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CONFORMAL DEFORMATIONS PRESERVING THE GAUSS MAP

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In this work, given a conformal immersion $f: M^n \to \mathbb{R}^N$ of a Riemannian manifold M^n into a euclidean space $\mathbb{R}^N$, we establish conditions for the existence of another conformal immersion $\overline{f}: M^n \to \mathbb{R}^N$ with the same Gauss map as f. In particular, for $n = 2$ and $N = 3$, these conditions are described by means of a partial differential equation on the principal curvatures of f.

0. Introduction. Let M^n be a connected n-dimensional Riemannian manifold and let $f: M^n \to \mathbb{R}^N$ be a conformal immersion. We denote by $F: M^n \to G_{n,N}$ the *Gauss map* of f, which assigns to each point $p \in M^n$ the n-dimensional tangent space $f_*(T_pM)$ in the Grassmannian $G_{n,N}$. We consider the following problem: Under what conditions does there exist another conformal immersion $\overline{f}: M^n \to \mathbb{R}^N$ such that the Gauss map of $\overline{f}$ coincides with the Gauss map of f, up to a congruence in $G_{n,N}$ induced by a congruence in $\mathbb{R}^N$? When this occurs we say that $\overline{f}$ is a *G-deformation of* f. This situation is equivalent to considering conformal immersions f and $\overline{f}$ with parallel tangent spaces $f_*(T_pM)$ and $\overline{f}_*(T_pM)$ in $\mathbb{R}^N$, which we will always assume. The analogous problem for isometric immersions f and $\overline{f}$ was considered by Dajczer and Gromoll [**D&G**].

In §1 we characterize our situation by means of a tensor field and a differentiable function satisfying certain conditions (see Proposition 1.5). This result will be used in §2, where we treat the above problem for $n = 2$.

For surfaces, we also consider the *oriented Gauss map* $F^*: M^2 \to G^*_{2,N}$, where now $f_*(T_pM)$ is seen as an oriented 2-plane in the oriented Grassmannian $G^*_{2,N}$. In regard to the above problem we have two different situations. The first one is when f and $\overline{f}$ have the same oriented Gauss map. In this case, it was shown by Hoffman and Osserman [**H&O-2**] that either f and $\overline{f}$ are minimal surfaces or $\overline{f}$ coincides with f up to homothety and translation in $\mathbb{R}^N$. The other situation is when, for any local orientation in M^2, the oriented Gauss maps of f and $\overline{f}$ differ by the orientation-reversing congruence in $G^*_{2,N}$. In this case we call $\overline{f}$ a G^*-*deformation* and say that

f is G^*-*deformable.* If f is not totally umbilic, we show that a G^*-deformation is unique up to homothety and translation (Theorem 2.1). When $N = 4$, we also prove that G^*-deformable immersions must have flat normal bundle. For $N = 3$, Theorem 2.4 characterizes G^*-deformable immersions by means of a condition on their principal curvatures. We apply Theorem 2.4 to obtain $\overline{f}$ when f is a rotation surface, a cyclid of Dupin or a surface with constant mean curvature. A similar result is obtained for constant mean curvature surfaces in the euclidean sphere S^3.

For hypersurfaces in $\mathbb{R}^{n+1}$, $n \geq 3$, the problem considered here will be treated in a forthcoming paper. Most of the results contained in these two works were announced in [Ve] and were obtained in my doctoral thesis. I wish to express my deep gratitude to Professor M. Dajczer for valuable advice and constant encouragement. I also thank the referee for many helpful suggestions.

1. Conformal deformation in $\mathbb{R}^N$ preserving the Gauss map. Let us denote by $\langle\cdot\rangle_\circ$ the Riemannian metric on M^n and by A_ξ the second fundamental form of the conformal immersion $f: M^n \to \mathbb{R}^N$ in the normal direction ξ, defined by

$$f_* A_\xi X = -(\tilde{\nabla}_X \xi)^t, \tag{1.1}$$

where $(\ )^t$ denotes the tangent projection along f and $\tilde{\nabla}$ is the Levi-Civita connection of the canonical metric $\langle\ ,\ \rangle$ on the euclidean space $\mathbb{R}^N$. We denote also by $\langle\ ,\ \rangle$ the metric on M^n induced by f, defined by $\langle\ ,\ \rangle = e^{2\varphi_1}\langle\ ,\ \rangle_\circ$, where $e^{2\varphi_1}: M^n \to \mathbb{R}$ is the conformal factor of f.

Let $\overline{f}: M^n \to \mathbb{R}^N$ be a G-deformation of f with conformal factor $e^{2\varphi_2}$. We define an orthogonal tensor field $T: TM \to TM$ by

$$T = e^{-\varphi} f_*^{-1} \circ P \circ \overline{f}_*, \tag{1.2}$$

where $\varphi = \varphi_2 - \varphi_1$ and, for each $q \in M^n$,

$$P_q: T_{\overline{f}(q)}\mathbb{R}^N \to T_{f(q)}\mathbb{R}^N \tag{1.3}$$

denotes the parallel transport in $\mathbb{R}^N$. For any vector field V along $\overline{f}$ we have

$$\tilde{\nabla}_X PV = P\tilde{\nabla}_X V, \tag{1.4}$$

where X is any tangent field on M^n. We denote by ∇ the Levi-Civita connection on M^n relative to the metric $\langle\ ,\ \rangle$ and by $\nabla\varphi$ the gradient of φ with respect to this metric. The following result gives necessary and sufficient conditions on T and φ for the existence (at least locally) of a G-deformation of f.

PROPOSITION 1.5. *Let $f: M^n \to \mathbb{R}^N$ be a conformal immersion.*

(i) *If $\overline{f}$ is a G-deformation of f, then*

$$\nabla_X T = T \circ (X \wedge \nabla\varphi) \tag{1.6}$$

and

$$A_\xi \circ T = T^{-1} \circ A_\xi \tag{1.7}$$

for any tangent field X and normal field ξ. Moreover the second fundamental form $\overline{A}$ of $\overline{f}$ is given by

$$\overline{A}_\xi = e^{-\varphi} T^{-1} \circ A_{P\xi}. \tag{1.8}$$

(ii) *If M^n is simply connected and there exist an orthogonal tensor field T and a differentiable function φ satisfying* (1.6) *and* (1.7), *then for any $q_0 \in M^n$,*

$$\overline{f}(q) = \int_{q_0}^{q} e^{\varphi} f_* T$$

defines a G-deformation of f.

Proof. We will make use of the Gauss formula

$$\widetilde{\nabla}_X f_* Y = f_* \nabla_X Y + \alpha(X, Y), \tag{1.9}$$

where $X, Y \in TM$ and $\alpha(X, Y)$ denotes the normal component of $\widetilde{\nabla}_X f_* Y$. Recall the relationship between α and A_ξ, given by

$$\langle \alpha(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle.$$

The Levi-Civita connection $\overline{\nabla}$ of the metric on M^n induced by $\overline{f}$ (see **[Ku]**, p. 316) is given by

$$\overline{\nabla}_X Y = \nabla_X Y + X(\varphi)Y + Y(\varphi)X - \langle X, Y\rangle \nabla\varphi.$$

Thus we can write

$$(\widetilde{\nabla}_X \overline{f}_* Y)^t = \overline{f}_*(\nabla_X Y + X(\varphi)Y + Y(\varphi)X - \langle X, Y\rangle\nabla\varphi). \tag{1.10}$$

From (1.4), (1.9) and (1.10) we obtain

$$\begin{aligned}
f_* \nabla_X T Y &= f_*(-X(\varphi)e^{-\varphi} f_*^{-1} P \overline{f}_* Y + e^{-\varphi} \nabla_X f_*^{-1} P \overline{f}_* Y) \\
&= -X(\varphi)e^{-\varphi} P\overline{f}_* Y + e^{-\varphi}(\widetilde{\nabla}_X P\overline{f}_* Y)^t \\
&= -X(\varphi)e^{-\varphi} P\overline{f}_* Y \\
&\quad + e^{-\varphi} P\overline{f}_*(\nabla_X Y + X(\varphi)Y + Y(\varphi)X - \langle X, Y\rangle\nabla\varphi) \\
&= f_*(T\nabla_X Y + Y(\varphi)TX - \langle X, Y\rangle T\nabla\varphi),
\end{aligned}$$

and this proves (1.6).

If ξ is a vector field normal to f, by (1.1) and (1.4) we have

$$\overline{f}_*\overline{A}_\xi X = -(\widetilde{\nabla}_X\xi)^t = -(P^{-1}\widetilde{\nabla}_X P\xi)^t = P^{-1}f_*A_{P\xi}X.$$

Thus $\overline{A}_\xi = e^{-\varphi}T^{-1}A_{P\xi}$. Now (1.7) follows from the fact that $\overline{A}_\xi$ and $A_{P\xi}$ are self-adjoint.

In order to prove (ii), we compute the exterior differential of the 1-form $e^\varphi f_*T$ defined on M^n with values in $\mathbb{R}^N$:

$$\begin{aligned} d(e^\varphi f_*T)(X, Y) &= \widetilde{\nabla}_X e^\varphi f_*TY - \widetilde{\nabla}_Y e^\varphi f_*TX - e^\varphi f_*T([X, Y]) \\ &= X(\varphi)e^\varphi f_*TY + e^\varphi\widetilde{\nabla}_X f_*TY - Y(\varphi)e^\varphi f_*TX \\ &\quad - e^\varphi\widetilde{\nabla}_Y f_*TX - e^\varphi f_*T(\nabla_X Y - \nabla_Y X). \end{aligned}$$

Now we use (1.9) to get

$$\begin{aligned} d(e^\varphi f_*T)(X, Y) = e^\varphi(\alpha(X, TY) - \alpha(Y, TX)) \\ + e^\varphi f_*(\nabla_X TY - T\nabla_X Y - \nabla_Y TX + T\nabla_Y X \\ + X(\varphi)TY - Y(\varphi)TX). \end{aligned}$$

By (1.6) the above equality becomes

$$d(e^\varphi f_*T)(X, Y) = e^\varphi(\alpha(X, TY) - \alpha(Y, TX)).$$

But, for each vector ξ normal to f we have

$$\begin{aligned} &\langle \alpha(X, TY) - \alpha(Y, TX), \xi\rangle \\ &\quad = \langle A_\xi X, TY\rangle - \langle A_\xi TX, Y\rangle = \langle (T^{-1}A_\xi - A_\xi T)X, Y\rangle \end{aligned}$$

and this vanishes by (1.7). Thus $e^\varphi f_*T$ is a closed 1-form on M^n. Since M^n is simply connected, we can define $\overline{f}\colon M^n \to \mathbb{R}^N$ by

$$\overline{f}(q) = \int_{q_0}^{q} e^\varphi f_*T.$$

Then $\overline{f}_* = e^\varphi f_*T$ and $\langle \overline{f}_*X, \overline{f}_*Y\rangle = e^{2\varphi}\langle X, Y\rangle$. So $\overline{f}$ is a G-deformation of f. □

REMARK 1.11. As an immediate consequence of (1.6), we see that φ is constant along M^n if and only if T is a parallel tensor field with respect to the metric $\langle\ ,\ \rangle$. When this occurs, f and $\overline{f}$ induce the same metric on M^n, up to a constant factor. Thus, in this case the problem considered here is equivalent to considering isometric immersions f and $\overline{f}$ with the same Gauss map. This was done by Dajczer and Gromoll in **[D&G]**, where the orthogonal tensor field (1.2) becomes $T = f_*^{-1}P\overline{f}_*$, is parallel and satisfies (1.7).

2. Conformal deformations of surfaces preserving the Gauss map. In this section we study conformal surfaces in $\mathbb{R}^N$ that are G^*-deformable. In this case, the above tensor field T must satisfy the additional condition $\det T = -1$ on M^2. We also obtain a result for surfaces with constant mean curvature in the euclidean sphere S^3. We begin with a uniqueness result.

THEOREM 2.1. *Let $f: M^2 \to \mathbb{R}^N$ be a conformal immersion which is not totally umbilic. If there exists a G^*-deformation $\bar{f}$, then $\bar{f}$ is unique up to homothety and translation in $\mathbb{R}^N$.*

Proof. Let $\overline{M}^2$ denote M^2 with the opposite orientation. Denote the Gauss maps of $f: M^2 \to \mathbb{R}^N$ and $\bar{f}: \overline{M}^2 \to \mathbb{R}^N$ by F and $\overline{F}$, respectively. Then $F = \overline{F}$ as maps of M^2 (without orientation) into $G^*_{2,N}$. Now apply Theorem 1.1 of **[H&O-1]** and the basic uniqueness result in **[H&O-2]**. □

Using some results of **[We-1]** and **[We-2]** we prove the following two theorems.

THEOREM 2.2. *Let $f: M^2 \to \mathbb{R}^4$ be a G^*-deformable conformal immersion. Then the normal bundle of f is flat.*

Proof. We may assume the Gauss map $F: M^2 \to G^*_{2,4}$ is an immersion since the curvature of the normal bundle is zero anywhere F fails to be regular. Then as in the proof of the previous theorem $F: M^2 \to G^*_{2,4}$ and $\overline{F}: \overline{M}^2 \to G^*_{2,4}$ are equal. Using Corollary 3 on p. 464 of **[We-2]**, and the notation there, the existence of $f: M^2 \to \mathbb{R}^4$ implies

$$\varepsilon_1(g) + \rho_1(g) = \varepsilon_2(g) + \rho_2(g);$$

from the existence of $\bar{f}: \overline{M}^2 \to \mathbb{R}^4$, it follows that

$$\varepsilon_1(g) - \rho_1(g) = \varepsilon_2(g) - \rho_2(g),$$

where g is the metric induced on M^2 by f. Thus $\varepsilon_1(g) = \varepsilon_2(g)$ and by Corollary 2 on p. 464 of **[We-2]** it follows that the normal bundle is flat. □

THEOREM 2.3. *Let $f: M^2 \to \mathbb{R}^N$ with $N \geq 5$ be a conformal immersion. If there exists a point of M^2 which is not an inflection point of f, then f is not G^*-deformable.*

Proof. This follows immediately from Proposition 5 of **[We-1]** and the observation that if $\overline{f}$ existed then as above $f: M^2 \to \mathbb{R}^N$ and $\overline{f}: \overline{M}^2 \to \mathbb{R}^N$ would have the same Gauss map. □

The main result of this section is the following.

THEOREM 2.4. *Let $f: M^2 \to \mathbb{R}^3$ be a conformal immersion without umbilic points and let v and w be unit principal vector fields of f, with eigenvalues λ and μ respectively.*

(i) *If f is G^*-deformable, then*

$$(\lambda - \mu)(v(w(\lambda)) + w(v(\mu))) + v(\mu)w(\mu) - v(\lambda)w(\lambda) = 0. \tag{2.5}$$

(ii) *If M^2 is simply connected and* (2.5) *is satisfied, then f is G^*-deformable.*

Proof. Let p be any point M^2 and ξ be a unit normal field defined on a neighborhood of p. Since f has no umbilic points, there exist differentiable functions λ and μ, and orthonormal tangent fields v and w, defined on a neighborhood of p, such that

$$A_\xi v = \lambda v, \quad A_\xi w = \mu w. \tag{2.6}$$

From Codazzi equation

$$\nabla_v(\mu w) - A_\xi \nabla_v w = \nabla_w(\lambda v) - A_\xi \nabla_w v$$

we get

$$v(\mu) = (\lambda - \mu)\langle \nabla_w v, w\rangle, \quad w(\lambda) = (\mu - \lambda)\langle \nabla_v w, v\rangle. \tag{2.7}$$

Let us suppose that f is G^*-deformable. Then there exist a differentiable function $\varphi: M^2 \to \mathbb{R}$ and an orthogonal tensor field T with $\det T = -1$, satisfying (1.6) and (1.7). Let v_1 and w_1 be orthonormal tangent fields on a neighborhood of p such that $Tv_1 = v_1$ and $Tw_1 = -w_1$. Then by (1.7) we have

$$A_\xi v_1 = A_\xi T^{-1} v_1 = T A_\xi v_1.$$

Thus $A_\xi v_1$ is parallel to v_1. So v_1 and w_1 are principal directions, and this determines T up to signal. From now on we will suppose that $Tv = v$ and $Tw = -w$. Now (1.6) is equivalent to

$$v(\varphi) = -2\langle \nabla_w v, w\rangle, \quad w(\varphi) = -2\langle \nabla_v w, v\rangle. \tag{2.8}$$

From (2.7) and (2.8) we have

$$v(\varphi) = -\frac{2v(\mu)}{\lambda - \mu}, \quad w(\varphi) = \frac{2w(\lambda)}{\lambda - \mu}. \tag{2.9}$$

By (2.8) we get

$$\begin{aligned}[v, w](\varphi) &= (\nabla_v w)(\varphi) - (\nabla_w v)(\varphi)\\ &= \langle \nabla_v w, v\rangle v(\varphi) - \langle \nabla_w v, w\rangle w(\varphi)\\ &= 0.\end{aligned}$$

Thus we must have

$$v(w(\varphi)) - w(v(\varphi)) = 0$$

or, using (2.9),

$$v\left(\frac{w(\lambda)}{\lambda - \mu}\right) + w\left(\frac{v(\mu)}{\lambda - \mu}\right) = 0, \tag{2.10}$$

which is equivalent to (2.5). Note that equation (2.5) is invariant by change of sign of the vector fields ξ, v and w. Thus it is valid everywhere on M^2.

Suppose now that (2.5) is satisfied on M^2. We define the tangent vector field

$$\delta = -2\langle \nabla_w v, w\rangle v - 2\langle \nabla_v w, v\rangle w$$

and observe that δ does not depend on the unit vector fields ξ, v and w satisfying (2.6). Now we define in M^2 the 1-form γ given by

$$\gamma(X) = \langle \delta, X\rangle. \tag{2.11}$$

Using (2.7) we compute

$$\begin{aligned}d\gamma(v, w) &= v(\gamma(w)) - w(\gamma(v)) - \gamma(\nabla_v w - \nabla_w v)\\ &= \frac{2}{\lambda - \mu}(v(w(\lambda)) + w(v(\mu)))\\ &\quad + \frac{2}{(\lambda - \mu)^2}(v(\mu)w(\mu) - v(\lambda)w(\lambda)).\end{aligned}$$

By (2.5) the 1-form γ is closed. Since M^2 is simply connected, there exists $\varphi : M^2 \to \mathbb{R}$ such that $\nabla\varphi = \delta$, that is,

$$v(\varphi) = -2\langle \nabla_w v, w\rangle, \quad w(\varphi) = -2\langle \nabla_v w, v\rangle. \tag{2.12}$$

We define the tensor field T by $Tv = v$ and $Tw = -w$. Then T is orthogonal, $\det T = -1$ and (1.7) is satisfied. From (2.12) it is easy to show that (1.6) is satisfied. By Proposition 1.5, f is G^*-deformable. □

The next result is needed for the proof of some of the corollaries to Theorem 2.4.

PROPOSITION 2.13. *If* (x, y) *are principal coordinates on an open set* $U \subset M^2$, *then on* U (2.5) *is equivalent to*

$$\left(\frac{(\lambda+\mu)_x}{\lambda-\mu}\right)_y + \left(\frac{(\lambda+\mu)_y}{\lambda-\mu}\right)_x = 0. \tag{2.14}$$

Proof. Let E and G be positive functions such that

$$\frac{\partial}{\partial x} = E\mathscr{v} \quad \text{and} \quad \frac{\partial}{\partial x} = G\mathscr{w}. \tag{2.15}$$

Since $[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}] = 0$ we obtain

$$E\mathscr{v}(G) - EG\langle \nabla_{\mathscr{w}}\mathscr{v}, \mathscr{w}\rangle = 0,$$
$$-G\mathscr{w}(E) + EG\langle \nabla_{\mathscr{v}}\mathscr{w}, \mathscr{v}\rangle = 0,$$

and then by (2.7) we have

$$G_x = \frac{G\mu_x}{\lambda-\mu}, \quad E_y = -\frac{E\lambda_y}{\lambda-\mu}. \tag{2.16}$$

From (2.15) and (2.16) it follows that

$$\mathscr{w}(\mathscr{v}(\mu)) = \frac{\mu_{xy}}{EG} + \frac{\mu_x\lambda_y}{(\lambda-\mu)EG},$$
$$\mathscr{v}(\mathscr{w}(\lambda)) = \frac{\lambda_{xy}}{EG} - \frac{\mu_x\lambda_y}{(\lambda-\mu)EG},$$

and then (2.5) and (2.14) are equivalent. □

In the next two corollaries, we will consider f as the inclusion map of an open subset of M^2 in $\mathbb{R}^3$.

COROLLARY 2.17. *Let* M^2 *be a rotation surface which does not meet its axis of symmetry. Then* f *is* G^**-deformable. If* M^2 *is not totally umbilic then* $\overline{f}$ *is unique up to homothety and translation; also,* $\overline{f}(M)$ *is again a rotation surface. If* M^2 *is part of a sphere and* $\overline{f}(M)$ *is a rotation surface, then* $\overline{f}(M)$ *is part of a catenoid.*

Proof. We take on $f(M^2)$ the parametrization $\phi: (0, 2\pi) \times I \to \mathbb{R}^3$ given by

$$\phi(x, y) = (\alpha(y)\cos x, \alpha(y)\sin x, \beta(y)),$$

where $(\alpha(y), \beta(y))$ is a plane curve defined on an open interval $I \subset \mathbb{R}$ and satisfying $\alpha(y) \neq 0$ for any $y \in I$.

Now define $\varphi: M^2 \to \mathbb{R}$ by $e^\varphi = 1/\alpha^2(y)$. One can immediately verify that

$$\overline{\phi}(x, y) = \left(\frac{1}{\alpha(y)}\cos x, \frac{1}{\alpha(y)}\sin x, \overline{\beta}(y)\right), \tag{2.18}$$

where

$$\overline{\beta}(y) = \int_{y_0}^{y} \frac{\beta'(t)}{\alpha^2(y)}\, dt,$$

satisfy $\overline{\phi}_x = e^\varphi \phi_x$ and $\overline{\phi}_y = -e^{\varphi(y)}\phi_y$. Thus $\overline{f}$ defined by $\overline{f}(\phi(x, y) = \overline{\phi}(x, y)$ is a G^*-deformation. Now, if M^2 is part of a sphere, then up to homothety and translation f is the normal Gauss map of f into the unit sphere S^2. So $\overline{f}(M^2)$ must be a minimal surface, hence it is part of a catenoid. □

A surface in $\mathbb{R}^3$ that is the envelope of a family of spheres tangent to three fixed spheres in $\mathbb{R}^3$ is called a cyclid of Dupin. These surfaces can be characterized by the fact that they are the surfaces without umbilic points whose principal curvatures are constant along the respective curvature lines (see **[C&R]**, pp. 151–166).

COROLLARY 2.19. *Let M^2 be a cyclid of Dupin and U an open simply connected subset of M^2. Then f restricted to U is G^*-deformable.*

Proof. If (x, y) are principal coordinates, then the respective principal curvatures λ and μ satisfy $\lambda_x = \mu_y = 0$. Thus (2.14) is verified and we can apply Theorem 2.4. □

REMARK 2.20. In the preceding corollary, we have by (2.9),

$$\varphi = \log(c(\lambda - \mu)^2)$$

for some positive constant $c \in \mathbb{R}$. By (1.8), the principal curvatures of $\overline{f}$ are

$$\overline{\lambda} = \frac{c\lambda}{(\lambda - \mu)^2}, \quad \overline{\mu} = \frac{-c\mu}{(\lambda - \mu)^2}.$$

Thus, in general, the new surface $\overline{f}(U)$ is not a cyclid of Dupin.

COROLLARY 2.21. *Let $f: M^2 \to \mathbb{R}^3$ be an oriented minimal surface without umbilic points and let $N: M^2 \to S^2 \subset \mathbb{R}^3$ be the normal Gauss map. Then f is G^*-deformable and $\overline{f} = N$ up to homothety and translation.*

Proof. Taking principal coordinates (x, y), we have

$$N_x = -\lambda \frac{\partial}{\partial x}, \quad N_y = \lambda \frac{\partial}{\partial y}.$$

Thus the corollary is a consequence of Theorem 2.1. □

COROLLARY 2.22. *Let $f: M^2 \to \mathbb{R}^3$ be an oriented surface free of umbilic points, with constant mean curvature $H \neq 0$, and let $N: M^2 \to S^2 \subset \mathbb{R}^3$ be the normal Gauss map. Then f is G^*-deformable and $\overline{f}$ is the parallel surface $g = f + \frac{1}{H}N$, up to homothety and translation.*

Proof. Taking principal coordinates (x, y), we have

$$g_x = f_x + \frac{1}{H}N_x = \left(1 - \frac{\lambda}{H}\right) f_x = \left(\frac{\mu - \lambda}{\lambda + \mu}\right) f_x$$

and

$$g_y = \left(\frac{\lambda - \mu}{\lambda + \mu}\right) f_y.$$

Thus g is a G^*-deformation of f. We observe that, since

$$N_x = -\lambda f_x = \lambda \left(\frac{\lambda + \mu}{\lambda - \mu}\right) \overline{f}_x$$

and

$$N_y = -\mu f_y = -\mu \left(\frac{\lambda + \mu}{\lambda - \mu}\right) \overline{f}_y,$$

one sees that the mean curvature of $\overline{f}$ is also H. □

We conclude this work with a result analogous to the preceding corollary, for a constant mean curvature surface in S^3.

PROPOSITION 2.23. *Let $f: M^2 \to S^3$ be an oriented surface free of umbilic points, with constant mean curvature H, and let $N: M^2 \to S^3$ be a vector field normal to f. Then f (seen as a surface in $\mathbb{R}^4$) is G^*-deformable and $\overline{f}$ is the parallel surface*

$$\overline{f} = \frac{1}{(H^2 + 1)^{1/2}}(N + Hf): M^2 \to S^3,$$

up to homothety and translation.

Proof. It is analogous to the preceding proof. □

REMARK 2.24. One can easily check that the above immersion $\overline{f}$ has constant mean curvature in S^3. When $H = 0$, $-\overline{f}$ is the polar map of the minimal immersion f, as defined by Lawson **[La]**.

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