -x\alpha(a)$; this implies $\alpha(a) = 1$.

We now derive a contradiction. It will be convenient to identify $W$ to the group or permutations of the set $[1, 2m]$; for each $i$ we will denote by $\alpha_i$ the simple root $a \mapsto a_i/a_{i+1}$. Since $w$ is the longest element of $W \cap M$, there is an index $i$ such that $wi = i$ or $wi = i + 1$. In the first case, the relation $wa = -a$ implies $a_i = -a_i$ or $a_i = 0$, a contradiction. In the second case, we get $a_i = -a_{i+1}$, that is, $\alpha_i(a) = -1$. However, we also have $w\alpha_i = -\alpha_i$ so $\alpha_i$ is in $\Theta$; then $\alpha_i(a) = 1$, a contradiction. Thus there is no relevant orbit in this case. An essentially equivalent result is the following:

PROPOSITION 3. *If $\theta$ is a generic character of $N$ then there is no non-zero distribution on $G$ which is invariant on the left under $H$ and transforms on the right under $\theta$.*

We sketch a proof. Any distribution on $G$ invariant under $H$ may be viewed as a distribution on $S$. Thus we have to prove there is no distribution $\mu$ on $S$ such that

$$\int f({}^t nsn)\, d\mu(s) = \theta(n) \int f(s)\, d\mu(s).$$

For each $w$ in $W$ with $w^2 = 1$, let $S_w$ be the set of matrices of the form ${}^t nwan$ with $waw = -a$. We can write $S$ as a finite union of an increasing sequence of open subsets $X_i$ such that each difference $X_{i+1} - X_i$ is one of the sets $S_w$. We prove by induction on $i$ that there is no non-zero distribution on $X_i$ which transforms under $\theta$. Assuming our assertion proved for $i$, we prove it for $i+1$. Consider the space $V$ of distributions on $X_{i+1}$ which transform under $\theta$. The restriction of such a distribution to $X_i$ is zero by assumption. Thus, we may view $V$ as the space of distributions on the manifold $S_w$ which transform under $\theta$. Let $T$ be the set of $a$ in $A$ such that $waw = -a$. Then $T$ is a manifold and there is a projection map

$$p: S_w \to T$$

defined by:

$$p({}^t nwan) = a.$$

We use it to regard the space of distributions on $S_w$ as a module over the space of smooth functions on $T$. In particular, $V$ is a submodule. By the localisation principle of Gel′fand-Kazhdan (see **[B]**), the elements of $V$ which are supported by a fiber form a total subspace in $V$, that is, their linear combinations are dense in $V$ for the weak topology. However, a fiber may be identified to the quotient of $N$ by the fixator $N_s$ of a point $s \in S$. A distribution supported by the fiber may be viewed as a distribution on this quotient. Since $\theta$ is non-trivial on $N_s$, a distribution on $N_s\backslash N$ which transforms under $\theta$ is necessarily 0 and we are done.

We note that, in turn, this implies that a generic representation of $G$ does not have a symplectic model, that is, does not imbed into the space of functions on $H\backslash G$ (**[H-R]**).

We now turn to the case of the degenerate character $\theta$ determined by formula (5).

PROPOSITION 4. *Among the elements of the system of representatives of Lemma 2, the ones relevant for $\theta$ are those of the form*

$$s = w_0 \begin{pmatrix} w'a' & 0 \\ 0 & -w'a' \end{pmatrix},$$

*where $w'$ is in $W'$, $a'$ in $A'$ and $w'a'$ is relevant in $G'$ for $\theta'$.*

First, we check that such an element is indeed relevant. Suppose that

$$n = \begin{pmatrix} n_1 & u \\ 0 & n_2 \end{pmatrix}$$

fixes $s$. Then $(n_1, n_2)$ fixes $w'a'$. Hence

$$\theta(n) = \theta'(n_1)\theta'(n_2) = 1.$$

So $s$ is relevant.

Conversely, let us show that every relevant element $wa$ has this form. We first observe the following: if $\alpha$ is a simple root not equal to $\alpha_m$ and $w\alpha$ is negative, then $w\alpha = -\beta$ with $\beta \in \Delta - \{\alpha_m\}$. Indeed, since $\theta$ is trivial on $N_{\alpha_m}$ and non-trivial on any $N_\alpha$ with $\alpha \neq \alpha_m$, the proof is the same as before. Moreover, just as before, $\alpha(a) = 1$ for such a root. Let $\Theta_1$ be the set of simple roots $\alpha \neq \alpha_m$ such that $w\alpha < 0$. Then $w\Theta_1 = -\Theta'$ where $\Theta'$ is a suitable subset of $\Delta - \{\alpha_m\}$. Since $w^2 = 1$, we have $\Theta' \subseteq \Theta_1$ and thus $\Theta' = \Theta_1$ or

$$w\Theta_1 = -\Theta_1.$$

Now suppose that $w\alpha_m > 0$. Then $\Theta_1$ is the set of simple roots $\alpha$ such that $w\alpha < 0$. Let $P_1 = M_1U_1$ be the parabolic subgroup attached to $\Theta_1$. Then by Lemma 1 $w$ is the longest element of $W \cap M_1$. As before, this gives a contradiction: either there is an $i$ such that $wi = i$ or there is a root $\alpha$ in $\Theta_1$ such that $w\alpha = -\alpha$. In the first case, we get $a_i = 0$; in the second case, we get $\alpha(a) = -1$. In either case, we get a contradiction.

Thus $w\alpha_m < 0$. Let $\Theta$ be the set $\Delta - \{\alpha_m\}$ and $\Theta_2$ be the set $\Delta - \Theta_1 - \{\alpha_m\}$. Let $P = MU$ be the parabolic subgroup attached to $\Theta$. Suppose $w\alpha_m$ is in $M$. Then:

$$w\alpha_m = -\sum_{\Theta_1} n_\alpha \alpha - \sum_{\Theta_2} n_\alpha \alpha,$$

with $n_\alpha \geq 0$. Applying $w$ to this formula, we get:

$$\alpha_m + \sum_{\Theta_2} n_\alpha w\alpha = \sum_{\Theta_1} n_\alpha(-w\alpha).$$

Since $\alpha \mapsto -w\alpha$ is a bijection of $\Theta_1$ onto itself, and $w\alpha > 0$ for $\alpha \in \Theta_2$, this implies $\alpha_m \in \Theta_1$, a contradiction. We conclude $w\alpha_m$ is in $\overline{U}$.

Next, let $\alpha_i$ be a root in $\Theta_2$. Thus $w\alpha_i > 0$. We are going to show that $w\alpha_i$ is in $M$. Suppose it is in $U$. Then if $i+1 \leq m$, we have $wi \leq m$ and $w(i+1) > m$. If $m < i$, then $wi \leq m, w(i+1) > m$. Let us show that the first case will lead to a contradiction. The argument would be similar for the second case. We have found the existence of an integer $j$ such that $j < m$ and $wj \leq m$; let us assume $j$ is the least such integer. Then $j = 1$. Otherwise, we would have $w(j-1) > m$ and the root $\alpha_{j-1}$ would be changed by $w$ into a negative root in $\overline{U}$. However, by definition of $\Theta_1$, $\alpha_{j-1}$ would be in $\Theta_1$ and $w(\alpha_{j-1})$ would be in $M_1 \subseteq M$, a contradiction. We thus have $k = w1 \leq m$. We now claim that $w$ transforms the interval $[1, k]$ into itself and reverses its order, that is, for $1 \leq i \leq k$, we have $i + wi = k + 1$. We have just proved this for $i = 1$. Assume it is true for $1, 2, \ldots, l$ with $2l \leq k$; if $2l = k$ or $2l + 1 = k$ we are done. So we assume this is not the case and we prove our assertion for $l+1$. We have in particular, $wl = k - l + 1$. Consider $w(k-l)$. Then $w(k-l) > l = w(k-l+1)$. Thus $w\alpha_{k-l}$ is a negative root; it follows that $\alpha_{k-l}$ is in $\Theta_1$ and $w\alpha_{k-l}$ is the opposite of a simple root. This implies that $w(k-l) = l+1$. Thus we obtain our assertion by induction on $l$. Now if $k$ is odd, we get $w((k+1)/2) = (k+1)/2$ and $a_{(k+1)/2} = 0$, a contradiction. If $k$ is even then $w\alpha_{k/2} = -\alpha_{k/2}$ and $\alpha_{k/2}$ is in $\Theta_1$ with $\alpha_{k/2}(a) = -1$. Again, this is a contradiction. We conclude that $w\alpha_i$ is in $M$.

At this point, we have

$$w\Theta_1 = -\Theta_1, \quad w\Theta_2 > 0, \quad w(\Theta_2) \subseteq M, \quad w(\alpha_m) \in \overline{U}.$$

Now consider the element $w_1 = w_0 w$, where $w_0$ is defined in (7). Since $w_0$ takes $\Theta$ to itself and $U$ to $\overline{U}$, we have

$$w_1\Theta_1 = -\Theta', \quad \text{with } \Theta' \subseteq \Theta,$$
$$w_1\Theta_2 > 0, \quad w_1(\Theta_2) \subseteq M.$$

On the other hand,

$$w_1\alpha_m \in U.$$

Let $P' = M'U'$ be the parabolic subgroup determined by $\Theta'$. Recall $P_1$ is the one determined by $\Theta_1$. We have $w_1 M_1 w_1^{-1} = M'$. Next, if $\beta$ is in $U_1$, then

$$\beta = \sum_{\Delta} n_\alpha \alpha$$

with $n_\alpha \geq 0$ for all $\alpha$ and $n_\alpha > 0$ for $\alpha = \alpha_m$ or some $\alpha \in \Theta_2$. This implies that $\gamma = w_1\beta$ is positive. Indeed:

$$\gamma + \sum_{\Theta_1} n_\alpha w_1(-\alpha) = \sum_{\Theta_2} n_\alpha w_1\alpha + n_{\alpha_m} w_1 \alpha_m .$$

The right-hand side is a sum of positive roots. Moreover $\alpha \mapsto w_1(-\alpha)$ is a bijection of $\Theta_1$ onto $\Theta'$. Thus if $\gamma$ is negative it must be in $M'$. Applying $w_1^{-1}$ to $\gamma$, we would find $\beta$ is in $M_1$, a contradiction. Thus $\gamma$ is positive and in fact in $U'$. We conclude that $w_1 P_1 w_1^{-1}$ is contained in $P'$, hence equal to it. Since $P_1$ and $P'$ are standard parabolic subgroups, it follows that $P_1 = P'$, $\Theta_1 = \Theta'$ and $w_1 = w_0 w$ transforms $\Theta_1$ into $-\Theta_1$. Thus $w_1$ is in fact the longest element of $W \cap M_1$. We have also $w_0\Theta_1 = \Theta_1$. This means that there is a Levi-subgroup $M'$ of $G' = \mathrm{GL}(m)$ such that $M_1$ is the group of matrices of the form

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix},$$

with $m_i \in M'$ and

$$w_1 = \begin{pmatrix} w' & 0 \\ 0 & w' \end{pmatrix},$$

where $w'$ is the longest element of $W' \cap M'$. Finally, the conditions $\alpha(a) = 1$ for $\alpha$ in $\Theta_1$ and $w(a) = -a$ mean that $a$ has the form:

$$\begin{pmatrix} a' & 0 \\ 0 & -a' \end{pmatrix},$$

where $a'$ is in the center of $M'$. So

$$wa = w_0 \begin{pmatrix} w'a' & 0 \\ 0 & -w'a' \end{pmatrix},$$

and the proposition is completely proved.

**4. Matching orbital integrals.** Let $\Phi$ be a smooth function of compact support on $S$. We will set

$$f'(g) = \int \Phi\left[w_0 \begin{pmatrix} g & v \\ 0 & -{}^t g \end{pmatrix}\right] dv |\det g|^{(1-m)/2}. \tag{13}$$

Here $v$ is integrated over the vector space $\mathrm{Sk}(m \times m)$ of skew $m \times m$ matrices. The Haar measure is self dual when we identify that vector space to its dual, via

$$(v, v') \mapsto \tfrac{1}{2}\mathrm{Tr}(vv').$$

We will write $\Phi \mapsto f'$. Clearly, $f'$ is a smooth function of compact support on $G'$. We claim that $\Phi$ and $f'$ have matching orbital integrals:

PROPOSITION 5. *Let* $g$ *be a relevant element in* $G'$ *and set*

$$s = w_0 \begin{pmatrix} g & 0 \\ 0 & -{}^t g \end{pmatrix}. \tag{14}$$

*Then*

$$I(s, \Phi) = J(g, f')|\det g|^{(m-1)/2}.$$

The proof of the proposition will indicate the correct choice of the Haar measures. Consider the orbital integral

$$I(s, \Phi) = \int \Phi({}^t nsn)\theta(n)\, dn.$$

Let us write:

$$n = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n_1 & 0 \\ 0 & n_2 \end{pmatrix}. \tag{15}$$

Then

$${}^t nsn = w_0 \begin{pmatrix} {}^t n_2 g n_1 & {}^t n_2 A_g(u) n_2 \\ 0 & -{}^t n_1 {}^t g n_2 \end{pmatrix} \tag{16}$$

where we have set:

$$A_g(u) = gu - {}^t u {}^t g. \tag{17}$$

Note that the image of $A_g$ is the space $\mathrm{Sk}(m \times m)$. We see that $n$ is in the fixator $N_s$ of $s$ if and only if $u$ is in the kernel of $A_g$ and the pair $(n_1, n_2)$ in the fixator $N'_g$ of $g$ in $N' \times N'$. Moreover, $N_s \backslash N$ is isomorphic to the product

$$(N'_g \backslash N' \times N') \times (\ker A_g \backslash M(m \times m)).$$

Let us choose a measure on the quotient $N'_g \backslash N' \times N'$ and use $A_g$ to transport the measure on $\mathrm{Sk}(m \times m)$ to

$$\ker A_g \backslash M(m \times m).$$

We obtain then a measure on $N_s \backslash N$. For that measure, we get:

(18) $I(s, \Phi)$

$$= \iint \Phi \left[ w_0 \begin{pmatrix} {}^t n_2 g n_1 & {}^t n_2 v n_2 \\ 0 & -{}^t n_1 {}^t g n_2 \end{pmatrix} \right] dv\, \theta'(n_1 n_2)\, dn_1\, dn_2.$$

Here $v$ is integrated over $\mathrm{Sk}(m \times m)$. After changing $v$ to ${}^t n_2 v n_2$, we obtain

$$I(s, \Phi) = \iint \Phi \left[ w_0 \begin{pmatrix} {}^t n_2 g n_1 & v \\ 0 & -{}^t n_1 {}^t g n_2 \end{pmatrix} \right] dv\, \theta'(n_1 n_2)\, dn_1\, dn_2$$

or

$$I(s, \Phi) = \int f'({}^t n_1 g n_2)\theta'(n_1 n_2)\, dn_1\, dn_2 |\det g|^{(m-1)/2}$$
$$= J(g, f')|\det g|^{(m-1)/2}.$$

Thus, we have proved the proposition.

Now we discuss orbital integrals for Hecke functions. We assume $F$ is non-Archimedean of odd residual characteristic. Let $R$ be the ring of integers of $F$. We choose a character $\psi$ with conductor $R$ and set $K = \mathrm{GL}(2m, R)$, $K' = \mathrm{GL}(m, R)$. Suppose that $f$ is a Hecke function, that is, a function of compact support, bi-invariant under $K = \mathrm{GL}(2m, R_F)$. As in the introduction, define:

$$\Phi_f({}^t g \varepsilon g) = \int f(h \varepsilon g)\, dh,$$

where $H$ is the fixator of $\varepsilon$ in $K$. It is easy to see that $K \cap S$ is one orbit of $K$. Because $f$ is $K$-invariant on the left, we could replace $\varepsilon$ by any other element of $K \cap S$ in this definition and yet arrive at the same function $\Phi_f$. On the other hand, the invariance of $f$ under $K$ on the right implies that $\Phi_f$ is invariant under $K$. For the function $f'$ corresponding to $\Phi_f$, this implies that $f'$ is bi-$K'$-invariant.

We claim that the map $f \mapsto f'$ is an homomorphism of the Hecke algebras. First suppose that $f$ is the characteristic function of $K$. Then $\Phi_f(s) = 0$ unless $s$ is in the orbit of $\varepsilon$ under $K$, that is, is in $K \cap S$. For such an $s$, we have $\Phi_f(s) = 1$. Thus $\Phi_f$ is the characteristic function of $K \cap S$. This implies that the function $f'$ is the characteristic function of $K'$.

Now let $\pi'$ be an unramified irreducible representation of $G_m$. Let $V'$ be its space, $\omega'$ the corresponding spherical function. Denote by $f' \mapsto \hat{f}'(\pi')$ the corresponding character of the Hecke algebra:

$$\int_{G'} f'(g')\omega'(g')\, dg' = \hat{f}'(\pi'). \tag{19}$$

Recall that the representation $\pi'$ is contragredient to the representation

$$g \mapsto \pi'({}^t g^{-1}).$$

In other words, there is a linear form $\beta \neq 0$ on the space $V' \otimes V'$ such that

$$\beta(\pi'(g) \otimes \pi'({}^t g^{-1})v) = \beta(v).$$

We choose a $K'$-invariant vector $\phi'_0$ in the representation such that

$$\beta(\phi'_0 \otimes \phi'_0) = 1.$$

We use $\pi'$ to construct a representation $\sigma$ of $G$ induced from the parabolic subgroup $P$ of type $(m, m)$. Its space is the space $V$ of smooth functions $\phi$ on $G$ with values in the space $V' \otimes V'$ such that

$$\phi\left[\begin{pmatrix} g_1 & x \\ 0 & g_2 \end{pmatrix} g\right] = \left|\frac{\deg g_1}{\det g_2}\right|^{(m+1)/2} \pi'(g_1) \otimes \pi'(g_2)\phi(g). \tag{20}$$

Let $\phi_0$ be the function in that space such that

$$\phi_0(k) = \phi_0', \qquad k \in K.$$

In particular, $\phi_0$ is $K$-invariant. We let $\pi$ be the irreducible component of $\sigma$ containing the unit representation of $K$. Our aim is to prove the following result:

PROPOSITION 6. *With the above notations,*

$$\hat{f}(\pi) = \hat{f}'(\pi').$$

*In particular, the map $f \mapsto f'$ is a morphism of the Hecke algebras.*

Before proving the proposition, we remark it is the fundamental lemma for the case at hand: we could define $f'$ by this property. Then the proposition states that the orbital integrals of $f$ and $f'$ match:

$$\int f(hgn)\, dh \theta(n)\, dn$$
$$= |\det g'|^{(m-1)/2} \iint f'({}^t n_1 g' n_2)\theta'(n_1)\theta'(n_2)\, dn_1\, dn_2$$

where $g'$ is relevant for $\theta'$ in $G'$ and $g$ is such that

$${}^t g \varepsilon g = w_0 \begin{pmatrix} g' & 0 \\ 0 & -{}^t g' \end{pmatrix}.$$

As in **[H-R]**, we construct a linear form $\gamma$ on $V$ which is invariant under translation by $H$. The group $P \cap H$ is a parabolic subgroup in $H$. A Levi factor consists of all matrices of the form

$$m = \begin{pmatrix} g & 0 \\ 0 & {}^t g^{-1} \end{pmatrix}.$$

The unipotent radical consists of all matrices of the form

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$$

where ${}^t u = u$. The module of $P \cap H$ is given by:

$$\delta_{P \cap H}(m) = |\det g|^{m+1}.$$

Thus the function $g \mapsto \beta(\phi(g))$ satisfies

$$\beta(\phi(pg)) = \delta_{P\cap H}(p)\beta(\phi(g)),$$

for $p \in P \cap H$. In particular, the linear form

$$\gamma(\phi) = \int_{K\cap H} \beta(\phi(k))\, dk \tag{21}$$

has the required invariance property:

$$\gamma(\sigma(h)\phi) = \gamma(\phi), \qquad h \in H.$$

Moreover

$$\gamma(\phi_0) = 1.$$

Now suppose $f$ is in the Hecke algebra $G$. Then we have:

$$\int \phi_0(gx)f(x)\, dx = \phi_0(g)\hat{f}(\pi).$$

Let us apply the linear form $\gamma$ to both sides. We obtain

$$\hat{f}(\pi) = \beta\left[\iint \phi_0(kx)f(x)\, dx\, dk\right] \tag{22}$$

the integral over $G\times(K\cap H)$. After a change of variables, the integral on the right can be written

$$\hat{f}(\pi) = \beta\left[\iint \phi_0(x)f(kx)\, dx\, dk\right]. \tag{23}$$

Set

$$x = \begin{pmatrix} g & 0 \\ 0 & {}^t g^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g' & 0 \\ 0 & 1 \end{pmatrix} k_1$$

where $u$ is symmetric, $v$ skew symmetric and $k_1 \in K$. Then

$$dx = dg\, dg'\, du\, dv\, dk_1 |\det g'|^{-m}.$$

Our integral becomes

$$\int f\left[k\begin{pmatrix} g & 0 \\ 0 & {}^t g^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g' & 0 \\ 0 & 1 \end{pmatrix}\right]$$
$$\cdot \beta\left[\pi' \otimes \pi' \begin{pmatrix} g' & 0 \\ 0 & 1 \end{pmatrix} \phi_0\right] |\det g'|^{(1-m)/2} |\det g|^{m+1}\, dg\, dg'\, du\, dv\, dk.$$

We recognize the Haar measure $dh$ on $H$ and we get:

$$\int f\left[h\begin{pmatrix}1 & v\\ 0 & 1\end{pmatrix}\begin{pmatrix}g' & 0\\ 0 & 1\end{pmatrix}\right]\beta(\pi'(g')\phi_0'\otimes\phi_0')|\det g'|^{(1-m)/2}\,dh\,dg'\,dv\,.$$

Now

$$\beta(\pi'(g')\phi_0'\otimes\phi_0')$$

is the spherical function $\omega'$ attached to $\pi'$. In terms of $\Phi_f$ the previous integral can be written as

$$\int\Phi_f\left[\begin{pmatrix}g' & 2v\\ 0 & -{}^tg'\end{pmatrix}\right]\omega'(g')|\det g'|^{\,(1-m)/2}\,dv\,dg'$$
$$=\int f'(g')\omega'(g')\,dg'=\hat{f}'(\pi')\,.$$

Thus, we find the right-hand side of (22) is equal to $\hat{f}'(\pi')$, as required.

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