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ORIENTATION AND STRING STRUCTURES ON LOOP SPACE

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This paper deals rigorously with the notion of a string structure and the topological obstruction to its existence. The question of orienting loop space is discussed and shown to be directly analogous to orientation of finite dimensional manifolds. Finally, equivariant string structures are considered.

1. Introduction. Physicists working on the grand unification program have recently been led to consider particles, not as points on some manifold M, but rather as loops on M. This novel idea has resulted in efforts to formulate a theory of spinors on LM, the free loop space of M [**3**]. The theory is called string theory.

Since LM is an infinite dimensional manifold, placing this on a rigorous mathematical footing is a real challenge. The first problem is to define the Dirac operator and involves constructing a spinor bundle on which it acts. E. Witten and T. P. Killingback have made considerable progress in this regard.

Witten in [**19**], [**20**] and Atiyah in [**2**] argued that LM should be considered orientable exactly when M is a spin manifold. Killingback [**10**] looked at bundles on LM whose structural groups are loop groups. He defined a string structure as a lifting of the structural group to a central extension of the loop group by a circle. The candidate for the spinor bundle on LM is then a certain infinite dimensional vector bundle associated with the string structure. Just as in finite dimensions, there is a topological obstruction to defining this bundle. Killingback argued that it is essentially the first Pontrjagin class of M. In this paper, we clarify these results and prove them rigorously.

In §2, we examine the orientability of loop space. Suppose that $P \to M$ is an $\mathrm{SO}(n)$-bundle. By taking free loops, we obtain an $L\,\mathrm{SO}(n)$-bundle $LP \to LM$ in a natural way. Assuming that M is simply connected, we show that it is possible to reduce the structural group of $LP \to LM$ to the connected component of the identity if and only if $P \to M$ admits a spin structure. The condition that M be simply connected is reasonable, since it is equivalent to LM

being connected. We will see that $L\,\mathrm{SO}(n)$ is the structural group of the tangent bundle to loop space. Combining this result with the observation of Atiyah and Witten, we find that orientability of loop space is exactly analogous to orientation in finite dimensions.

Now suppose that $P \to M$ has a spin structure $Q \to M$. A string structure for this bundle is a lifting of the structural group of $LQ \to LM$ to a central extension of $L\,\mathrm{Spin}(n)$ by the circle. In §3, we prove (compare **[10]**)

THEOREM. *Suppose that M is simply connected and $n \geq 5$. Then $LQ \to LM$ has a string structure if $\frac{1}{2}p_1(P) = 0$, where $p_1(P)$ is the first Pontrjagin class of $P \to M$. The converse is also true if we further assume that $\pi_2(M) = 0$.*

In §4, we consider the case where a compact, abelian group G acts on M. It is easy to see that there exists a G-equivariant spin structure if and only if the equivariant Stiefel-Whitney class $w_2(P)_G$ vanishes. As noted in **[19]**, the analogous result for string theory would be that a G-equivariant string structure exists if $\frac{1}{2}p_1(P)_G$ is zero. We will prove this statement and show also that the converse is, in general, false. Finally, we show that the spinor bundle on LM is equivariant under the action of rotating loops (at least if $\pi_2(M) = 0$). The novelty here is that this action is not induced from one on M.

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2. Orientability of loop space. Let M be a compact, connected, orientable, Riemannian manifold of dimension n. LM will denote the space of smooth loops on M. It is an infinite dimensional, paracompact manifold modelled on the topological vector space $L\mathbb{R}^n$ (with the topology of uniform convergence of the functions and all their derivatives) **[14**, Chapter 3]. If M happens to be a Lie group, then LM is an infinite dimensional Lie group—a loop group.

A tangent vector to a loop γ in M is an "infinitesimal deformation" of γ and therefore can be regarded as a map $v\colon \theta \to T_{\gamma(\theta)}M$. Thus, we see that the space $T_\gamma LM$ of all such vectors is precisely the space of sections of the pullback bundle $\gamma^*TM \to S^1$. But this is a trivial bundle since M is orientable. Therefore, we can identify $T_\gamma LM$ with $L\mathbb{R}^n$. In this way, we see that the tangent bundle to loop space is an infinite dimensional, locally trivial vector bundle with fiber $L\mathbb{R}^n$. It is associated to the $L\,\mathrm{SO}(n)$-bundle $LFM \to LM$, obtained in a natural way by taking free loops on the frame bundle $FM \to M$, (compare [10]). Thus, $LFM \to LM$ plays the role of the frame bundle on loop space.

However, as topological spaces, $L\,\mathrm{SO}(n) \cong \Omega\,\mathrm{SO}(n) \times \mathrm{SO}(n)$, where $\Omega\,\mathrm{SO}(n)$ denotes the pointed loop space. It follows that $L\,\mathrm{SO}(n)$ has two connected components, $n \geq 3$. The question arises whether the structural group of $LFM \to LM$ can be reduced to $L^0\,\mathrm{SO}(n)$, the connected component of the identity. A complete answer is given by

PROPOSITION 2.1. *Suppose that M is simply connected and that $P \to M$ is an $\mathrm{SO}(n)$-bundle, $n \geq 4$. The following are equivalent:*

(1) *The structural group of $LP \to LM$ is reducible to $L^0\,\mathrm{SO}(n)$.*
(2) *$P \to M$ has a spin structure.*
(3) *The structural group of $LP \to LM$ can be lifted to $L\,\mathrm{Spin}(n)$.*

Proof. (1) $\Leftrightarrow$ (2). First we characterize the obstruction to reducing the structural group of $LP \to LM$ to $L^0\,\mathrm{SO}(n)$. There are classifying spaces $BL\,\mathrm{SO}(n)$ and $BL^0\,\mathrm{SO}(n)$, for $L\,\mathrm{SO}(n)$ and $L^0\,\mathrm{SO}(n)$-bundles respectively. Moreover, the inclusion map $i\colon L^0\,\mathrm{SO}(n) \to L\,\mathrm{SO}(n)$ induces a double covering $B(i)\colon BL^0\,\mathrm{SO}(n) \to BL\,\mathrm{SO}(n)$ [4]. Reducing the structural group of $LP \to LM$ corresponds to lifting its classifying map in the following diagram.

$$\begin{array}{ccc} & & BL^0\,\mathrm{SO}(n) \\ & & \big\downarrow {\scriptstyle B(i)} \\ LM & \longrightarrow & BL\,\mathrm{SO}(n) \end{array}$$

Since $BL^0\,\mathrm{SO}(n)$ is connected, the double covering $B(i)$ is non-trivial. There is a unique, well-defined obstruction to finding a section of $B(i)$ and it lies in $H^1(BL\,\mathrm{SO}(n);\mathbb{Z}_2)$. From the Hurewicz Theorem, we see that this group is $\mathbb{Z}_2$. Therefore, the non-zero element must generate the obstruction. By functoriality, we conclude that the obstruction to reducing the structural group of $LP \to LM$ is the pull-back of this element by f. We denote it by $\lambda(P)$.

The evaluation map $ev\colon LM \times S^1 \to M$ defined by evaluating a loop against an angle, induces a map $ev^*\colon H^2(M) \to H^2(LM \times S^1)$. Composing this with "integration over S^1" (denoted $\int_{S^1}$), we obtain a map $\int_{S^1} \circ\, ev^*\colon H^2(M) \to H^1(LM)$. To complete the proof, we show that this composition carries the second Stiefel-Whitney class of $P \to M$ isomorphically to $\lambda(P)$.

Taking free loops on the universal $\mathrm{SO}(n)$-bundle $E\,\mathrm{SO}(n) \to B\,\mathrm{SO}(n)$ yields a bundle with contractible total space, on which $L\,\mathrm{SO}(n)$ acts freely. This means that $LE\,\mathrm{SO}(n) \to LB\,\mathrm{SO}(n)$ is actually a model for the universal $L\,\mathrm{SO}(n)$-bundle. Furthermore, we see that by "looping" the classifying map of $P \to M$, we obtain the classifying map of $LP \to LM$. But, $w_2(P)$ is the pullback of the generator of $H^2(B\,\mathrm{SO}(n); \mathbb{Z}_2)$ under the classifying map of $P \to M$. Therefore, the assertion will follow, if we show that $\int_{S^1} \circ ev^*$ is an isomorphism when M is simply connected.

We work on the level of homology. By the Hurewicz Theorem, $H_2(M)$ is generated by some map $f: S^2 \to M$. But, S^2 can be covered by loops which meet only at one point, and the parameter space for this set of loops is a copy of S^1. Using this, we obtain a map $g: S^1 \to LM$ and a corresponding element $[g]$ in $H_1(LM)$. The map g has the property that its evaluation against the circle is exactly f. Thus, ev_* maps $[g] \otimes [\mathrm{id}] \in H_2(LM \times S^1)$ to $[f]$, the generator of $H_2(M)$. Conversely, starting with a map $g: S^1 \to LM$, we can realize S^1 as the parameter space for such loops and then evaluate to produce $f: S^2 \to M$. We conclude that $\int_{S^1} \circ ev^*$ is an isomorphism if $\pi_1(M) = 0$.

(2) $\Leftrightarrow$ (3). There is another evaluation map $ev_0 : LM \to M$ which maps a loop to its basepoint. It induces a bundle morphism

$$\begin{array}{ccc} LP & \longrightarrow & P \\ \downarrow & & \downarrow \\ LM & \longrightarrow & M \end{array}$$

which in turn induces a morphism of the associated spectral sequences. This leads to the following commutative diagram.

$$\begin{array}{ccccccc} \cdots \longrightarrow & H^1(P;\mathbb{Z}_2) & \longrightarrow & H^1(\mathrm{SO}(n);\mathbb{Z}_2) & \longrightarrow & H^2(M;\mathbb{Z}_2)\cdots \\ & \downarrow & & \downarrow & & \downarrow \\ \cdots \longrightarrow & H^1(LP;\mathbb{Z}_2) & \longrightarrow & E^{0,1} & \longrightarrow & E^{2,0}\ldots \end{array}$$

Since $H^1(L\,\mathrm{SO}(n);\mathbb{Z}_2) \cong \mathbb{Z}_2$, it follows that $E^{0,1} \cong \mathbb{Z}_2$. Lifting the structural group of $LP \to LM$ to $L\,\mathrm{Spin}(n)$, corresponds to finding an element of $H^1(LP;\mathbb{Z}_2)$ which restricts to the generator of $E^{0,1}$. By exactness, the image of this element in $E^{2,0}$ is the obstruction. We denote it by $\nu(P)$.

We know from **[1]** that $w_2(P)$ is the image of the generator of $H^1(\mathrm{SO}(n); \mathbb{Z}_2)$ in $H^2(M; \mathbb{Z}_2)$. Moreover, ev_0 has an obvious section obtained by regarding M as the constant loops in LM. Therefore, ev_0^* is injective, and we conclude that it carries $w_2(P)$ isomorphically to $\nu(P)$, completing the proof.

Witten **[20]** and Atiyah **[2]** both argued that LM should be considered orientable exactly when M is spin. Combining this with the above proposition, we see that orientation of loop space is directly analogous to orientation of finite dimensional manifolds.

COROLLARY 2.2. *LM is orientable if and only if the structural group of the tangent bundle can be reduced to the connected component of the identity.*

3. String structures on loop space. To define a spin structure in finite dimensions, one lifts the structural group of the frame bundle $FM \to M$ to $\mathrm{Spin}(n)$, the universal cover of $\mathrm{SO}(n)$ for $n \geq 3$. The corresponding notion for loop space is called a string structure, which we now describe.

Let $Q \to M$ be a spin structure for the $\mathrm{SO}(n)$-bundle $P \to M$. Following Killingback in **[10]**, we define a string structure to be a lifting of the structural group of $LQ \to LM$ to $\widetilde{L\mathrm{Spin}}(n)$, a non-trivial, central extension of $L\,\mathrm{Spin}(n)$ by S^1. We will be especially interested in $\widehat{L\mathrm{Spin}}(n)$, the universal such extension. For the existence of such extensions and the fact that they are completely determined by their topological class as S^1-bundles, see **[14**, Chapter 4].

The motivation for this definition is the following: In trying to formally define the Dirac operator on loop space, one is led to an infinite dimensional Clifford algebra. The spinor representation of this algebra is then a representation of some extension of $L\,\mathrm{SO}(n)$. The novelty here is that it is an extension by a circle, rather than a discrete group as one might expect **[14**, Chapter 12]. It follows that the spinor representation is also a representation of some extension of $L\,\mathrm{Spin}(n)$ by the circle. We will return to this later. We want to prove (compare **[10]**)

THEOREM 3.1. *Let $P \to M$ be an $\mathrm{SO}(n)$-bundle over a simply connected manifold, $n \geq 5$. Suppose that $Q \to M$ is a spin structure for this bundle. Then $LQ \to LM$ has a string structure if $\frac{1}{2}p_1(P)$ vanishes, where $p_1(P)$ denotes the first Pontrjagin class of $P \to M$. The converse is also true, if we further assume that M is doubly connected.*

First, we explain what we mean by $\frac{1}{2}p_1(P)$.

LEMMA 2.2. *A spin structure $Q \to M$ for the* $\mathrm{SO}(n)$*-bundle $P \to M$ determines a class in $H^4(M;\mathbb{Z})$, which when multiplied by 2 is exactly $p_1(P)$. It is the pullback of the generator of $H^4(B\,\mathrm{Spin}(n);\mathbb{Z})$ by the classifying map of $Q \to M$.*

Proof. The projection π: $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ induces a fibration $B(\pi)$: $B\,\mathrm{Spin}(n) \to B\,\mathrm{SO}(n)$ with fiber $B\mathbb{Z}_2$ [4]. The classifying maps of $P \to M$ and $Q \to M$ induce the following commutative diagram.

$$\begin{array}{ccc} H^4(B\,\mathrm{SO}(n);\mathbb{Z}) & \longrightarrow & H^4(M;\mathbb{Z}) \\ \big\downarrow{\scriptstyle B(\pi)^*} & & \big\downarrow{\scriptstyle \mathrm{id}} \\ H^4(B\,\mathrm{Spin}(n);\mathbb{Z}) & \longrightarrow & H^4(M;\mathbb{Z}) \end{array}$$

We will show that the image of $B(\pi)^*$ has index 2 and that both groups in the left-hand column are $\mathbb{Z}$. Since $p_1(P)$ is the image of the generator of $H^4(B\,\mathrm{SO}(n);\mathbb{Z})$ under the classifying map of $P \to M$, the result will follow.

The integral cohomology of $B\,\mathrm{SO}(n)$ can be computed using the following facts [4], [13]: The only torsion in $H^*(B\,\mathrm{SO};\mathbb{Z})$ is 2-torsion and $H^*(B\,\mathrm{SO};\mathbb{Z}_2)$ is a polynomial algebra generated by the Stiefel-Whitney classes $w_2, w_3, \ldots$. Moreover, $H^*(B\,\mathrm{SO};\mathbb{Q})$ is a polynomial algebra generated by elements in degree $4i$, the Pontrjagin classes. We deduce that $H^i(B\,\mathrm{SO}(n);\mathbb{Z})$ is zero in degrees 1 and 2, $\mathbb{Z}_2$ in degrees 3 and 5, and $\mathbb{Z}$ in degree 4. On the other hand, it follows from the Hurewicz Theorem that $H^4(B\,\mathrm{Spin}(n);\mathbb{Z}) \cong \mathbb{Z}$.

It is well known [7] that $H^*(B\mathbb{Z}_2;\mathbb{Z}) \cong \mathbb{Z}_2[a]/(2a)$. Therefore, in the spectral sequence associated to $B(\pi)$, d_3: $E^{0,4} \to E^{3,2}$ maps a^2, the generator of $H^4(B\mathbb{Z}_2;\mathbb{Z})$, to $2a \cdot da$ and so must be zero. The only other relevant differential is d_3: $E^{2,2} \to E^{5,0}$. This is actually an isomorphism. To see this, note that both groups are $\mathbb{Z}_2$. If d_3 were the zero map, $E^{5,0}$ would survive in the spectral sequence giving something non-zero in $H^5(B\,\mathrm{Spin}(n))$. But $B\,\mathrm{Spin}(n)$ is 3-connected so the Hurewicz homomorphism is surjective in degree 5. Since $\pi_5(B\,\mathrm{Spin}(n)) \cong \pi_4(\mathrm{SO}(n))$ is at most 2-torsion [17, Part 2], we must have $H^5(B\,\mathrm{spin}(n);\mathbb{Z}) = 0$—a contradiction.

These facts yield a short exact sequence

$$0 \to H^4(B\,\mathrm{SO}(n);\mathbb{Z}) \to H^4(B\,\mathrm{Spin}(n);\mathbb{Z}) \to \mathbb{Z}_2 \to 0,$$

and the lemma follows.

Now we discuss the obstruction to defining a string structure. Since $L\,\mathrm{Spin}(n)$ is connected and simply connected, the spectral sequence of the bundle $LQ \to LM$ degenerates in low degrees and we obtain an exact sequence

$$\begin{aligned} 0 &\to H^2(LM;\mathbb{Z}) \to H^2(LQ;\mathbb{Z}) \\ &\to H^2(L\,\mathrm{Spin}(n);\mathbb{Z}) \to H^3(LM;\mathbb{Z}) \to \cdots . \end{aligned}$$

Lifting the structural group of $LQ \to LM$ to $\widehat{L\mathrm{Spin}}(n)$ corresponds to finding a circle bundle $\widehat{LQ}$ over LQ which restricts to the bundle $\widehat{L\mathrm{Spin}}(n) \to L\,\mathrm{Spin}(n)$ in each fiber. Circle bundles over LQ and $L\,\mathrm{Spin}(n)$ are classified by elements of $H^2(LQ;\mathbb{Z})$ and $H^2(L\,\mathrm{Spin}(n);\mathbb{Z})$. The latter group is $\mathbb{Z}$ and the bundle $\widehat{L\mathrm{Spin}}(n) \to L\,\mathrm{Spin}(n)$ corresponds to the generator. We conclude that the existence of a string structure corresponding to $\widehat{L\mathrm{Spin}}(n)$ is equivalent to the existence of an element of $H^2(LQ;\mathbb{Z})$, which restricts to the generator of $H^2(L\,\mathrm{Spin}(n);\mathbb{Z})$. From the exact sequence above, the image of the generator in $H^3(LM;\mathbb{Z})$ is the obstruction to defining this string structure. We denote it by $\mu(Q)$. It is the pullback of the generator of $H^3(LB\,\mathrm{Spin}(n);\mathbb{Z})$ by the classifying map of $LQ \to LM$. The obstruction to defining a string structure corresponding to any other central extension, is just some multiple of μ. Note also, that inequivalent string structures are classified by elements of $H^2(LM;\mathbb{Z})$.

We prove Theorem 3.1 by showing that $\int_{S^1} \circ\, ev^*$ maps $\frac{1}{2}p_1(P)$ to μ and does so injectively when M is 2-connected. By functoriality, the first assertion will follow from the fact that

$$\int_{S^1} \circ\, ev^* \colon H^4(B\,\mathrm{Spin}(n);\mathbb{Z}) \to H^3(LB\,\mathrm{Spin}(n);\mathbb{Z})$$

is an isomorphism for $n \geq 5$. To see this, we use a similar argument to the one given in Proposition 2.1. By the Hurewicz Theorem, $H_4(B\,\mathrm{Spin}(n))$ is generated by a map $f\colon S^4 \to B\,\mathrm{Spin}(n)$. If we cover S^4 by loops meeting at only one point, the parameter space for such loops is S^3. Using this, we produce a map $g\colon S^3 \to L\,\mathrm{Spin}(n)$ which "evaluates" to f. Proceeding exactly as before yields the result. In fact this argument shows that $\int_{S^1} \circ\, ev^*$ is injective on those classes which are cohomologous to maps of spheres into our manifold. If M is assumed to be 2-connected, this will certainly be the case, since then the Hurewicz homomorphism is an isomorphism in degree 3 and surjective in degree 4.

REMARK. The case $n = 4$ must be treated separately, since $\mathrm{SO}(4)$ is no longer simple. But $\pi_i(B\,\mathrm{Spin}(4)) = 0$ for $i = 1, 2, 3$, so that $\int_{S^1} \circ ev^*$ is still an isomorphism. But now,

$$H^4(B\,\mathrm{Spin}(n)\,;\,\mathbb{Z}) \cong H^3(LB\,\mathrm{Spin}(n)\,;\,\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

As before, this result allows us to study the obstruction to lifting the structural group of $LQ \to LM$ to a central extension of $L\,\mathrm{Spin}(4)$ by S^1. However, the universal central extension is now an extension by a 2-torus, not a circle, since $\mathrm{SO}(4)$ has two simple factors [**14**, Chapter 4].

The spinor bundle on loop space is a certain infinite dimensional vector bundle associated with a string structure. To define it, we need the appropriate representation of $\widetilde{L\mathrm{Spin}(n)}$. Let H be the Hilbert space $L^2(S^1\,;\,\mathbb{R}^{2n})$ and $e_1, \ldots, e_{2n}$ the standard basis for $\mathbb{R}^{2n}$. If $\varepsilon_k = \frac{1}{2}(e_{2k-1} + ie_{2k})$, the elements $\varepsilon_k z^m$ and $\bar{\varepsilon}_k z^m$ form a basis for the complexification $H_{\mathbb{C}}$. Let W denote the subspace of $H_{\mathbb{C}}$ spanned by $\varepsilon_k z^r$ and $\bar{\varepsilon}_l z^s$, for $r \geq 0$ and $s \geq 1$. Then it is shown in [**14**, Chapter 12], that $L^0\,\mathrm{SO}(n)$ (and hence $L\,\mathrm{Spin}(n)$) acts projectively on the Hilbert space completion of $\bigwedge(W)$ and on the completions of $\bigwedge^{\mathrm{even}}(W)$, $\bigwedge^{\mathrm{odd}}(W)$ by two inequivalent, irreducible representations. Therefore, they are actual representations of some central extension $\widetilde{L\mathrm{Spin}(n)}$ by S^1. If $LQ \to LM$ admits a string structure, we can form the associated bundles $\widetilde{LQ} \times_{\widetilde{L\mathrm{Spin}(n)}} \bigwedge^{\pm}(W) \to LM$. Sections of these bundles are called strings.

REMARK. One would now like to define the Dirac operator acting on these bundles. This is discussed in [**15**] and involves difficult analytical problems. However, it is possible to give such a definition in a neighbourhood of the constant loops. This is done in [**18**].

4. Equivariant string structures. Suppose that a compact, abelian group G acts on M and let $EG \to BG$ denote the universal G-bundle. We can then form the space $EG \times_G M$, the quotient of $EG \times M$ by the diagonal action of G. This space is fibered over BG with fiber M and its cohomology $H^*_G(M)$ is, by definition, the equivariant cohomology of M. If $P \to M$ is a G-equivariant $\mathrm{SO}(n)$-bundle, then this construction yields an $\mathrm{SO}(n)$-bundle $EG \times_G P \to EG \times_G M$. There is a commutative diagram of classifying maps,

$$\begin{array}{ccc} M & \longrightarrow & B\,\mathrm{SO}(n) \\ \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle \mathrm{id}} \\ EG\times_G M & \longrightarrow & B\,\mathrm{SO}(n) \end{array}$$

where i is the inclusion of the fiber. If, in addition, $P \to M$ has a G-equivariant spin structure $Q \to M$, then there is a similar diagram involving $B\,\mathrm{Spin}(n)$ and the classifying map of $Q \to M$. We conclude that the bundle $EG\times_G Q \to EG\times_G M$ is a spin structure for $EG\times_G P \to EG\times_G M$. Therefore, the second Stiefel-Whitney class of the latter bundle must be zero. This class is denoted by $w_2(P)_G$.

Conversely, suppose that $P \to M$ is G-equivariant and that $w_2(P)_G = 0$. Then, the bundle $EG\times_G P \to EG\times_G M$ admits a spin structure $\widetilde{Q} \to EG\times_G M$. But $\widetilde{Q} \to EG\times_G P$ is a double cover which by **[11]** corresponds to some G-equivariant double cover $Q \to P$. Note that the main result of **[11]** applied here because by **[9]** P has the homotopy type of a G-CW complex. Since $Q \to P$ is the pullback of $\widetilde{Q} \to EG\times_G P$ by the inclusion $P \hookrightarrow EG\times_G P$, we conclude that $Q \to M$ is a G-equivariant spin structure for $P \to M$.

As noted in **[19]**, the analogous result for string theory would be that a G-equivariant string structure exists if and only if the equivariant class $\frac{1}{2}p_1(P)_G$ vanishes—the "rigidity condition" (see also **[6]**). In fact, this is not completely true, as we will now see. First observe that the action of G on M induces an action on LM. If $Q \to M$ is G-equivariant, then $LQ \to LM$ is an equivariant $L\,\mathrm{Spin}(n)$-bundle under this induced action. As above, we obtain a commutative diagram,

$$\begin{array}{ccc} LM & \longrightarrow & BL\,\mathrm{Spin}(n) \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{id}} \\ EG\times_G LM & \longrightarrow & BL\,\mathrm{Spin}(n). \end{array}$$

Suppose now that $LQ \to LM$ has a G-equivariant string structure $\widetilde{LQ} \to LM$. Then arguing as before, we can conclude that $EG\times_G \widetilde{LQ} \to EG\times_G LM$ is a string structure for the bundle $EG\times_G LQ \to EG\times_G LM$. Thus, the obstruction μ_G to lifting the structural group of $EG\times_G LQ \to EG\times_G LM$ to $L\widetilde{\mathrm{Spin}}(n)$ must vanish. The reasoning used in §3 shows that μ_G lies in $H^3_G(LM; \mathbb{Z})$, and is the pullback of the generator of $H^3(BL\,\mathrm{Spin}(n); \mathbb{Z})$ by the classifying map of $EG\times_G LQ \to EG\times_G LM$.

Conversely, suppose that $\mu_G = 0$. Then the structural group of $EG \times_G LQ \to EG \times_G LM$ can be lifted to $L\widetilde{\mathrm{Spin}}(n)$, yielding a new bundle $\widetilde{W} \to EG{\times_G}LM$. The map $\widetilde{W} \to EG{\times_G}LQ$ has the structure of an S^1-bundle. According to **[12]**, the space LQ has the homotopy type of a locally finite CW complex. Therefore, we can apply the main result of **[8]** to obtain a corresponding G-equivariant circle bundle $W \to LQ$. As before, we conclude that $W \to LM$ is a G-equivariant string structure for $LQ \to LM$.

The evaluation map $ev\colon LM \times S^1 \to M$ is G-equivariant (with the trivial action on S^1). Therefore it induces a map

$$ev_G : (EG \times_G LM) \times S^1 \to EG \times_G M.$$

Composing with integration over S^1, we obtain a map from $H^4_G(M;\mathbb{Z})$ to $H^3_G(LM;\mathbb{Z})$, and a commutative diagram:

$$\begin{array}{ccc} H^4_G(M;\mathbb{Z}) & \xrightarrow{\int_{S^1} \circ\, ev_G^*} & H^3_G(LM;\mathbb{Z}) \\ \uparrow & & \uparrow \\ H^4(B\,\mathrm{Spin}(n);\mathbb{Z}) & \xrightarrow{\int_{S^1} \circ\, ev^*} & H^3(LB\,\mathrm{Spin}(n);Z)\,. \end{array}$$

The vertical maps are induced from the classifying maps of $EG \times_G Q \to EG{\times_G}M$ and $EG{\times_G}LQ \to EG{\times_G}LM$. By definition, $\frac{1}{2}p_1(P)_G$ is the pullback of the generator of $H^4(B\,\mathrm{Spin}(n);\mathbb{Z})$ by the classifying map of $EG \times_G Q \to EG \times_G M$. From the diagram, we see that it is mapped to μ_G. From this, we conclude that if $\frac{1}{2}p_1(P)_G$ vanishes then $LQ \to LM$ has a G-equivariant string structure. Conversely, suppose that there exists an equivariant string structure, so that μ_G is zero. To conclude that $\frac{1}{2}p_1(P)_G$ is zero, we need to know whether $\int_{S^1} \circ\, ev_G^*$ is injective. There is a commutative diagram,

$$\begin{array}{ccc} H^4_G(M;\mathbb{Z}) & \xrightarrow{\int_{S^1} \circ\, ev_G^*} & H^3_G(LM;\mathbb{Z}) \\ \downarrow & & \downarrow \\ H^4(M;\mathbb{Z}) & \xrightarrow{\int_{S^1} \circ\, ev^*} & H^3(LM;\mathbb{Z}) \end{array}$$

where the vertical arrows are induced from the homotopy inclusion of the fiber. Thus, we see that injectivity will depend on the cohomology of BG and the injectivity of $\int_{S^1} \circ ev^*$. In the important case where $G = S^1$, the map is not generally injective. For instance, if M is 3-connected, then the spectral sequence associated to $ES^1 \times_{S^1} M \to BS^1$ yields an exact sequence

$$0 \to \mathbb{Z} \to H^4_{S^1}(M; \mathbb{Z}) \to H^4(M; \mathbb{Z}) \to 0.$$

Summarizing, we have proved

PROPOSITION 4.1. *Let G be a compact, abelian group and $P \to M$ a G-equivariant* SO(n)*-bundle.*

(1) *$P \to M$ admits a G-equivariant spin structure if and only if $w_2(P)_G$ vanishes.*

(2) *Suppose that $n \geq 5$ and that $w_2(P)_G = 0$. Then there exists a G-equivariant string structure if $\frac{1}{2}p_1(P)_G$ vanishes. The converse is not true in general.*

The proposition does not generalize to non-compact groups, since the result of **[11]** does not apply in this case. If $G = \mathbb{Z}$, the above argument does show that the existence of a $\mathbb{Z}$-equivariant spin structure forces $w_2(P)_{\mathbb{Z}}$ to vanish. Moreover, if we assume M is 2-connected, then $\int_{S^1} \circ ev^*_{\mathbb{Z}}$ is injective, since $B\mathbb{Z} = S^1$. Therefore, the existence of a $\mathbb{Z}$-equivariant string structure also forces $\frac{1}{2}p_1(P)_{\mathbb{Z}}$ to be zero.

Closely related to this is the question of whether a given diffeomorphism f preserves spin and string structures. The map f generates a $\mathbb{Z}$-action on M. The space $E\mathbb{Z} \times_{\mathbb{Z}} M$ can then be identified with the "mapping cylinder"

$$M_f = \frac{M \times I}{(x, 0) \sim (f(x), 1)}.$$

The bundle $E\mathbb{Z} \times_{\mathbb{Z}} P \to E\mathbb{Z} \times_{\mathbb{Z}} M$ is just $P_f \to M_f$. If $P \to M$ is $\mathbb{Z}$-equivariant, there is a diagram of bundle morphisms,

$$\begin{tikzcd}
\mathrm{SO}(n) \arrow[r, equal] \arrow[d] & \mathrm{SO}(n) \arrow[r] \arrow[d] & pt. \arrow[d] \\
P \arrow[r] \arrow[d] & P_f \arrow[r] \arrow[d] & S^1 \arrow[d, equal] \\
M \arrow[r] & M_f \arrow[r] & S^1.
\end{tikzcd}$$

The induced morphisms of spectral sequences yield a commutative array of cohomology groups in low degrees. In the Gysin sequence associated to $P_f \to S^1$, the arrow $H^1(P) \to H^1(P)$ can be identified with $f^* - \mathrm{id}^*$. Thus, we can conclude that there is a spin structure invariant under f if and only if $w_2(P_f) = 0$.

Suppose now that such a bundle $Q \to M$ exists. The induced action of $\mathbb{Z}$ on LM is generated by a map F, related in the obvious manner to f. Arguing in a similar fashion, we see that $LQ \to LM$ has a string structure invariant under F if and only if $(LQ)_F \to (LM)_F$ admits a string structure. Of course the latter bundle is $E\mathbb{Z} \times_{\mathbb{Z}} LQ \to E\mathbb{Z} \times_{\mathbb{Z}} LM$. Assuming that M is 2-connected, the proof of Proposition 4.1 shows that $\int_{S^1} \circ ev_{\mathbb{Z}}^*$ is injective. We conclude that there is an invariant string structure if and only if $\frac{1}{2}p_1(P)_{\mathbb{Z}} = \frac{1}{2}p_1(P_f) = 0$. This was noted in **[10]**.

The most natural action on LM is that of rotating loops. It differs from the above situation in that it is not inherited from an action on M. We now ask whether a given string structure $\widetilde{LQ} \to LM$ is equivariant under this action. This is equivalent to requiring that the S^1-action on LQ lifts to the S^1-bundle $\widetilde{LQ} \to LQ$. Again, we apply the main result of **[8]**. We conclude that the S^1-action lifts if and only if there is a commutative diagram of classifying maps,

$$\begin{array}{ccc} LQ & \longrightarrow & BS^1 \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \mathrm{id}} \\ ES^1 \times_{S^1} LQ & \longrightarrow & BS^1. \end{array}$$

But, $\widetilde{LQ} \to LQ$ corresponds to some element μ in $H^2(LQ; \mathbb{Z})$. Therefore, such a diagram exists if and only if μ lies in the image of $i^*: H^2_{S^1}(LQ; \mathbb{Z}) \to H^2(LQ; \mathbb{Z})$. From the spectral sequence associated to $ES^1 \times_{S^1} LQ \to BS^1$, we see that the obstruction to this diagram is $d_2(\mu) \in H^2(BS^1; \mathbb{Z}) \otimes H^1(LQ; \mathbb{Z})$. This will be zero, for instance, when $\pi_2(M)$ is pure torsion, since then $H^1(LM; \mathbb{Z}) = 0$, forcing $H^1(LQ; \mathbb{Z}) = 0$. At least in this case, we can state

PROPOSITION 4.2. *Suppose M is simply connected and that $\pi_2(M)$ is torsion. Then every string structure for the* $\mathrm{SO}(n)$*-bundle $P \to M$ is necessarily equivariant under the action of rotating loops.*

CONCLUDING REMARKS. Witten has shown in **[19]** that the natural action of S^1 on LM carries deep topological information. Suppose the spin manifold M admits an S^1-action and $p_1(M)$ is torsion. Then, by formally applying the Atiyah-Bott-Lefschetz fixed point formula, he showed that the index of the Dirac operator on LM is a modular form.

The group, $\mathrm{Diff}^+(S^1)$, of orientation-preserving diffeomorphisms, also acts on loop space, in a more refined way than the circle. Moreover, this action lifts to a projective (infinitesimal) action on the bundle of spinors, and gives some geometrical insight into elliptic cohomology [**5**].

Finally, we could have defined the notion of a string structure for any compact, semi-simple Lie group G and its universal cover $\widetilde{G}$, rather than $\mathrm{SO}(n)$ and $\mathrm{Spin}(n)$. Exactly the same methods apply to study this situation.

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