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CHAOS IN TERMS OF THE MAP $x \to \omega(x, f)$

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Let $\mathscr{K}$ be the class of compact subsets of $I = [0, 1]$, furnished with the Hausdorf metric. Let $f \in C(I, I)$. We study the map $\omega_f : I \to \mathscr{K}$ defined as $\omega_f(x) = \omega(x, f)$, the ω-limit set of x under f. This map is rarely continuous, and is always in the second Baire class. Those f for which ω_f is in the first Baire class exhibit a form of nonchaos that allows scrambled sets but not positive entropy. This class of functions can be characterized as those which have no infinite ω-limit sets with isolated points. We also discuss methods of constructing functions with zero topological entropy exhibiting infinite ω-limit sets with various properties.

Introduction. One finds a variety of definitions of the notion of chaos for self-maps of an interval in the mathematical literature. While these definitions differ they all carry the idea, in some form or other, that points arbitrarily close together can have orbits or ω-limit sets (attractors) that spread out or are far apart. The works **[D]**, **[LY]** and **[BC]**, for example, provide three such definitions.

In the present paper we address this idea directly. We furnish the family of ω-limit sets of a continuous function f with the Hausdorff metric and ask questions related to the continuity of the map ω_f : $x \to \omega(x, f)$. While one could phrase the questions in terms of the size of the set of points of continuity of ω_f we found a more cohesive development is possible if the questions are phrased in terms of the Baire class of ω_f. This allows us to obtain results concerning continuity as corollaries, to obtain a notion of chaos strictly between the notions involving scrambled sets **[LY]** and positive entropy **[BC]**, and to obtain a complete characterization in terms of the types of ω-limit set that f possesses.

In §1 we find that ω_f is rarely continuous. We obtain several characterizations of continuity for ω_f. In particular, we find that ω_f is continuous if and only if each ω-limit set for f has cardinality 1 or 2 and the union of all ω-limit sets is connected.

In §2 we obtain some general theorems relating the Baire class of ω_f to its Borel class and to certain notions of semi-continuity of ω_f as a set valued mapping. In particular, we find that ω_f is always in (at most) the second Baire class and if all ω-limit sets are finite, then ω_f is in the first Baire class.

Our main results are found in §3. There we show that for functions possessing infinite ω-limit sets, ω_f is Baire 1 if and only if each infinite ω-limit set is perfect. As corollaries, one finds that a function f which is chaotic in the sense of Li and Yorke can have ω_f Baire 1, that functions with zero topological entropy may or may not have ω_f Baire 1, and that functions with positive topological entropy cannot have ω_f Baire 1. Thus the condition that ω_f is Baire 1 is a notion of nonchaos strictly between the notions that involve scrambled sets or entropy.

In the final section we reverse a process used in §3 to obtain a method of constructing functions with zero entropy and infinite ω-limit sets. The method has the advantage that it shows how certain variations in the construction lead to examples exhibiting various features.

We would like to point out that we could have framed our development in terms of the continuity of the map: $x \to \overline{\text{orb } x}$. It is not difficult to verify that this notion is equivalent to the one we chose.

NOTATION AND TERMINOLOGY. In the sequel a function is understood to be a continuous function from $[0, 1]$ into $[0, 1]$ unless clearly specified otherwise. For $x \in [0, 1]$ we define $f^0(x) = x$, $f^1(x) = f(x)$ and $f^{n+1}(x) = f(f^n(x))$ in general. By the *orbit* of a set I, written orb I we mean $\bigcup_{k=0}^{\infty} f^k(I)$. When I is a singleton we will often view orb x as a sequence. The *ω-limit set*, $\omega(x, f)$, is the set of all subsequential limits of the sequence $\{f^n(x)\}_{n=0}^{\infty}$.

We say that x is a *periodic* (or *cyclic*) *point of order* n if $f^{i+n}(x) = f^i(x)$ for all i and no smaller value for n has that property. If x is periodic of order n we say that the set $\{f(x), f^2(x), \dots, f^n(x)\}$ is a *periodic orbit of order* n or an *n-cycle*. Let Fix(f) denote the set of fixed points for f.

If f has zero topological entropy we write $h(f) = 0$. The reader may refer to the literature for the definition. A list of useful characterizations of zero topological entropy is found in **[FShS]**. For our purposes we find it convenient to use the terminology of entropy and we mention only the following characterization: *$h(f) = 0$ if and only if each periodic point has order a power of* 2 **[FShS]**.

We say that f is a *2^∞-function* if f has cycles of order equal to each power of 2 and no others. We say that f is a *2^n-function* if f has cycles of order equal to each 2^k for $k \leq n$ and no others. Then, as is well known, $h(f) = 0$ *if and only if* f *is a* 2^∞*-function or a* 2^n*-function for some* n.

A function f is *chaotic* (in the sense of Li and Yorke) if there exists an uncountable set S such that

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0$$

and

$$\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0$$

for each distinct x and y in S. In **[FShS]** there appears a list of useful characterizations of non-chaotic functions. In particular, any non-chaotic function has zero topological entropy.

By int A and $\overline{A}$ or cl A we mean the interior and closure of A respectively. Let $\mathscr{K}$ denote the class of non-empty compact subsets of $[0, 1]$. Let H be the Hausdorff metric in $\mathscr{K}$. Then $(\mathscr{K}, H)$ becomes a compact metric space.

A function $f: X \to Y$ is a Baire 1 function if it is a pointwise limit of a sequence of continuous functions from X to Y. Denote the class of Baire 1 functions by $\mathscr{B}_1$. Continuing inductively $\mathscr{B}_{n+1}$ is the set of pointwise limits from $\mathscr{B}_n$, the class of Baire n functions.

A function $f: X \to Y$ is Borel $1, 2, 3$, etc., if the inverse image of an open set is an F_σ, $G_{\delta\sigma}$, $F_{\sigma\delta\sigma}$, etc., respectively.

A function $f: X \to Y$ is a Baire* 1 function if for each nonempty perfect subset P of X there exists an open V such that $P \cap V \neq \varnothing$ and f restricted to $P \cap V$ is continuous. Each Baire* 1 function is Baire 1 since $\mathscr{B}_1$ can be characterized as those f whose restriction to any nonempty perfect set P has a point of continuity. In particular, if f is Baire 1, then f is continuous on a dense G_δ. On the other hand if f is Baire* 1, then f is continuous on a dense open set.

If Φ is a function from X into the class of non-empty subsets of Y, then we say that Φ is *lower semi-continuous* or l. s. c. (*upper semi-continuous* or *u.s.c.*) if for each closed (resp. open) subset V of Y the set $\{x : \Phi(x) \subseteq V\}$ is closed (resp. open) in X. If we impose the condition that this set is a G_δ (resp. F_σ), then we say that Φ is *lower semi-continuous of class* 1 or l. s. c. (1) (resp. *upper semi-continuous of class* 1 or *u.s.c.* (1)).

When X and Y are metric then each of l. s. c. (1) and *u.s.c.* (1) is implied by l. s. c. or *u.s.c.* Otherwise there are no general relationships between these semi-continuous functions and Baire 1 or Borel 1 functions.

1. Continuity of ω_f. The desirable situation that ω_f be continuous puts considerable restrictions on f. Theorem 1.2 provides characterizations of ω_f being continuous in several forms. In terms of possible ω-limit sets condition (5) is the most relevant.

We begin with a lemma.

LEMMA 1.1. *If ω_g is continuous, then* Fix(g) *is connected.*

Proof. Suppose Fix(g) is not connected. Pick a, $b \in \text{Fix}(g)$ such that $g(x) \neq x$ for $x \in (a, b)$. Without loss of generality we may assume $g(x) > x$ for $a < x < b$. Then we have 2 cases:

Case 1. $x \in (a, b)$ implies $g(x) < b$. Then for any $x \in (a, b)$, $\{g^n(x)\}_{n=1}^{\infty}$ is an increasing sequence converging to b. Hence $\omega(x, g) = \{b\}$. Since $\omega(a, g) = \{a\}$, ω_g is discontinuous at a.

Case 2. There exists $c \in (a, b)$ such that $g(c) = b$. Choose $x_1 \in (a, b)$ such that $g(x_1) = b$. Choose $x_2 \in (a, x_1)$ such that $g(x_2) = x_1$. Continuing in this way we obtain a decreasing sequence $\{x_n\}_{n=1}^{\infty}$ converging to a for which $g(x_{n+1}) = x_n$ for each n. Since $g^n(x_n) = b$, $\omega(x_n, g) = \{b\}$ for each n. Since $\omega(a, g) = \{a\}$, ω_g will then be discontinuous at a.

THEOREM 1.2. *The following conditions are equivalent*:

(1) ω_f *is continuous.*
(2) $\{f^n\}_{n=1}^{\infty}$ *is equicontinuous.*
(3) ω_{f^2} *is continuous.*
(4) $\text{Fix}(f^2) = \bigcap_{n=1}^{\infty} f^n(I)$.
(5) $\text{Fix}(f^2)$ *is connected and for all* x, $\{f^{2n}(x)\}_{n=1}^{\infty}$ *converges to a point of* $\text{Fix}(f^2)$.
(6) $\text{Fix}(f^2)$ *is connected.*
(7) ω_f *is lower semi-continuous.*
(8) ω_f *is upper semi-continuous.*

(1)$\Rightarrow$(2): Let $J = \bigcap_{n=1}^{\infty} f^n(I)$. Then J is a compact interval and $f(J) = J$. Suppose ω_f is continuous and that $\{f^n\}_{n=1}^{\infty}$ is not equicontinuous. If $J = \{y\}$ for some y, then $\{y\} = \text{Fix}(f)$ and for each x $f^n(x) \to y$. By Theorem 11 of **[BH]** this implies equicontinuity. Hence, J is a non-degenerate closed interval.

By corollary 12 of **[BH]**, J is not $\text{Fix}(f^2)$. Since $\text{Fix}(f^2) \subseteq J$ this means $f^2(z) \neq z$ for some $z \in J$. Since $f(J) = J$ and Fix(f)

is connected by Lemma 1.1 it is easy to see that f^2 has more than one fixed point in J. Hence, there exist $a, b \in J \cap \operatorname{Fix}(f^2)$ such that $(a, b) \cap \operatorname{Fix}(f^2) = \varnothing$. Without loss of generality we may assume $f^2(x) > x$ for all $x \in (a, b)$. According to the proof of Lemma 1.1 there exists a sequence $\{x_n\}_{n=1}^{\infty}$ approaching a for which $\omega(a, f^2) = \{a\}$ and $\omega(x_n, f^2) = \{b\}$. Hence $\omega(a, f) = \{a, f(a)\}$ and $\omega(x_n, f) = \{b, f(b)\}$.

Since ω_f is continuous $\{a, f(a)\} = \{b, f(b)\}$ so that $a = f(b)$ and $b = f(a)$. Therefore f, and f^2 too, has a fixed point in (a, b), a contradiction.

(2) $\Rightarrow$ (1): Let $\varepsilon > 0$. Choose $\delta > 0$ such that $|x - y| < \delta$ implies $|f^n(x) - f^n(y)| < \varepsilon/3$ for all n. Let $x_0 \in \omega(x, f)$. There exists $\{n_k\}_{k=1}^{\infty}$ such that $|f^{n_k}(x) - x_0| < \varepsilon/3$ for all k. Then $|f^{n_k}(y) - x_0| < 2\varepsilon/3$ for all k. Thus $\omega(y, f)$ contains a point within ε of x_0. Likewise if $y_0 \in \omega(y, f)$ then $\omega(x, f)$ contains a point within ε of y_0. Thus $H(\omega(x, f), \omega(y, f)) < \varepsilon$ whenever $|x - y| < \delta$ and ω_f is continuous.

(3) $\Rightarrow$ (1): For all y we have $\omega(y, f) = \omega(f(y), f^2) \cup \omega(f^2(y), f^2)$. But in the Hausdorff metric $A_\alpha \to A$ and $B_\alpha \to B$ imply $A_\alpha \cup B_\alpha \to A \cup B$. It follows that ω_f is continuous when ω_{f^2} is continuous.

(2) $\Rightarrow$ (3): By (2) $\{f^{2n}\}_{n=1}^{\infty}$ is equicontinuous and by (1) ω_{f^2} is continuous.

(2) $\Leftrightarrow$ (4): This is Corollary 12 of **[BH]**.

(2) $\Rightarrow$ (5): By (4) $\operatorname{Fix}(f^2)$ is connected. Now applying Corollary 10 of **[BH]** $\{f^{2n}(x)\}_{n=1}^{\infty}$ converges to some point of $\operatorname{Fix}(f^2)$ for each x.

(5) $\Rightarrow$ (4): Let $J = \bigcap_{n=1}^{\infty} f^n(I)$, $E = \operatorname{Fix}(f^2)$. It is clear that $E \subset J$. Suppose $E \neq J$. Let $E = [a, b]$ and $J = [c, d]$. If $c = a$ or $b = d$ we have an immediate contradiction since $f(J) = J$ and $f(x) < x$ on $(b, d]$, $(f(x) > x$ on $[c, a)$, respectively). Thus, suppose $c < a < b < d$. Now $f^2(x) < d$ for $x \in [b, d]$ and $f(J) = J$. Thus $f^2([c, a]) \supset [a, d]$. Since $f^2(a) = a$, there exists a_1 with $c < a_1 < a$ such that $f^2([c, a_1]) = [b, d]$. Similarly there exists b_1 with $b < b_1 < d$ such that $f^2([b_1, d]) \supset [c_1, a]$. Thus $f^4([c, a_1]) \supset f^2([b, d]) \supset f^2([b_1, d]) \supset [c, a_1]$ and the interval $[c, a_1]$ has a periodic point. But $[c, a_1]$ is disjoint from E, a contradiction. If $a = b$ the same argument applies unless $f^2([c, a]) = [c, a]$, but in that case there is $x \in [c, a)$ such that $f^2(x) = x$, again a contradiction.

(6) $\Rightarrow$ (5): If $\operatorname{Fix}(f^2) = \{a\}$, then by Propostion 6 of **[BH]** applied to f^2 we have $\{f^{2n}\}_{n=1}^{\infty}$ is equicontinuous. Hence, by the equivalence of (1), (2) and (3) ω_f is continuous.

So let $\mathrm{Fix}(f^2) = [a, b]$ where $a < b$. We will first show that $\omega(x, f) \subseteq [a, b]$ for all x. We have two cases: (1) $0 < a < b < 1$ and (2) $a = 0$ or $b = 1$. We will carry out the proof when $0 < a < b < 1$; the other case will require a simple modification. Let $E = \{x : \omega(x, f) \subseteq [a, b]\}$. Then $f(E) = E$. By Lemma 2 of **[BH]** there exist α and β such that $0 < \alpha < a < b < \beta < 1$ and for all $x \in (\alpha, \beta)$, $\{f^{2n}(x)\}_{n=1}^{\infty}$ converges to some point in $[a, b]$. Hence, $(\alpha, \beta) \subseteq E$. Let (c, d) be the union of all open intervals J such that $[a, b] \subseteq J \subseteq E$.

We will show $c = 0$ and $d = 1$. Suppose $c \neq 0$. Then $f(c) > c$ since $f(0) > 0$ and there are no fixed points of f in $(0, c)$. If $f(c) < d$, then some neighborhood of c maps into E, contradicting the maximality of (c, d). Hence, $f(c) \geq d$. On the other hand if $f(c) > d$, then $[b, f(c)] \subseteq f([c, b])$ since f has a fixed point in $[a, b]$. But d is interior to $[b, f(c)]$ so that there is a $z \in [b, f(c)] - E$ with $z \in f(E) = E$, a contradiction. Therefore we conclude $f(c) = d$. Likewise $f(d) = c$. Hence, $f^2(c) = c$ contradicting the fact that $c \notin [a, b]$. Hence, $c = 0$ and $d = 1$. From this it follows that $\omega(x, f) \subseteq [a, b]$ for all $x \in [0, 1]$.

This implies that for a given x, $\omega(x, f^2)$ is a singleton set or $\omega(x, f) = \{a, b\}$.

(5) $\Rightarrow$ (6): obvious

(5) $\Rightarrow$ (7): Since (5) $\rightarrow$ (1) $\leftrightarrow$ (2) has already been established there exists a continuous λ such that $\omega(x, f) = \{\lambda(x), f(\lambda(x))\}$ for each x. If F is closed we need to show that $\{x : \omega(x, f) \subseteq F\}$ is closed. Let $x_\alpha \rightarrow x$ where $\{\lambda(x_\alpha), f(\lambda(x_\alpha))\} \subseteq F$. By continuity of f and λ we have $\{\lambda(x), f(\lambda(x))\} \subseteq F$.

(5) $\Rightarrow$ (8): As above we need to show $\{x : \{\lambda(x), f(\lambda(x))\} \subseteq G\}$ is open whenever G is open. This is immediate from the continuity of λ and f.

(8) $\Rightarrow$ (6): Suppose $\mathrm{Fix}(f^2)$ is not connected. Choose $a, b \in \mathrm{Fix}(f^2)$ such that $(a, b) \cap \mathrm{Fix}(f^2) = \varnothing$. We may assume that $f^2(x) > x$ for $a < x < b$. Then as in the proof of Lemma 1.1 there exists a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \rightarrow a$ and $\omega(x_n, f^2) = \{b\}$. Consequently $\omega(a, f) = \{a, f(a)\}$ and $\omega(x_n, f) = \{b, f(b)\}$. Since f has no fixed point in (a, b) we must have $\{a, f(a)\} \neq \{b, f(b)\}$.

Let G be an open set such that $\{a, f(a)\} \subseteq G$ and $\{b, f(b)\} \not\subseteq G$. Then $\omega(a, f) \subseteq G$ and for each n, $\omega(x_n, f) \not\subseteq G$. Hence, $\{x : \omega(x, f) \subseteq G\}$ is not open and ω_f is not *u.s.c.*

(7) $\Rightarrow$ (6): Suppose $\mathrm{Fix}(f^2)$ is not connected. Then we have the

information in the first paragraph of the proof of (8) $\Rightarrow$ (6).

Let $F = \{b, f(b)\}$. Then $\{x: \omega(x, f) \subseteq F\}$ contains the sequence $\{x_n\}_{n=1}^{\infty}$ but not its limit point a. Therefore $\{x : \omega(x, f) \subseteq F\}$ is not closed and ω_f is not l. s. c.

COROLLARY 1.3. *If ω_f is continuous, then f is a 2^0-function or a 2^1-function and in particular* $\operatorname{card} \omega(x, f) \leq 2$ *for all* x.

This is an immediate consequence of (5) of Theorem 1.2. The converse is not true as shown by the squaring function on $[0, 1]$.

By Lemma 1.1 and Theorem 1.2 it follows that $\operatorname{Fix}(f)$ is connected whenever $\operatorname{Fix}(f^2)$ is connected. However, $\operatorname{Fix}(f)$ can be connected with $\operatorname{Fix}(f^2)$ not connected. For example put $f(x) = 2(x - \frac{1}{2})^2 + \frac{1}{2}$, if $0 \leq x \leq \frac{1}{2}$ and $= -2(x-\frac{1}{2})^2+\frac{1}{2}$ if $\frac{1}{2} \leq x \leq 1$. Then $\operatorname{Fix}(f) = \{\frac{1}{2}\}$ and $\operatorname{Fix}(f^2) = \{0, \frac{1}{2}, 1\}$. This example also shows the converse of Lemma 1.1 is false.

We end this section by noting that equicontinuity of the sequence $\{f^n\}_{n=1}^{\infty}$ implies that orbits of nearby points x and y stay close together, while the continuity of ω_f implies only that the sets $\omega(x, f)$ and $\omega(y, f)$ are close. Theorem 1.2 shows these notions are equivalent. However equicontinuity of $\{f^n\}_{n=1}^{\infty}$ on a set S may be a stronger condition than continuity of ω_f restricted to S. For example, for the hat function,

$$h(x) = \begin{cases} 2x, & 0 \leq x < \frac{1}{2}, \\ 2(1-x), & x \geq \frac{1}{2}, \end{cases} \qquad S = \{x : \omega(x, h) = [0, 1]\}$$

is a dense set of type G_δ. The function ω_h restricted to S is constant, but the sequence $[h^n]_{n=1}^{\infty}$ is not equicontinuous on S.

2. Baire classes and semi-continuity of ω_f. We have seen that ω_f is continuous only under very restrictive circumstances. One might seek less restrictive conditions that would imply that ω_f possesses large sets of continuity points. One might expect results such as the following:

(1) ω_f is continuous on a dense *open* set if f has only finitely many ω-limit sets.

(2) ω_f is continuous on a dense set if all ω-limit sets are finite.

(3) If h is the hat function, then ω_h is discontinuous everywhere. In fact ω_h takes all of its values in each subinterval.

Indeed the first two results are true and are corollaries of Theorems 2.9 and 2.8 below. (The second will be improved in Theorem 3.8.) The third result follows readily from the analysis of the hat function done in **[BCR]**.

The hypotheses of (1) and (2) actually yield stronger conclusions; namely, that ω_f is Baire* 1 and Baire 1 respectively. Our emphasis in this section, as well as in §3, is on studying the Baire class of ω_f under various hypotheses on f. We shall see that the Baire class is closely related to the kinds of ω-limit sets f has. We shall also see that, although ω_f may be discontinuous everywhere, ω_f is always Baire 2. A useful tool involves the notions of semi-continuity.

In this section we obtain some general theorems relating semi-continuity of ω_f to the Baire class of ω_f and we obtain some easy results on 2^n-functions. We also show that ω_f is not Baire 1 when $h(f) > 0$. We defer the somewhat deeper analysis of 2^∞-functions to §3.

We begin with a theorem that allows us to interchange Baire with Borel.

THEOREM 2.1. *For functions from* $[0, 1]$ *into* $(\mathscr{K}, H)$ *the Borel and Baire classes agree for finite ordinals.*

Proof. The proof that the Baire and Borel classes agree for finite ordinals for functions from a metric space X into $[0, 1]^m$, m some ordinal, which is found in $[\mathbf{K}_2]$ can be modified to fit the case when $X = [0, 1]$ and $(\mathscr{K}, H)$ is the range space. The only fact needed is the validity of a Tietze extension theorem namely: if $F_1, \dots, F_n$ are mutually disjoint non-void closed sets in $[0, 1]$ and $Y_1, \dots, Y_n$ are in $\mathscr{K}$ then there exists a continuous function $f\colon [0, 1] \to (\mathscr{K}, H)$ such that $f(F_i) = Y_i$ for all i.

To show this suppose (a, b) is a component of $(0, 1) - \bigcup_{m=1}^n F_m$ with $a \in F_i$ and $b \in F_j$. Let $\mathscr{L}$ consist of all line segments in R^2 from (a, u) to (b, v) where $u \in Y_i$ and $v \in Y_j$. For $x \in (a, b)$ define

$$f(x) = \{y : \text{ there exists } L \in \mathscr{L} \text{ such that } (x, y) \in L\}$$

If $x \in F_i$, put $f(x_i) = Y_i$. If $[0, b)$ is a component of $[0, 1] - \bigcup_{m=1}^n F_m$ and $b \in F_i$ put $f(x) = F_i$ for $x \in [0, b)$. Similarly for a component of the form $(a, 1]$. It is easily checked that f is the desired continuous function.

THEOREM 2.2. *If* ω_f *is Baire* 1, *then* $h(f) = 0$.

Proof. Suppose $h(f) > 0$. We show there exists a perfect set K such that $\omega_f \mid K$ is everywhere discontinuous. This will imply that ω_f is not Baire 1, since a Baire 1 function from $[0, 1]$ into a separable metric space must have points of relative continuity in each perfect set.

One finds in [SS] that there exists a perfect ω-limit set K for f such that (i) the set of periodic points in K is dense in K and (ii) if $L \subset K$ is an ω-limit set for f, then $\{x \in K : \omega(x, f) = L\}$ is dense in K.

Thus the function $\omega_f \mid K$ takes each of its infinitely many values on a dense subset of K. It follows that $\omega_f \mid K$ is everywhere discontinuous, completing the proof of the theorem.

THEOREM 2.3. *For all* f, ω_f *is u.s.c.* (1).

Proof. Let W be open. Without loss of generality we may assume that there exists a sequence of mutually disjoint open intervals $\{(a_i, b_i)\}_{i=1}^{\infty}$ such that $W = \bigcup_{i=1}^{\infty}(a_i, b_i)$. Let F be the family of all finite subsets of N. Then

$$\{x : \omega(x, f) \subseteq W\} = \bigcup_{S \in F} \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} \left\{x : f^k(x) \in \bigcup_{i \in S} \left[a_i + \frac{(b_i - a_i)}{n}, b_i - \frac{(b_i - a_i)}{n}\right]\right\}$$

which is an F_σ set.

THEOREM 2.4. *For all* f, ω_f *is Baire* 2.

Proof. Let K be a compact set and $\{a_i\colon\ i = 1, 2, \ldots\}$ be a countable dense subset of K. Let $\varepsilon > 0$. It will suffice to show that $\{x\colon H(\omega_f(x), K) < \varepsilon\}$ is a $G_{\delta\sigma}$ set. For any C let $S_\varepsilon(C) = \{y\colon |z - y| < \varepsilon \text{ for some } z \in C\}$. Put $A = \{x : \omega_f(x) \subseteq S_\varepsilon(K)\}$ and $B = \{x\colon K \subseteq S_\varepsilon(\omega_f(x))\}$. By definition $H(\omega_f(x), K) < \varepsilon$ if and only if $x \in A \cap B$. It is easily verified that

$$A = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} \left\{x\colon \operatorname{dist}(f^k(x), K) \le \frac{n\varepsilon}{n+1}\right\}$$

and

$$B = \bigcup_{n=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} \left\{x\colon |f^k(x) - a_j| < \frac{\varepsilon n}{n+1}\right\}.$$

Clearly A is an F_σ set and B is a $G_{\delta\sigma}$ set so that $A \cap B$ is a $G_{\delta\sigma}$ set, completing the proof.

LEMMA 2.5. *Let $I_1, \dots, I_n$ be open intervals (relative to $[0, 1]$). Let $B(I_1, \dots, I_n)$ be the set of all $K \in \mathscr{K}$ such that*
(1) $K \subseteq \bigcup_{i=1}^n I_i$ and
(2) $K \cap I_i \neq \varnothing$ for each i.
Then $B(I_1, \dots I_n)$ is open in $(\mathscr{K}, H)$.

Proof. Let $K \in B = B(I_1, \dots, I_n)$. It suffices to find $\varepsilon > 0$ such that if $H(K, J) < \varepsilon$ then $J \in B$. For each $i = 1, \dots, n$, choose $x_i \in K \cap I_i$ and let

$$\varepsilon_i = \begin{cases} \operatorname{dist}(\{x_i\}, R - I_i), & \text{if } x_i \neq 0, 1, \\ \operatorname{diam} I_i, & \text{if } x_i = 0 \text{ or } x_i = 1. \end{cases}$$

Let

$$\varepsilon_0 = \begin{cases} 1, & \text{if } \bigcup_{i=1}^n, I_i = I, \\ \operatorname{dist}(K, I - \bigcup_{i=1}^n I_i), & \text{if } \bigcup_{i=1}^n I_i \neq I. \end{cases}$$

One verifies directly that if $H(K, J) < \varepsilon_i$, $(i > 0)$, then $J \cap I_i \neq \varnothing$, and if $H(K, J) < \varepsilon_0$, then $J \subset \bigcup_{i=1}^n I_i$. It follows that $\varepsilon = \min\{\varepsilon_i : 0 \le i \le n\}$ works.

LEMMA 2.6. *Let E be the set of rationals in $[0, 1]$. The metric space $(\operatorname{range} \omega_f, H)$ is separable and has a countable basis, $\mathscr{B}$, consisting of all sets of the form $B(I_1, \dots, I_n) \cap \operatorname{range} \omega_f$ where each I_i has end points in E.*

Proof. Let K be an ω-limit set and $\varepsilon > 0$. Since $\mathscr{B}$ is countable it suffices to show there exist $I_1, \dots, I_n$ with end points in E such that $K \in B(I_1, \dots, I_n) \subseteq \{J : H(J, K) < \varepsilon\}$. We may cover K with finitely many intervals $I_1, \dots, I_n$ with end points in E all having length $\le \varepsilon$ such that $K \subseteq B(I_1, \dots, I_n)$. Clearly $B(I_1, \dots, I_n) \subseteq \{J : H(K, J) < \varepsilon\}$.

Perhaps the main result of this section is Theorem 2.7 below. It provides a characterization of our form of nonchaos, namely that ω_f be Baire 1. This characterization will allow us to simplify arguments in the sequel.

THEOREM 2.7. *ω_f is Baire 1 if and only if ω_f is l.s.c. (1).*

Proof. $\Rightarrow$ It suffices to show that $\{x : \omega(x, f) \cap J \neq \varnothing\}$ is an F_σ set whenever J is an open interval (relative to $[0, 1]$) with rational end points. Let the countable base $\mathscr{B}$ be stipulated by Lemma 2.6 and let $\mathscr{A}$ consist of all $B = B(I_1, I_2, \dots, I_n)$, such that J is some I_j. Then

$$\{x\colon \omega(x, f) \cap J \neq \varnothing\} = \bigcup\{\omega_f^{-1}(B)\,;\, B \in \mathscr{A}\}.$$

This is an F_σ set because $\mathscr{A}$ is countable and each $\omega_f^{-1}(B)$ is an F_σ set since ω_f is Baire 1.

($\Leftarrow$) Let $B = B(I_1, \dots, I_n) \cap \operatorname{range} \omega_f$. It suffices to show $\omega_f^{-1}(B)$ is an F_σ set. Clearly

$$\omega_f^{-1}(B) = \left\{x\colon \omega(x, f) \subseteq \bigcup_{i=1}^{n} I_i\right\} \cap \left(\bigcap_{i=1}^{n}\{x\colon \omega(x, f) \cap I_i \neq \varnothing\}\right).$$

By Theorem 2.3 $\{x\colon \omega(x, f) \subseteq \bigcup_{i=1}^{n} I_i\}$ is an F_σ set and since ω_f is l.s.c. (1) each of the sets $\{x\colon \omega(x, f) \cap I_i \neq \varnothing\}$ is an F_σ set. Hence $\omega_f^{-1}(B)$ is an F_σ set.

We now provide two simple sufficient conditions for ω_f to be Baire* 1 and Baire 1.

THEOREM 2.8. *If f is a 2^n-function, then ω_f is Baire* 1.

Proof. We show ω_f is l.s.c. (1). For each x, $\omega(x, f)$ is a 2^k-cycle for some $k \leq n$. Hence $\omega(x, f^{2^n}) = \{g(x)\}$ for some $g(x)$. Then $g = \lim_{i\to\infty} f^{i2^n}$ and therefore g is Baire 1. For any x there is a k such that

$$\omega(x, f) = \{f(g(x)), \dots, f^{2^k}(g(x))\}.$$

Let F be closed. Then

$$\{x : \omega(x, f) \subseteq F\} = \bigcap_{i=1}^{2^n}\{x : f^i(g(x)) \in F\}.$$

This set is a G_δ set since each $f^i g$, being a composition of a continuous function with a Baire 1 function, is a Baire 1 function. Therefore, ω_f is l.s.c. (1) and Baire 1.

We shall extend Theorem 2.8 in §3 (Theorem 3.8) to certain 2^∞-functions.

THEOREM 2.9. *If the family of all ω-limit sets is finite, then ω_f is Baire* 1.*

Proof. It is clear that f is a 2^j-function for some j and each ω-limit set is a 2^k-cycle where $k \leq j$.

Suppose $A = \{a_i, \dots, a_{2^k}\}$ is an ω-limit set. Then

$$\{x : \omega(x, f) = A\} = \bigcup_{i=1}^{2^k} \{x : \omega(x, f^{2^k}) = \{a_i\}\}$$
$$= \bigcup_{i=1}^{2^k} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} (\{x : |f^{n2^k}(x) - a_i| \leq \delta/2\})$$

where δ is less than the Hausdorff distance between any two of the finite family of ω-limit sets for f. This set is an F_σ set. Hence, $[0, 1]$ is a union of finitely many F_σ sets on each of which ω_f is constant. It follows that ω_f is Baire* 1.

The identity function shows that the converse of Theorem 2.9 is false. Theorem 2.9 does not extend to the family of ω-limit sets being countable as shown by

EXAMPLE 2.10. There exists a 2^0-function f whose family of ω-limit sets is countable and ω_f is not Baire* 1.

Proof. We may choose sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ for which $a_1 = 0$, $\lim a_n = 1$ and $a_n < b_n < a_{n+1}$ for all n and having the additional property that if U_n is the line segment joining (a_n, a_n) to $(b_n, 1)$, U_n has slope 3, and if D_n is the line segment joining (a_n, a_n) to $(b_{n-1}, 1)$ then D_n has slope -3. Let f be the function whose graph is $(\bigcup_{n=2}^{\infty} D_n) \cup (\bigcup_{n=1}^{\infty} U_n) \cup \{(1, 1)\}$. Clearly f is continuous and $f(x) \geq x$ for all x. Hence, f is a 2^0-function. Then clearly all ω-limit sets are orbits of the fixed points, which are 1 and a_n for $n \geq 1$.

If J is an interval contained in some (a_{i-1}, a_i), then clearly $|f(J)| \geq (3/2)|J|$. Then some iterate of J will contain some a_j. To see this, suppose no iterate of J contains an a_j. Then there exists a sequence $\{(a_{n_k}, a_{n_k+1})\}_{k=1}^{\infty}$ for which $f^k(J) \subseteq (a_{n_k}, a_{n_k+1})$ and $|a_{n_k+1}, a_{n_k}| \gtrsim |f^k(J)| \supseteq (3/2)^k|J| \to \infty$ a contradiction.

In fact the orbit of J will contain a subsequence of $\{a_i\}_{i=1}^{\infty}$. Hence, each interval contains points y and x for which $\omega(y, f) = \{1\}$ and $\omega(x, f) =$ some $\{a_n\}$.

Let J be open and pick $x \in J$ such that $\omega(x, f) = \{a_n\}$ for some n. Choose $y_\alpha \in J$ such that $y_\alpha \to x$ and $\omega(y_\alpha, f) = \{1\}$. Then $\omega(y_\alpha, f) \not\to \omega(x, f)$ so that each interval contains a point of discontinuity of ω_f and f cannot be Baire* 1.

We conclude this section by noting that while the condition that ω_f be continuous is rare, the condition that ω_f be Baire* 1, which implies ω_f is continuous on a dense open set, is not. For the well-studied logistic family: $f_k(x) = kx(1-x)$, $0 \le x \le 1$, $0 < k \le 4$, Theorem 2.9 implies ω_{f_k} is Baire* 1 whenever f_k is a 2^n-function. This will occur as long as k is less than a certain k_0 (approximately equal to 3.5699 **[P]**).

On the other hand when $k = k_0$, f has an infinite ω-limit set. We shall see in §3 that this implies that ω_f is not Baire* 1. Incidentally, $\omega_f \in B_1$ when $k = k_0$, but $\omega_f \notin B_1$ when $k > k_0$.

3. 2^∞-functions. In the previous section we saw that ω_f is always Baire 2, that ω_f is Baire 1 if each ω-limit set is finite and that ω_f is never Baire 1 when $h(f) > 0$. There remains the case when $h(f) = 0$ and f has an infinite ω-limit set so that f is a 2^∞-function.

We shall see that if all infinite ω-limit sets of f are perfect, then ω_f is Baire 1, but if f has an infinite ω-limit set with isolated points, then ω_f is not Baire 1. It will follow that the Baire class of ω_f provides a measure of chaos strictly intermediate to the two common notions: the existence of scrambled sets and positive topological entropy.

Our program will be as follows: We first develop some properties of "simple systems" associated with infinite ω-limit sets for 2^∞-functions (Proposition 3.1). These properties will be used repeatedly in the sequel. We then obtain a few lemmas that indicate how the sets of points attracted to various ω-limit sets "intermingle." These results are useful in the proofs of two of our main results. Finally we obtain some results related to chaos. In particular, we prove for a function f, all of whose infinite ω-limit sets are perfect, ω_f is Baire 1. Thus there are chaotic functions for which ω_f is Baire 1.

We begin with a discussion of a notion that others have used in various forms for various purposes. Smital **[S]** has shown that if Ω is an infinite ω-limit set for a 2^∞-function f, then there exists a sequence of closed intervals $\{T_k\}_{k=1}^\infty$ such that

(i) for each k, $\{f^i(T_k)\}_{i=1}^{2^k}$ are pairwise disjoint and $T_k = f^{2^k}(T_k)$

(ii) for each k, $T_{k+1} \cup f^{2^k}(T_{k+1}) \subseteq T_k$

(iii) for each k, $\Omega \subseteq \bigcup_{i=1}^{2^k} f^i(T_k)$
(iv) for each k and i, $\Omega \cap f^i(T_k) \neq \varnothing$.

We call the set $\mathscr{J}$ of all such $f^i(T_k)$ a *simple system for* Ω *relative to* f.

Our purpose is to analyse simple systems, to derive some new results and to shed light on previously known results. Moreover, in §4 we will reverse Smital's construction of simple systems by constructing 2^∞-functions from certain "systems" of intervals.

In order to accomplish this project we use a device suggested in [**D**; p. 136]. We code the sets $f^i(T_k)$ with finite tuples of zeros and ones. To this end, let N denote the set of positive integers and let $\mathscr{N}$ be the set of sequences of zeros and ones. If $\mathbf{n} \in \mathscr{N}$ and $\mathbf{n} = \{n_j\}_{j=1}^\infty$ we write $\mathbf{n} \upharpoonright k = (n_1, n_2, \dots, n_k)$. By $\mathbf{0}$ (res. $\mathbf{1}$) we mean that $\mathbf{n} \in \mathscr{N}$ such that $n_i = 0$ (resp. 1) for all i.

Define a function $A : \mathscr{N} \to \mathscr{N}$ by

$$A(\mathbf{n}) = \mathbf{n} + \mathbf{10}$$

where addition is modulus 2 from left to right.

For each $k \in N$ and $i \in N$ put

$$J_{\mathbf{1}\upharpoonright k} = T_k \quad \text{and} \quad J_{A^i(\mathbf{1})\upharpoonright k} = f^i(T_k).$$

It easily follows from (i) that for any $\mathbf{m}, \mathbf{n} \in \mathscr{N}$ and $k \in N$ there exists $j \in N$ such that $A^j(\mathbf{m} \upharpoonright k) = \mathbf{n} \upharpoonright k$. Hence the above relations actually define $J_{\mathbf{n}\upharpoonright k}$ for all $\mathbf{n} \in \mathscr{N}$ and $k \in N$ so that the collection of all $J_{\mathbf{n}\upharpoonright k}$ coincides with the simple system $\mathscr{J}$.

Recasting (i) through (iv) into the new notation we have the following:

(a) For each $\mathbf{n} \in \mathscr{N}$ and $k \in N$, $J_{\mathbf{n}\upharpoonright k,0}$ and $J_{\mathbf{n}\upharpoonright k,1}$ are disjoint closed subintervals of $J_{\mathbf{n}\upharpoonright k}$ which f^{2^k} interchanges.

(b) For each $k \in N$, f maps the collection $\{J_{\mathbf{n}\upharpoonright k} : \mathbf{n} \in \mathscr{N}\}$ onto itself.

(c) For each $k \in N$, $\Omega \subseteq \bigcup\{J_{\mathbf{n}\upharpoonright k} : \mathbf{n} \in \mathscr{N}\}$

(d) For each $\mathbf{n} \in \mathscr{N}$ and $k \in N$, $\Omega \cap J_{\mathbf{n}\upharpoonright k} \neq \varnothing$.

Now put

$$K = \bigcup_{\mathbf{n} \in \mathscr{N}} \bigcap_{k=1}^{\infty} J_{\mathbf{n}\upharpoonright k}$$

and

$$J_{\mathbf{n}} = \bigcap_{k=1}^{\infty} J_{\mathbf{n}|k}$$

Then K and each $J_{\mathbf{n}}$ are compact. For a fixed $\mathbf{n} \in \mathscr{N}$, $\{J_{\mathbf{n}|k}\}_{k=1}^{\infty}$ is descending. Therefore the components of K consist of the $J_{\mathbf{n}}$ sets. Moreover

$$K = \bigcap_{k=1}^{\infty} \bigcup_{\mathbf{n} \in \mathscr{N}} J_{\mathbf{n}|k} = \bigcap_{k=1}^{\infty} \bigcup_{i=1}^{2^k} f^i(T_k).$$

Let S consist of all x for which there exists $\mathbf{n} \in \mathscr{N}$ such that $\{x\} = J_{\mathbf{n}}$. Let

$$Q = \overline{S} \quad \text{and} \quad C = (K - \operatorname{int} K) - Q.$$

Then Q is a Cantor set (i.e. perfect, nowhere dense and nonvoid). Also C is countable and possibly empty.

Let G be the component of $[0, 1] - K$ which contains the interval between J_0 and J_1. In general, let $G_{\mathbf{n}|k}$ be that component of $[0, 1] - K$ which contains the interval between $J_{\mathbf{n}|k0}$ and $J_{\mathbf{n}|k1}$. Let

$$\mathscr{G} = \{G\} \cup \{G_{\mathbf{n}|k} : \mathbf{n} \in \mathscr{N}, k \in N\}$$

and

$$G^0 = G \cup [0, \inf K) \cup (\sup K, 1]$$

and

$$G^k = \bigcup \{G_{\mathbf{n}|k} : \mathbf{n} \in \mathscr{N}\}.$$

Note that $[\inf K, \sup K] = K \cup (\bigcup \mathscr{G})$ and $[0, 1] = K \cup \bigcup_{j=0}^{\infty} G^j$.

In the rest of §3 the symbols A, K, Q, S, C, $\mathscr{G}$, $G_{\mathbf{n}|k}$, G^k, $J_{\mathbf{n}|k}$, etc. will always mean those sets defined above associated with a particular $\mathscr{J}$ arising from a 2^{∞}-function f and one of its infinite ω-limit sets Ω. Hence, we give no further explanations for these symbols when they appear in the sequel.

Proposition 3.1 below lists some properties of the system we have described. Some of these properties are essentially known but are scattered throughout the literature and are sometimes stated in different forms (see [S]). For completeness we sketch the proofs.

In particular 3.1 part (1) implies that if we identify $\mathbf{n}$ with $J_{\mathbf{n}}$ then "$f(\mathbf{n}) = A(\mathbf{n})$ for $\mathbf{n} \in S$". Hence the coding by $\mathscr{N}$ allows us to represent f on S by the fixed function A.

Proposition 3.1. *Let Ω be an infinite ω-limit set for a 2^∞-function f with $\mathscr{J} = \{J_{\mathbf{n}|k} : \mathbf{n} \in \mathscr{N}, k \in N\}$ a simple system for Ω relative to f. Then*

(1) *for each component $J_{\mathbf{n}}$ of K, $f(J_{\mathbf{n}}) = J_{A(\mathbf{n})}$*

(2) *for each j, $f^j[a, b] \cap [a, b] = \varnothing$ for each component (a, b) of* $\operatorname{int} K$.

(3) $\Omega \cap \operatorname{int} K = \varnothing$. *In fact,* $\operatorname{int} K$ *contains no points in any ω-limit set.*

(4) *for some $B \subseteq C$, $\Omega = Q \cup B$.*

(5) *if $c \in C$, then c is an endpoint of a component of* $\operatorname{int} K$ *and c is isolated in $Q \cup C$.*

(6) *if (a, b) is a component of* $\operatorname{int} K$, *then either both a and b are in Q or one is in Q and the other in C*

(7) *for all $x \in K$, $\omega(x, f) = Q$.*

(8) *if* $\operatorname{int} K \neq \varnothing$, *then* $\overline{\operatorname{int} K} = K$.

(9) *if $B \neq \varnothing$, then $Q \subseteq \overline{B}$.*

(10) *C can have at most 2 points in any component of* $[0, 1] - Q$ *and at most one point in* $[0, \inf Q)$ *and* $(\sup Q, 1]$.

(11) *if Ω' is an ω-limit set which intersects Ω, then $Q \subseteq \Omega' \subseteq Q \cup C$.*

(12) *For each x either $Q \subseteq \omega(x, f) \subseteq K$ or $\omega(x, f) \subseteq \bigcup_{j=0}^{k} G^j$ for some $k \in N$.*

(13) *if $J_{\mathbf{n}} \cap B \neq \varnothing$, then $J_{\mathbf{n}} \subseteq \operatorname{int}(J_{\mathbf{n}|k})$ for all k.*

Proof. (1) Each equality below is easy to verify:

$$f(J_{\mathbf{n}}) = f\left(\bigcap_{k=1}^{\infty} J_{\mathbf{n}|k}\right) = \bigcap_{k=1}^{\infty} f(J_{\mathbf{n}|k}) = \bigcap_{k=1}^{\infty} A(\mathbf{n}) \restriction k = J_{A(\mathbf{n})}.$$

(2) Let $[a, b] = J_{\mathbf{n}}$. We have $f^j(J_{\mathbf{n}}) = J_{A^j(\mathbf{n})}$. The set $J_{A^j(\mathbf{n})}$ is disjoint from $J_{\mathbf{n}}$ by a).

(3) It follows from (2) that $\operatorname{int} K$ can contain no points of any ω-limit set.

(4) From (3) and the definition of simple system, $\Omega \subseteq K - \operatorname{int} K = Q \cup C$. Let $B = \Omega \cap C$.

(5) Since $c \notin S$, c is contained in a nondegenerate component of K. The definition of C shows c is an endpoint of the component. Let U be a neighborhood of c disjoint from Q. If c is a limit point of C, there exists $\mathbf{n} \in \mathscr{N}$, $k \in N$ such that $J_{\mathbf{n}}$ is a component of $\operatorname{int} K$ and $J_{\mathbf{n}|k} \subset U$. But $J_{\mathbf{n}|k} \cap \Omega$ is uncountable and therefore

contains points of Q. Thus c is a limit point of Q. Since Q is closed, $c \in Q$, a contradiction. Thus c is isolated in $Q \cup C$.

(6) Choose $\mathbf{n} \in \mathcal{N}$ such that $[a, b] = J_{\mathbf{n}}$. Then $a, b \in J_{\mathbf{n}\upharpoonright k}$ for all $k \in N$. But $J_{\mathbf{n}\upharpoonright k} \cap Q \neq \varnothing$, while $(a, b) \cap Q = \varnothing$. Thus either a or b must be a limit point of Q and therefore in Q.

(7) Since K is invariant, $\omega(x, f) \subset K$. By (3), int $K \cap \omega(x, f) = \varnothing$. It follows from (2) and (5) that $C \cap \omega(x, f) = \varnothing$. Thus $\omega(x, f) \subset Q$. To establish the reverse inclusion let $q \in Q$, U a neighborhood of q, and $s \in S \cap U$. There exists $\mathbf{n} \in \mathcal{N}$ such that $\{s\} = J_{\mathbf{n}}$, and there exists $k \in N$ such that $J_{\mathbf{n}\upharpoonright k} \subset U$. Choose $\mathbf{m} \in \mathcal{N}$ such that $x \in J_{\mathbf{m}}$. There exists $j \in N$ such that $A^j(\mathbf{m}) \upharpoonright k = \mathbf{n} \upharpoonright k$. Thus $f^j(J_{\mathbf{m}}) \subset f^j(J_{\mathbf{m}\upharpoonright k}) = J_{\mathbf{n}\upharpoonright k}$, and $f^j(x) \in U$. Repetitions of this argument show $q \in \omega(s, f)$.

(8) Let U be an open set intersecting K. Suppose $U \cap \text{int } K = \varnothing$. By (5) $U \cap K = U \cap Q$. There exists $k_0 \in N$, $\mathbf{m} \in \mathcal{N}$ such that $J_{\mathbf{m}\upharpoonright k} \subset U$ for all $k \geq k_0$. Let (a, b) be a component of int K, $[a, b] = J_{\mathbf{n}}$. Choose $j \in N$ such that $f^j(J_{\mathbf{m}\upharpoonright k_0}) = J_{\mathbf{n}\upharpoonright k_0}$. For each $k \geq k_0$ there exists $J \in \mathcal{J}$ such that $J \subset J_{\mathbf{m}\upharpoonright k}$ and $f^j(J) \supset J_{\mathbf{n}}$. Inductively we obtain a sequence $\{J_i\} \subset \mathcal{J}$ such that $J_{i+1} \subset J_i$ and $f^j(J_i) \supset J_{\mathbf{n}}$ for all $i \in N$. Since f^j is a continuous function and $J_{\mathbf{n}}$ is a nondegenerate interval, the lengths of the intervals J_i cannot approach 0. Thus $\bigcap_{i=1}^{\infty} J_i$ is a nondegenerate interval I. It is clear that $I \subset U$ and that I is a component of K.

(9) Let $q \in Q$, $b \in B$ and U be a neighborhood of q. There exists $\mathbf{n} \in \mathcal{N}$ such that $b \in J_{\mathbf{n}}$ and there exists $k \in N$ and $\mathbf{m} \in \mathcal{N}$ such that $J_{\mathbf{m}\upharpoonright k} \subseteq U$. Choose $j \in N$ and $\mathbf{p} \in \mathcal{N}$ such that $\mathbf{p}_i = \mathbf{m}_i$ for all $i \leq k$ and $A^j(\mathbf{p}) = \mathbf{n}$. Then int $J_{\mathbf{p}}$ is a component of int K since $f^j(J_{\mathbf{p}}) = J_{\mathbf{n}}$ and $J_{\mathbf{n}}$ is non-degenerate. Choose $c \in J_p \cap \Omega$ such that $f^j(c) = b$. Since Q is an ω-limit set by (7), Q is invariant so $c \notin Q$. Thus $c \in B$ and the conclusion follows from $J_{\mathbf{p}} \subseteq U$.

(10) Observe first that if (a, b) is contiguous to Q and (a, c) is a component of int K with $b \neq c$, then c must be in C. Thus c is either an isolated point in Ω or $c \notin \Omega$. (Either situation can occur.) From this the assertion follows easily. This result was proved by different methods in **[Sh$_1$]**.

(11) Suppose Ω' is an ω-limit set such that $\Omega \cap \Omega' \neq \varnothing$. Let $z \in \Omega \cap \Omega'$. Thus Ω' is infinite. Let Q' and K' have the obvious meanings. Since $z \in K'$, $\omega(z, f) = Q'$ by 3.1(7). Since $z \in K$, $\omega(z, f) = Q$. Thus $Q' = Q$. Suppose $b \in \Omega' - \Omega$. Then $\omega(b, f) = Q'$. For each $\mathbf{n} \in \mathcal{N}$, and $k \in N$, $J_{\mathbf{n}\upharpoonright k}$ intersects Q in an

uncountable set. Thus $\operatorname{orb} b$ intersects the *interior* of $J_{\mathbf{n}\upharpoonright k}$. Thus, if $b \in \omega(x, f)$, then $\omega(x, f)$ intersects $J_{\mathbf{n}\upharpoonright k}$. This is true for all $\mathbf{n} \in \mathscr{N}$ and $k \in N$. Thus $\omega(x, f) \subset K$, so $b \in K$. Since $\operatorname{int} K$ contains no ω-limit points, $b \in C$. Hence, *if Ω' is an infinite ω-limit set that intersects another infinite ω-limit set Ω, then Ω' and Ω have a common simple system and differ at most on a countable set* contained in $C = (K - \operatorname{int} K) - Q$. This improves a result in [$\mathbf{Sh}_1$].

(12) If $\operatorname{orb} x$ hits K, then we have $\omega(x, f) = Q$ by part (7). So we may assume $\operatorname{orb} x \subseteq \bigcup_{j=0}^{\infty} G^j$. If $\operatorname{orb} x$ hits infinitely many members of $\mathscr{G}$, then $\omega(x, f) \cap Q \neq \varnothing$ and by part (11), $Q \subseteq \omega(x, f) \subseteq K$. Hence, $\operatorname{orb} x \subseteq \bigcup_{j=0}^{k} G^j$ for some k. Then either $\omega(x, f) \subseteq \bigcup_{j=0}^{k} G^j$ or $\omega(x, f)$ contains a boundary point of $\bigcup_{j=0}^{k} G^j$ in which case $\omega(x, f) \cap K \neq \varnothing$ and $Q \subseteq \omega(x, f)$ by part (11).

(13) Suppose $J_{\mathbf{n}} = [c, d]$ where c is an isolated point of Ω. Then $d \in Q$. Since $J_{\mathbf{n}\upharpoonright k} \cap Q$ is uncountable for any k, it follows that d is interior to each $J_{\mathbf{n}\upharpoonright k}$. Let $\omega(x, f) = \Omega$. Fix k. Let V be an open interval about c which misses d and all $J_{\mathbf{m}\upharpoonright k}$ for $\mathbf{m} \upharpoonright k \neq \mathbf{n} \upharpoonright k$. Pick $f^i(x) \in J_{\mathbf{n}\upharpoonright k}$. Because $J_{\mathbf{m}\upharpoonright k}$ is periodic $f^j(x) \in \bigcup\{J_{\mathbf{m}\upharpoonright k} : \mathbf{m} \in \mathscr{N}\}$ for all $j \geq i$. Then there are infinitely many values of j for which $f^j(x) \in V$ and, hence, $f^j(x) \in J_{\mathbf{n}\upharpoonright k}$. By part (2) there is at most one orbit point in $[c, d]$. Therefore, there is some $f^t(x) \in J_{\mathbf{n}\upharpoonright k} \cap [0, c)$ and c is interior to $J_{\mathbf{n}\upharpoonright k}$.

The next three lemmas give some information about the intermingling of ω-limit sets, certain orbits and members of simple systems. These together with the topological Lemma 3.5 are the foundation of the proofs of Theorems 3.6 and 3.7.

LEMMA 3.2. *Let $\mathscr{J}$ be a simple system for Ω relative to f. Suppose Ω_1 is an ω-limit set containing Q and Ω_2 is any ω-limit set different from Ω_1, say $\Omega_1 = \omega(x, f)$ and $\Omega_2 = \omega(y, f)$. Then there exists $J \in \mathscr{J}$ and $i \in N$ such that J is between $f^i(x)$ and $f^i(y)$.*

Proof. Suppose first that $\Omega_2 \cap K = \varnothing$. Since Ω_2 is closed, there exist neighborhoods U_1 of Ω_1, and U_2 of Ω_2 such that $U_1 \cap U_2 = \varnothing$. Choose $q \in U_1$, q a bilateral limit point of Q. There exist $L, M \in \mathscr{J}$ such that $L \subset U_1$, $M \subset U_1$, and q is between L and M. There exists $i_0 \in N$ such that $f^i(y) \in U_2$ for all $i \geq i_0$. Choose $i \geq i_0$ such that $f^i(x)$ is between L and M. Then one of the two sets L and M is between $f^i(x)$ and $f^i(y)$.

Now suppose $\Omega_2 \cap K \neq \varnothing$. By the proof of part 11 of 3.1, Ω_1, Ω_2 and Ω have the same simple system $\mathscr{J}$. Then $Q \subseteq \Omega_2$. Since $\Omega_1 - \Omega_2$ or $\Omega_2 - \Omega_1$ is non void, say $\Omega_2 - \Omega_1 \neq \varnothing$, we may choose $c \in C$ with $c \in \Omega_2 - \Omega_1$. Then there exists $J_{\mathbf{n}}$ with c as an endpoint. Hence, say $J_{\mathbf{n}} = [c, d]$ where $d \in Q$. Choose $k \in N$ so that no member of $\mathscr{J}$ is contained in $[0, c] \cap J_{\mathbf{n}\upharpoonright k}$.

By 3.1(3), orb x is eventually out of $[0, d] \cap J_{\mathbf{n}\upharpoonright k}$. Hence there is an s such that $i \geq s$ and $f^i(x) \in J_{\mathbf{n}\upharpoonright k}$ imply $f^i(x) > d$. However, orb y is frequently in $J_{\mathbf{n}\upharpoonright k} \cap [0, c]$ so there is $j > s$ such that $f^j(y) \in J_{\mathbf{n}\upharpoonright k} \cap [0, c]$. In case $f^j(x) \in J_{\mathbf{n}\upharpoonright k}$ we have $f^j(x) > d$ and $[c, d] \subseteq (f^j(y), f^j(x))$. Hence, there exists $t > k$, such that $J_{\mathbf{n}\upharpoonright t} \subseteq (f^j(y), f^j(x))$. In case $f^j(x) \in J_{\mathbf{m}\upharpoonright k}$ where $\mathbf{m} \upharpoonright k \neq \mathbf{n} \upharpoonright k$ we may use the function A to find $v \geq s$ to obtain a member of $\mathscr{J}$ between $f^v(x)$ and $f^v(y)$.

We say that two non-void mutually disjoint subsets X and Y of some real interval are *intertwined* if between each point of X (resp. Y) and each point not in X (resp. Y) there exist points of both X and Y.

LEMMA 3.3. *Let $\mathscr{J}$ be a simple system for Ω relative to f. Let Ω_1 and Ω_2 be different ω-limit sets each containing Q. If $\Omega_1^* = \{x : \omega(x, f) = \Omega_1\}$ and $\Omega_2^* = \{y : \omega(y, f) = \Omega_2\}$, then Ω_1^* and Ω_2^* are intertwined.*

Proof. If $x \in \Omega_1^*$ (resp. $x \in \Omega_2^*$) and $y \notin \Omega_1^*$ (resp. $y \notin \Omega_2^*$), then Lemma 3.2 gives a member $J \in \mathscr{J}$ between $f^i(x)$ and $f^i(y)$ for some $i \in N$. If $Q \subseteq \omega(z, f)$, then orb z hits each member of $\mathscr{J}$. If this were not the case there would exist a k such that orb z $\cap \bigcup\{J_{\mathbf{n}\upharpoonright k} : \mathbf{n} \in \mathscr{N}\} = \varnothing$ by the periodicity of the intervals in $\mathscr{J}$. This would mean that $\omega(z, f)$ is disjoint from $\bigcup\{\operatorname{int} J_{\mathbf{n}\upharpoonright k} : \mathbf{n} \in \mathscr{N}\}$ and $\omega(z, f) \cap Q$ is finite, a contradiction.

Obviously, orb $x \subseteq \Omega_1^*$ and orb $y \subseteq \Omega_2^*$. Therefore J hits both Ω_1^* and Ω_2^*. Let $w \in \Omega_1^* \cap J$. Then pick v between x and y such that $f^i(v) = w$. Clearly $v \in \Omega_1^*$. Likewise there is a point of Ω_2^* between x and y.

LEMMA 3.4. *Let $\mathscr{J}$ be simple system for Ω relative to f. Let*

$$X = \{x : \omega(x, f) \cap K = \varnothing\},$$
$$Y = \{x : Q \subseteq \omega(x, f) \subseteq K\}.$$

Then X and Y are intertwined.

Proof. Obviously X and Y are nonvoid and mutually disjoint. By 3.1 part 12 $[0, 1] = X \cup Y$. Hence it suffices to show that if $x \in X$ and $y \in Y$ then there are points of both X and Y between x and y. Apply Lemma 3.2 to $\omega(x, f)$ and $\omega(y, f)$ to get $i \in N$ and $J \in \mathscr{J}$ such that J is between $f^i(x)$ and $f^i(y)$. But J intersects Q and J contains a member of $\mathscr{G}$ which in turn contains a periodic point. Pick $a \in J \cap X$ and $b \in J \cap Y$. Pick z and ω between x and y so that $f^i(z) = a$ and $f^i(\omega) = b$. Clearly $z \in X$ and $b \in Y$.

The next lemma is probably known in some form or another. Its proof is straightforward and will be omitted.

LEMMA 3.5. *Suppose X and Y are intertwined. Then*

(a) *X and Y have the same boundary P, which is a perfect set.*
(b) *$X \cap P$ and $Y \cap P$ are dense in P.*
(c) *Each of X and Y is dense-in-itself and bilaterally dense-in-itself when restricted to* $(\inf X \cup Y, \sup X \cup Y)$.

Now we present two consequences of the preceding lemmas.

THEOREM 3.6. *If f has an infinite ω-limit set, then ω_f is not Baire* 1.*

Proof. If $h(f) > 0$ we know that ω_f is not Baire 1 by Theorem 2.2. So we may assume that f is a 2^∞-function with an infinite ω-limit set Ω. Let $\mathscr{J}$ be a simple system for Ω relative to f. Let X and Y be the sets of Lemma 3.4. Let P be the common boundary of X and Y by 3.5.

Suppose $x \in X$. Then $\omega(x, f) \subseteq \bigcup_{j=0}^{k} G^j$ for some k by 3.1 part 12. Let $\varepsilon = \operatorname{dist}(\omega(x, f), K)$. Then for all $y \in Y$ we must have $H(\omega(x, f), \omega(y, f)) \geq \varepsilon$. Therefore $\omega_f \upharpoonright P$ is discontinuous at each point of the set $X \cap P$ which is dense in P. Hence ω_f is not Baire* 1.

Example 2.10 shows that the converse of Theorem 3.6 is false. We have been unable to characterize Baire* 1 functions in terms of their ω-limit sets.

THEOREM 3.7. *If f has an infinite ω-limit set with an isolated point, then ω_f is not Baire 1.*

Proof. If $h(f) > 0$, then ω_f is not Baire 1 by Theorem 2.2. Hence, we may assume that f is a 2^∞-function. Let Ω be an infinite ω-limit set with an isolated point. Let $\mathscr{J}$ be a simple system for Ω relative to f. Then $Q \subseteq \Omega$ and by Lemma 3.3 Q^* and Ω^* are intertwined. And by lemma 3.5 $P \cap Q^*$ and $P \cap \Omega^*$ are dense in the perfect set P. If ω_f is Baire 1, then clearly $P \cap \Omega^*$ and $P \cap Q^*$ are both G_δ sets. This is impossible by the Baire Category theorem. Hence ω_f is not Baire 1.

The converse of Theorem 3.7 is also true.

THEOREM 3.8. *If every infinite ω-limit set for f is perfect, then ω_f is Baire* 1.

Proof. Suppose M is a perfect ω-limit set for f. If int $M \neq \varnothing$, then $h(f) > 0$ (see **[FShS**; Theorem A]). But if $h(f) > 0$, then f has a countably infinite ω-limit set **[HOLE]**, a contradiction. Therefore, each ω-limit set for f is either a Cantor set or a 2^k-cycle.

By Theorem 2.7 it suffices to show that ω_f is l.s.c. (1) and thus to show the set $A = \{x : \omega(x, f) \cap W \neq \varnothing\}$ is an F_σ set whenever W is an open interval.

Choose a sequence of closed intervals $\{W_n\}_{n=1}^{\infty}$ such that $W = \bigcup_{n=1}^{\infty} W_n$. Form the set

$$E = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{m=1}^{\infty} \{x : f^{j+m2^k}(x) \in W_n\}$$

which is clearly an F_σ set.

We will show $A = E$. The inclusion $E \subseteq A$ is clear. Suppose now that $x \in A$. If $\omega(x, f)$ is a cycle, then obviously $x \in E$. Suppose, then, that $\omega(x, f)$ is a Cantor set. Let $\mathscr{J} = \{J_{\mathbf{n}|k} : \mathbf{n} \in \mathscr{N}, k \in N\}$ be a simple system for $\omega(x, f)$ relative to f. Thus $\omega(x, f) = Q$. Since $Q \cap W \neq \varnothing$, we see from Proposition 3.1, that there exists $\mathbf{n} \in \mathscr{N}$ and $k \in N$ such that $J_{\mathbf{n}|k} \subseteq W$. Since $J_{\mathbf{n}|k}$ is closed, there exists $n \in N$ such that $J_{\mathbf{n}|k} \subseteq W_n$. Since orb x intersects all intervals in $\mathscr{J}$, there exists $j \in N$ such that $f^j(x) \in J_{\mathbf{n}|k}$. But $J_{\mathbf{n}|k}$ is periodic of period 2^k so $f^{j+m2^k}(x) \in J_{\mathbf{n}|k} \subseteq W_n$ for all $m \in N$, that is, $A \subseteq E$. Thus $A = E$.

Combining Theorems 3.7 and 3.8 we obtain

THEOREM 3.9. *ω_f is Baire* 1 *if and only if any infinite ω-limit set for f is perfect.*

If Ω is any ω-limit set for f, then $\Omega^* = \{x : \omega(x, f) = \Omega\}$ is a level set for the function ω_f. Since ω_f is always Baire 2 each Ω^*

is an $F_{\sigma\delta}$ set. This is well-known **[Sh2]**. In case ω_f is Baire 1 (or equivalently all infinite ω-limit sets are perfect) then clearly each Ω^* is a G_δ. What follows from Lemmas 3.3 and 3.5 and Theorem 3.7 is that *if* $h(f) = 0$ *and* Ω_1 *and* Ω_2 *are different and intersect, then* Ω_1^* *and* Ω_2^* *can't both be* G_δ *sets.*

We close this section with some applications of the previous results to non-chaotic functions. The first is a new characterization of non-chaotic functions.

COROLLARY 3.10. *Let* $h(f) = 0$. *Then* f *is non-chaotic if and only if there are periodic points between any two points of any infinite* ω*-limit set.*

Proof. Obviously we can assume that f is a 2^∞-function. Suppose Ω is any infinite ω-limit set and $\mathscr{J}$ is a simple system for Ω relative to f. Since f^{2^k} interchanges each $J_{\mathbf{n}|k,0}$ and $J_{\mathbf{n}|k,1}$ there is a periodic point between them. In general then each interval contiguous to K contains a periodic point. By 3.1 part (2) no point in int K is periodic. Thus if a component of int K has both end points in Ω, then there is no periodic point between them. If each component of int K has an endpoint not in Ω, then each two points in Ω can be separated by a periodic point. Hence, the stated condition is equivalent to no component of int K having both end points in Ω. This in turn is equivalent to each two points in any infinite Ω being separated by periodic intervals. But this is a known characterization of non-chaotic functions **[FShS]**.

COROLLARY 3.11. *If* f *is non-chaotic, then any infinite* ω*-limit set for* f *is perfect and* ω_f *is Baire* 1.

Proof. That any infinite ω-limit set is perfect follows from the argument in the proof of Corollary 3.10. That ω_f is Baire 1 now follows from Theorem 3.8.

We summarize our results relating the Baire class of ω_f to the types of ω-limit sets possessed by f.

(1) ω_f is continuous if and only if each ω-limit set for f has cardinality one or two and the union of all ω-limit sets is connected.

(2) ω_f is Baire 1 if and only if all ω-limit sets are either finite sets or Cantor sets.

(3) ω_f is Baire 2 but not in $\mathscr{B}_1$ if and only if f possesses an infinite ω-limit set with isolated points.

Regarding the Baire class of ω_f and forms of chaos we have the following chain of implications

ω_f is continuous $\Rightarrow f$ is nonchaotic $\Rightarrow \omega_f$ is Baire 1 $\Rightarrow h(f) = 0$.

None of the reverse implications is valid.

4. Construction of examples. In this section we will attempt to reverse our construction of simple systems from a given 2^∞-function. We will begin with a certain collection of sets coded by finite tuples of 0's and 1's with properties similar to those of simple systems and produce a 2^∞-function f and corresponding infinite ω-limit set Ω relative to which this collection is a simple system. The results of this construction are stated as Theorem 4.1. We then modify the construction to obtain a 2^∞-function possessing an infinite ω-limit set with isolated points.

A system $\mathscr{J} = \{J_{\mathbf{n}\upharpoonright k} : \mathbf{n} \in \mathscr{N}, k \in N\}$ of non-degenerate closed subintervals of [0,1] is called a *primitive system* if

(P_1) J_1 and J_0 are disjoint with $0 \in J_1$ and $1 \in J_0$. In general $J_{\mathbf{n}\upharpoonright k,1}$ and $J_{\mathbf{n}\upharpoonright k,0}$ are disjoint subsets of $J_{\mathbf{n}\upharpoonright k}$ containing the left and right endpoints of $J_{\mathbf{n}\upharpoonright k}$ respectively.

(P_2) If $\bigcap_{k=1}^\infty J_{\mathbf{n}\upharpoonright k}$ is a singleton, then $\bigcap_{k=1}^\infty J_{A(\mathbf{n})\upharpoonright k}$ is a singleton too.

Observe that in comparison to the arbitrary simple system of §3 there is now an order in that $J_{\mathbf{n}\upharpoonright k,1}$ always lies to the left of $J_{\mathbf{n}\upharpoonright k,0}$ and that the endpoints of each $J_{\mathbf{n}\upharpoonright k}$ must belong to K.

Suppose that $\mathscr{J} = \{J_{\mathbf{n}\upharpoonright k} : \mathbf{n} \in \mathscr{N}, k \in N\}$ is a primitive system. Put

$$K = \bigcup_{\mathbf{n}\in\mathscr{N}} \bigcap_{k=1}^{\infty} J_{\mathbf{n}\upharpoonright k}$$

and

$$J_{\mathbf{n}} = \bigcap_{k=1}^{\infty} J_{\mathbf{n}\upharpoonright k}$$

Then K is perfect and $\{J_{\mathbf{n}} : \mathbf{n} \in \mathscr{N}\}$ is the set of components of K. Let S consist of all those x for which $\{x\}$ is a component of K.

Define f on S as follows: if $\{x\} = J_{\mathbf{n}}$, then $\{f(x)\} = J_{A(\mathbf{n})}$.

First we show that for each $x \in \overline{S}$, $\lim_{s\to x} f(s)$ exists. Let $x \in \overline{S}$. Then there exists $\mathbf{n} \in \mathscr{N}$ such that one of the following is true: $\{x\} = J_{\mathbf{n}}$; or $[x, b] = J_{\mathbf{n}}$ for some $b > x$ with $x \in \overline{S} - S$; or

$[a, x] = J_{\mathbf{n}}$ for some $a < x$ with $x \in \overline{S} - S$. Let $[a_k, b_k] = J_{\mathbf{n}|k}$ and $[c_k, d_k] = J_{A(\mathbf{n})|k}$ and $[c, d] = J_{A(\mathbf{n})}$. There are several cases

(*Case* 1). $\mathbf{n} = \mathbf{1}$. Then $J_{\mathbf{n}} = [0, x]$ where $0 \le x$ and $x \in \overline{S}$. Let $s \in S \cap (x, b_k)$. Pick $m > k$ so that $[0, x] \subseteq J_{\mathbf{n}|m,1}$ and $s \in J_{\mathbf{n}|m,0}$. Let $[c, 1] = J_{A(\mathbf{n})} = J_{\mathbf{0}}$. Then $f(s) \in J_{\mathbf{0}|m,1}$ so that $c_k < f(s) < c$. Since $b_k \to x$ and $c_k \to c$, we have $\lim_{s \to x^+} f(s) = c$.

(*Case* 2). $\mathbf{n} = \mathbf{0}$. This is similar to Case 1 and we get $\lim_{s \to x^-} f(s)$ exists again.

(*Case* 3). $J_{\mathbf{n}} = [a, x]$ where $a \le x$ and $\mathbf{n}$ is not constant. Suppose $n_i = 0$ and let $k > i$. Let $s \in (x, b_k)$. Then there exists $m > k$ such that $[a, x] \subseteq J_{\mathbf{n}|m,1}$ and $s \in J_{\mathbf{n}|m,0}$. Since $A(\mathbf{n} | m, j) = (A(\mathbf{n} | m), j)$ we have $f(s) \in J_{A(\mathbf{n})|m,0}$ so that $d < f(s) < d_k$. Since $b_k \to x$ and $d_k \to d$ it follows that $\lim_{s \to x^+} f(s) = d$.

(*Case* 4). $J_{\mathbf{n}} = [x, b]$ where $x \le b$ and $\mathbf{n}$ is not constant. Similarly we can prove that $\lim_{s \to x^-} f(s) = c$.

(*Case* 5). $J_{\mathbf{n}} = \{x\}$ and $\mathbf{n}$ is not constant. Using Cases 3 and 4 we obtain $\lim_{s \to x^+} f(x) = d = c = \lim_{s \to x^-} f(s)$ so that $\lim_{s \to x} f(s) = f(x)$.

Since no non-degenerate component contains a point of S interior to it we conclude from the above cases that $\lim_{s \to x} f(s)$ exists for all $x \in \overline{S}$. This means that f is uniformly continuous on S and can be extended to be continuous on $\overline{S}$.

Next we extend f continuously to K as follows: Suppose $J_{\mathbf{n}} = [a, b]$ with $a \ne b$. Then a or b belongs to $\overline{S}$. Suppose, for example, that $a \in \overline{S}$ and $b \notin \overline{S}$. Let $J_{A(\mathbf{n})} = [c, d]$ where $c \le d$. From Cases 2 and 4 above $f(a) = c$. Define $f(b) = d$. Similarly with the other possibility. Now having defined f at both endpoints of $J_{\mathbf{n}}$ we define it on $J_{\mathbf{n}}$ by linearity.

Finally we define f on the components of $[0, 1] - K$ by linearity. It is easily checked that this extension is continuous.

We now examine the behavior of f on the components of $[0, 1] - K$.

Let $G_{\mathbf{n}|k}$ be the open interval between $J_{\mathbf{n}|k,1}$ and $J_{\mathbf{n}|k,0}$ and let G denote the interval between J_1 and J_0. Thus $J_{\mathbf{n}|k}$ is a disjoint union of $J_{\mathbf{n}|k,1}$, $G_{\mathbf{n}|k}$, and $J_{\mathbf{n}|k,0}$ by condition (P_1). It also follows from (P_1) that for every $\mathbf{n} \in \mathscr{N}$, $k \in N$, $G_{\mathbf{n}\upharpoonright k}$ is a component of the complement of K.

Notice that these definitions agree with those of G and $G_{\mathbf{n}\upharpoonright k}$ found in §3. Also, put $\mathscr{G} = \{G\} \cup \{G_{\mathbf{n}\upharpoonright k} : \mathbf{n} \in \mathscr{N}, k \in N\}$, $G^0 = G$ and $G^k = \bigcup\{G_{\mathbf{n}\upharpoonright k} : \mathbf{n} \in \mathscr{N}\}$.

We will say that each $G_{\mathbf{n}\upharpoonright k}$ has *rank* k.

Suppose $G_{\mathbf{n}\upharpoonright k} = (a, b)$. Then a is the right endpoint of the component $J_{\mathbf{n}\upharpoonright k, \mathbf{10}}$ and b is the left endpoint of the component $J_{\mathbf{n}\upharpoonright k, \mathbf{01}}$. (Note: If $a \in S$, consider a as both a right and left endpoint.)

If $\mathbf{n} \upharpoonright k \neq \mathbf{1} \upharpoonright k$, then Cases 3 and 4 above show that $f(a)$ is the right endpoint of $J_{A(\mathbf{n}\upharpoonright k, \mathbf{10})} = J_{A(\mathbf{n})\upharpoonright k, \mathbf{10}}$ and $f(b)$ is the left endpoint of $J_{A(\mathbf{n}\upharpoonright k, \mathbf{01})} = J_{A(\mathbf{n})\upharpoonright k, \mathbf{01}}$. Therefore, f is increasing on $G_{\mathbf{n}\upharpoonright k}$ and

$$(\alpha) \qquad f(G_{\mathbf{n}\upharpoonright k}) = G_{A(\mathbf{n})\upharpoonright k}$$

On the other hand if $\mathbf{n} \upharpoonright k = \mathbf{1} \upharpoonright k$ Cases 3 and 4 again show that $f(b)$ is the left endpoint of $J_{\mathbf{0}\upharpoonright k, \mathbf{1}}$ and $f(a)$ is the right endpoint of $J_{\mathbf{0}\upharpoonright k, 010}$. Hence f is decreasing on $G_{\mathbf{1}\upharpoonright k}$ and

$$(\beta) \qquad f(G_{\mathbf{1}\upharpoonright k}) = \operatorname{int}(J_{\mathbf{0}\upharpoonright k, 1} \cup G_{\mathbf{0}\upharpoonright k} \cup J_{\mathbf{0}\upharpoonright k, 01}).$$

In particular, $f(G_{\mathbf{1}\upharpoonright k})$ contains intervals of arbitrarily high rank but none with rank lower than k.

Finally it is easy to see that f is decreasing on G with slope < -1 and $f(G) \supsetneqq G$.

Moreover, similar arguments to those above show that when $\mathbf{n} \upharpoonright k \neq \mathbf{1} \upharpoonright k$ the endpoints of $J_{\mathbf{n}\upharpoonright k}$ map onto the endpoints of $J_{A(\mathbf{n})\upharpoonright k}$. It follows that when $\mathbf{n} \upharpoonright k \neq \mathbf{1} \upharpoonright k$, $f(J_{\mathbf{n}\upharpoonright k}) = J_{A(\mathbf{n})\upharpoonright k}$, .

If $[0, a] = J_{\mathbf{1}\upharpoonright k}$, then $f(0) = 1 \in J_{\mathbf{0}\upharpoonright k, 0}$ and $f(J_{\mathbf{1}\upharpoonright k, 0}) = J_{\mathbf{0}\upharpoonright k, 1}$. The interval $J_{\mathbf{1}\upharpoonright k}$ is the union of $J_{\mathbf{1}}$ and all the sets $J_{\mathbf{n}\upharpoonright m}$ and $G_{\mathbf{n}\upharpoonright m}$ where $m \geq k$ and $n_i = 1$ for all $i \leq k$. However, by (α) and (β) and the above relations each of these sets is mapped into $J_{\mathbf{0}\upharpoonright k}$. It follows that $f(J_{\mathbf{1}\upharpoonright k}) = J_{\mathbf{0}\upharpoonright k}$. Therefore, for all $\mathbf{n}$, j, k and i

$$(\gamma) \qquad f(J_{\mathbf{n}\upharpoonright k}) = J_{A(\mathbf{n})\upharpoonright k}$$

and in general

$$f^i(J_{\mathbf{n}\upharpoonright k}) = J_{A^i(\mathbf{n})\upharpoonright k}$$

Now fix $k \in N$. The calculations above show that f is linear on each of the intervals $G_{\mathbf{n}\upharpoonright k}$ and that $f^i(G_{\mathbf{0}, \upharpoonright k}) = G_{A^j(\mathbf{0}, \upharpoonright k)}$ for $j < 2^k$ while $f^{2^k}(G_{\mathbf{0}\upharpoonright k})$ contains $G_{\mathbf{0}\upharpoonright k}$ properly. Thus f^{2^k} is linear on $G_{\mathbf{0}\upharpoonright k}$

with slope $\lambda > 1$. It follows that f *has a periodic point* p_k *of period* 2^k *in* G_{01k}.

Let $x \neq p_k$ and $x \in G_{01k}$. Then $|f^{2^k}(x) - f^{2^k}(p_k)| = \lambda|x - p_k| > |x - p_k| = |x - f^{2^k}(p_k)|$. Thus $f^{2^k}(x) \neq x$. This shows that p_k is the only periodic point of order 2^k in G_{01k}. Since $f^j(G_{01k})$ is disjoint form G_{01k} for $j < 2^k$, p_k is in fact *the only periodic point of any period in* G_{01k}. Moreover it is easily seen that there is only one fixed point p_0 in G. Since it is clear that f has no periodic points in K, this means that f *is a* 2^∞*-function.*

Now put $\Omega = Q = \overline{S}$. Then it is easily seen that $\mathscr{J}$ is a simple system for Q relative to the 2^∞-function f by verifying conditions (a) through (d) of §3.

Let E_n be the set of points which are eventually periodic of order 2^n. Then $E_n = \bigcup_{m=1}^\infty f^{-m}(p_n)$. From ($\alpha$) and ($\beta$) it follows that if $M \in \mathscr{G}$ and $m \in N$ then $f^{-m}(M)$ is the union of countably many open intervals each contained in some member of $\mathscr{G}$. Moreover, upon each of these intervals, f^m is linear and non-constant. Now suppose E_n were uncountable. Then some $f^{-m}(p_n)$ is uncountable and there exists an interval T such that $f^m \upharpoonright T$ is linear and $T \cap f^{-m}(p_n)$ is uncountable. This implies f^m is constant somewhere, a contradiction. Therefore E_n *is countable.* We also remark that E_n *is a* G_δ *set* because it can be represented as

$$E_n = \bigcap_{j=1}^\infty \bigcap_{m=1}^\infty \bigcup_{k=m}^\infty \{x : |f^k(x) - p_n| < 1/j\}.$$

Hence, E_n *is nowhere dense in each perfect set* by the Baire Category Theorem.

Now put $E = \bigcup_{n=1}^\infty E_n$. Then *the set* E *of eventually periodic points is countable* From 3.1 we know that each member of $\mathscr{G}$ hits E and $E \cap K = \varnothing$.

Moreover, since the derivative of f^{2^k} on G_{01k} is λ and $\lambda > 1$ the point p_k is a repelling periodic point for $k \geq 1$. In addition it is easily verified that p_0 is repelling. Hence, *any asymptotically periodic point is eventually periodic.* That is, if $\lim_{i\to\infty} f^{j+i2^k}(x) = p_k$ then $f^m(x) = p_k$ for some m. By 3.1, part 12, we know for each x, $\omega(x, f) = Q$ or $\omega(x, f) \subseteq \bigcup_{j=0}^k G^j$ for some k. In the latter case $\omega(x, f)$ must be a cycle. Therefore it follows that *for each* x *either* $\omega(x, f) = Q$ *or* x *is eventually periodic.*

Now apply Lemmas 3.4 and 3.5 to E and its complement to obtain that E *is bilaterally dense-in-itself.*

Let V be any open subinterval of $G_{\mathbf{0}|k}$ and put $I = f^{-2^k}(V) \cap G_{\mathbf{0}|k}$. Then I is an open interval which f^{2^k} maps linearly onto V. Since f^{2^k} has slope $\lambda > 1$ on $G_{\mathbf{0}|k}$ the iterates $f^{i2^k}(I)$, $i = 1, 2, 3, \ldots$ will increase in length with magnification factor λ as long as they remain inside $G_{\mathbf{0}|k}$. Therefore there is a first j such that $f^{j2^k}(I)$ is not inside $G_{\mathbf{0}|k}$. Moreover, $f^{j2^k}(I)$ is open and intersects K.

If $\operatorname{int} K \neq \varnothing$, then since $\overline{\operatorname{int} K} = K$ by 3.1(8), $f^{j2^k}(I)$ hits $\operatorname{int} K$. But since $\operatorname{int} K \cap E = \varnothing$ by 3.1(7), $f^{j2^k}(I)$ contains an interval missing E. Then V contains an interval W missing E. On the other hand, if $\operatorname{int} K = \varnothing$, then $f^{j2^k}(I)$ hits Q and $f^{j2^k}(I)$ contains members of $\mathscr{G}$ of arbitrarily large rank. Since each of these hit E it follows that $V \cap E \neq \varnothing$.

Therefore, it follows that *if* $\operatorname{int} K \neq \varnothing$, *then* E *is nowhere dense; and if* $\operatorname{int} K = \varnothing$, *then* E *is everywhere dense.*

By Corollary 3.10 and its proof f *is chaotic if and only if some component of* $\operatorname{int} K$ *has both endpoints in* Q.

Since the only infinite ω-limit set for f is the Cantor set Q, we have ω_f is Baire 1 by Theorem 3.9 and ω_f is not Baire* 1 by Theorem 3.6. Note that ω_f is continuous on a dense open set if $\operatorname{int} K \neq \varnothing$.

We summarize the foregoing results in the following theorem.

THEOREM 4.1. *Suppose* $\mathscr{J} = \{J_{\mathbf{n}|k} : \mathbf{n} \in \mathscr{N}, k \in N\}$ *is a primitive system with* $K = \bigcup_{\mathbf{n}\in\mathscr{N}} \bigcap_{k=1}^{\infty} J_{\mathbf{n}|k}$.

Then $K - \operatorname{int} K = Q \cup C$ *where* Q *is a Cantor set and* C *is a countable set disjoint from* Q.

Moreover, there exists a 2^∞*-function* f *such that*

(1) $\mathscr{J}$ *is a simple system for* Q *relative to* f.

(2) *For each* k, f *has a single periodic orbit of order* 2^k.

(3) *For each* x, *either* x *is eventually periodic or* $\omega(x, f) = Q$.

(4) *The set* E *of eventually periodic points is countable and bilaterally dense-in-itself. Also* E *is everywhere dense in* $[0, 1]$ *whenever* $\operatorname{int} K = \varnothing$; *and* E *is nowhere dense if* $\operatorname{int} K \neq \varnothing$.

(5) f *is chaotic if and only if some component of* $\operatorname{int} K$ *has both end points in* Q.

(6) ω_f *is Baire* 1 *but not Baire** 1 *and* ω_f *is continuous on a dense open set if* $\operatorname{int} K \neq \varnothing$.

Now we would like to apply the foregoing construction to several examples which illustrate various possibilities. We can either explicitly specify the primitive system (in which case the details may be cumbersome) or we can indirectly obtain the primitive system by specifying the sequences in $\mathcal{N}$ corresponding to the set of singleton components of K. Theorem 4.2 below indicates that there is an abundance of such correspondences possible.

THEOREM 4.2. *Suppose $\mathcal{S} \subseteq \mathcal{N}$ such that* (1) $\mathcal{N} - \mathcal{S}$ *is countable and* (2) $A(\mathcal{S}) \subseteq \mathcal{S}$.

Then there exists a primitive system $\{J_{\mathbf{n}|k} : \mathbf{n} \in \mathcal{N}, k \in N\}$ *such that* $\mathbf{n} \in \mathcal{S}$ *if and only if* $J_{\mathbf{n}}$ *is a singleton.*

Moreover, the function f given by Theorem 4.1 *is non-chaotic if and only if* $\mathbf{n}$ *is eventually constant for all* $\mathbf{n} \notin \mathcal{S}$.

Proof. Case 1. $\mathcal{N} = \mathcal{S}$. Let Γ by the Cantor set in [0,1] and $\{B_m\}_{m=1}^{\infty}$ be an enumeration of the components of $[0, 1]-\Gamma$. Suppose $B_1 = (a, b)$. Put $J_1 = [0, a]$ and $J_0 = [b, 1]$. Having chosen $J_{\mathbf{n}|k}$ for $k \leq m$, consider a fixed $J_{\mathbf{n}|m} = [c, d]$. Let $B_j = (a, b)$ be the first member of $\{B_n\}_{n-1}^{\infty}$ inside $J_{\mathbf{n}|m}$. Put $J_{\mathbf{n}|m,1} = [c, a]$ and $J_{\mathbf{n}|m,0} = [b, d]$. It is easily checked that $\{J_{\mathbf{n}|k} : \mathbf{n} \in \mathcal{N}, k \in N\}$ is a primitive system.

Case 2. $\mathcal{N} \neq \mathcal{S}$. In this case we see from (2) that $\mathcal{N} - \mathcal{S}$ is countably infinite. There is a homeomorphism h from $\mathcal{N}$ (considered as $\{0, 1\}^N$ with the product topology), onto the Cantor set Γ such that $h(\mathbf{1}) = 0$ and $h(\mathbf{0}) = 1$. Let $\{v_k\}_{k=1}^{\infty}$ be an enumeration of $\mathcal{N} - \mathcal{S}$. Let $X = [0, 1] \cup \bigcup_{k=1}^{\infty} \{h(v_k)\} \times [0, 2^{-k}]$. Thus X consists of the unit interval together with vertical segments of length 2^{-k} over $h(v_k)$. We shall define the required primitive system in such a way that these vertical segments "transform" into component intervals of int K.

For each k, let $x_k = h(v_k)$. For $(x, y) \in X$, let $g(x, y) = x + y + \sum_{x_k < x} 2^{-k}$. Then g maps X onto $[0, 2]$ in an order-preserving manner when X is furnished with the lexicographic order.

Now let (a_k, b_k) be a component interval of $[0, 1] - \Gamma$. Let

$$\alpha_k = \frac{1}{2} a_k + \sum_{x_j < a_k} \frac{1}{2^{j+i}},$$

$$\gamma_k = \alpha_k + \theta_k \quad \text{where} \quad \theta = \begin{cases} \dfrac{1}{2^{i+1}} & \text{if } \alpha_k = x_i, \\ 0 & \text{otherwise}, \end{cases}$$

$$\beta_k = \frac{1}{2} b_k + \sum_{x_j < b_j} \frac{1}{2^{j+1}}.$$

Let $B_k = (\gamma_k, \beta_k)$.

Carrying out the inductive construction in Case 1 with the present intervals B_k, we arrive at a system $\{J_{\mathbf{n}|k} : \mathbf{n} \in \mathscr{N}, k \in N\}$ that satisfies condition (P_1). It is easy to check that $J_{\mathbf{n}}$ is a singleton if and only if $\mathbf{n} \in \mathscr{S}$. Since $A(\mathscr{S}) \subseteq \mathscr{S}$, condition (P_2) is also satisfied.

Finally we recall from Theorem 4.1, part 5, that f is nonchaotic if and only if each nondegenerate component of K has an endpoint in C. In our present setting such a component corresponds to an $\mathbf{n}$ which is eventually constant. By taking specific subsets $\mathscr{S}$ of $\mathscr{N}$ and applying Theorems 4.1 and 4.2 *we find both chaotic and nonchaotic* 2^∞*-functions for which* ω_f *is Baire* 1 (and for which the other conclusions of Theorem 4.1 are also valid). The following table of examples illustrates this.

	S consists of	f is	special features
1	all $\mathbf{n}$	non-chaotic	(1) $C = \varnothing$
2	all $\mathbf{n}$ with no "01" tail	chaotic	(1) no left or right tails (2) $C = \varnothing$
3	all $\mathbf{n}$ such that $n_i = 0$ for infinitely many i	non-chaotic	(1) no right tails (2) C consists of the left ends of components of K (3) each $J_{\mathbf{n}}$ eventually maps onto a singleton
4	all $\mathbf{n}$ not eventually constant	non-chaotic	(1) C consists of all left ends of left tails and all right ends of right tails (2) each non-degenerate $J_{\mathbf{n}}$ eventually maps onto a right tail

By a *left tail* (resp. *right tail*) we mean any non-degenerate component $J_{\mathbf{n}}$ such that $\mathbf{n}$ is eventually 1 (resp. 0) or equivalently the left end (resp. right) point of $J_{\mathbf{n}}$ abuts the complement of K.

Theorem 4.1 is not a full reversal of our construction of a simple system in the sense that the ω-limit set Ω of Proposition 3.1 can have isolated points whereas the unique infinite ω-limit set Q of Theorem 4.1 is a Cantor set. For a full reversal we would have to first know what are the possible infinite ω-limit sets for (chaotic) 2^∞ functions. This is presently unknown. However as a sufficient condition we know from parts 9 and 10 of 3.1 that the set of isolated points of any infinite ω-limit set, if non-void, is dense in Q and intersects each interval contiguous to Q in at most two points.

However, Kirchheim [$\mathbf{K}_1$] has recently provided an example of a 2^∞-function with an infinite ω-limit set having isolated points and mentioned that certain previously alleged examples were not completely correct.

We will now give another example of a 2^∞-function with an infinite ω-limit set with isolated points using the methods of §§3 and 4. This construction is radically different from Kirchheim's. It involves a system $\{F_{\mathbf{n}\upharpoonright k} : \mathbf{n} \in \mathscr{N}, k \in N\}$ which differs from a primitive system in that part (13) of Proposition 3.1 is taken in account.

THEOREM 4.3. *There exists a* 2^∞*-function having an* ω*-limit set of the form* $Q \cup C$ *where* Q *is a Cantor set in* $(0,1)$ *and* C *consists of one point taken from each component of* $[0, \sup Q] - Q$ *and* $C \cap (\sup Q, 1] = \varnothing$. *Moreover,* f *has exactly one* 2^k*-cycle for each* K.

Proof. Let Q be any Cantor set in $(0,1)$ and C consist of exactly one point from each interval continuous to Q together with the point $\frac{1}{2}\inf Q$. Let $\mathscr{M}$ consist of all $\mathbf{n} \in \mathscr{N}$ having a tail of 1's.

Similar to previous constructions we may define by induction a system $\mathscr{F}$ of closed intervals $\{F_{\mathbf{n}\upharpoonright k} : \mathbf{n} \in \mathscr{N}, k \in N\}$ such that for each $\mathbf{n}$ and k, $F_{\mathbf{n}\upharpoonright k,1}$ and $F_{\mathbf{n}\upharpoonright k,0}$ are disjoint subintervals of int $F_{\mathbf{n}\upharpoonright k}$ for which the non-degenerate components of $K = \bigcup_{\mathbf{n}\in\mathscr{N}} \bigcap_{k=1}^\infty F_{\mathbf{n}\upharpoonright k}$ coincide with all $F_{\mathbf{n}} = \bigcap_{k=1}^\infty F_{\mathbf{n}\upharpoonright k}$ with $\mathbf{n} \in \mathscr{M}$ which in turn coincides with all $[c, q]$ where $c \in C$ and q is the nearest point of Q to the right of c. Moreover, we may choose F_1 and F_0 so that $0 = \inf F_1$ and $1 = \sup F_0$.

For each $\mathbf{n} \in \mathscr{N}$ and $k \in N$ let $F_{\mathbf{n}\upharpoonright k} = [a_{\mathbf{n}\upharpoonright k}, b_{\mathbf{n}\upharpoonright k}]$. If $\mathbf{n} \in \mathscr{M}$, then $\bigcap_{k=1}^\infty [a_{\mathbf{n}\upharpoonright k}, b_{\mathbf{n}\upharpoonright k}] = [a_{\mathbf{n}}, b_{\mathbf{n}}]$ where $a_{\mathbf{n}}$ and $b_{\mathbf{n}}$ are the endpoints of $F_{\mathbf{n}}$. Then C consists of all $a_{\mathbf{n}}$ for $\mathbf{n} \in \mathscr{M}$.

Let S consist of all x such that $\{x\} = F_{\mathbf{n}}$ for some $\mathbf{n} \in \mathscr{N}$. Clearly $\overline{S} = Q$. Let B consist of all $b_{\mathbf{n}}$ for $\mathbf{n} \in \mathscr{M}$. Then $Q = S \cup B$. Obviously $F_{A(\mathbf{n})}$ is a singleton whenever $F_{\mathbf{n}}$ is a singleton. Let $L = Q \cup C \cup \{a_{\mathbf{n}\upharpoonright k} : \mathbf{n} \in \mathscr{N}, k \in N\} \cup \{b_{\mathbf{n}\upharpoonright k} : \mathbf{n} \in \mathscr{N}, k \in N\}$.

We will now define a function f on L as follows:

If $x \in S$, define $f(x)$ so that $\{f(x)\} = F_{A(\mathbf{n})}$ when $\{x\} = F_{\mathbf{n}}$.

On $C \cup B$ define f by

$$f(a_{\mathbf{n}}) = a_{A(\mathbf{n})} \quad \text{and} \quad f(b_{\mathbf{n}}) = b_{A(\mathbf{n})}.$$

Finally define f on the remaining points of L when $\mathbf{n} \upharpoonright k \neq \mathbf{1} \upharpoonright k$ by

$$f(a_{\mathbf{n}\upharpoonright k}) = a_{A(\mathbf{n})\upharpoonright k} \quad \text{and} \quad f(b_{\mathbf{n}\upharpoonright k}) = b_{A(\mathbf{n})\upharpoonright k}$$

and

$$f(a_{1|k}) = a_{0|k+1} \quad \text{and} \quad f(b_{1|k}) = b_{0|k+1}$$

Note that $f(a_1) = a_0 = \sup Q = b_0 = f(b_1)$. The following is easily verified; whenever $j \geq k+2$

$$(1) \qquad f(a_{n|k}) < f(a_{n|j}) < f(b_{n|j}) < f(b_{n|k}).$$

We remark that $k+2$ cannot be replaced by $k+1$. Then f is continuous on L. To show this it suffices to show continuity at each point of C, B and S since the other points of L are isolated. We will carry out the proof of continuity only at a_1 and b_1. The general proof will be essentially the same.

To show that f is continuous at a_1, let U be a neighborhood of $f(a_1) = a_0 = \sup Q$. There exists $k \in N$ such that $F_{0|k} \subset U$. Let $V = (a_{1|k+1}, b)$ where $b \in (a_1, b_1)$. Then $V \cap L = \{a_{1|j} : j > k+1\} \cup \{a_1\}$, and $f(V \cap L) = \{a_{0|j+1} : j > k+1\} \cup \{a_0\} \subset U$. Thus f is continuous at a_1.

To show that f is continuous at b_1, let U be a neighborhood of $f(b_1) = b_0 = a_0 = \sup Q$. Choose $k \in N$ such that $F_{0|k} \subset U$. Let $V = (b, b_{1|k+1})$, $b \in (a_1, b_1)$. The set $V \cap L$ consists of end points of intervals in $\mathscr{F}$ contained in $F_{0|k+1}$ and limits of sequences of these end points. It follows from our observation (1) that if $x \in V \cap L$, then $a_{0|k} < f(x) < b_{0|k}$ so $f(x) \in F_{0|k} \subset U$. Thus f is continuous at b_1.

We now extend f linearly on the intervals contiguous to the closed set L obtaining a function also denoted by f that is continuous on all of [0,1].

It is clear from our definition of f that orb $a_0 = \{a_{n|k} : n \in \mathscr{N}, k \in N\}$ so that $\omega(a_0, f) = Q \cup C$.

We now proceed to show the required periodic behaviour.

Let G be the open interval between F_1 and F_0 and $G_{n|k}$ to be the open interval between $F_{n|k,1}$ and $F_{n|k,0}$. Hence

$$G = (b_1, a_0)$$
$$G_{n|k} = (b_{n|k,1}, a_{n|k,0})$$

Put $G^0 = G$, $G^k = \bigcup\{G_{n|k} : n \in \mathscr{N}\}$ and $F^k = \bigcup\{F_{n|k} : n \in \mathscr{N}\}$.

It is easily verified that

$$f(G_{\mathbf{n}\upharpoonright k}) = G_{A(\mathbf{n})\upharpoonright k} \quad \text{if } \mathbf{n}\upharpoonright k \neq \mathbf{1}\upharpoonright k$$
$$\overline{G} \subseteq f(G) = (a_1, b_{00})$$
$$\overline{G_{\mathbf{0}\upharpoonright k}} \subseteq f(G_{\mathbf{1}\upharpoonright k}) = (a_{\mathbf{0}\upharpoonright k,1}, b_{\mathbf{0}\upharpoonright k+2}).$$

From this, it follows that f^{2^k} is linear on $G_{\mathbf{0}\upharpoonright k}$ and has slope greater than 1. Hence we have the same situation as we had in the construction of Theorem 4.1 so that $\overline{G}$ contains exactly one periodic point p_0, which is a repelling fixed point and each $G_{\mathbf{n}\upharpoonright k}$ contains exactly one periodic point which is repelling and has order 2^k. As before any asymptotically periodic point must be eventually periodic. We may also verify that

$$f(F_{\mathbf{n}\upharpoonright k}) = F_{A(\mathbf{n})\upharpoonright k} \quad \text{if } \mathbf{n}\upharpoonright k \neq \mathbf{1}\upharpoonright k$$

$f(F_{\mathbf{1}\upharpoonright k}) = [a_{\mathbf{0}\upharpoonright k,1}, b_{\mathbf{0}\upharpoonright k,0}] = \operatorname{conv}[F_{\mathbf{0}\upharpoonright k,1} \cup F_{\mathbf{0}\upharpoonright k+1}]$, which is a proper subset of $F_{\mathbf{0}\upharpoonright k}$.

It then follows that $f(F_{\mathbf{n}\upharpoonright k})$ contains no $F_{\mathbf{m}\upharpoonright k}$ or $G_{\mathbf{m}\upharpoonright j}$ where $j \leq k$ and $f(G_{\mathbf{n}\upharpoonright k})$ contains no $F_{\mathbf{m}\upharpoonright k}$ or $G_{\mathbf{m}\upharpoonright j}$ for $j < k$. From the above facts we may establish that when k is fixed, for each i there exists m_i such that

$$f^i(F_{\mathbf{1}\upharpoonright k}) \subseteq F^{m_i} \cup \left(\bigcup_{j=0}^{m_i} G^j \right) \tag{2}$$

and such that $m_i \to \infty$ when $i \to \infty$.

Now consider any x. If orb x hits F^k, then clearly $\omega(x, f) \subseteq F^k$. Hence, if orb x hits each F^k, then $\omega(x, f) \subseteq \bigcap_{k=1}^{\infty} F_k = K$.

Suppose then that orb x misses some F^k and that $\omega(x, f) \not\subseteq K$. Let m be the first k such that $F^k \cap \operatorname{orb} x = \varnothing$. If $m = 1$, then $\operatorname{orb} x \subseteq G$. If $m > 1$, then $\operatorname{orb} x \cap F^{m-1} \neq \varnothing$. By (2) either orb x hits each F^j or orb $x \subseteq \bigcup_{j=0}^{i} G^j$ for some i. In the former case we have $\omega(x, f) \subseteq K$, a contradiction. Therefore, there is an i such that $\operatorname{orb} x \subseteq \bigcup_{j=0}^{i} G^j$.

Then $\omega(x, f) \subseteq \bigcup_{j=0}^{i} \overline{G}^j$, and it follows that $\omega(x, f)$ is a cycle of order 2^k with $k \leq i$. Therefore, we have shown that for each x either $\omega(x, f)$ is a 2^k-cycle for some k or $\omega(x, f) \subseteq K$. Since K clearly has no cycles f must be a 2^∞-function.

We end with several remarks.

(1) Let $\mathscr{H} = \{H_{\mathbf{n}\upharpoonright k} : \mathbf{n} \in \mathscr{N}, k \in N\}$ be the family of intervals contiguous to K coded so that $G_{\mathbf{n}\upharpoonright k} \subseteq H_{\mathbf{n}\upharpoonright k}$ for all $\mathbf{n}$ and k. In our two constructions of this section, those of Theorems 4.1 and 4.3, we

saw that if $x \in G_{\mathbf{n}|k}$, then orb $x \cap G_{\mathbf{m}|j} = \varnothing$ for all $\mathbf{m} \in \mathcal{N}$ and $j < k$. The families $\mathcal{G}$ and $\mathcal{H}$ coincided in the first construction but not in the second.

For members of $\mathcal{H}$ transportation between a set $H_{\mathbf{n}|k} \in \mathcal{H}$ to a set of lower rank is possible because of the "wings" created by the requirement that each $F_{\mathbf{n}|k+i}$ is interior to $F_{\mathbf{n}|k}$. For example, $a_{110} \in H_{11}$ but $f(a_{110}) = a_{001} \in H_0$. This transportation from a point in one rank relative to $\mathcal{H}$ to one of lower rank is obviously necessary for there to be isolated points in an infinite ω-limit set.

(2) It was notationally convenient in our second construction (i.e., Theorem 4.3) to have $F_{\mathbf{n}|k+1} \subseteq \operatorname{int} F_{\mathbf{n}|k}$ but it was necessary only that $F_{\mathbf{n}|k,1}$ be in the interior for us to draw the same conclusions about f. A disadvantage of the method we actually chose is that the sets do not quite form a simple system for $\Omega = Q \cup C$ relative to f. What fails is that $f(F_{\mathbf{1}|k})$ is properly contained in $F_{\mathbf{0}|k}$. It is very easy, however, to obtain a simple system $\mathcal{J}$. One simply has the right end point $r_{\mathbf{n}|k}$ of $J_{\mathbf{n}|k}$ coincide with the right end point of $F_{\mathbf{n}|k,\mathbf{0}}$, i.e., $r_{\mathbf{n}|k} = b_{\mathbf{n}|k,\mathbf{0}}$. More specifically $J_1 = [a_{11}, b_{1\mathbf{0}}]$ and in general $J_{\mathbf{1}|k} = [a_{\mathbf{1}|k+1}, b_{\mathbf{1}|k\mathbf{0}}]$. Then $J_{\mathbf{n}|k}$ is defined as usual via the function A. The verification that $\mathcal{J} = \{J_{\mathbf{n}|k} : \mathbf{n} \in \mathcal{N}, k \in N\}$ is a simple system for Ω relative to f is straightforward. We mention only that the fact that $f^{2^k}(J_{\mathbf{1}|k}) = J_{\mathbf{1}|k}$ follows from the observations that the end points of $J_{\mathbf{1}|k}$ are the images under f^{2^k} of $a_{\mathbf{1}|k,0}$ and $b_{\mathbf{1}}$; that is $f^{2^k}(a_{\mathbf{1}|k,0}) = a_{\mathbf{1}|k+1}$ and $f^{2^k}(b_{\mathbf{1}}) = b_{\mathbf{1}k,\mathbf{0}}$.

(3) Since each of the members in $\mathcal{H}$ contains exactly one periodic point, no isolated point of Ω is in the closure of the set of periodic points. In addition, the isolated point $a_{\mathbf{0}}$ of Ω has an orbit which includes all the isolated points. Similar statements are true for the example of [$\mathbf{K}_1$]. We have not seen an example of a 2^∞-function having an infinite ω-limit set with isolated points not exhibiting these features.

(4) The lemmas in §3 that deal with intertwining sets provide additional information about the distribution of points attracted to given ω-limit sets. In the notation of those lemmas, each pair selected from the sets Q^*, Ω^* and E (the eventually periodic points) is intertwined. Thus each of these sets is bilaterally dense-in-itself. One can also verify easily that the set Q^* contains a dense open set.

One can show as in Theorem 4.1 that E is countable. Consider now the set Ω^*. Let c be an isolated point of Ω and let $x \in [0, 1]$. It is clear that $c \in \omega(x, f)$ if and only if $\omega(x, f) = \Omega$. But

$$\{x : c \in \omega(x, f)\} = \bigcap_{j=1}^{\infty} \bigcup_{k=1}^{\infty} f^{-k}\left(\left(c - \frac{1}{j}, c\right)\right)$$

a G_δ set. Thus Ω^* is a G_δ set. (This also follows from **[Sh 2]**). Since Ω^* is dense-in-itself Ω^* is uncountable.

The set Q^* cannot be a G_δ set as we saw in §3. Since $Q^* = [0, 1] - [E \cup \Omega^*)$, Q^* is both an $F_{\sigma\delta}$ and $G_{\delta\sigma}$. We have not determined whether Q^* is an F_σ set.

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