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# NONSPLIT RING SPECTRA AND PRODUCTS OF $\beta$-ELEMENTS IN THE STABLE HOMOTOPY OF MOORE SPACES

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**This paper proves trivialities and nontrivialities of some products of higher order $\beta_{(tp^n/s)}$ elements in the stable homotopy of Moore spaces. The proof is based mainly on properties of nonsplit ring spectra $K_r$ (the cofibre of $r$-iterated Adams map with $r$ not divisible by prime $p \geq 5$) which are given in the rest of the paper.**

**1. Introduction.** Let $S$ be the sphere spectrum and $M$ the Moore spectrum modulo a prime $p \geq 5$ given by the cofibration $S \xrightarrow{p} S \xrightarrow{i} M \xrightarrow{j} \Sigma S$. Consider the Brown-Peterson spectrum $BP$ at $p$; it is known that there is a map $\alpha: \Sigma^q M \to M$ such that the induced $BP_*$ homomorphism $\alpha_* = v_1: BP_*/(p) \to BP_*/(p)$, $q = 2(p-1)$.

Let $K_r$ be the cofibre of $\alpha^r$ given by the cofibration

$$(1.1) \qquad \Sigma^{rq} M \xrightarrow{\alpha^r} M \xrightarrow{i'_r} K_r \xrightarrow{j'_r} \Sigma^{rq+1} M.$$

In [4] and [6], S. Oka showed that $K_r$ is a ring spectrum for $r \geq 1$; if $r \equiv 0 \pmod p$ it is called a split ring spectrum since $K_r \wedge K_r$ splits into four summands $K_r$, $\Sigma K_r$, $\Sigma^{rq+1} K_r$, $\Sigma^{rq+2} K_r$. If $r \not\equiv 0 \pmod p$, it is called a nonsplit ring spectrum since $K_r \wedge K_r$ splits only into three summands $K_r$, $\Sigma L \wedge K_r$, $\Sigma^{rq+2} K_r$, where $L$ is the cofibre of $\phi_1 = j\alpha^r i \in \pi_{rq-1} S$.

In the nonsplit case, S. Oka showed in [4] that there is a direct summand decomposition

$$(1.2) \qquad [\Sigma^* K_r, K_r] = \mathrm{Mod} \oplus \mathrm{Der} \oplus \mathrm{Mod}\, \delta_0$$

where Mod consists of right $K_r$-module maps, Der consists of elements which behave as a derivation on the cohomology defined by $K_r$ and $\delta_0 = i'_r i j j'_r \in [\Sigma^{-rq-2} K_r, K_r]$. Moreover, Mod is a commutative subring, $\ker\{(i'_r i)^*: [\Sigma^* K_r, K_r] \to \pi_* K_r\} = \mathrm{Der} \oplus \mathrm{Mod}\, \delta_0$ and $(i'_r i)^*: \mathrm{Mod} \to \pi_* K_r$ is an isomorphism.

One of the most important properties which are shown in [4] is $\delta' f - f\delta' \in \mathrm{Mod}$ for any $f \in \mathrm{Mod}$, $\delta' = i'_r j'_r \in [\Sigma^{-rq-1} K_r, K_r]$ and the commutativity $\delta' f^p = f^p \delta'$ for any $f \in \mathrm{Mod}$ having even degree.

This has been found very useful in the detection of higher order $\beta_{tp^n/s}$ elements in $\pi_*S$ (cf. [8]).

From [8] and [9], there exist $f_s \in \mathrm{Mod} \cap [\Sigma^* K_s, K_s]$ for $p \geq 5$, $s \leq p^n$ if $p \nmid t \geq 2$ or $s \leq p^n - 1$ if $t = 1$ such that the induced $BP_*$ homomorphism $(f_s)_* = v_2^{tp^n}$, $\beta_{(tp^n/s)} = j_s' f_s i_s'$ is known to be a $\beta$-element in $[\Sigma^* M, M]$ such that

$$\beta'_{tp^n/s} \in \mathrm{Ext}^{1,*} M = \mathrm{Ext}^{1,*}_{BP_*BP}(BP_*, BP_*M)$$

converges to $\beta_{(tp^n/s)} i \in \pi_* M$ in the Adams-Novikov spectral sequence $\mathrm{Ext}^{*,*} M \Rightarrow \pi_* M$.

In this paper, we will prove the following trivialities and nontrivialities of products of $\beta_{(tp^n/s)}$ elements in $[\Sigma^* M, M]$.

THEOREM I. *Let $p \geq 5$. The following relations on products of $\beta$-elements in $[\Sigma^* M, M]$ hold:*

(1) $\beta_{(ktp^n/s)} \cdot \beta_{(tp^n/s)} = 0$ *for* $s \leq p^n$ *if* $p \nmid t \geq 2$, $s \leq p^n - 1$ *if* $t = 1$ *and* $k \not\equiv -1 \pmod p$.

(2) $\beta_{(ktp^n/s)} \delta \beta_{(tp^n/s)} = 0$ *for* $s \leq p^{n-1}$ *if* $p \nmid t \geq 2$, $s \leq p^{n-1} - 1$ *if* $t = 1$ *and* $k \not\equiv -1 \pmod p$, *where* $\delta = ij \in [\Sigma^{-1} M, M]$.

(3) $\beta_{(ap^m/s)} \delta \beta_{(tp^n/s)} = -\beta_{(tp^n/s)} \delta \beta_{(ap^m/s)}$ *if one of the following conditions holds*

(i) $s \leq \min(p^{n-1}, p^{m-1})$ *if* $p \nmid t \geq 2$ *and* $p \nmid a \geq 2$.
(ii) $s \leq \min(p^{n-1}, p^{m-1} - 1)$ *if* $p \nmid t \geq 2$ *and* $a = 1$.
(iii) $s \leq \min(p^{n-1} - 1, p^{m-1})$ *if* $t = 1$ *and* $p \nmid a \geq 2$.
(iv) $s \leq \min(p^{n-1} - 1, p^{m-1} - 1)$ *if* $t = a = 1$.

(4) *Suppose that* $s \leq p^n$ *if* $p \nmid t \geq 2$ *or* $s \leq p^n - 1$ *if* $t = 1$, $r \leq p^m$ *if* $p \nmid a \geq 2$ *or* $r \leq p^m - 1$ *if* $a = 1$; *then*

$$\beta_{(ap^m/r)} \cdot \beta_{(tp^n/s)} \neq 0, \quad \beta_{(ap^m/r)} \delta \beta_{(tp^n/s)} \neq 0$$

*if* $r + s \geq p^n + p^{n-1}$ *and one of the following conditions holds:*

(i) $m = n$, $a + t \equiv 0 \pmod p$.
(ii) $m = n - 1$, $a \not\equiv 1 \pmod p$.
(iii) $m < n - 1$, $a \not\equiv -1 \pmod p$.

Theorem I is proved by using some results on nonsplit ring spectra $K_r$ given in S. Oka [4] and some results on $\mathrm{Ext}^{1,*} M$ given in Miller and Wilson [1]. The proof also needs some further properties of $K_r$ which are not in [4], mainly the following fact on commutativity of some elements in $[\Sigma^* K_r, K_r]$.

THEOREM II. *If* $r \not\equiv 0 \pmod p$ *and* $g, f \in \mathrm{Mod} \cap [\Sigma^* K_r, K_r]$, *then*

$$g^p(\delta_0 f^p - f^p \delta_0) = (-1)^{|f|\cdot|g|}(\delta_0 f^p - f^p \delta_0) g^p$$

*and* $\delta_0 f^{p^2} = f^{p^2}\delta_0$ *if* $f$ *has even degree, where* $\delta_0 = i'_r i j j'_r$ *is the unique generator in* $[\Sigma^{-rq-2} K_r, K_r]$. *If* $r \equiv 0 \pmod p$, $\delta_0 f^p - f^p \delta_0$ *belongs to the commutative subring* $\mathscr{C}_*$ *of* $[\Sigma^* K_r, K_r]$ *and the above two equalities also hold.*

The proof of Theorem I will be given in §2. In §3, we first recall some results on $K_r$ given in [4], then develop some further technical results on $K_r$ and prove Theorem II.

**2. Proof of Theorem I.** From [8] and [9], there exists $f \in [\Sigma^{tp^n(p+1)q} K_s, K_s]$ for $s \le p^n$ if $p \nmid t \ge 2$ or $s \le p^n - 1$ if $t = 1$ such that the induced $BP_*$ homomorphism $f_* = v_2^{tp^n} : BP_*/(p, v_1^s) \to BP_*/(p, v_1^s)$. We may assume $f \in \mathrm{Mod}$ (or $f \in \mathscr{C}_*$ in case $s \equiv 0 \pmod p$) since the components of $f$ in Der and Mod $\delta_0$ induce the zero homomorphism. Then $j'_s f i'_s = \beta_{(tp^n/s)} \in [\Sigma^* M, M]$ and $\beta_{(ktp^n/s)} \beta_{(tp^n/s)} = j'_s f^k i'_s j'_s f i'_s$.

Recall that $\delta' = i'_s j'_s \in [\Sigma^{-sq-1} K_s, K_s]$ and $\delta' f - f\delta' \in \mathrm{Mod}$. From commutativity of Mod, we have $f(\delta' f - f\delta') = (\delta' f - f\delta')f$ or equivalently $f^2\delta' - \delta' f^2 = 2(f^2\delta' - f\delta' f)$. Composing $f$ with the above equation, inductively we have

$$f^r\delta' - \delta' f^r = r(f^r\delta' - f^{r-1}\delta' f), \qquad r \ge 1,$$

and $f^k\delta' f = \frac{1}{k+1}(\delta' f^{k+1} + k f^{k+1}\delta')$ if we let $r - 1 = k \not\equiv -1 \pmod p$. So $\beta_{(ktp^n/s)} \cdot \beta_{(tp^n/s)} = j'_s f^k \delta' f i'_s = 0$; this proves Theorem I (1).

(2) From [8], there exists $f \in [\Sigma^{tp^{n-1}(p+1)q} K_s, K_s]$ such that the induced $BP_*$ homomorphism $f_* = v_2^{tp^{n-1}}$ and $f \in \mathrm{Mod}$. Hence $f_*^p = v_2^{tp^n}$ and $\beta_{(ktp^n/s)} \delta \beta_{(tp^n/s)} = j'_s f^{kp} i'_s i j j'_s f^p i'_s = j'_s f^{kp} \delta_0 f^p i'_s$. From Theorem II, $f^p(\delta_0 f^p - f^p\delta_0) = (\delta_0 f^p - f^p\delta_0) f^p$ or equivalently $f^{2p}\delta_0 - \delta_0 f^{2p} = 2(f^{2p}\delta_0 - f^p\delta_0 f^p)$. By induction we have $f^{rp}\delta_0 - \delta_0 f^{rp} = r(f^{rp}\delta_0 - f^{(r-1)p}\delta_0 f^p)$ for $r \ge 1$. Thus

$$f^{kp}\delta_0 f^p = \frac{1}{k+1}(\delta_0 f^{(k+1)p} + k f^{(k+1)p}\delta_0)$$

for $k \not\equiv -1 \pmod p$ and so $\beta_{(ktp^n/s)} \delta \beta_{(tp^n/s)} = j'_s f^{kp} \delta_0 f^p i'_s = 0$.

(3) In all cases, there exists $f \in \mathrm{Mod} \cap [\Sigma^{tp^{n-1}(p+1)q} K_s, K_s]$ and $g \in \mathrm{Mod} \cap [\Sigma^{ap^{m-1}(p+1)q} K_s, K_s]$ such that $f_* = v_2^{tp^{n-1}}$ and $g_* = v_2^{ap^{m-1}}$. Then $\beta_{(ap^m/s)} \delta \beta_{(tp^n/s)} = j'_s g^p i'_s i j j'_s f^p i'_s = j'_s g^p \delta_0 f^p i'_s$.

From Theorem II, $g^p(\delta_0 f^p - f^p\delta_0) = (\delta_0 f^p - f^p\delta_0)g^p$ or equivalently $g^p\delta_0 f^p + f^p\delta_0 g^p = \delta_0 f^p g^p + g^p f^p \delta_0$. Hence $\beta_{(ap^m/s)}\delta\beta_{(tp^n/s)} + \beta_{(tp^n/s)}\delta\beta_{(ap^m/s)} = j_s'(g^p\delta_0 f^p + f^p\delta_0 g^p)i_s' = 0$.

(4) From [**4**, p. 422], $i_r'j_s': K_s \to \Sigma^{sq+1}K_r$ induces a cofibration

$$\Sigma^{sq}K_r \xrightarrow{\psi_{r,r+s}} K_{r+s} \xrightarrow{\rho_{r+s,s}} K_s \xrightarrow{i_r'j_s'} \Sigma^{sq+1}K_r$$

which realizes the short exact sequence

$$0 \to BP_*/(p, v_1^r) \xrightarrow{\psi_*} BP_*/(p, v_1^{r+s}) \xrightarrow{\rho_*} BP_*/(p, v_1^s) \to 0$$

such that $\psi_* = v_1^s$ and then induces Ext exact sequence

$$\cdots \to \mathrm{Ext}^{k,t-sq}K_r \xrightarrow{\psi_*} \mathrm{Ext}^{k,t}K_{r+s} \xrightarrow{\rho_*} \mathrm{Ext}^{k,t}K_s \xrightarrow{(i_r'j_s')_*} \mathrm{Ext}^{k+1,t-sq}K_r \to \cdots$$

where we briefly write $\mathrm{Ext}^{k,*}X = \mathrm{Ext}^{k,*}_{BP_*BP}(BP_*, BP_*X)$ and $(i_r'j_s')_*$ as the boundary homomorphism. Moreover, we have (cf. [**8**] (3.23))

$$\psi_{r,r+s}i_r' = i_{r+s}'\alpha^s, \quad \rho_{r+s,s}i_{r+s}' = i_s', \quad j_s'\rho_{r+s,s} = \alpha^r j_{r+s}'.$$

Note that the behavior of $\psi_*$, $\rho_*$, $(i_r'j_s')_*$ in the above Ext exact sequence is compatible with that of $\psi$, $\rho$, $i_r'j_s'$ in the cofibration, i.e., we also have $\psi_*(i_r')_* = (i_{r+s}')_* v_1^s$, $\rho_*(i_{r+s}')_* = (i_s')_*$ in the Ext stage, where $(i_r')_*: \mathrm{Ext}^{k,*}M \to \mathrm{Ext}^{k,*}K_r$ is the reduction in the following exact sequence

$$\cdots \to \mathrm{Ext}^{k,t-rq}M \xrightarrow{v_1^r} \mathrm{Ext}^{k,t}M \xrightarrow{(i_r')_*} \mathrm{Ext}^{k,t}K_r \xrightarrow{(j_r')_*} \mathrm{Ext}^{k+1,t-rq}M \to \cdots$$

*Case* (A). $r + s = p^n + p^{n-1}$. Let $g \in \mathrm{Mod} \cap [\Sigma^* K_r, K_r]$ and $f \in \mathrm{Mod} \cap [\Sigma^* K_s, K_s]$ such that $g_* = v_2^{ap^m}$ and $f_* = v_2^{tp^n}$. Consider $\beta_{(ap^m/r)}\beta_{(tp^n/s)} = j_r' g i_r' j_s' f i_s' \in [\Sigma^* M, M]$.

Suppose that $j_r' g i_r' j_s' f i_s' = 0$; then $g i_r' j_s' f i_s' = i_r' k$ for some $k \in \pi_* M$ and the arguments below show that it yields a contradiction.

Since $j_s' f i_s' i \in \pi_* M$ is detected by $\beta'_{tp^n/s} \in \mathrm{Ext}^1 M$, then $i_r' j_s' f i_s' i \in \pi_* K_r$ is detected by

$$\begin{aligned}(i_r')_*(\beta'_{tp^n/s}) &= (i_r')_*(v_1^{r-1}\beta'_{tp^n/r+s-1}) \\ &= (\psi_{1,r})_* i_*'(\beta'_{tp^n/p^n+p^{n-1}-1}) \in \mathrm{Ext}^1 K_r.\end{aligned}$$

From [**1**, p. 132 Theorem 1.1(b)(iii)],

$$i_*'(c_1(tp^n)) = 2tv_2^{tp^n-p^{n-1}}h_0 \in \mathrm{Ext}^1 K_1,$$

where $c_1(tp^n)$ in [1] actually is $\beta'_{tp^n/p^n+p^{n-1}-1} \in \mathrm{Ext}^1 M$ and $h_0 \in \mathrm{Ext}^1 K_1$ is the $v_2$-torsion free generator. Hence $i'_r j'_s f i'_s i \in \pi_* K_r$ is detected by $2t(\psi_{1,r})_*(v_2^{tp^n - p^{n-1}} h_0) \in \mathrm{Ext}^1 K_r$.

Since $g \in \mathrm{Mod} \cap [\Sigma^* K_r, K_r]$ and $(g i'_r i)_* = v_2^{ap^m} \in \mathrm{Ext}^0 K_r$, then $g i'_r j'_s f i'_s i \in \pi_* K_r$ is detected by the product

$$\begin{aligned} & v_2^{ap^m} \cdot 2t(\psi_{1,r})_*(v_2^{tp^n-p^{n-1}} h_0) \\ & \qquad = 2t(\psi_{1,r})_*(v_2^{ap^m+tp^n-p^{n-1}} h_0) \neq 0 \in \mathrm{Ext}^1 K_r \end{aligned}$$

(if it is zero, then $v_2^{ap^m+tp^n-p^{n-1}} h_0 = (i'_1 j'_{r-1})_*(x)$ for some $x \in \mathrm{Ext}^{0,(ap^m+tp^n-p^{n-1})(p+1)q+rq} K_{r-1}$, but the group vanishes for degree reasons, cf. [1, p. 140 Prop. 6.3]).

Hence $i'_r k \in \pi_* K_r$ and so $k \in \pi_* M$ has $BP$ filtration 1, i.e. $k$ is detected by some $y \in \mathrm{Ext}^1 M$ and $(i'_r)_*(y) = 2t(\psi_{1,r})_*(v_2^{ap^m+tp^n-p^{n-1}} h_0) \neq 0 \in \mathrm{Ext}^1 K_r$. Thus $(i'_{r-1})_*(y) = (\rho_{r,r-1})_*(i'_r)_*(y) = 0$ and $y = v_1^{r-1}\bar{y}$ for some $\bar{y} \in \mathrm{Ext}^{1,(ap^m+tp^n-p^{n-1})(p+1)q+q} M$.

From [1, p. 132 Theorem 1.1], $\mathrm{Ext}^1 M$ is generated by $v_1^u h_0$ $(u \geq 0)$ and $v_1^u c_1(bp^s)$ $(0 \leq u < p^s + p^{s-1} - 1$ if $p \nmid b \geq 2$, $0 \leq u < p^s$ if $b = 1)$ additively, where $h_0 \in \mathrm{Ext}^1 M$ is the $v_1$-torsion free generator and $c_1(bp^s) \in \mathrm{Ext}^1 M$ is the $v_1$-torsion generator whose internal degree is $(bp^s - p^{s-1})(p+1)q + q$.

It is impossible for $\bar{y} = v_1^u h_0$ since then $(i'_r)_*(y) = (i'_r)_*(v_1^{r-1}\bar{y}) = 0$ which yields a contradiction.

If $\bar{y} = v_1^u c_1(bp^s)$ with $u > 0$, then $y = v_1^{r-1}\bar{y} = v_1^r z$ for $z = v_1^{u-1} c_1(bp^s)$ and so $(i'_r)_*(y) = 0$ which yields a contradiction.

If $\bar{y} = c_1(bp^s)$, then for degree reasons $(bp-1)p^{s-1} = ap^m + tp^n - p^{n-1}$. If $m = n$, $a + t \equiv 0 \pmod p$, then $b = a + t \equiv 0 \pmod p$ which yields a contradiction. If $m = n-1$ and $a \not\equiv 1 \pmod p$, $(bp-1)p^{s-1} = (a + tp - 1)p^{n-1}$ and so $bp - 1 \equiv 0 \pmod p$ if $s < n$, $a \equiv 1$ if $s > n$ and $a \equiv 0 \pmod p$ if $s = n$ all of which yields contradictions. Similarly, there is a contradiction if $m < n-1$ and $a \not\equiv -1 \pmod p$. Thus we have $\beta_{(ap^m/r)} \cdot \beta_{(tp^n/s)} \neq 0$ for $r + s = p^n + p^{n-1}$ and one of the conditions (i)–(iii) holds.

Case (**B**). $r + s > p^n + p^{n-1}$.

Let $u = (r+s) - (p^n + p^{n-1})$; then there are $c$ and $d$ such that $u = c + d$ and $c < r$, $d < s$. From [6, p. 277 Lemma 2.4], $d(i'_r) = 0 = d(j'_r)$. Moreover, $\mathrm{Mod} \subset \ker d$, so $\beta_{(ap^m/r)} = j'_r g i'_r$, $\beta_{(tp^n/s)} = j'_s f i'_s$ all belong to ker $d$ which is a commutative subring of $[\Sigma^* M, M]$.

Since $\alpha^d j'_s f i'_s \delta = j'_{s-d}\rho_{s,s-d} f i'_s i j$, there exists $\overline{f} \in \mathrm{Mod}\cap[\Sigma^* K_{s-d}, K_{s-d}]$ such that $\rho_{s,s-d} f i'_s i = \overline{f} i'_{s-d} i$ and $\overline{f}_* = v_2^{tp^n}$; then $\alpha^d \beta_{(tp^n/s)}\delta = \alpha^d j'_s f i'_s \delta = j'_{s-d}\overline{f} i'_{s-d}\delta = \beta_{(tp^n/s-d)}\delta$.

Suppose that $\beta_{(ap^m/r)} \cdot \beta_{(tp^n/s)} = 0$. Then

$$\begin{aligned}\beta_{(ap^m/r-c)}\beta_{(tp^n/s-d)}\delta &= \beta_{(ap^m/r-c)}\alpha^d\beta_{(tp^n/s)}\delta \\ &= -\alpha^d\beta_{(tp^n/s)}\beta_{(ap^m/r-c)}\delta = \alpha^{c+d}\beta_{(ap^m/r)}\beta_{(tp^n/s)}\delta = 0.\end{aligned}$$

By applying the derivation $d$ to the above equation we have $\beta_{(ap^m/r-c)}\beta_{(tp^n/s-d)} = 0$ which contradicts case (A) when one of the conditions (i)–(iii) holds.

Hence we have $\beta_{(ap^m/r)}\beta_{(tp^n/s)} \neq 0$ for $r + s \geq p^n + p^{n-1}$ and one of the conditions (i)–(iii) holds. $\beta_{(ap^m/r)}\beta_{(tp^n/s)} \neq 0$ implies $\beta_{(ap^m/r)}\delta\beta_{(tp^n/s)} \neq 0$ since by applying the derivation $d$ to the equation $\beta_{(ap^m/r)}\delta\beta_{(tp^n/s)} = 0$ we will have $\beta_{(ap^m/r)}\beta_{(tp^n/s)} = 0$. □

**3. Structure of nonsplit ring spectra.** In this section, we will develop some technical results on nonsplit ring spectra $K_r$ and prove Theorem II.

We first recall some facts on $K_r$ given in [**4**]. A spectrum $X$ is called a $Z_p$ spectrum if there are two maps $m_X: M \wedge X \to X$, $\overline{m}_X: \Sigma X \to M \wedge X$ such that

$$\begin{aligned}(3.1)\qquad m_X(i \wedge 1_X) &= 1_X, \qquad (j \wedge 1_X)\overline{m}_X = 1_X, \\ m_X\overline{m}_X &= 0, \qquad (i \wedge 1_X)m_X + \overline{m}_X(j \wedge 1_X) = 1_{M\wedge X},\end{aligned}$$

where $M$ is the mod $p$ Moore spectrum and $m_X$ is called an $M$-module action of $X$. For $Z_p$ spectra $X$, $Y$, $Z$, we define $d: [\Sigma^r X, Y] \to [\Sigma^{r+1} X, Y]$ to be $d(f) = m_Y(1_M \wedge f)\overline{m}_X$. If $m_X$ is associative, then $d$ is a derivation, i.e.

$$(3.2)\qquad d^2 = 0, \quad d(fg) = (-1)^t d(f)g + f d(g)$$

for $g \in [\Sigma^* X, Y]$, $f \in [\Sigma^* Y, Z]$ and $\deg g = t$.

We briefly write $K_r$, $i'_r$, $j'_r$ as $K$, $i'$, $j'$. Since $p \wedge 1_K = 0: S \wedge K \to S \wedge K$, then there is a homotopy equivalence $M \wedge K = K \vee \Sigma K$. From [**4**, p. 432], there is an associative $M$-module action $m: M \wedge K \to K$ and $\overline{m}: \Sigma K \to M \wedge K$ is an associated element such that

$$\begin{aligned}(3.3)\qquad m(i \wedge 1_K) &= 1_K, \quad (j \wedge 1_K)\overline{m} = 1_K, \\ m\overline{m} &= 0, \quad (i \wedge 1_K)m + \overline{m}(j \wedge 1_K) = 1_{M\wedge K}.\end{aligned}$$

So (3.2) also holds in case $X = Y = Z = K$.

Let $\phi = \alpha^r \in [\Sigma^{rq}M, M]$ and $\phi_1 = j\alpha^r i \in \pi_{rq-1}S$, $\overline{\phi} = \phi_1 \wedge 1_K \in [\Sigma^{rq-1}K, K]$, then [4, p. 431 (5.14) and p. 432 Remark 5.7] showed that

$$\overline{\phi} = r\overline{\alpha}^{r-1}\alpha', \quad \overline{\phi}i' = i'\delta\phi, \quad j'\overline{\phi} = -\phi\delta j', \quad \overline{\phi}\delta_0 = \delta_0\overline{\phi}, \tag{3.4}$$

where $\delta = ij \in [\Sigma^{-1}M, M]$, $\delta_0 = i'ijj' \in [\Sigma^{-rq-2}K, K]$, $\overline{\alpha} = \lambda(\alpha\delta) \in [\Sigma^q K, K]$, $\alpha' = \lambda(\delta\alpha\delta) \in [\Sigma^{q-1}K, K]$ and $\lambda: [\Sigma^r M, M] \to [\Sigma^{r+1}K, K]$ is defined to be $\lambda(f) = m(f \wedge 1_K)\overline{m}$. [4, p. 432 (6.2)] also showed that

$$\phi \wedge 1_K = \overline{m}\overline{\phi}m. \tag{3.5}$$

Then there is a homotopy equivalence

$$K \wedge K = K \vee \Sigma L \wedge K \vee \Sigma^{rq+1}K \tag{3.6}$$

where $L$ is the cofibre of $\phi_1 = j\phi i$ given by the cofibration

$$\Sigma^{rq-1}S \xrightarrow{\phi_1} S \xrightarrow{i''} L \xrightarrow{j''} \Sigma^{rq}S \tag{3.7}$$

and there exist

$$\mu: K \wedge K \to K, \quad \mu_2: K \wedge K \to \Sigma L \wedge K, \quad \mu_3: K \wedge K \to \Sigma^{rq+2}K$$
$$\nu_3: K \to K \wedge K, \quad \nu_2: \Sigma L \wedge K \to K \wedge K, \quad \nu: \Sigma^{rq+2}K \to K \wedge K$$

such that (cf. [4, p. 433])

(3.8) (A) $\mu(i' \wedge i_K) = m, \quad (j' \wedge 1_K)\nu = \overline{m}$,

(B) $\mu_2(i' \wedge 1_K) = (i'' \wedge 1_K)(j \wedge 1_K)$,
$(j' \wedge 1_K)\nu_2 = (i \wedge 1_K)(j'' \wedge 1_K)$,

(C) $(j'' \wedge 1_K)\mu_2 = m(j' \wedge 1_K), \quad \nu_2(i'' \wedge 1_K) = (i' \wedge 1_K)\overline{m}$,

(D) $\mu\nu_2 = 0, \quad \mu\nu = 0, \quad \mu_2\nu = 0, \quad \mu_2\nu_2 = 1_{L\wedge K}$.

Let $\mu_3 = jj' \wedge 1_K$, $\nu_3 = i'i \wedge 1_K$, (A) and (B) imply

(3.9) (A)′ $\mu\nu_3 = 1_K, \quad \mu_3\nu = 1_K$,

(B)′ $\mu_2\nu_3 = 0, \quad \mu_3\nu_2 = 0$,

(C)′ $\nu\mu_3 + \nu_2\mu_2 + \nu_3\mu = 1_{K\wedge K}$.

Recall that $\delta' = i'j' \in [\Sigma^{-rq-1}K, K]$, $\delta_0 = i'ijj' \in [\Sigma^{-rq-2}K, K]$ and $\delta = ij \in [\Sigma^{-1}M, M]$; they satisfy (cf. [**4**, p. 434])

$$(3.10) \qquad d(\delta) = -1_M, \quad d(\delta') = 0, \quad d(\delta_0) = \delta'.$$

LEMMA 3.11 ([**4**, p. 434 Lemma 6.2]). *There exist elements*

$$\widetilde{\Delta} \in [\Sigma^{-1}K, L \wedge K], \quad \overline{\Delta} \in [\Sigma^{-rq-1}L \wedge K, K]$$

*such that*

(i) $(j'' \wedge 1_K)\widetilde{\Delta} = \delta'$, $\overline{\Delta}(i'' \wedge 1_K) = \delta'$,
(ii) $\widetilde{\Delta}i' = (i'' \wedge 1_K)i'\delta$, $j'\overline{\Delta} = \delta j'(j'' \wedge 1_K)$,
(iii) $(1_L \wedge j')\widetilde{\Delta} = -(i'' \wedge 1_M)\delta j'$, $\overline{\Delta}(1_L \wedge i') = -i'\delta(j'' \wedge 1_M)$,
(iv) $\overline{\Delta}\widetilde{\Delta} = 2\delta_0$.

THEOREM 3.12 ([**4**, p. 438 Theorems 6.5 and 6.6]). *There is a choice of* $(\mu, \mu_2, \nu, \nu_2)$ *such that*

$$\mu T = \mu, \qquad T\nu = \nu,$$
$$\mu_2 T = -\mu_2 + \widetilde{\Delta}\mu, \quad T\nu_2 = -\nu_2 + \nu\overline{\Delta}$$

*and any such* $\mu$ *is an associative multiplication of* $K$, *where* $T: K \wedge K \to K \wedge K$ *is the switching map.*

DEFINITION 3.13 ([**4**, p. 423 Def. 2.2)].

$$\mathrm{Mod} = \{f \in [\Sigma^* K, K] | \mu(f \wedge 1_K) = f\mu\},$$
$$\mathrm{Der} = \{f \in [\Sigma^* K, K] | f\mu = \mu(f \wedge 1_K) + \mu(1_K \wedge f)\}.$$

That is, Mod consists of right $K$-module maps and Der consists of elements which behave as a derivation on the cohomology defined by $K$.

THEOREM 3.14 ([**4**, p. 424 Remark 2.4 and p. 423 Lemma 2.3]). *There is a direct summand decomposition*

$$[\Sigma^* K, K] = \mathrm{Mod} \oplus \mathrm{Der} \oplus \mathrm{Mod}\,\delta_0$$

*and* $\ker i_0^* = \mathrm{Der} \oplus \mathrm{Mod}\,\delta_0$, $[\mathrm{Der}, \mathrm{Mod}] \subset \mathrm{Mod}$, *where* $i_0 = i'i: S \to K$ *is injection of the bottom cell and* $[f, g]$ *denotes the graded commutator* $fg - (-1)^{|f|\cdot|g|}gf$.

*By using Theorem* 3.12 *and* (3.8) (A) (B) (D), *we can easily check that* $h\nu = 0$, $h\nu_2 = 0$, $h\nu_3 = 0$ *for* $h = \mu(\delta' \wedge 1_K) + \mu(1_K \wedge \delta') - \delta'\mu$. *Hence it follows from* (3.9)(C)$'$ *that* $\delta'\mu = \mu(\delta' \wedge 1_K) + \mu(1_K \wedge \delta')$ *and* *so* $\delta' \in \mathrm{Der}$. *From Theorem* 3.14, $[\delta', f] \in \mathrm{Mod}$ *for* $f \in \mathrm{Mod}$ *and in particular we have* $\delta' f^p = f^p \delta'$ *for* $f \in \mathrm{Mod}$ *having even degree.*

*Now we consider further properties of* $[\Sigma^* K, K]$ *which are not in* [**4**]. *Define*

$$d_0 : [\Sigma^s K, K] \to [\Sigma^{s+rq+2} K, K]$$

*to be* $d_0(f) = \mu(f \wedge 1_K)\nu$. $d_0$ *has the following important properties.*

Proposition 3.15. (1) $d_0(\delta_0) = 1_K$, $d_0(g\delta_0) = g$ *for* $g \in \mathrm{Mod}$.
(2) $\ker d_0 = \mathrm{Mod} \oplus \mathrm{Der}$, $\operatorname{im} d_0 \subset \mathrm{Mod}$.

*Proof*. (1) From (3.9) (A)$'$,

$$d_0(\delta_0) = \mu(\delta_0 \wedge 1_K)\nu = \mu(i'i \wedge 1_K)(jj' \wedge 1_K)\nu = 1_K$$

and $d_0(g\delta_0) = \mu(g\delta_0 \wedge 1_K)\nu = g\mu(\delta_0 \wedge 1_K)\nu = g$.

(2) It is easily seen that $\mathrm{Mod} \subset \ker d_0$ and for $f \in \mathrm{Der}$

$$\begin{aligned} d_0(f) &= \mu(f \wedge 1_K)\nu = f\mu\nu - \mu(1_K \wedge f)\nu \\ &= -\mu T(1_K \wedge f)\nu = -\mu(f \wedge 1_K)\nu = -d_0(f) = 0; \end{aligned}$$

then $\mathrm{Der} \subset \ker d_0$. On the other hand, if $f \in \ker d_0$, let $f = f_1 + f_2 + f_3\delta_0$ with $f_1, f_3 \in \mathrm{Mod}$ and $f_2 \in \mathrm{Der}$, (cf. Thm. 3.14), then $0 = d_0(f) = d_0(f_3\delta_0) = f_3$ and so $f \in \mathrm{Der} \oplus \mathrm{Mod}$. $\operatorname{im} d_0 \subset \mathrm{Mod}$ is immediate. □

Proposition 3.16. (1) *If* $h \in \mathrm{Mod}$, $u \in \mathrm{Der}$, *then* $hu \in \mathrm{Der}$; *in particular,* $\mathrm{Mod}\,\delta' \subset \mathrm{Der}$.
(2) $d_0(\delta' g) = (-1)^{t+1} d(g) + \delta' d_0(g)$, $d_0(g\delta') = -d(g_2)$, *where* $t = \deg g$ *and* $g_2$ *is the component of* $g$ *in Der in the decomposition in Theorem* 3.14.

*Proof*. (1) If $h \in \mathrm{Mod}$ and $u \in \mathrm{Der}$, then $h\mu = \mu(h \wedge 1_K)$ and $u\mu = \mu(u \wedge 1_K) + \mu(1_K \wedge u)$. Hence

$$\begin{aligned} hu\mu &= h\mu(u \wedge 1_K) + h\mu(1_K \wedge u) \\ &= \mu(hu \wedge 1_K) + h\mu T(1_K \wedge u), \quad (\mu T = \mu \text{ from Thm. 3.12}) \\ &= \mu(hu \wedge 1_K) + \mu(h \wedge 1_K)T(1_K \wedge u) \\ &= \mu(hu \wedge 1_K) + \mu T(1_K \wedge hu) \\ &= \mu(hu \wedge 1_K) + \mu(1_K \wedge hu) \end{aligned}$$

and so $hu \in \mathrm{Der}$. Since $\delta' \in \mathrm{Der}$, then $\mathrm{Mod}\,\delta' \subset \mathrm{Der}$.

(2) If $g_1 \in \mathrm{Mod}$, then $d_0(g_1\delta') = \mu(g_1\delta' \wedge 1_K)\nu = g_1\mu(\delta' \wedge 1_K)\nu = 0$. Since $[\delta', g_1] \in \mathrm{Mod}$, then $d_0(\delta' g_1) = d_0(g_1\delta') = 0$.

Let $g = g_1 + g_2 + g_3\delta_0$ with $g_1, g_3 \in \mathrm{Mod}$ and $g_2 \in \mathrm{Der}$; then

$$\begin{aligned} d_0(\delta' g) &= d_0(\delta' g_2) + d_0(\delta' g_3\delta_0) \\ &= d_0(\delta' g_2) + \delta' g_3 - (-1)^t g_3\delta'. \end{aligned}$$

Moreover,

$$\begin{aligned} d_0(\delta' g_2) &= \mu(1_K \wedge \delta')\nu\mu_3(1_K \wedge g_2)\nu + \mu(1_K \wedge \delta')\nu_2\mu_2(1_K \wedge g_2)\nu \\ &\quad + \mu(1_K \wedge \delta')\nu_3\mu(1_K \wedge g_2)\nu, \quad (\text{cf. } (3.9)(\mathrm{C})') \\ &= \mu(\delta' \wedge 1_K)T\nu_2\mu_2(1_K \wedge g_2)\nu, \\ &\qquad\qquad (\text{since 1st and 3rd terms are zero}) \\ &= -\mu(\delta' \wedge 1_K)\nu_2\mu_2(1_K \wedge g_2)\nu, \quad (T\nu_2 = -\nu_2 + \nu\overline{\Delta}) \\ &= -m(i \wedge 1_K)(j'' \wedge 1_K)\mu_2(1_K \wedge g_2)\nu, \\ &\qquad\qquad ((j' \wedge 1_K)\nu_2 = (ij'' \wedge 1_K)) \\ &= -m(j' \wedge 1_K)(1_K \wedge g_2)\nu, \quad ((j'' \wedge 1_K)\mu_2 = m(j' \wedge 1_K)) \\ &= (-1)^{t+1}m(1_M \wedge g_2)\overline{m}, \quad (\overline{m} = (j' \wedge 1_K)\nu) \\ &= (-1)^{t+1}d(g_2). \end{aligned}$$

Hence

$$\begin{aligned} d_0(\delta' g) &= (-1)^{t+1}d(g_2) + \delta' g_3 - (-1)^t g_3\delta' \\ &= (-1)^{t+1}d(g) + \delta'(d_0(g); \end{aligned}$$

note that $d(g) = d(g_2) + g_3\delta'$ and $d_0(g) = g_3$.

The proof of $d_0(g\delta') = -d(g_2)$ is similar. □

PROPOSITION 3.17. *If* $g \in \mathrm{Der}$, *then* $g\delta' \in \mathrm{Mod}\,\delta_0$ *and* $d(g) \in \mathrm{Mod}$. *Moreover,* $g \in \mathrm{Mod}\,\delta'$ *if* $d(g) = 0$.

*Proof*. Since $g \in \mathrm{Der}$, then $gi'i = 0$ (cf. Thm. 3.14) and so $gi' = \eta j$ for some $\eta \in \pi_*K$. $\eta$ can be extended to $\overline{\eta} \in [\Sigma^*K, K]$ such that $\eta = \overline{\eta}i'i$ and $\overline{\eta} \in \mathrm{Mod}$. Then $g\delta' = \overline{\eta}i'ijj' = \overline{\eta}\delta_0 \in \mathrm{Mod}\,\delta_0$.

On the other hand, $\overline{\eta} = d_0(\overline{\eta}\delta_0) = d_0(g\delta') = -d(g)$, so $d(g) \in \mathrm{Mod}$. Moreover, if $d(g) = 0$, then $gi' = \overline{\eta}i'ij = -d(g)i'ij = 0$ and so $g = \overline{g}j'$ for some $\overline{g} \in [\Sigma^*M, K]$. Since $g\delta_0 = 0$, then

$$\begin{aligned}
0 &= \mu(1_K \wedge g)(1_K \wedge \delta_0)\nu \\
&= \mu(1_K \wedge g)\nu\mu_3(1_K \wedge \delta_0)\nu \\
&\quad + \mu(1_K \wedge g)\nu_2\mu_2(1_K \wedge \delta_0)\nu + \mu(1_K \wedge g)\nu_3\mu(1_K \wedge \delta_0)\nu \\
&= \mu(1_K \wedge g)\nu_2\mu_2(1_K \wedge \delta_0)\nu + g \\
&\qquad\qquad (\mu(1_K \wedge g)\nu = 0,\ \mu(1_K \wedge \delta_0)\nu = 1_K) \\
&= \mu(1_K \wedge g)\nu_2\mu_2 T(\delta_0 \wedge 1_K)\nu + g \qquad (T\nu = \nu) \\
&= -\mu(1_K \wedge g)\nu_2\mu_2(\delta_0 \wedge 1_K)\nu + \mu(1_K \wedge g)\nu_2\widetilde{\Delta}\mu(\delta_0 \wedge 1_K)\nu + g \\
&\qquad\qquad (\mu_2 T = -\mu_2 + \widetilde{\Delta}\mu) \\
&= g + \mu(1_K \wedge g)\nu_2\widetilde{\Delta} \quad (\mu_2(\delta_0 \wedge 1_K) = (i''j \wedge 1_K)(ijj' \wedge 1_K) = 0) \\
&= g - \mu(g \wedge 1_K)\nu_2\widetilde{\Delta} \qquad (\mu T = \mu,\ T\nu_2 = -\nu_2 + \nu\overline{\Delta}) \\
&= g - \mu(\overline{g} \wedge 1_K)(j' \wedge 1_K)\nu_2\widetilde{\Delta} \quad (\text{since } g = \overline{g}j') \\
&= g - \mu(\overline{g} \wedge 1_K)(i \wedge 1_K)(j'' \wedge 1_K)\widetilde{\Delta} \qquad ((j' \wedge 1_K)\nu_2 = (ij'' \wedge 1_K)) \\
&= g - \mu(\overline{g}i \wedge 1_K)\delta' \qquad ((j'' \wedge 1_K)\widetilde{\Delta} = \delta').
\end{aligned}$$

Thus $g = u\delta'$, where $u = \mu(\overline{g}i \wedge 1_K) \in \mathrm{Mod}$. □

PROPOSITION 3.18. *$\overline{\phi} \in \mathrm{Mod}$ and there exists $\varepsilon \in \mathrm{Der}$ such that $d(\varepsilon) = \overline{\phi}$.*

*Proof*. Recall (3.4), $\overline{\phi} = r\overline{\alpha}^{r-1}\alpha'$, where $\overline{\alpha} = \lambda(\alpha\delta)$ and $\alpha' = \lambda(\delta\alpha\delta)$. Hence, it follows from $\mathrm{im}\,\lambda \subset \mathrm{Mod}$ that $\overline{\phi} \in \mathrm{Mod}$.

From Lemma 3.11(i) and (3.4), $\overline{\phi\Delta}(i'' \wedge 1_K) = \overline{\phi}\delta' = i'\delta\phi j' = 0$; then $\overline{\phi\Delta} = u(j'' \wedge 1_K)$ for some $u \in [\Sigma^* K, K]$. Hence it follows from Lemma 3.11(iv) and (i) that

$$2\overline{\phi}\delta_0 = \overline{\phi\Delta}\widetilde{\Delta} = u(j'' \wedge 1_K)\widetilde{\Delta} = u\delta'$$

and so $2\overline{\phi} = 2d_0(\overline{\phi}\delta_0) = d_0(u\delta') = -d(u_2)$ (cf. Prop. 3.16(2)). Thus $\overline{\phi} = d(\varepsilon)$ if we let $\varepsilon = -\frac{1}{2}u_2$. □

PROPOSITION 3.19. (1) *If $g \in \mathrm{Mod}$ and $g\delta' = 0$ (resp. $\delta' g = 0$), then $g = \eta\overline{\phi}$ resp. $g = \overline{\phi}\eta$) for some $\eta \in \mathrm{Mod}$.*

(2) *If $\eta \in \mathrm{Mod}$, then $\eta\overline{\phi} = 0$ if and only if $\eta = d(u)$ for some $u \in \mathrm{Der}$.*

*Proof*. (1) Since $g\delta_0(j'' \wedge 1_K) = gi'\delta j'(j'' \wedge 1_K) = gi'j'\overline{\Delta} = 0$ (cf. Lemma 3.11(ii)), then $g\delta_0 = \overline{\eta}(j\phi i \wedge 1_K) = \overline{\eta}\overline{\phi}$ for some $\overline{\eta} \in [\Sigma^* K, K]$. Let $\overline{\eta} = \eta_1 + \eta_2 + \eta_3\delta_0$ with $\eta_1, \eta_3 \in \mathrm{Mod}$ and $\eta_2 \in \mathrm{Der}$. Then $g\delta_0 = \eta_1\overline{\phi} + \eta_2\overline{\phi} + \eta_3\delta_0\overline{\phi}$ and $g = d_0(g\delta_0) = d_0(\eta_2\overline{\phi}) + d_0(\eta_3\delta_0\overline{\phi})$. However, $d_0(\eta_3\delta_0\overline{\phi}) = d_0(\eta_3\overline{\phi}\delta_0) = \eta_3\overline{\phi}$ (cf. (3.4)) and

$\eta_2\overline{\phi}-(-1)^t\overline{\phi}\eta_2 \in \text{Mod}$, $d_0(\eta_2\overline{\phi}) = \pm d_0(\overline{\phi}\eta_2) = 0$ (note that $\overline{\phi}\eta_2 \in \text{Der}$ from Prop. 3.16(1)); then $g = \eta_3\overline{\phi}$ with $\eta_3 \in \text{Mod}$.

If $g \in \text{Mod}$ and $\delta' g = 0$, then $g\delta' = g\delta' - (-1)^{|g|}\delta' g \in \text{Mod} \cap \text{Mod}\,\delta' \subset \text{Mod} \cap \text{Der} = 0$. So $g = \eta\overline{\phi} = \pm\overline{\phi}\eta$ for some $\eta \in \text{Mod}$.

(2) $d(u)\overline{\phi}m = m(1_M \wedge u)\overline{m}\overline{\phi}m = m(1_M \wedge u)(\phi \wedge 1_K) = m(\phi \wedge \cdot 1_K) \cdot (1_K \wedge u) = 0$. Then $d(u)\overline{\phi} = d(u)\overline{\phi}m(i \wedge 1_K) = 0$.

Conversely, if $\eta\overline{\phi} = 0$ for $\eta \in \text{Mod}$, then $\eta\overline{\phi}i'i = 0 = \eta i'ij\phi i$ and so $\eta i'ij\phi = uj$ for some $u \in \pi_*K$. $u$ can be extended to $\overline{u} \in [\Sigma^*K, K]$ such that $\overline{u}i'i = u$ and $\overline{u} \in \text{Mod}$. Then $\eta i'ij\phi = \overline{u}i'ij$ and $\overline{u}\delta_0 = 0$, $\overline{u} = d_0(\overline{u}\delta_0) = 0$. Hence $\eta i'ij\phi = 0$ and $\eta i'ij = wi'$ for some $w \in [\Sigma^*K, K]$. Thus $\eta\delta_0 = w\delta'$, $\eta = d_0(\eta\delta_0) = d_0(w\delta') = -d(w_2)$, where $w_2$ is the component of $w$ in Der, see Proposition 3.16(2). □

Proposition 3.20. *If* $g \in \text{Mod}$, *then* $d_0(\delta_0 g) = g$ *and* $\delta_0 g - g\delta_0 \in \text{Mod} \oplus \text{Der}$.

*Proof.*

$$\begin{aligned} d_0(\delta_0 g) &= \mu(\delta_0 \wedge 1_K)(g \wedge 1_K)\nu \\ &= \mu(\delta_0 \wedge 1_K)T\nu\mu_3(1_K \wedge g)\nu + \mu(\delta_0 \wedge 1_K)T\nu_2\mu_2(1_K \wedge g)\nu \\ &\quad + \mu(\delta_0 \wedge 1_K)T\nu_3\mu(1_K \wedge g)\nu \quad \text{(cf. (3.9)(C)')} \\ &= (jj' \wedge 1_K)(1_K \wedge g)\nu - \mu(\delta_0 \wedge 1_K)\nu_2\mu_2(1_K \wedge g)\nu \\ &\quad + \mu(\delta_0 \wedge 1_K)\nu\overline{\Delta}\mu_2(1_K \wedge g)\nu \\ &\qquad\qquad (\text{since } \mu(1_K \wedge g)\nu = 0,\ T\nu_2 = -\nu_2 + \nu\overline{\Delta}) \\ &= g + \overline{\Delta}\mu_2(1_K \wedge g)\nu \quad (\text{since } \mu(\delta_0 \wedge 1_K)\nu_2 = 0,\ \text{cf. (3.8)}). \end{aligned}$$

Let $h = d_0(\delta_0 g) - g = \overline{\Delta}\mu_2(1_K \wedge g)\nu$. Then $h \in \text{Mod}$ and

$$\begin{aligned} j'h &= j'\overline{\Delta}\mu_2(1_K \wedge g)\nu = \delta j'(j'' \wedge 1_K)\mu_2(1_K \wedge g)\nu \\ &= \delta j' m(j' \wedge 1_K)(1_K \wedge g)\nu = \delta j' m(1_M \wedge g)\overline{m} = j'd(g) = 0. \end{aligned}$$

So $\delta' h = 0$ and from Prop. 3.19(1) we have $h = \overline{\phi}g_1$ for some $g_1 \in \text{Mod}$, i.e. there is some $g_1 \in \text{Mod}$ such that

$$d_0(\delta_0 g) - g = \overline{\phi}g_1 \quad \text{and} \quad j'\overline{\phi}g_1 = 0.$$

Thus inductively we have $g_s, g_{s+1} \in \text{Mod}$ ($s \geq 0$ with $g_0 = g$) such that $d_0(\delta_0 g_s) - g_s = \overline{\phi}g_{s+1}$ and $j'\overline{\phi}g_{s+1} = 0$ ($s \geq 0$). It is easily seen for degree reasons that $g_{s+1} = 0$ for $s$ large and so $d_0(\delta_0 g_s) = g_s$ for some fixed large $s$.

Since $j'\overline{\phi}g_s = 0$, then $\phi\delta j'g_s = 0$ (cf. (3.4)) and so $\delta j'g_s = j'k$ for some $k \in [\Sigma^* K, K]$. Hence $\delta_0 g_x = \delta' k$ and $g_s = d_0(\delta_0 g_s) = d_0(\delta' k) = \pm d(k) + \delta' d_0(k)$ (cf. Prop. 3.16(2)). Thus $\overline{\phi}g_s = 0$ since $\overline{\phi}d(k) = 0$ and $\overline{\phi}\delta' = 0$ (cf. Prop. 3.19(2) and (3.4)). Hence $d_0(\delta_0 g_{s-1}) - g_{s-1} = \overline{\phi}g_s = 0$ and inductively we have $d_0(\delta_0 g) = g$.

Since $d_0(\delta_0 g - g\delta_0) = g - g = 0$, then $\delta_0 g - g\delta_0 \in \ker d_0 = \mathrm{Mod} \oplus \mathrm{Der}$. □

Now we are ready to prove Theorem II stated in §1.

*Proof of Theorem* II. Let $f, g \in \mathrm{Mod} \cap [\Sigma^* K_r, K_r]$ and $r \not\equiv 0 \pmod p$. From Prop. 3.20 we may assume $\delta_0 f^p - f^p \delta_0 = h_1 + h_2$ with $h_1 \in \mathrm{Mod}$ and $h_2 \in \mathrm{Der}$. By applying the derivation $d$, $d(h_2) = d(\delta_0 f^p - f^p \delta_0) = \delta' f^p - f^p \delta' = 0$ (cf. Thm. 3.14). Hence $h_2 = u\delta'$ for some $u \in \mathrm{Mod}$ (cf. Prop. 3.17). Hence

$$\begin{aligned} g^p(\delta_0 f^p - f^p \delta_0) &= g^p h_1 + g^p u \delta' = (-1)^{|f|\cdot|g|}(h_1 + u\delta) g^p \\ &= (-1)^{|f|\cdot|g|}(\delta_0 f^p - f^p \delta_0) g^p \end{aligned}$$

since $g^p$ commutes with $\delta'$ and $h_1, u \in \mathrm{Mod}$.

Moreover, if $f$ has even degree, $f^p(\delta_0 f^p - f^p\delta_0) = (\delta_0 f^p - f^p \delta_0) f^p$ and by induction we have $f^{kp}\delta_0 - \delta_0 f^{kp} = k(f^{kp}\delta_0 - f^{(k-1)p}\delta_0 f^p)$ for $k \geq 1$. In particular we have $f^{p^2}\delta_0 \equiv \delta_0 f^{p^2}$.

If $r \equiv 0 \pmod p$, [**6**] showed that there exists $\overline{\delta} \in [\Sigma^{-1} K_r, K_r]$ such that $\overline{\delta} i'_r = i'_r ij$, $j'_r \overline{\delta} = -ijj'_r$ and apart from the derivation $d: [\Sigma^s K_r, K_r] \to [\Sigma^{s+1} K_r, K_r]$ there is another derivation $d'$: $[\Sigma^s K_r, K_r] \to [\Sigma^{s+rq+1} K_r, K_r]$ such that

$$d'(\delta') = -1_{K_r}, \quad d'(\overline{\delta}) = 0, \quad d(\overline{\delta}) = -1_{K_r}, \quad d(\delta') = 0.$$

Moreover, there is a direct summand decomposition

$$[\Sigma^* K_r, K_r] = \mathscr{C}_* \oplus \mathscr{C}_*\overline{\delta} \oplus \mathscr{C}_*\delta' \oplus \mathscr{C}_*\overline{\delta}\delta'$$

such that $\mathscr{C}_* = \ker d \cap \ker d'$ is a commutative subring (cf. [**6**, p. 297 Thm. 5.5, 5.6]) and $\overline{\delta} f^p = f^p \overline{\delta}$, $\delta' f^p = f^p \delta'$ for $f \in \mathscr{C}_*$ having even degree (cf. [**6**, p. 298 Cor. 5.7]).

Hence $\delta_0 = \overline{\delta}\delta'$, $d(\delta_0 f^p - f^p\delta_0) = \delta' f^p - f^p \delta' = 0$, $d'(\delta_0 f^p - f^p \delta_0) = \overline{\delta} f^p - f^p \overline{\delta} = 0$ and so $\delta_0 f^p - f^p \delta_0 \in \ker d \cap \ker d' = \mathscr{C}_*$. □

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