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THE ENDLICH BAER SPLITTING PROPERTY

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It is well known that projective modules P are characterized by the property that each surjection $M \to P$ of modules splits. For arbitrary modules A one can ask for conditions under which each surjection $A^{(c)} \to A^{(d)}$ will split where c and d are cardinals. Modules with this property are said to have the *Baer splitting property*. If the surjection $A^{(c)} \to A^{(d)}$ splits whenever d is a finite cardinal then A is said to have the *finite Baer splitting property*. If the surjection $A^{(c)} \to A^{(d)}$ splits whenever c and d are finite cardinals then A is said to have the *endlich Baer splitting property*. Albrecht generalizes a theorem of Arnold and Lady by showing that if A satisfies mild hypotheses, then A has the Baer splitting property iff $IA \neq A$ for each proper right ideal $I \subset \mathrm{End}(A)$.

The goal of this paper is to organize what is known about the (finite, endlich) Baer splitting property by generalizing to pairs (A, P) that have the (endlich) Baer splitting property. (See definitions below.) As an application, we show that the torsion-free abelian group of finite rank has the finite Baer splitting property iff it has the endlich Baer splitting property. We cite examples to show that this result is not true of countable modules.

1. Introduction. Throughout this paper, R denotes a fixed but otherwise arbitrary associative ring, A is a right R-module, and $E = \mathrm{End}_R(A)$ denotes the ring of R-module endomorphisms of A. The term *module* will mean right R-module, $\mathscr{M}_R$ denotes the category of modules, and $\mathscr{M}_E$ denotes the category of right E-modules. Let $T_A(\cdot) = \cdot \otimes_E A$ and let $H_A(\cdot) = \mathrm{Hom}_R(A, \cdot)$. The module G is (*finitely*) A*-generated* if there is a (finite) subset $H \subset H_A(G)$ such that $G = \sum\{f(A) | f \in H\}$.

Fix a pair (A, P) of modules, and consider the statements

(I) If $g: G \to P$ is a surjection of modules such that $G' + \ker g = G$ for some A-generated submodule $G' \subset G$, then g is a split surjection.

(I_0) If $g: G \to P$ is a surjection of modules such that $G' + \ker g = G$ for some finitely A-generated submodule $G' \subset G$, then g is a split surjection.

Reinhold Baer has proved that if A is a subgroup of the abelian group $\mathbf{Q}$ of rational numbers then (A, A) satisfies (I), [**15**, Proposition 86.5]. This result, known as Baer's Lemma, has assumed an important role in the study of the transfer of properties between abelian groups and their endomorphism rings.

For example, (I) is used to characterize the modules with Noetherian hereditary endomorphism rings [**3**, **7**, **13**], (I_0) is used to characterize the modules with semi-hereditary endomorphism rings, [**13**, **18**], and Hausen [**18**] uses (I_0) to study modules A with the finite summand intersection property. See the references and 8.1 for more examples.

The module P is *finitely A-projective* if P is a direct summand of $A^{(n)}$ for some integer n. Following [**4**] we say that A has the *finite Baer splitting property* if (A, P) satisfies (I) for each finitely A-projective module P. We further specialize this concept as follows. The pair (A, P) of modules has the (*endlich*) Baer splitting property if it satisfies (I) (if it satisfies (I_0)). Fix the module A. If (A, P) has the endlich Baer splitting property for each finitely A-projective module P then we say that A has the *endlich Baer splitting property.*

Now consider the properties

(II) $IA \neq A$ for each proper right ideal $I \subset E$,

(II_0) $IA \neq A$ for each proper finitely generated right ideal $I \subset E$.

Examples of modules that satisfy (II) and (II_0) include the torsion-free abelian groups A of finite rank such that E is either (semi-) hereditary, [**3**, **7**, **13**, **18**], (sub-)commutative [**7**, **13**], or local [**13**]. Arnold and Lady [**7**, Theorem 2.1] show that the torsion-free abelian group A of finite rank has the finite Baer splitting property iff it satisfies (II). Albrecht [**2**, Corollary 2.2] extends this result by showing that the self-small module A has the finite Baer splitting property iff A satisfies

(III) $T_A(M) \neq 0$ for each nonzero finitely generated $M \in \mathcal{M}_E$.

(A is a *self-small module* if for each cardinal c the canonical imbedding $\mathrm{Hom}_R(A, A)^{(c)} \to \mathrm{Hom}_R(A, A^{(c)})$ is an isomorphism.) In [**13**, Propositions 2.3, 2.4] we show that (A, P) has the endlich Baer splitting property iff A satisfies (II_0) iff $KA \neq P$ for each finitely A-projective module P and proper finitely generated E-submodule $K \subset H_A(P)$. It is then natural to consider the property

(III_0) $T_A(M) \neq 0$ for each nonzero finitely presented $M \in \mathcal{M}_E$.

Each of the aforementioned results is proved by passing to E via the functors T_A and H_A. The use of the working hypotheses in [**2**] and [**7**] leads us to believe that [**2**, Theorem 2.1] is true for a larger class of modules. Furthermore, the properties (II), (II_0), (III), and (III_0) are torsion theoretic in nature, so the logical relationships among them should be proved without invoking the Baer splitting property.

The goal of this paper is to study the relationship between the above six properties. Our approach will differ from those in [**2**], [**7**], and [**13**] in that we discuss the splitting properties without passing to the endomorphism ring, and we discuss (II) through (III_0) without explicitly mentioning the splitting properties. This approach allows us to extend the results from the literature by deleting unnecessary hypotheses.

A detailed description of the sections follows.

In §2 we prove that the endlich Baer splitting property is passed onto finite direct sums. Thus, (A, A) has the (endlich) Baer splitting property iff A has the finite (endlich) Baer splitting property 2.4, 2.6. The proof avoids the endomorphism ring.

In §3 we prove (II) $\Rightarrow$ (III) without referring to the Baer splitting property. This extends [**2**, Corollary 2.2], [**7**, Theorem 2.1] by removing the self-small hypothesis on A. We also prove (II_0) $\Rightarrow$ (III_0) which extends [**13**, Proposition 2.3] and [**18**, Theorem 2.2].

In §4 we prove that if P is an A-projective module then (A, P) has the Baer splitting property if $KA \neq P$ for each proper E-submodule $K \subset H_A(P)$, 4.2. Thus, (II) is generally a sufficient condition to imply the finite Baer splitting property for A.

The pairs (A, P) of modules possessing the endlich Baer splitting property are characterized in §5. We prove that if P is a finitely A-projective module then (A, P) has the endlich Baer splitting property iff $T_A(M) \neq 0$ for each nonzero finitely $H_A(P)$-presented $M \in \mathscr{M}_E$.

In §6 we reintroduce the self-small hypothesis on A, and then extend [**2**, Corollary 2.2] and [**7**, Theorem 2.1] to include pairs of modules (A, P). We also give a new proof of [**2**, Corollary 2.2] in which the use of the self-small hypothesis is minimized, 6.3.

Finally, in §7 we give an example of a class of modules for which the endlich and finite Baer splitting properties are equivalent. If A is a torsion-free abelian group of finite rank then A has the endlich Baer splitting property iff A has the finite Baer splitting property, 7.2.

We greatfully acknowledge the assistance of Professor C. Vinsonhaler whose insights were used to produce the given proof of 2.3.

We use the notation and terminology given in [**5**] and [**15**]. Given $M \in \mathscr{M}_E$ and $G \in \mathscr{M}_R$ there are natural homomorphisms $\Phi_M \colon M \to H_A T_A(M)$ and $\Theta_G \colon T_A H_A(G) \to G$ given by $[\Phi_M(m)](a) = m \otimes a$ and $\Theta_G(f \otimes a) = f(a)$ for each $m \in M$, $f \in H_A(G)$, and $a \in A$. Furthermore, given a projective right E-module Q and a subset $X \subset Q$ then we will identify $X = \Phi_Q(X) \subset H_A T_A(Q)$. We let M^* denote the E-dual $\mathrm{Hom}_E(M, E)$ of the right E-module M. We also make use of

THEOREM 1.1 [7, *Theorem* 1.1]. *Let* A *be a module.*

(a) H_A *and* T_A *restrict to inverse equivalences between the category of finitely* A*-projective modules, and the category of finitely generated projective right* E*-modules.*

(b) Θ_P *and* Φ_Q *are isomorphisms for each finitely* A*-projective module* P *and finitely generated projective right* E*-module* Q. □

2. Direct sums and direct summands. We show that the Baer splitting property is inherited by direct summands and passed onto direct sums.

The first lemma appears in [2].

LEMMA 2.1. *Let* A *and* P *be modules. The following are equivalent.*

(a) (A, P) *has the (endlich) Baer splitting property.*

(b) *Given a surjection* $g: G \to P$ *in* $\mathcal{M}_R$ *such that* G *is a (finitely)* A*-projective module, then* g *is a split surjection.*

(c) *Given a surjection* $g: G \to P$ *in* $\mathcal{M}_R$ *such that* G *is a (finitely)* A*-projective module, then* $\ker g$ *is a direct summand of* G. □

LEMMA 2.2. *Let* A *and* P *be modules and assume* (A, P) *has the (endlich) Baer splitting property.*

(a) *If* P *is a (finitely)* A*-generated module then* P *is a (finitely)* A*-projective module.*

(b) *If* P' *is a direct summand of* P *then* (A, P') *has the (endlich) Baer splitting property.*

Proof. (a) Is an easy exercise.

(b) Let $g: G \to P'$ be a surjection in $\mathcal{M}_R$ such that $G' + \ker g = G$ for some (finitely) A-generated submodule $G' \subset G$. Write $P = P' \oplus X$ for some submodule $X \subset P$ and let $\jmath_{P'}: P' \to P$, $p_{P'}: P \to P'$ be the canonical injection, projection. Then $g \oplus 1_X: G \oplus X \to P$ is a surjection in $\mathcal{M}_R$ such that $(G' \oplus X) + \ker(g \oplus 1_X) = G \oplus X$. Observe that X is a (finitely) A-generated module. Because (A, P) has the (endlich) Baer splitting property there is a map $j: P \to G \oplus X$ such that $(g \oplus 1_X)j = 1_P$. The usual elementary argument shows that $\jmath = \jmath' \oplus 1_X$ for some map $\jmath': P' \to G$. Then

$$g\jmath' = p_{P'}(g \oplus 1_X)(\jmath' \oplus 1_X)_{\jmath_{P'}} = p_{P'}(g \oplus 1_X)\jmath\jmath_{P'} = p_{P'}1_P\jmath_{P'} = 1_{P'}$$

as required by (b). □

PROPOSITION 2.3. *Let A be a module and let $\{P_1, \ldots, P_n\}$ be a finite set of modules such that (A, P_i) has the endlich Baer splitting property for each $i = 1, \ldots, n$. Then $(A, \bigoplus_{n=1}^n P_i)$ has the endlich Baer splitting property.*

Proof. Let $g\colon G \to \bigoplus_{i=1}^n P_i$ be a surjection such that G is a finitely A-projective module. By 2.1 it suffices to show that $\ker g$ is a direct summand of G.

We proceed by induction on n. By hypothesis (A, P_1) has the endlich Baer splitting property. Assume we have shown that $(A, \bigoplus_{i=1}^{n-1} P_i)$ has the Baer splitting property. Let $e_n\colon \bigoplus_{i=1}^n P_i \to \bigoplus_{i=1}^{n-1} P_i$ be the canonical projection, and let $G_n = \ker e_n g$. Because $e_n g\colon G \to \bigoplus_{i=1}^{n-1} P_i$ is a surjection, the induction hypothesis yields a direct sum decomposition

$$(1) \qquad G = G'_n \oplus G_n$$

for some submodule $G'_n \subset G$.

Notice that $\ker e_n g = g^{-1}(P_n)$, so $P_n = g(G_n) = (1 - e_n)g(G_n)$. By hypothesis (A, P_n) has the Baer splitting property and by (1) G_n is a finitely A-projective module, so we can write

$$(2) \qquad G_n = G''_n \oplus [G_n \cap \ker(1 - e_n)g]$$

for some submodule $G''_n \subset G_n$. Next, observe that

$$(3) \qquad G_n \cap \ker(1 - e_n)g = \ker g.$$

Finally, a combination of (1), (2), and (3) shows that $G = G'_n \oplus G''_n \oplus \ker g$, which completes the proof. □

Let P be an A-projective module. If in 2.3 we choose $\{P_1, \ldots, P_n\}$ to be distinct copies of P then using 2.2 we have proved

COROLLARY 2.4. *Let A be a module.*

(a) *The pair (A, P) of modules has the endlich Baer splitting property iff $(A, P^{(n)})$ has the endlich Baer splitting property for each integer $n > 0$.*

(b) *(A, A) has the endlich Baer splitting property iff A has the endlich Baer splitting property.* □

The same argument shows that the Baer splitting property is passed onto direct sums.

PROPOSITION 2.5. *Let A be a module and let $\{P_1, \dots, P_n\}$ be a finite set of modules such that (A, P_i) has the Baer splitting property for each $i = 1, \dots, n$. Then $(A, \bigoplus_{i=1}^n P_i)$ has the Baer splitting property.* □

COROLLARY 2.6. *Let A be a module.*

(a) *(A, P) has the Baer splitting property iff $(A, P^{(n)})$ has the Baer splitting property for each integer $n > 0$.*

(b) *(A, A) has the Baer splitting property iff A has the finite Baer splitting property.* □

REMARK 2.7. In the proof of [7, Theorem 2.1] the authors leave it to the reader to prove that the torsion-free abelian group A of finite rank has the finite Baer splitting property if (A, A) has the Baer splitting property. In proving [2, Corollary 2.2] the author states that when A is a self-small module, then this implication follows from an induction on the A-rank of P. (The A-rank of P is the least cardinal number c such that P is a direct summand of $A^{(c)}$.) In [18, Theorem 2.2] it is stated that the proof given in [2] works equally well to prove that the module A has the endlich Baer splitting property if (A, A) has the endlich Baer splitting property. Thus 2.4 and 2.6 extend these results and fill a small gap.

3. $\mathcal{T}_A$-compressed E-modules. We use a divisibility property for finitely generated projective left E-modules to characterize the finitely generated $M \in \mathcal{T}_A$.

Let $\mathcal{T}_A$ denote the class of $M \in \mathcal{M}_E$ such that $T_A(M) = 0$, let $\mathcal{D}(A)$ denote the set of right ideals $I \subset E$ such that $IA = A$. Let $K \subset M$ be an E-submodule. If $M/K \in \mathcal{T}_A$ then K is called *$\mathcal{T}_A$-dense in M*. Observe that $I \in \mathcal{D}(A)$ iff I is $\mathcal{T}_A$-dense in E. If M does not contain a proper (finitely generated) $\mathcal{T}_A$-dense E-submodule then M is *(finitely) $\mathcal{T}_A$-compressed*.

The following lemma contains some elementary facts that we will use throughout this section. The proof is left to the reader.

LEMMA 3.1. *Let A be a module, and let $M \in \mathcal{M}_E$.*

(a) *$\mathcal{T}_A$ is closed under direct sums, homomorphic images, and extensions (i.e. $\mathcal{T}_A$ is a torsion class in $\mathcal{M}_E$).*

(b) *Let $K \subset K' \subset M$ be E-submodules, and let K be $\mathcal{T}_A$-dense in M. Then K' is $\mathcal{T}_A$-dense in M.*

(c) *If M is (finitely) $\mathcal{T}_A$-compressed and if N is a direct summand of M then N is (finitely) $\mathcal{T}_A$-compressed.* □

The first result shows that (finite) $\mathcal{T}_A$-compression is passed onto direct sums.

LEMMA 3.2. *Let Q be a finitely generated projective right E-module.*

(a) *Q is finitely $\mathcal{T}_A$-compressed iff $Q^{(n)}$ is finitely $\mathcal{T}_A$-compressed for each integer $n > 0$.*

(b) *Q is $\mathcal{T}_A$-compressed iff $Q^{(n)}$ is $\mathcal{T}_A$-compressed for each integer $n > 0$.*

Proof. (a) Assume Q is finitely $\mathcal{T}_A$-compressed. This is the basis for an induction on n. Assume that we have shown that $Q^{(n-1)}$ is finitely $\mathcal{T}_A$-compressed.

Let $K \subset Q^{(n)}$ be a finitely generated $\mathcal{T}_A$-dense E-submodule, fix a canonical direct sum decomposition $Q \oplus Q^{(n-1)} = Q^{(n)}$, let K' be the projection of K into Q, and let $K'' = K \cap Q^{(n-1)}$. There is an exact sequence

$$0 \to \frac{Q^{(n-1)}}{K''} \to \frac{Q^{(n)}}{K} \to \frac{Q}{K'} \to 0 \tag{4}$$

of right E-modules. By 3.1(a) $Q/K' \in \mathcal{T}_A$, so K' is a finitely generated $\mathcal{T}_A$-dense E-submodule of Q, and hence $K' = Q$ by hypothesis. Then (4) implies that

$$\frac{Q^{(n-1)}}{K''} \cong \frac{Q^{(n)}}{K},$$

so that K'' is a $\mathcal{T}_A$-dense submodule of $Q^{(n-1)}$. Inasmuch as K and $Q^{(n-1)}$ are finitely generated, Schanuel's Lemma shows that K'' is a finitely generated E-submodule of $Q^{(n-1)}$. Then $K'' = Q^{(n-1)}$ by induction, and therefore $K = Q^{(n)}$. The converse is clear so the proof is complete.

(b) is proved in an analogous manner. □

The following result shows that (finite) $\mathcal{T}_A$-compression is a generalization of (III) and (III_0). The right E-module N is called *finitely M-generated* if there is a surjection $M^{(n)} \to N$ for some integer n. The right E-module N is called *finitely M-presented* if there is an integer $n > 0$ and a finitely generated E-submodule $K \subset M^{(n)}$ such that $N \cong M^{(n)}/K$.

COROLLARY 3.3. *Let Q be a finitely generated projective right E-module.*

(a) *Q is finitely $\mathcal{T}_A$-compressed iff $T_A(M) \neq 0$ for each nonzero finitely Q-presented $M \in \mathcal{M}_E$.*

(b) *Q is $\mathcal{T}_A$-compressed iff $T_A(M) \neq 0$ for each nonzero finitely Q-generated $M \in \mathcal{M}_E$.*

Proof. (a) By 3.2(a) and the definitions, Q is finitely $\mathcal{T}_A$-compressed iff $Q^{(n)}$ is finitely $\mathcal{T}_A$-compressed for each integer $n > 0$ iff $T_A(M) \neq 0$ for each nonzero finitely Q-presented $M \in \mathcal{M}_E$.

(b) follows in a manner similar to (a) but appeal to 3.2(b) instead of 3.2(a). □

The *trace ideal of* Q in E is $\tau_Q = \sum f(Q)$ where the sum is indexed by $f \in \operatorname{Hom}_E(Q, E)$. Our characterization of the projective finitely $\mathcal{T}_A$-compressed right E-modules Q is in terms of τ_Q.

Proposition 3.4. *Let A be a module, and let Q be a finitely generated projective right E-module. The following are equivalent.*

(a) *Q is not finitely $\mathcal{T}_A$-compressed.*

(b) *There exists a finitely generated $I \in \mathcal{D}(A)$ such that $(E/I)\tau_Q = E/I$.*

Proof. (a) $\Rightarrow$ (b) Assume there is a finitely generated $\mathcal{T}_A$-dense E-submodule $K \subset Q$. Let $\{x_1, \dots, x_n\} \subset Q$ map onto a minimal set of generators for Q/K, and let $K' = K + \sum_{i=2}^n x_i E$. Then K' is a finitely generated $\mathcal{T}_A$-dense E-submodule of Q, 3.1(b), and because $\{\overline{x}_1, \dots, \overline{x}_n\}$ is a minimal set of generators of Q/K'', $Q/K' \in \mathcal{T}_A$ is a nonzero cyclic right E-module. There is a right ideal $I \in \mathcal{D}(A)$ such that $Q/K' \cong E/I$, and Schanuel's Lemma shows that I is finitely generated. Inasmuch as $Q\tau_Q = Q$ we have $(E/I)\tau_Q = E/I$, which proves (b).

(b) $\Rightarrow$ (a) Assume there is a proper finitely generated $I \in \mathcal{D}(A)$ such that $(E/I)\tau_Q = E/I$. Then $E/I = (\tau_Q + I)/I$ so that there is a cardinal c and a surjection $f\colon Q^{(c)} \to E/I$. Because E/I is a cyclic right E-module we may assume that c is finite. Then $\ker f$ is finitely generated by Schanuel's Lemma. Furthermore, $Q^{(c)}/\ker f \cong E/I \in \mathcal{T}_A$, so that $\ker f$ is a proper finitely generated $\mathcal{T}_A$-dense E-submodule of $Q^{(c)}$, and hence $Q^{(c)}$ is not finitely $\mathcal{T}_A$-compressed. Then by 3.1(c) Q is not finitely $\mathcal{T}_A$-compressed. □

In contrast to 3.4, the $\mathcal{T}_A$-compressed property in Q is characterized in terms of the dual $Q^* = \operatorname{Hom}_E(Q, E)$ of Q. Let $\mathbf{m}_A$ denote the set of maximal right ideals $I \in \mathcal{D}(A)$. The *left* E-module L is *$\mathbf{m}_A$-divisible* if $IL = L$ for each $I \in \mathbf{m}_A$.

PROPOSITION 3.5. *Let A be a module, let Q be a finitely generated projective right E-module. The following are equivalent.*

(a) *Q is $\mathcal{T}_A$-compressed.*

(b) *$\tau_Q \subset I$ for each $I \in \mathbf{m}_A$.*

(c) *Q^* is $\mathbf{m}_A$-divisible.*

Proof. (a) $\Rightarrow$ (b) We prove the contrapositive. Assume there is an $I \in \mathbf{m}_A$ such that $\tau_Q \not\subset I$, and let $p\colon E \to E/I$ be the natural projection map. There is a map $f\colon Q \to E$ such that $f(Q) \not\subset I$, and because I is a maximal right ideal of E, $f(Q) + I = E$. Hence $pf\colon Q \to E/I$ is a surjection. Furthermore, $E/I \in \mathcal{T}_A$, so $\ker pf$ is a proper $\mathcal{T}_A$-dense E-submodule of Q, and hence Q is not $\mathcal{T}_A$-compressed.

(b) $\Leftrightarrow$ (c) It is well known that $IQ^* = Q^*$ implies $\tau_{Q^*} \subset I$, and that $\tau_Q = \tau_{Q^*}$. (See e.g. [1].) Thus (b) is equivalent to (c).

(b) $\Rightarrow$ (a) Again, we prove the contrapositive. Assume there is an integer $n > 0$ and a proper $\mathcal{T}_A$-dense E-submodule $K \subset Q$. Because Q is finitely generated there is a maximal right E-submodule $K \subset K' \subset Q$, and by 3.1(b) K' is $\mathcal{T}_A$-dense in Q. Since $Q/K' \neq 0$ is a simple module there is a maximal right ideal $I \subset E$ such that $Q/K' \cong E/I$. Since I is then $\mathcal{T}_A$-dense in E, $I \in \mathbf{m}_A$. Finally, $Q\tau_Q = Q$ so $(E/I)\tau_Q = E/I \neq 0$. Hence $\tau_Q \not\subset I$, which completes the proof. □

We are interested in when the projective modules of the form $H_A(P)$ are finitely $\mathcal{T}_A$-compressed right E-modules. The next result shows that the (finitely) $\mathcal{T}_A$-compressed property is a generalization of (II) and (II_0).

PROPOSITION 3.6. *Let A be a module and let P be a finitely A-projective module. The following are equivalent.*

(a) *$H_A(P)$ is finitely $\mathcal{T}_A$-compressed.*

(b) *$T_A(M) \neq 0$ for each nonzero finitely $H_A(P)$-presented $M \in \mathcal{M}_E$.*

(c) *$KA \neq P$ for each proper finitely generated E-submodule $K \subset H_A(P)$.*

Proof. (a) $\Leftrightarrow$ (b) follows from 3.3 because $H_A(P)$ is a finitely generated projective right E-module, 1.1.

(a) $\Leftrightarrow$ (c) Let $K \subset H_A(P)$ be an E-submodule, let $\iota\colon K \to H_A(P)$ be the inclusion map, and observe that $\Theta_P T_A(\iota)\colon T_A(K) \to P$ has image KA. Because Θ_P is an isomorphism, 1.1(b), it follows that K

is $\mathcal{T}_A$-dense in $H_A(P)$ iff $0 = T_A(H_A(P)/K) = \operatorname{coker} T_A(\iota)$ iff $T_A(\iota)$ is a surjection iff $KA = P$. □

The $\mathcal{T}_A$-compressed E-modules of the form $H_A(P)$ are characterized in

PROPOSITION 3.7. *Let A be a module and let P be a finitely A-projective module. The following are equivalent.*

(a) *$H_A(P)$ is $\mathcal{T}_A$-compressed.*

(b) *$T_A(M) \neq 0$ for each nonzero finitely $H_A(P)$-generated $M \in \mathcal{M}_E$.*

(c) *$KA \neq P$ for each proper E-submodule $K \subset H_A(P)$.*

(d) *$\mathrm{Hom}_R(P, A)$ is an $\mathbf{m}_A$-divisible left E-module.*

Proof. (a) $\Leftrightarrow$ (b) is 3.3(b), and (a) $\Leftrightarrow$ (c) follows as in 3.6.

(a) $\Leftrightarrow$ (d) By 3.5 it suffices to show that $\mathrm{Hom}_R(P, A)$ is the E-dual of $H_A(P)$. But this follows from the isomorphism $T_A H_A(P) \cong P$, 1.1(b), and the adjoint isomorphism

$$\begin{aligned}\mathrm{Hom}_R(P, A) &\cong \mathrm{Hom}_R(T_A H_A(P), A)\\ &\cong \mathrm{Hom}_E(H_A(P), H_A(A)) = H_A(P)^*.\end{aligned}$$ □

If $\mathcal{T}_A$ is a hereditary torsion class, (i.e. if $\mathcal{T}_A$ is closed under E-submodules), then $\mathcal{T}_A = \{0\}$ iff A satisfies (III). Thus the following is not without interest.

COROLLARY 3.8. *Let A be a module.*

(a) *A satisfies* (II_0) *iff A satisfies* (III_0).

(b) *A satisfies* (II) *if A satisfies* (III).

Proof. (a) follows immediately from 3.6, (from 3.7), since each finitely presented right E-module is finitely $H_A(A)$-presented.

(b) is proved in a similar manner, but appeal to 3.7 instead of 3.6. □

4. Sufficient conditions. The $\mathcal{T}_A$-compressed condition provides a test for the (endlich) Baer splitting property.

LEMMA 4.1. *Let A be a module and let P be a finitely A-projective module. If $H_A(P)$ is finitely $\mathcal{T}_A$-compressed then (A, P) has the endlich Baer splitting property.*

Proof. Let G be a finitely A-generated module, let $g: G \to P$ be an epimorphism in $\mathcal{P}_A$, and let $M = \operatorname{coker} H_A(g)$. By 2.1 it suffices to show that g is a split surjection. An application of T_AH_A to g yields a commutative diagram

$$\begin{array}{ccccccc} T_AH_A(G) & \longrightarrow & T_AH_A(P) & \longrightarrow & T_A(M) & \longrightarrow & 0 \\ \Theta_G\downarrow & & \downarrow\Theta_P & & \downarrow\theta & & \\ G & \underset{g}{\longrightarrow} & P & \longrightarrow & 0 & & \end{array}$$

with exact rows in $\mathscr{M}_R$. Now Θ_P is an isomorphism, 1.1(b), and because G is a finitely A-generated module Θ_G is a surjection. Then a diagram chase shows that $\ker\theta = 0$, and hence $T_A(M) = 0$. Because $H_A(P)$ is finitely $\mathscr{T}_A$-compressed $\operatorname{coker} H_A(g) = M = 0$, so that $H_A(g)$ is a surjection. Since $H_A(P)$ is a projective right E-module, 1.1(a), there is a map $\imath\colon H_A(P) \to H_A(C)$ such that $H_A(g)\imath = 1_{H_A(P)}$. Then

$$T_AH_A(g)T_A(\imath) = T_A(1_{H_A(P)}) = 1_{T_AH_A(P)},$$

and because the diagram commutes

$$[g\Theta_G]T_A(\imath)\Theta_P^{-1} = [\Theta_P T_AH_A(g)]T_A(\imath)\Theta_P^{-1} = \Theta_P 1_{T_AH_A(P)}\Theta_P^{-1} = 1_P.$$

Therefore, (A, P) has the endlich Baer splitting property. □

A similar argument proves

LEMMA 4.2. *Let A be a module and let P be an A-projective module such that $H_A(P)$ is a projective right E-module and such that Θ_P is an isomorphism. If $H_A(P)$ is $\mathscr{T}_A$-compressed then (A, P) has the Baer splitting property.* □

COROLLARY 4.3. *Let A be a module.*
(a) *If A satisfies* (II_0) *then A has the endlich Baer splitting property.*
(b) *If A satisfies* (II) *then A has the finite Baer splitting property.*

Proof. (a) Let P be a finitely A-projective module, and let $P\oplus P' = A^{(n)}$ for some module P' and integer $n > 0$. Because $IA \neq A$ for each proper finitely generated right ideal $I \subset E$, $H_A(A^{(n)})$ is finitely $\mathscr{T}_A$-compressed, 3.6, so that $H_A(P)$ is finitely $\mathscr{T}_A$-compressed, 3.1(c). Thus (A, P) has the endlich Baer splitting property, 4.1, and hence A has the endlich Baer splitting property.
(b) Proceed as in part (a). □

5. The endlich Baer splitting property. Given a finitely A-projective module P let τ_P be the trace ideal of the finitely generated projective right E-module $H_A(P)$ in E.

THEOREM 5.1. *Let A be a module, and let P be a finitely A-projective module. The following are equivalent for the pair (A, P).*

(a) *(A, P) has the endlich Baer splitting property.*

(b) *$H_A(P)$ is finitely $\mathcal{T}_A$-compressed.*

(c) *If $E \neq I \in \mathcal{D}(A)$ is finitely generated then $(E/I)\tau_P \neq E/I$.*

Proof. (b) $\Leftrightarrow$ (c) is 3.4, and (b) $\Rightarrow$ (a) follows from 4.1.

(a) $\Rightarrow$ (b) Let $K \subset H_A(P)$ be a finitely generated $\mathcal{T}_A$-dense E-submodule, and choose a finitely generated projective module Q and a map $k: Q \to H_A(P)$ such that Image $k = K$. Because

$$\operatorname{coker} T_A(k) = T_A(\operatorname{coker} k) = T_A(H_A(P)/K) = 0,$$

$T_A(k): T_A(Q) \to T_AH_A(P)$ is a surjection in $\mathcal{M}_R$. Furthermore, by 1.1(a) $T_AH_A(P) \cong P$ and $T_A(Q)$ is a finitely A-projective module, so (a) implies that $T_A(k)$ is a split surjection. Finally, an application of H_A to $T_A(k)$ yields a commutative diagram

$$\begin{array}{ccc} Q & \xrightarrow{k} & H_A(P) \\ \Phi_Q \downarrow & & \downarrow \Phi_{H_A(P)} \\ H_AT_A(Q) & \xrightarrow[H_AT_A(k)]{} & H_AT_AH_A(P) \end{array}$$

in $\mathcal{M}_E$ whose bottom row is a (split) surjection. Inasmuch as Φ_Q and $\Phi_{H_A(P)}$ are isomorphisms, 1.1(b), k is a surjection in $\mathcal{M}_E$, and hence $K = H_A(P)$. This proves (b) and completes the proof. □

COROLLARY 5.2. *The following are equivalent for a module A.*

(a) *(A, A) has the endlich Baer splitting property.*

(b) *A has the endlich Baer splitting property.*

(c) *A satisfies* (II_0).

(d) *A satisfies* (III_0).

Proof. (a) $\Leftrightarrow$ (b) is 2.4(b), (a) $\Leftrightarrow$ (c) is 5.1, and (c) $\Leftrightarrow$ (d) is 3.8(a). □

6. Self-small modules. We extend [**2**, Corollary 2.2] to include pairs of modules (A, P).

Arnold and Murley [**8**] prove that if A is a self-small module then H_A and T_A restrict to inverse equivalences between the category of A-projective modules, and the category of projective right E-modules. Furthermore, Θ_P and Φ_Q are isomorphisms for each A-projective module P and projective right E-module Q.

THEOREM 6.1. *Let A be a self-small module and let P be an A-projective module. The following are equivalent for the pair (A, P).*

(a) *(A, P) has the Baer splitting property.*

(b) *$H_A(P)$ is $\mathcal{T}_A$-compressed.*

Proof. (a) $\Rightarrow$ (b) Let $K \subset H_A(P)$ be a $\mathcal{T}_A$-dense E-submodule. Proceed as in 5.1(a) $\Rightarrow$ (d) to show that $K = H_A(P)$ but appeal to **[8]** instead of 1.1 to prove that $\Phi_{H_A(P)}$ and Φ_Q are isomorphisms. (b) $\Rightarrow$ (a) is 4.2. □

COROLLARY 6.2. *Let A be a self-small module, and let P be a finitely A-projective module. Then (A, P) has the Baer splitting property iff* $\mathrm{Hom}_R(P, A)$ *is $\mathbf{m}_A$-divisible.*

Proof. Because P is finitely A-projective, $\mathrm{Hom}_R(P, A)$ is $\mathbf{m}_A$-divisible iff $H_A(P)$ is $\mathcal{T}_A$-compressed, 3.7, iff (A, P) has the Baer splitting property, 6.1. □

The next result is **[2**, Corollary 2.2**]**. The self-small hypothesis is only used to prove (a) $\Rightarrow$ (d).

COROLLARY 6.3. *The following are equivalent for a self-small module A.*

(a) *(A, A) has the Baer splitting property.*

(b) *A has the finite Baer splitting property.*

(c) *A satisfies* (II).

(d) *A satisfies* (III).

Proof. (d) $\Rightarrow$ (c) is clear, (c) $\Rightarrow$ (a) is 4.3(b), and (a) $\Rightarrow$ (b) is 2.6(b). (b) $\Rightarrow$ (d) is 6.1 since each $M \in \mathcal{M}_E$ is $H_A(A)$-generated. □

7. Torsion-free Abelian groups. We show that for torsion-free abelian groups of finite rank the endlich Baer splitting property and the finite Baer splitting property are equivalent.

THEOREM 7.1. *Let A be a torsion-free abelian group of finite rank, and let P be a finitely A-projective module. Then (A, P) has the endlich Baer splitting property iff (A, P) has the Baer splitting property.*

Proof. Assume (A, P) has the endlich Baer splitting property and recall that a torsion-free abelian group of finite rank is self-small. Let $I \in \mathbf{m}_A$. Then **[13**, Lemma 3.1a**]** states that E/I is a finite group, so there is an integer $n > 0$ such that $nE \subset I$. Since E/nE is bounded and since E has finite rank, E/nE is a finite group. Choose representatives $x_1, \ldots, x_k$ of the finitely many cosets in I/nE, and observe that $\{x_1, \ldots, x_k, n1_A\}$ is a finite set of generators for I.

Now let τ_P denote the trace ideal of $H_A(P)$ in E and let $E \neq I \in \mathbf{m}_A$. Because (A, P) has the endlich Baer splitting property and because $E \neq I$ is finitely generated $(E/I)\tau_P \neq E/I$, 5.1. Then $\tau_P \subset I$ because I is a maximal right ideal of E. Thus 3.5 and 6.1 imply that (A, P) has the Baer splitting property. The converse is clear so the proof is complete. □

COROLLARY 7.2. *The following are equivalent for the torsion-free abelian group A of finite rank.*

(a) *A has the endlich Baer splitting property.*

(b) *A has the finite Baer splitting property.*

(c) *A satisfies* (III_0).

(d) *A satisfies* (III).

Proof. Use 5.2, 6.3, and 7.1. □

8. Examples.

REMARK 8.1. (a) Let c be an infinite cardinal and let $A = R^{(c)}$. Then A has the Baer splitting property, and by 5.2 A satisfies (III_0). However, let Δ be the ideal consisting of the $f \in E$ such that $f(A)$ is a finitely generated module. Then Δ is a proper ideal of E such that $\Delta A = A$. Thus the converse to 4.3 is not true in general.

(b) $A = \mathbf{Q}$ has the Baer splitting property, but $A = \mathbf{Z}_{p^\infty}$ does not have the endlich Baer splitting property.

(c) The main theorems in [**10**, **11**, **14**] show that each cotorsion-free (respectively, countable, reduced torsion-free finite rank) ring E is the endomorphism ring of a cotorsion-free (respectively, countable, reduced torsion-free finite rank) E-flat abelian group A that satisfies (II), and hence satisfies (II_0).

(d) In [**14**, Example 4.7] there is constructed an E-flat torsion-free abelian group A of rank 8 such that E is a (noncommutative) domain, and such that the least right ideal Δ in $\{I \subset E | IA = A\}$ is a proper idempotent ideal of finite index in E. Arnold and Lady [7] give an example of a completely decomposable abelian group A of rank 2 such that $IA = A$ for some proper pure ideal $I \subset E$.

(e) Let $\Omega(E)$ be the class of torsion-free abelian groups A of finite rank such that $E \cong \mathrm{End}(A)$. [**12**, Theorem 7.1] characterizes the reduced torsion-free finite rank rings E such that each $A \in \Omega(E)$ satisfies (II).

(f) In **[11]** there is given an example of a reduced self-small torsion-free abelian group A such that $\mathbf{Z}_p[X] = E$, (a countable commutative Noetherian integral domain), A satisfies (II), but $T_A(M) = 0$ for some nonzero $M \in \mathscr{M}_E$. The module M is not finitely generated.

(g) In **[11]** there is given an example of a reduced self-small torsion-free abelian group A such that E is a countable commutative Noetherian integral domain but $IA = A$ for each of the infinitely many maximal right ideals $I \subset E$.

REMARK 8.2. Statements (II) and (III) appear as working hypotheses in many works concerning properties of endomorphism rings. For example, Arnold **[6]** calls A *finitely faithful* if $IA \neq A$ for each maximal right ideal I of finite index in E. The module A satisfying statement (III) is called *completely faithful* by Fuller **[16]**, a *weak generator* by Azumaya **[9]** and Sato **[19]**, and *fully faithful* in **[2]**. Garcia and Saorin **[17]** call A *intrinsically projective* if $K = \mathrm{Hom}_E(A, KA)$ for each E-submodule K of a finitely generated projective right E-module, and Wisbauer **[21]** calls the module A an *ideal module* if the assignment $I \mapsto IA$ defines a bijection from the set of right ideals of E onto the set of A-generated submodules of A. These properties are equivalent to (II) when A is a flat left E-module, **[13]**, or a Σ-quasi-projective module, **[17]**.

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