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UNIVERSAL CLASSES OF ORLICZ FUNCTION SPACES

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It is shown that for each $0 < p < q < \infty$ the space $L^p(0, \infty) + L^q(0, \infty)$, defined as in Interpoliation Theory, is universal for the class of all Orlicz function spaces L^ψ with Boyd indices strictly between p and q (i.e. every Orlicz function space L^ψ is order-isomorphically embedded into $L^p(0, \infty) + L^q(0, \infty)$). The extreme case of spaces having Boyd indices equal to p or q is also studied. In particular every space $L^r(0, \infty) + L^s(0, \infty)$ embeds isomorphically into the sum $L^p(0, \infty)+L^q(0, \infty)$ for any $0 < p \leq r \leq s < q < \infty$.

0. Introduction. It is a well-known fact from Interpolation Theory (cf. [L-T II] Proposition 2.b.3) that given $1 \leq p < q \leq \infty$, every rearrangement invariant (r.i.) Banach function space X on the interval $(0, \infty)$ with Boyd indices between p and q, is an intermediate space between the spaces $L^p(0, \infty) \cap L^q(0, \infty)$ and $L^p(0, \infty) + L^q(0, \infty)$. This means that $L^p(0, \infty) \cap L^q(0, \infty) \subset X \subset L^p(0, \infty) + L^q(0, \infty)$ with continuous inclusions.

One of the purposes of this paper is to study the universality of the spaces $L^p(0, \infty) + L^q(0, \infty)$ with respect to classes of intermediate r.i. function spaces X, in the sense of whether the above inclusion $X \subset L^p(0, \infty)+L^q(0, \infty)$ can be replaced by suitable isomorphic embeddings of X into $L^p(0, \infty) + L^q(0, \infty)$. At the same time, these spaces $L^p(0, \infty)+L^q(0, \infty)$ can be regarded as Orlicz function spaces $L^\varphi(0, \infty)$, taking the Orlicz function $\varphi(x) = x^p \wedge x^q = \min(x^p, x^q)$, and, consequently, we are also interested in finding universal Orlicz function spaces for classes of quasi-Banach Orlicz function spaces $L^\psi(0, \infty)$.

These questions are motived in part by the existence of positive results of universality in the context of Orlicz sequence spaces: J. Lindenstrauss and L. Tzafriri (cf. [L-T I] Theorem 4.b.12) proved that there exist universal Orlicz sequence spaces l^φ with arbitrary prefixed indices $1 \leq p < q < \infty$ such that every Orlicz sequence space l^ψ with indices between p and q is isomorphic to a (complemented) subspace of l^φ.

We show here that, in general, for $0 < p < q < \infty$, the spaces $L^p(0, \infty) + L^q(0, \infty)$ are universal for the class of all Orlicz function spaces $L^{\psi}(0, \infty)$ with Boyd indices strictly between p and q. On the other hand, this universality of the spaces $L^p(0, \infty) + L^q(0, \infty)$ does not hold for every intermediate r.i. function space X: there are Lorentz function spaces which cannot be embedded into $L^p(0, \infty) + L^q(0, \infty)$.

These results are deduced from the two main theorems presented in §2 (Theorems 3.A and 3.B), which are obtained through results of representing the Orlicz function ψ, between p and q, in an integral form with respect to the function $x^p \wedge x^q = \varphi(x)$, i.e. there exists a probability measure ν on $(0, \infty)$ such that, up to equivalence,

$$\psi(x) = \int_0^\infty \frac{\varphi(xs)}{\varphi(s)}\, d\nu(s).$$

Several remarkable consequences are given in §3. Thus, in Proposition 6 we obtain sufficient conditions on the embedding of Orlicz function spaces $L^{\psi}(0, \infty)$ into Orlicz spaces over finite measure $L^{\varphi}[0, 1]$, which were obtained by Bretagnolle and Dacunha-Castelle (**[B-DC]**) by using probabilistic tools (see also **[J-M-S-T]**).

Corollary 8 extends a recent result of S. J. Dilworth (**[D]**) about the scale of spaces $L^2(0, \infty) + L^q(0, \infty)$ for $2 < q < \infty$. We show that for any $0 < p \le r \le s < q < \infty$ the spaces $L^r(0, \infty) + L^s(0, \infty)$ are order-isomorphic to sublattices of the space $L^p(0, \infty) + L^q(0, \infty)$.

We would like to thank Ives Raynaud for helpful suggestions on this topic.

1. Preliminaries. Let us start with some notations and definitions. By φ we denote an Orlicz function, i.e. a non-decreasing continuous function defined for $x \ge 0$ so that $\varphi(0) = 0$ and $\varphi(1) = 1$. Two Orlicz functions φ and ψ are equivalent at ∞, denoted by $\psi \overset{\infty}{\sim} \psi$ (resp. at 0, $\varphi \overset{0}{\sim} \psi$) if there exist a constant $C \ge 1$ and $x_0 > 0$ such that $C^{-1}\varphi(x) \le \psi(x) \le C\varphi(x)$ for every $x \ge x_0$ (resp. $0 < x \le x_0$). When φ and ψ are equivalent at ∞ and at 0 we say that φ and ψ are equivalent, i.e. $\varphi \sim \psi$. Given $0 < p < \infty$, an Orlicz function φ is p-convex (resp. p-concave) if $\varphi(x^{1/p})$ is a convex function of x (resp. a concave function).

(I, μ) means the Lebesgue measure space for $I = [0, 1]$ or $(0, \infty)$. The Orlicz function space $L^{\varphi}(I)$ is defined as the set of equivalence classes of μ-measurable scalar functions on I such that

$$m(f/u) = \int_I \varphi(|f(t)|/u)\, d\mu(t) < \infty \quad \text{for some } u > 0.$$

The space $L^\varphi(I)$ is an F-space when we consider the F-norm

$$|f|_\varphi = \inf\{u > 0 \colon m(f/u) \le u\}.$$

It is well-known ([**M-O**]) that an Orlicz space $L^\varphi(0, \infty)$ is p-normable $(0 < p \le 1)$ if and only if there exists a p-convex function ψ such that $L^\varphi(0, \infty) = L^\psi(0, \infty)$, where $L^\psi(0, \infty)$ is endowed with the p-norm,

$$\|f\|_\psi = \inf\left\{u > 0 \colon \int_0^\infty \psi(|f(t)|/u^{1/p})\, d\mu(t) \le 1\right\},$$

so $(\|f\|_\psi)^{1/p}$ is a quasi-norm (in the case $p = 1$, $\| \ \|_\psi$ is the usual Luxemburg norm). Moreover, recall that if two Orlicz functions φ and ψ are equivalent (resp. $\varphi \overset{\infty}{\approx} \psi$), then $L^\varphi(0, \infty) = L^\psi(0, \infty)$ (resp. $L^\varphi[0, 1] = L^\psi[0, 1]$) and the identity map is an isomorphism.

An Orlicz function φ satisfies the Δ_2-condition (resp. at ∞) if there exists a constant $M > 0$ such that $\varphi(2x) \le M\varphi(x)$ for $x > 0$ (resp. $x > x_0$ for some $x_0 > 0$). $L^\varphi(0, \infty)$ is separable iff φ verifies the Δ_2-condition.

For basic properties of Orlicz spaces we refer to [**Lu**] and [**Mu**]. For Banach structure properties see [**L-T II**], [**J-M-S-T**], [**H-Ro**] and [**R**].

Following the terminology of Woo ([**W**]) we have the following:

Definition 1. Let $0 < p < q < \infty$. An Orlicz function φ is said to be between p and q if $\varphi(x)/x^p$ is non-decreasing on $\mathbb{R}^+$ and $\varphi(x)/x^q$ is non-increasing on $\mathbb{R}^+$. By $\mathscr{K}(p, q)$ is denoted the set of all Orlicz functions φ between p and q.

It is clear that every Orlicz function φ which is p-convex and q-concave belongs to $\mathscr{K}(p, q)$. If $p \le p_1$ and $q_1 \le q$, then $\mathscr{K}(p_1, q_1) \subset \mathscr{K}(p, q)$. Also it is very easy to show that a function φ with continuous derivative belongs to $\mathscr{K}(p, q)$ iff

$$p \le x\varphi'(x)/\varphi(x) \le q \quad \text{for every } x > 0.$$

Proposition 2 ([**M**] and [**W**]). *Let $0 < p < q < \infty$. For every Orlicz function $\varphi \in \mathscr{K}(p, q)$ there exists a p-convex and q-concave Orlicz function ψ with continuous second derivative, such that φ and ψ are equivalent.*

The following result of universality for Orlicz sequence spaces is given in [**L-T I**], Theorem 4.b.12: Given $1 \le p < q < \infty$ there exists an Orlicz function $\varphi_{p,q} = \varphi$ which is p-convex and q-concave at 0, such that for every Orlicz function ψ p-convex and q-concave at 0, the Orlicz sequence space l^{ψ} embeds isomorphically into l^{φ}. Moreover l^{φ} has a 1-complemented copy of l^{ψ}.

We analyze here possible results of universality like the above one in the context of Orlicz function spaces $L^{\varphi}(0, \infty)$. In **[J-M-S-T]**, Theorem 7.1, it is proved that if a rearrangement invariant function space $X([0, 1])$, in particular an Orlicz function space $L^{\psi}[0, 1]$, is isomorphic to a complemented subspace of a reflexive Orlicz function space $L^{\varphi}(0, \infty)$, then $L^{\psi}[0, 1] = L^{\varphi}[0, 1]$ or $L^{\psi}[0, 1] = L^{2}[0, 1]$, up to an equivalent renorming. It follows from this result that there are no Orlicz function spaces $L^{\varphi}(0, \infty)$ which are complementably universal for classes of function spaces $L^{\psi}(0, \infty)$. This leads us to study the existence of universal spaces $L^{\varphi}(0, \infty)$ just for (non-complemented) isomorphic embedding.

We will make use of the following remarkable result given in **[J-M-S-T]**: Every reflexive Orlicz function space $L^{\varphi}[0, 1]$ can be represented isomorphically as the Orlicz space $L^{\overline{\varphi}}(0, \infty)$, where $\overline{\varphi}$ is the function defined by x^2 at 0 and by $\varphi(x)$ at ∞.

2. Main results. Given $0 < p < q < \infty$, let us denote by $x^p \wedge x^q$ the Orlicz function defined by $\min\{x^p, x^q\} = \varphi(x)$ for every $x \geq 0$. Obviously the function $x^p \wedge x^q$ belongs to the class $\mathscr{K}(p, q)$.

The main results of this paper are the following:

THEOREM 3.A. *Let $0 < p < q < \infty$. For every Orlicz function $\psi \in \mathscr{K}(p, q)$, which is non-equivalent to the function x^p and x^q neither at 0 nor at ∞, the Orlicz space $L^{\psi}(0, \infty)$ is order-isomorphic to a sublattice of $L^{x^p \wedge x^q}(0, \infty)$.*

THEOREM 3.B. *Let $0 < p < q < \infty$. For every Orlicz function $\psi \in \mathscr{K}(p, q)$ such that $\psi(x) = x^p$ at ∞ and ψ is non-equivalent to the function x^p and x^q at 0, the Orlicz space $L^{\psi}(0, \infty)$ is order-isomorphic to a sublattice of $L^{x^p \wedge x^q}(0, \infty)$.*

In order to prove these theorems we give some preliminary results. Given a positive function g of an Orlicz space $L^{\varphi}(0, \infty)$ with $\|g\|_{\varphi} = 1$, we consider the Orlicz function Φ_g defined by

$$\Phi_g(x) = \int_0^{\infty} \varphi(x g(s))\, ds \quad \text{for } x \geq 0.$$

It holds that the Orlicz space $L^{\Phi_g}(0, \infty)$ is order-isomorphic (and isometric) to a sublattice of $L^{\varphi}(0, \infty)$. This follows from the fact that the map $T\colon L^{\Phi_g}(0, \infty) \to L^{\varphi}((0, \infty) \times (0, \infty))$ defined by $T(f) =$

$f \otimes g$, where $f \otimes g(t, s) = f(t)g(s)$, is an (isometric) order-isomorphism into $L^{\varphi}((0, \infty) \times (0, \infty))$ (see **[J-M-S-T]** pp. 189 for the proof in the case of convex functions φ, which also works in the p-convex case).

PROPOSITION 4. *Let φ be an Orlicz function verifying the Δ_2-condition and ν be a probability measure on $(0, \infty)$. If ψ is the Orlicz function defined by*

$$(1) \qquad \psi(x) = \int_0^{\infty} \varphi(xs)/\varphi(s)\, d\nu(s) \quad \text{for } x \geq 0$$

then the Orlicz space $L^{\psi}(0, \infty)$ is order-isometric to a sublattice of $L^{\varphi}(0, \infty)$.

Proof. It is similar to the one given in **[J-M-S-T]**, Theorem 7.7, in the case of normed Orlicz spaces over the $[0, 1]$ interval. Let us sketch it for the sake of completeness.

Let us define a positive function g by its distribution function:

$$\mu\{s\colon g(s) \in A\} = \int_A 1/\varphi(s)\, d\nu(s)$$

for every measurable set A in $(0, \infty)$. For each positive μ-measurable function h on $(0, \infty)$

$$\int_0^{\infty} h(s)/\varphi(s)\, d\nu(s) = \int_0^{\infty} h(g(t))\, d\mu(t).$$

In particular for $h = \varphi$, we get $\int_0^{\infty} \varphi(g(t))\, d\mu(t) = 1$, so $g \in L^{\varphi}(0, \infty)$ and $\|g\|_{\varphi} = 1$. Now, taking $h(s) = \varphi(xs)$ for $x \geq 0$, we obtain that $\psi(x) = \Phi_g(x)$. Hence we conclude that

$$L^{\psi}(0, \infty) = L^{\Phi_g}(0, \infty) \subsetneq L^{\varphi}(0, \infty). \qquad \square$$

The next results allow us to represent Orlicz functions of the class $\mathscr{K}(p, q)$ in an intergral form of type (1):

PROPOSITION 5.A. *Given $0 < p < q < \infty$ and the Orlicz function $\varphi(x) = x^p \wedge x^q$. Every Orlicz function $\psi \in \mathscr{K}(p, q)$, which is nonequivalent to the functions x^p and x^q neither at 0 nor at ∞, can be represented, up to equivalence, as*

$$\psi(x) = \int_0^{\infty} \varphi(xs)/\varphi(s)\, d\nu(s)$$

for some probability measure ν on $(0, \infty)$.

PROPOSITION 5.B. *Given $0 < p < q < \infty$ and $\varphi(x) = x^p \wedge x^q$. Every Orlicz function $\psi \in \mathscr{K}(p, q)$ such that $\psi(x) = x^p$ at ∞,* *and which is non-equivalent to the function* x^p *and* x^q *at* 0, *can be represented, up to equivalence, as*

$$\psi(x) = \int_0^\infty \varphi(xs)/\varphi(s)\, d\nu(s)$$

for some probability measure ν *on* $(0, \infty)$.

Proof (5.A). We will take some ideas of the proof of Theorem 4.5 in [W]. We assume by Proposition 2 that ψ is a continuous second derivative. Consider the function

$$N(x) = \frac{\psi(x^{1/q-p})}{x^{p/q-p}}.$$

It follows easily from the properties of the function ψ that N is a non-decreasing function, $N(0) = 0$, and satisfies

$$\lim_{x\to 0} xN'(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} N'(x) = 0.$$

We define the function

$$\begin{aligned}\Phi(x) &= \int_0^\infty t^{p-2q-1} N''(t^{p-q})(p-q)\varphi(xt)\, dt \\ &= x^q \int_0^{1/x} t^{p-2q-1} N''(t^{p-q})(p-q)t^q\, dt \\ &\quad + x^p \int_{1/x}^\infty t^{p-2q-1} N''(t^{p-q})(p-q)t^p\, dt.\end{aligned}$$

Making the variable change $u = t^{p-q}$,

$$\begin{aligned}\Phi(x) &= x^q \int_\infty^{(1/x)^{p-q}} N''(u)\, du + x^p \int_{(1/x)^{p-q}}^0 uN''(u)\, du \\ &= x^q N((1/x)^{p-q}) = \psi(x).\end{aligned}$$

Now, denoting by $\widetilde{N}$ the function

$$\widetilde{N}(t) = t^{2p-2q-1} N''(t^{p-q})(p-q),$$

we have the expressions

$$1 = \psi(1) = \int_0^1 t^{q-p} \widetilde{N}(t)\, dt + \int_1^\infty \widetilde{N}(t)\, dt = \int_0^\infty \frac{t^p \wedge t^q}{t^p} \widetilde{N}(t)\, dt$$

and in general for $x > 0$

$$\begin{aligned}\psi(x) &= x^q \int_0^{1/x} t^q \frac{\widetilde{N}(t)}{t^p}\, dt + x^p \int_{1/x}^\infty t^p \frac{\widetilde{N}(t)}{t^p}\, dt \\ &= \int_0^\infty \frac{(xt)^q \wedge (xt)^p}{t^q \wedge t^p} \, \frac{t^q \wedge t^p}{t^p} \widetilde{N}(t)\, dt.\end{aligned}$$

Hence, taking the probability measure ν on $(0, \infty)$ defined by

$$\nu(A) = \int_A \frac{t^q \wedge t^p}{t^p} \widetilde{N}(t)\, dt$$

we are done. □

Proof (5.B). We will keep the above notation. So, reasoning as before, we consider the function

$$\Phi(x) = \frac{1}{K} \int_1^\infty (p-q) t^{p-2q-1} N''(t^{p-q})(xt)^p \wedge (xt)^q dt$$

for every $x \geq 0$, where N is defined as in the above proof, but only on the interval $[0, 1]$. And the constant K is

$$K = \int_1^\infty (p-q) t^{p-2q-1} N''(t^{p-q}) t^p \, dt .$$

By integrating, we get

$$\Phi(x) = 1/K(x^p N((1/x)^{p-q}) - x^q N'(1)) .$$

Now, from the definition of N, it follows that there exists $0 < x_0 \leq 1$ such that

$$\frac{x^p N((1/x)^{p-q}) - x^q N'(1)}{\psi(x)} = 1 - N'(1) \frac{x^q}{\psi(x)} \geq 1/2$$

for every $x \in [0, x_0]$. Therefore $\psi \overset{0}{\sim} \Phi$.

Proceeding as in 5.A we consider the function

$$\widetilde{N}(t) = \frac{1}{K} t^{2p-2q-1} N''(t^{p-q})(p-q)$$

obtaining that

$$\psi(x) \sim \int_0^\infty \frac{(xt)^p \wedge (xt)^q}{t^p \wedge t^q} \, d\nu(t) ,$$

where ν is a probability measure on $(0, \infty)$ defined by $\nu([0, 1]) = 0$ and for every measurable set A in $(1, \infty)$

$$\nu(A) = \int_A \frac{t^q \wedge t^p}{t^p} \widetilde{N}(t)\, dt . \qquad \square$$

Proof of Theorems 3.A *and* 3.B. It follows directly from Propositions 5.A and 5.B together with Proposition 4. □

3. Consequences. We start this section giving an embedding result for Orlicz spaces over finite measure space.

PROPOSITION 6. *Let φ be a convex Orlicz function at ∞ which is q-concave at ∞ for some $q < 2$. Then the Orlicz function space $L^\varphi[0, 1]$ is universal for the class of all Orlicz function spaces $L^\psi(0, \infty)$ with $\psi \in \mathscr{K}(r, s)$ and $q < r \leq s < 2$.*

Proof. From $\varphi \in \mathscr{K}(1, q)$, it follows that $\varphi(x)/x^r$ is a decreasing function for $x \geq 1$. So

$$\int_0^1 \varphi((1/t)^{1/r})\,dt \leq \int_0^1 (1/t)^{s/r}\,dt < \infty$$

and, by Proposition 8.9 in **[J-M-S-T]** (see also **[L-T II]** Theorem 2.f.4) we obtain that the space $L^r[0, 1]$ is isometrically isomorphic to a subspace of $L^\varphi[0, 1]$. Now, using the important isomorphic representation of $L^r[0, 1]$ as the space $L^{x^r \wedge x^2}(0, \infty)$ (**[J-M-S-T]** Theorem 8.6) together with Theorem 3.A we conclude that

$$L^\psi(0, \infty) \subsetneq L^r[0, 1] \subsetneq L^\varphi[0, 1]. \qquad \square$$

REMARK. The above result was stated implicitly by Bretagnolle and Dacunha-Castelle in **[B-DC]**.

Notice that Proposition 6 extends a well-known result of Bretagnolle, Dacunha-Castelle and Krivine (**[B-DC-K]**), obtained by using p-stable random variables: If $1 \leq p \leq r < 2$, then the space $L^r[0, 1]$ is (isometrically) isomorphic to a subspace of $L^p[0, 1]$.

Let us remark that here we have not used probability tools in the proof of Theorems 3.A and 3.B, but when we prove Proposition 6 we do implicitly because we make use of the Theorem 8.6 of **[J-M-S-T]** which requires Poisson processes. (Also notice that is the reason of losing the isometric property.)

We turn now to give a representation of the universal Orlicz spaces in Theorems 3.A and 3.B as a sum $L^p(0, \infty) + L^q(0, \infty)$, which are well-known spaces in Interpolation Theory (see **[L-T II]**). As a consequence we will extend a recent result of S. Dilworth (**[D]**) on the scale of spaces $L^2(0, \infty) + L^q(0, \infty)$.

Given $L^\varphi(0, \infty)$ and $L^\psi(0, \infty)$ the space sum $L^\varphi(0, \infty) + L^\psi(0, \infty)$ is the space of all functions f on $(0, \infty)$ which can be written as $g + h$ such that $g \in L^\varphi(0, \infty)$ and $h \in L^\psi(0, \infty)$, endowed with the norm (or quasi-norm) defined by

$$\|f\|_{\varphi+\psi} = \inf\{\|g\|_\varphi + \|h\|_\psi : f = g + h\}.$$

PROPOSITION 7. *If φ and ψ are Orlicz functions verifying the Δ_2-condition, $\varphi(x) \leq \psi(x)$ for $0 \leq x \leq 1$, and $\psi(x) \leq \varphi(x)$ for $x \geq 1$, then $L^\varphi(0, \infty) + L^\psi(0, \infty) = L^{\varphi\wedge\psi}(0, \infty)$, the identity being an isomorphism.*

Proof. Given $f \in L^{\varphi\wedge\psi}(0, \infty)$, we consider the functions $f^1 = f\chi_{A_1}$ and $f^2 = f\chi_{A_2}$ where $A_1 = |f|^{-1}([0, 1])$ and $A_2 = |f|^{-1}(1, \infty)$. Hence $f^1 \in L^\varphi(0, \infty)$ and $f^2 \in L^\psi(0, \infty)$, so the inclusion $L^{\varphi\wedge\psi}(0, \infty) \subset L^\varphi(0, \infty) + L^\psi(0, \infty)$ is continuous. Indeed, if

$$\|f_n\|_{\varphi\wedge\psi} \xrightarrow[n\to\infty]{} 0,$$

then

$$\|f_n^1\|_\varphi \xrightarrow[n\to\infty]{} 0 \quad \text{and} \quad \|f_n^2\|_\psi \xrightarrow[n\to\infty]{} 0$$

(cf. [**Mu**], Theorem 1.6). Hence,

$$\|f_n\|_{\varphi+\psi} \xrightarrow[n\to\infty]{} 0.$$

Now, we show that the above inclusion is also onto, which finishes the proof by the Open Mapping Theorem.

If $f \in L^\varphi(0, \infty) + L^\psi(0, \infty)$ with $f = g + h$, $g \in L^\varphi(0, \infty)$ and $h \in L^\psi(0, \infty)$, consider g^1, g^2, h^1, and h^2 as before. Then $g^2 \in L^\psi(0, \infty)$, $h^1 \in L^\varphi(0, \infty)$, and $f = g^1 + g^2 + h^1 + h^2$. It is clear that $g^1 + h^1 \in L^{\varphi\wedge\psi}(0, \infty)$, since $\frac{|g^1(t)+h^1(t)|}{2} \leq 1$ for every $t \geq 0$. Let us see that $g^2 + h^2 \in L^{\varphi\wedge\psi}(0, \infty)$. If $A = |g^2 + h^2|^{-1}([0, 1])$, we have $|g^2 + h^2|\chi_A \leq |g^2|\chi_A$; hence

$$\int_A \varphi(|g^2 + h^2|)\, dt \leq \int_A \varphi(|g^2|)\, dt < \infty,$$

so $(g^2 + h^2)\chi_A \in L^{\varphi\wedge\psi}(0, \infty)$.

Finally, as $g^2 + h^2 \in L^\psi(0, \infty)$, we have

$$(g^2 + h^2)\chi_{(0,\infty)\setminus A} \in L^{\varphi\wedge\psi}(0, \infty),$$

and we can conclude that $f \in L^{\varphi\wedge\psi}(0, \infty)$. □

A direct consequence of the above proposition and Theorems 3.A and 3.B is the following:

COROLLARY 8. *Let $0 < p \leq r \leq s < q$. The space $L^r(0, \infty) + L^s(0, \infty)$ is order-isomorphic to a sublattice of $L^p(0, \infty) + L^q(0, \infty)$.*

The case $p = r = s$ follows from the fact

$$L^p(0, \infty) \approx L^p[0, 1] \subsetneq L^{x^p\wedge x^q}(0, \infty).$$

REMARK. The above result in the particular case of $p = r = 2$ has been obtained by S. Dilworth in **[D]**, Theorem 5.7, using a different argument, and giving several structure properties of the scale of spaces $L^2(0, \infty) + L^q(0, \infty)$ for $2 < q < \infty$.

We pass now to give more consequences of Theorems 3.A and 3.B, related with the Boyd indices of rearrangement invariant (r.i.) function spaces. Let us recall their definition, following **[L-T II]**, where they are given as the converses of the original Boyd indices in **[Bo]**.

Let X be a r.i. Banach function space. For $0 < s < \infty$, we define the dilation operators D_s on X by $(D_s f)(t) = f(t/s)$ for $0 \leq t < \infty$, which are linear and continuous. Now, the Boyd indices p_X and q_X are defined by

$$p_X = \lim_{s\to\infty} \frac{\log(s)}{\log(\|D_s\|_X)} = \sup_{s>1} \frac{\log(s)}{\log(\|D_s\|_X)}$$

and

$$q_X = \lim_{s\to 0^+} \frac{\log(s)}{\log(\|D_s\|_X)} = \inf_{0<s<1} \frac{\log(s)}{\log(\|D_s\|_X)}.$$

It holds that $1 \leq p_X \leq q_X \leq \infty$.

In Interpolation Theory the following result is known (see **[L-T II]**, Proposition 2.b.3), showing the r.i. Banach spaces are intermediate spaces between the spaces $L^p(0, \infty) \cap L^q(0, \infty)$, with the norm of the maximum, and $L^p(0, \infty) + L^q(0, \infty)$.

PROPOSITION 9. *Let X be an r.i. Banach function space on $(0, \infty)$. Then, for every $1 < p < p_X \leq q_X < q \leq \infty$, we have*

$$L^p(0, \infty) \cap L^q(0, \infty) \subset X \subset L^p(0, \infty) + L^q(0, \infty)$$

with the inclusion map being continuous.

Let us show here that when we restrict ourselves to considering Orlicz function spaces $L^\varphi(0, \infty) = X$, we can substitute the above inclusion $X \subset L^p(0, \infty) + L^q(0, \infty)$ by an order-isomorphism.

When a r.i. function space X is an Orlicz function space $L^\varphi(0, \infty)$, the associated Boyd indices p_X and q_X coincide with the Matuszewska-Orlicz indices σ_φ^a and s_φ^a respectively (see **[Bo]**, also **[Ma]**). Let us recall that these indices σ_φ^a and s_φ^a, related with the growing behavior of the Orlicz function φ in the positive real line, are defined by

$$\sigma_\varphi^a = \lim_{\lambda\to\infty} \frac{\log\left(\inf_{u>0}\{\varphi(\lambda u)/\varphi(u)\}\right)}{\log(\lambda)}$$

and

$$s_\varphi^a = \lim_{\lambda\to\infty} \frac{\log(\sup_{u>0}\{\varphi(\lambda u)/\varphi(u)\})}{\log(\lambda)}.$$

Hence, given $0 < p < q < \infty$, for any Orlicz function space $L^\varphi(0, \infty)$ with Boyd indices verifying

$$p < p_X = \sigma_\varphi^a \leq s_\varphi^a = q_X < q,$$

we get that the function φ is, up to equivalence, a strictly p-convex function and a strict q-convex function (see f.i. **[Ma]** pp. 22–24). Thus, the next proposition follows now from Theorem 3.A.

PROPOSITION 10. *Let $0 < p < q < \infty$. If an Orlicz function space $L^\varphi(0, \infty) = X$ has Boyd indices $p < p_X \leq q_X < q$, then $L^\varphi(0, \infty)$ is order-isomorphic to a sublattice of $L^p(0, \infty) + L^q(0, \infty)$.*

In the case $1 \leq p < q = \infty$, it also holds that the space $L^p(0, \infty) + L^\infty(0, \infty)$ is universal for every Orlicz function space $L^\varphi(0, \infty) = X$ with Boyd indices $p < p_X \leq q_X < \infty$. This follows from the fact that the space $L^p(0, \infty) + L^\infty(0, \infty)$ contains an isomorphic copy of l^∞ and the spaces $L^\varphi(0, \infty)$ are separable.

REMARK. In general the continuous inclusion map in Proposition 9, $L^p(0, \infty) \cap L^q(0, \infty) \subset X$, cannot be replaced by an isomorphic embedding as in the above proposition. For example, if $p = 2 < r < q$, then the space $L^2(0, \infty) \cap L^q(0, \infty)$ which is isomorphic to $L^q(0, \infty)$ cannot be embedded into the space $L^r(0, \infty)$.

Finally, let us show that the last proposition does not hold in general for intermediate r.i. Banach spaces different from Orlicz function spaces:

EXAMPLE. Consider the Lorentz function space $X = L_{2,1}(0, \infty)$ defined as the space of all measurable functions f on $(0, \infty)$ such that its decreasing rearrangement f^* verifies

$$\int_0^\infty (f^*(t))t^{-1/2}\,dt < \infty.$$

It holds that its associated Boyd indices are $p_X = q_X = 2$; hence by Proposition 9, we have $L_{2,1}(0, \infty) \subset L^p(0, \infty) + L^q(0, \infty)$ for $1 < p < 2 < q < \infty$.

However, the space $L_{2,1}(0, \infty)$ does not embed isomorphically into $L^p(0, \infty) + L^q(0, \infty)$. Indeed,

$$L^p(0, \infty) + L^q(0, \infty) = L^{x^p \wedge x^q}(0, \infty)$$

is a reflexive space, but the space $L_{2,1}(0, \infty)$ is not reflexive (it contains an l^1-complemented subspace, see **[F-J-T]**).

REMARK. We do not know whether there are Orlicz function spaces over the $[0, 1]$-interval universal for intermediate classes of Orlicz function spaces $L^\psi[0, 1]$, i.e. similar results to Theorems 3.A and 3.B for Orlicz spaces over finite measurable spaces $L^\varphi[0, 1]$. Related to this question, it holds that the universal spaces $L^p(0, \infty) + L^q(0, \infty)$, for $1 < p < q < 2$, cannot be represented as a r.i. Banach function space on the $[0, 1]$-interval (see **[J-M-S-T]** p. 230).

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