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ON THE INCIDENCE CYCLES OF A CURVE: SOME GEOMETRIC INTERPRETATIONS

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In this paper, we note that the incidence cycles of a seminormal curve X intervene in the calculation of the arithmetic genus $p_a(X)$, of the algebraic fundamental group $\pi_1^{\text{alg}}(X)$ and of the Picard group $\text{Pic}(X)$ of X. Really we do not consider only seminormal curves, but more generally varieties obtained from a smooth variety by glueing a finite set of points.

0. Introduction. By a curve we mean a dimension 1 quasi-projective scheme over an algebraically closed field k.

Let X be a connected reduced seminormal curve (see **[T]**, **[P]** and **[D]** for the definition and the geometric meaning of seminormality).

Let $X_1, \dots, X_n$ be the irreducible components of X; the normalization $\overline{X}$ of X is isomorphic to the disjoint union $\bigsqcup_{i=1}^n \overline{X}_i$ of the normalizations $\overline{X}_i$ of the curves X_i. Let $\pi: \overline{X} \to X$ denote the normalization morphism.

Let $P_1, \dots, P_m$ be the singular points of X and let $x_1, \dots, x_M$ be the branches of X ($x \in \overline{X}$ is a branch of X over a singular point P of X if $x \in \pi^{-1}(P)$).

We define $\nu(X) = M - m - n + 1$. In **[R]** one can find a geometric characterization of the number $\nu(X)$ in terms of the incidence cycles of X. One associates to the curve X the graph Γ whose vertices are $P_1, \dots, P_m, X_1, \dots, X_n$ and whose edges represent the M branches of X in this way: if x_r is a branch over P_i and $x_r \in \overline{X}_j$, an edge joining P_i and X_j is constructed. Any cycle of the graph Γ associated to X is said to be an *incidence cycle* of X.

In **[R]** it is proved that the graph Γ associated to X is connected, the number of the independent cycles of Γ is $\nu(X)$ and Γ contains cycles if and only if X satisfies one of the following conditions:

(a) an irreducible component of X is not locally unibranch,

(b) two irreducible components of X meet in more than one point,

(c) X contains polygons.

In the present paper we'll consider more generally a class of varieties X of dimension $r \geq 1$ and we'll see that the number $\nu(X)$

intervenes in the calculus of the arithmetic genus $p_a(X)$, of the algebraic fundamental group $\pi_1^{\mathrm{alg}}(X)$ and of the Picard group $\mathrm{Pic}(X)$ of X.

By a *variety* we mean a reduced quasi-projective scheme over an algebraically closed field k.

Now we recall the definition of glueing of varieties and of k-algebras.

DEFINITION 0.1. Let X and X' be two varieties, let $x_1, \dots, x_M$ be closed points of X' and let P be a closed point of X. We say that X *is obtained from* X' *by glueing* $x_1, \dots, x_M$ *over* P if there exists a morphism $f: X' \to X$, called a *glueing morphism*, making cocartesian the following square:

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle i} \\ \mathrm{Spec}(k_1 \oplus \cdots \oplus k_M) & \xrightarrow{\delta} & \mathrm{Spec}(k) \end{array}$$

where k_i is the residue field of x_i, the residue fields of x_i and P are isomorphic to k, δ is induced by the diagonal morphism, i and j are the canonical injections.

Algebraically Definition 0.1 is equivalent to the following

DEFINITION 0.2 (see [T] §1 and [P] §1). Let A and B be two finitely generated k-algebras, with k an algebraically closed field, let $\mathfrak{m}_1, \dots, \mathfrak{m}_M$ be maximal ideals of B and let $\mathfrak{m}$ be a maximal ideal of A. We say that A *is obtained from* B *by glueing the maximal ideals* $\mathfrak{m}_1, \dots, \mathfrak{m}_M$ *over* $\mathfrak{m}$ if A is the fibered product of B and k over $k_1 \oplus \cdots \oplus k_M$, i.e. if we have the following cartesian square:

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ k & \xrightarrow{\delta} & k_1 \oplus \cdots \oplus k_M \end{array}$$

where α is the canonical projection $A \to A/\mathfrak{m} \cong k$, β is the canonical projection $B \to B/\mathfrak{m}_1 \oplus \cdots \oplus B/\mathfrak{m}_M = k_1 \oplus \cdots \oplus k_M$, $k_i \cong k$ and δ is the diagonal morphism.

We recall that a seminormal curve X is obtained from the normalization $\overline{X}$ by a finite number of glueing morphisms (see [T], Theorem 2.1).

Note that Mestrano in [Me] used Severi curves, which are curves obtained from a finite (disjoint) union of projective lines by a finite number of glueing morphisms, to study the Picard group of the rational points of the Picard scheme of C_g, where C_g is the universal curve over the function field of the coarse moduli space M_g of the curves of genus g.

In what follows X *denotes* a connected variety of pure dimension r whose singular locus $\operatorname{Sing}(X)$ consists of a finite set of points $P_1, \dots, P_m$, such that the normalisation $\overline{X}$ of X is a smooth variety having n connected components, every one of them of dimension r, $\overline{X}_1, \dots, \overline{X}_n$ and the normalisation morphism $\pi: \overline{X} \to X$ is the composition of a finite number of glueing morphisms satisfying the conditions of Definition 0.1. Let M be the number of points of $\pi^{-1}(\operatorname{Sing}(X))$; we define $\nu(X) = M - m - n + 1$.

We'll prove the following results:

THEOREM 0.3. *If X is projective, we have*

$$p_a(X) = p_a(\overline{X}_1) + \cdots + p_a(\overline{X}_n) + (-1)^{r-1}\nu(X).$$

THEOREM 0.4. *We have*

$$\pi_1^{\mathrm{alg}}(X) \cong (\pi_1^{\mathrm{alg}}(\overline{X}_1) * \cdots * \pi_1^{\mathrm{alg}}(\overline{X}_n) * L_{\nu(X)})^{\wedge},$$

where L_ν denotes the free group with ν generators, $*$ denotes the free product of groups and $\wedge$ denotes the completion of the group.

THEOREM 0.5. *We have* $\operatorname{Pic}(X) \cong \operatorname{Pic}(\overline{X}_1) \oplus \cdots \oplus \operatorname{Pic}(\overline{X}_n) \oplus \nu(X)k^*$, *where k^* is the multiplicative group $k - \{0\}$ and νk^* denotes the direct sum of ν copies of k^*.*

Theorem 0.3 is an easy calculation.

Theorem 0.4 was obtained by Vistoli in [V] for X irreducible or having a unique singular point. He proved his result by obtaining any étale covering of X from an étale covering of the normalization $\overline{X}$ by glueing the fibres of the branches of X.

By generalizing Vistoli's constructions described in [V], one can prove that any étale covering of X is obtained from the étale coverings of $\overline{X}_1, \dots, \overline{X}_n$ by a finite number of glueing morphisms.

But in a shorter way we'll prove Theorem 0.4 by induction on n and by using Vistoli's results on varieties having only one singular point.

Theorem 0.5 generalizes a result of Roberts contained in **[Ro1]** and in **[Ro2]**; by using the Mayer-Vietoris sequences, Roberts calculated the Picard group of an affine curve $X = \mathrm{Spec}(A)$ having the irreducible components X_i rational, i.e. $\overline{X}_i = \mathrm{Spec}(k[t])$.

In order to calculate the Picard group $\mathrm{Pic}(X)$ of X, we construct the line bundles of X by glueing line bundles of $\overline{X}$, by using a similar method as the one employed in **[Mi]** to construct the projective modules over a ring A satisfying the conditions of Definition 0.2.

1. The arithmetic genus. The arithmetic genus of a projective variety X of dimension r is the number $p_a(X) = (-1)^r(\chi(O_X) - 1)$, where $\chi(O_X)$ is the Euler-Poincaré characteristic of O_X.

1.1. *Proof of Theorem* 0.3. There is the following exact sequence of sheaves on X: $0 \to O_X \to \pi_* O_{\overline{X}} \to \sum_{P\in X} \overline{O}_{X,P}/O_{X,P} \to 0$, where $\overline{O}_{X,P}$ is the integral closure of $O_{X,P}$. Since $O_{X,P}$ is obtained from $\overline{O}_{X,P}$ by glueing a finite number of maximal ideals, we have length $(\overline{O}_{X,P}/O_{X,P}) = M_P - 1$, where M_P is the number of points x of X lying over P $(x \in \pi^{-1}(P))$. Since the morphism π is affine, then $\chi(\pi_* O_{\overline{X}}) = \chi(O_{\overline{X}})$ and therefore $\chi(O_X) = \chi(O_{\overline{X}}) - \sum_{P\in X}(M_P^{-1})$.

Let us suppose r odd.

We prove first that $p_a(\overline{X}) = p_a(\overline{X}_1) + \cdots + p_a(\overline{X}_n) - n + 1$. We proceed by induction on n. For $n = 1$ it is true. Now we suppose that the statement is true for $n-1$ and we consider $Y = \bigsqcup_{i=1}^{n-1} \overline{X}_i$; then we have

$$\begin{aligned} p_a(\overline{X}) &= 1 - \chi(O_{\overline{X}}) = 1 - \chi(O_{\overline{X}_n}) - \chi(O_Y) = p_a(\overline{X}_n) + p_a(Y) - 1 \\ &= p_a(\overline{X}_1) + \cdots + p_a(\overline{X}_n) - (n-1). \end{aligned}$$

Then

$$\begin{aligned} p_a(X) &= 1 - \chi(O_X) = 1 - \chi(O_{\overline{X}}) + \sum_{P\in X}(M_P - 1) = p_a(\overline{X}) + M - m \\ &= p_a(\overline{X}_1) + \cdots + p_a(\overline{X}_n) + \nu(X). \end{aligned}$$

If r is even, the calculation is similar.

2. The algebraic fundamental group. If X is connected, there exists a profinite topological group G such that the category $\mathrm{Et}(X)$ of the étale coverings of X is equivalent to the category $\mathrm{Ac}(G)$ of the finite sets on which G acts continuously. G is unique up to unique isomorphism; it is denoted $\pi_1^{\mathrm{alg}}(X)$ and it is defined the algebraic fundamental group of X.

Vistoli proved in **[V]** the following propositions:

Proposition 2.1 (*see* [V], *Teorema* II.12). *Let X and X' be connected varieties and let $f: X' \to X$ be a composition of a finite number of glueing morphisms; if $x \in X$, let $p(x)$ denote the cardinality of the fibre $f^{-1}(x)$.*

We have $\pi_1^{\text{alg}}(X) = (\pi_1^{\text{alg}}(X') * L_p)^\frown$, where $p = \sum_{x\in X}(p(x)-1)$.

Proposition 2.2 (*see* [V] *Corollario* II.11). *Let $X_1, \ldots, X_n$ be disjoint connected varieties, let $x_1 \in X_1, \ldots, x_n \in X_n$ be n closed points. Let X denote the variety obtained by glueing the points $x_1, \ldots, x_n$.*

Then we have $\pi_1^{\text{alg}}(X) = (\pi_1^{\text{alg}}(X_1) * \cdots * \pi_1^{\text{alg}}(X_n))^\frown$.

2.3. *Proof of Theorem* 0.4. We proceed by induction on the number n of the irreducible components of X. If $n = 1$, the claim follows from Proposition 2.1.

Now we suppose that the theorem is true for $n-1$.

Let X' be the variety $\pi(\bigcup_{i=1}^{n-1}\overline{X}_i)$; we can suppose that X' is connected.

Furthermore we can suppose $P_1 \in X' \cap X_n$, so there exist a point $a \in X'$ and a point $b \in X_n$ such that $\pi(a) = \pi(b) = P_1$. Let X'' denote the variety obtained from $X' \sqcup X_n$ by glueing a and b over P_1.

The variety X can be obtained from X'' by a finite number of glueing morphisms.

Then we can factor the morphism π as:

$$\pi: \coprod_{i=1}^{n} \overline{X}_i \xrightarrow{\varphi_1} X' \sqcup \overline{X}_n \xrightarrow{\varphi_2} X' \sqcup X_n \xrightarrow{\varphi_3} X'' \xrightarrow{\varphi_4} X.$$

From the inductive hypothesis we have

$$\pi_1^{\text{alg}}(X') = (\pi_1^{\text{alg}}(\overline{X}_1) * \cdots * \pi_1^{\text{alg}}(\overline{X}_{n-1}) * L_{\nu(X')})^\frown.$$

From Proposition 2.2 we have $\pi_1^{\text{alg}}(X'') = (\pi_1^{\text{alg}}(X') * \pi_1^{\text{alg}}(X_n))^\frown$ and from Proposition 2.1 $\pi_1^{\text{alg}}(X) = (\pi_1^{\text{alg}}(X'') * L_p)^\frown$, where $p = \sum_{i=1}^{m} p(P_i) - m$ and $p(P_i)$ denotes the cardinality of the fibre $\varphi_4^{-1}(P_i)$.

We must prove $\nu(X) = \nu(X') + \nu(X_n) + p$.

If $\overline{Y}$ is a union of connected components of $\overline{X}$ and $Y = \pi(\overline{Y})$, we denote by m_Y and M_Y the number of the singular points of Y and the number of the points of $\overline{Y}$ lying over the singular points of Y respectively. We note that $\nu(X') = M_{X'} - m_{X'} - n + 2$ and $\nu(X_n) = M_{X_n} - m_{X_n}$.

Let us consider the last morphism φ_4; we find

$$M = M_{X''} + \sum_{i=1}^{m} p(P_i) - m_{X''}.$$

Moreover the glueing morphism φ_3 gives the equalities $M_{X''} = M_{X'} + M_{X_n} + 2$ and $m_{X''} = m_{X'} + m_{X_n} + 1$.

So, after easy calculations, we can conclude.

3. Line bundles obtained by glueing. We begin with a lemma.

LEMMA 3.1. *Let X be a (connected) quasi-projective variety and let F be a locally free sheaf on X of rank r. If $x_1, \dots, x_M$ are M closed points of X, then there exists an affine open U of X containing $x_1, \dots, x_M$ such that the $O_X(U)$-module $F(U)$ is free of rank r.*

Proof. For any (standard) affine open $V = \operatorname{Spec} A$ of X we have that the sheaf $F_{|V}$ is isomorphic to the sheaf $\widetilde{N}$ associated to the A-module $N = F(V)$ (see **[H]**, Chapter II, §5).

N is a projective A-module of rank r (see **[Bo]**, Chapter II, §5, Theorem 1).

Let us choose V containing the points $x_1, \dots, x_M$; let $\mathfrak{m}_1, \dots, \mathfrak{m}_M$ be the maximal ideals of A corresponding to the points $x_1, \dots, x_M$ respectively.

If $S = \bigcap_{i=1}^{M}(A - \mathfrak{m}_i) = A - (\bigcup_{i=1}^{M} \mathfrak{m}_i)$, the ring A_S is semi-local, then the A_S-module $N_S = N \otimes_A A_S$ is free of rank r (see **[Bo]**, Chapter II, §5, Proposition 5) and there exists $f \in S$ such that N_f is a free A_f-module (see **[Bo]**, Chapter II, §2, Corollary 2 and the proof of the Proposition 2 of Chapter II, §5). We take $U = \operatorname{Spec} A_f$.

Let X be a connected variety obtained from a variety X' by glueing the points $x_1, \dots, x_M$ of X' over a point P of X. The glueing morphism $f: X' \to X$ induces a group homomorphism $f^*: \operatorname{Pic}(X) \to \operatorname{Pic}(X')$. We want to see how a line bundle on X originates from a line bundle on X'.

In what follows we confuse a line bundle L on X with the locally free sheaf of rank 1 associated to it, but we denote by L_x the fibre of the line bundle L at the point $x \in X$ ($L_x \cong k$) and by $L_\mathfrak{m}$ the fibre of the locally free sheaf L at the point x if $\mathfrak{m}$ denotes the maximal ideal of the local ring $O_{X,x}$ ($L_\mathfrak{m} \cong O_{X,x}$).

PROPOSITION 3.2 (*We use the notations of Definition* 0.1). *Let* L *be a line bundle on* X'. *We have* $j^*(L) = L_{x_1} \oplus \cdots \oplus L_{x_M}$, $L_{x_i} \cong k_i \cong k$, *and let* $h\colon L_{x_1} \oplus \cdots \oplus L_{x_M} \xrightarrow{\sim} k_1 \oplus \cdots \oplus k_M$ *be an isomorphism of* $(k_1 \oplus \cdots \oplus k_M)$-*modules. Then the couple* (L, h) *gives canonically a line bundle* L_h *on* X *such that* $f^*(L_h) = L$.

Proof (*see* **[Mi]**, §2). If U is an affine open of X containing P, we have $U = \operatorname{Spec}(A)$ and $f^{-1}(U) = \operatorname{Spec}(B)$, A and B are two k-algebras satisfying the conditions of Definition 0.2.

Let $L_h(U)$ be the group fibred product of k and $L(f^{-1}(U))$ over $k_1 \oplus \cdots \oplus k_M$, making cartesian the following square of groups:

$$\begin{array}{ccc} L_h(U) & \longrightarrow & L(f^{-1}(U)) \\ \downarrow & & \downarrow h \\ k & \xrightarrow{\delta} & k_1 \oplus \cdots \oplus k_M \end{array}$$

$L_h(U)$ is in a natural way an A-module and it is projective of rank 1.

If U is an (affine) open of X not containing P, we put $L_h(U) = L(f^{-1}(U))$. That defines a line bundle L_h on X (see **[Bo]**, Chapter II, §5, Theorem 1) and we have $f^*(L_h) = L$.

DEFINITION 3.3. (a) The couple (L, h) of Proposition 3.2 is said to be *the glueing of* L *by* h.

(b) Two glueings of line bundles (L, h) and (L', h') are said to be *isomorphic* if there exists an isomorphism $\lambda\colon L \to L'$ such that the following diagram

$$\begin{array}{ccc} L_{x_1} \oplus \cdots \oplus L_{x_M} & \xrightarrow{h} & k_1 \oplus \cdots \oplus k_M \\ {\scriptstyle \lambda \otimes 1_{k_1 \oplus \cdots \oplus k_M}} \downarrow & & \| \\ L'_{x_1} \oplus \cdots \oplus L'_{x_M} & \xrightarrow{h'} & k_1 \oplus \cdots \oplus k_M \end{array}$$

is commutative.

(c) We define $(L, h) \cdot (L', h') = (L \otimes L', h \otimes h')$, where

$$(h \otimes h')(u \otimes u') = h(u)h'(u').$$

In this way the isomorphism classes of the couples (L, h) form an abelian group H_f.

THEOREM 3.4. *The Picard group* Pic(X) *of* X *is isomorphic to the group* H_f *defined as above.*

Proof. We can define a natural group homomorphism $\Phi: H_f \to \text{Pic}(X)$ that to the class of (L, h) associates the class of the line bundle L_h constructed in the proof of Proposition 3.2, Φ is injective; in fact if $\Phi(L, h) = O_X$, we have that the couple (L, h) is isomorphic to the couple $(O_X, \text{id}_{k_1 \oplus \cdots \oplus k_M})$.

Now let F be a line bundle on X. Then $L = f^*(F)$ is a line bundle on X' and from the square of Definition 0.1, we see that $L_{x_1} \oplus \cdots \oplus L_{x_M} = j^*(f^*(F)) = \delta^*(i^*(F)) = \delta^*(F_P)$, $F_P \cong k$.

F induces an isomorphism $h: \delta^*(F_P) \xrightarrow{\sim} k_1 \oplus \cdots \oplus k_M$. The couple $(f^*(F), h)$ gives with the above construction a line bundle over X isomorphic to F (see **[Mi]**, §2). Hence Φ is surjective.

4. The Picard group.

PROPOSITION 4.1. *Let* $f: X' \to X$ *be a glueing morphism of* M *points* $x_1, \ldots, x_M$ *of a connected quasi-projective variety* X' *over a point* P *of* X. *Then* $\text{Pic}(X) \cong \text{Pic}(X') \oplus (M-1)k^*$.

Proof. It is sufficient to consider $M = 2$. We'll prove the proposition by defining an isomorphism Ψ from H_f to $\text{Pic}(X') \oplus k^*$ (cf. Theorem 3.4).

Let L be a line bundle on X' and let h be an isomorphism from $L_{x_1} \oplus L_{x_2}$ to $k_1 \oplus k_2$. Let us consider an open affine U of X' containing x_1 and x_2 such that there exists an isomorphism from $O_{X'}(U)$ to $L(U)$ (see Lemma 3.1); let e be the image of a unit u of $O_{X'}(U)$ satisfying the following condition:

(*) u is such that $\beta(u)$ is contained in the image of the diagonal morphism δ (see Definition 0.2).

$e_i = e \otimes 1_{k_i}$ is a generator of the k-vector space L_{x_i}, $i = 1, 2$. If $h(e_1, e_2) = (a, b)$, we define $\Psi((L, h)) = (L, \frac{a}{b})$.

We note that if V and e' are an affine open of X' and a generator of $L(V)$ respectively satisfying the same conditions that U and e satisfy respectively, then we have $e' = ce$, where c is a unit of $O_{X'}(U)$ satisfying the condition (*). Then $h(e'_1, e'_2) = h(\bar{c}e_1, \bar{c}e_2) = (\bar{c}a, \bar{c}b)$, $\bar{c} \in k^*$ and $\Psi((L, h))$ does not depend on the choice of U and e.

If (L, h) is isomorphic to (L', h'), there exists an isomorphism λ

from L to L' such that $h(e_1, e_2) = h'(e'_1, e'_2)$, where e'_1, e'_2 are the images in L'_{x_1} and L'_{x_2} respectively of $\lambda_U(e)$, λ_U is the isomorphism from $L(U)$ to $L'(U)$ induced by λ. Then $\Psi((L, h)) = \Psi((L', h'))$.

It is easy to verify that the map Ψ is a group isomorphism.

PROPOSITION 4.2. *Let X' be a quasi-projective variety having n connected components $X_1, \dots, X_n$, let $x_i \in X_i$ for every $i = 1, \dots, n$. Let $f\colon X' \to X$ be the glueing morphism of the points $x_1, \dots, x_n$. then* $\mathrm{Pic}(X) \cong \mathrm{Pic}(X_1) \oplus \cdots \oplus \mathrm{Pic}(X_n)$.

Proof. We may assume $n = 2$. From Theorem 3.4 it is sufficient to prove that the group H_f is isomorphic to $\mathrm{Pic}(X') \cong \mathrm{Pic}(X_1) \oplus \mathrm{Pic}(X_2)$.

Let $L = L_1 \oplus L_2$ be a line bundle on X'. Let U_i be an affine open of X_i containing x_i, such that there exists an isomorphism $O_{X_i}(U_i) \to L_i(U_i)$, let e_i denote the image of 1, we denote the element $e_i \otimes 1_{k_i} \in L_{x_i}$ by e_i also, $i = 1, 2$.

Let $i_L\colon (L_1)_{x_1} \oplus (L_2)_{x_2} \xrightarrow{\sim} k_1 \oplus k_2$ denote the isomorphism defined by $i_L(e_1, e_2) = (1, 1)$.

Two couples (L', h) and (L, i_L) of H_f are isomorphic if and only if L and L' are isomorphic; in fact, we can suppose $L' = L$, if $h(e_1, e_2) = (a_1, a_2)$, a_i determines an isomorphism of L_i into itself, $i = 1, 2$.

LEMMA 4.3. *Let $f\colon X' \to X$ be a morphism of connected quasi-projective varieties which is a composition of a finite number of glueing morphisms.*

Let $\rho = \sum_{P \in X}(\rho(P) - 1)$, where $\rho(P)$ is the cardinality of $f^{-1}(P)$. Then $\mathrm{Pic}(X) \cong \mathrm{Pic}(X') \oplus \rho k^*$.

Proof. Let $P_1, \dots, P_m$ be the points of X having $\rho(P) > 1$. We proceed by induction on m. If $m = 1$ the result follows from Proposition 4.1.

Now we suppose the lemma true for $m - 1$. We can factor the morphism f by $X' \xrightarrow{f'} X'' \xrightarrow{f''} X$, where f' is the composition of the glueing morphisms over the points $P_1, \dots, P_{m-1}$ only and f'' is the glueing over P_m.

By the induction hypothesis we have $\mathrm{Pic}(X'') \cong \mathrm{Pic}(X') \oplus \rho' k^*$, $\rho' = \sum_{P \in X''}(\rho'(P) - 1)$, where $\rho'(P)$ is the cardinality of $f'^{-1}(P)$. By Proposition 4.1 we have $\mathrm{Pic}(X) \cong \mathrm{Pic}(X'') \oplus (\rho''(P_m) - 1)k^*$, $\rho''(P_m)$ is the cardinality of $f''^{-1}(P_m)$.

4.4. *Proof of Theorem* 0.5. By using Proposition 4.2 and Lemma 4.3, we can proceed by induction on the number n of the irreducible components of X as in the proof of Theorem 2.

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