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DETERMINANT IDENTITIES

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A number of determinants are evaluated in closed form including

$$\det\left(\binom{i+j+x}{2i-j}+\binom{i+j+y}{2i-j}\right)_{0\le i,j\le n-1}.$$

1. Introduction. In one of their series of papers on plane partitions and related questions, Mills, Robbins and Rumsey [**9**; p. 53] prove the following determinant formula.

$$m_n(x)=\det\left(\binom{i+j+x}{2i-j}\right)_{0\le i,j\le n-1}=\frac{1}{2^n}\prod_{k=0}^{n-1}\Delta_{2k}(2x),\tag{1.1}$$

where $\Delta_0(u)=2$ and for $j>0$

$$\Delta_{2j}(u)=\frac{(u+2j+2)_j(\frac{1}{2}u+2j+\frac{3}{2})_{j-1}}{(j)_j(\frac{1}{2}u+j+\frac{3}{2})_{j-1}}\tag{1.2}$$

with

$$(A)_j=A\cdot(A+1)\cdots(A+j-1).\tag{1.3}$$

Our object here is primarily to prove the following generalization of (1.1).

THEOREM 1. *Let*

$$M_n(x,y)=\det\left(\binom{i+j+x}{2i-j}+\binom{i+j+y}{2i-j}\right)_{0\le i,j\le n-1};\tag{1.4}$$

$$N_n(x,y)=\det\left(\frac{2}{x+1-y}\left\{\binom{i+j+x+1}{2i-j+1}-\binom{i+j+y}{2i-j+1}\right\}\right)_{0\le i,j\le n-1}.\tag{1.5}$$

Then

$$M_n(x,y)=N_n(x,y)=\prod_{k=0}^{n-1}\Delta_{2k}(x+y).\tag{1.6}$$

Sections 2 and 3 will be devoted to the proof of this result. In §§4 and 5 we shall show how our work leads to two alternative proofs of the T.S.S.C.P.P. conjecture [2], and we shall mention a related application of Ishikawa [6].

2. Bailey's balanced ${}_4F_3$ summation. In this section, we consider summation formulas for hypergeometric series [4; p. 8], [10, p. 41]:

$$(2.1)\qquad {}_{n+1}F_n\left[\begin{matrix} a_0, a_1, \ldots, a_n; t \\ b_1, \ldots, b_n \end{matrix}\right] = \sum_{j=0}^{\infty} \frac{(a_0)_j (a_1)_j \cdots (a_n)_j t^n}{j!(b_1)_j \cdots (b_n)_j}.$$

The formula of Bailey [3; p. 512, (c)] [10; p. 245, (III.20)] alluded to in the title of this section is

$$(2.2)\qquad {}_4F_3\left[\begin{matrix} \frac{a}{2}, \frac{a+1}{2}, b+n, -n; 1 \\ \frac{b}{2}, \frac{b+1}{2}, a+1 \end{matrix}\right] = \frac{(b-a)_n}{(b)_n},$$

and a closely related companion is

$$(2.3)\qquad {}_4F_3\left[\begin{matrix} \frac{a}{2}, \frac{a+1}{2}, b+n, -n; 1 \\ \frac{b+1}{2}, \frac{b+2}{2}, a \end{matrix}\right] = \frac{(b-a+1)_n}{(b+1)_{n-1}(b+2n)}.$$

Also useful for our work is a transformation due to F.J.W. Whipple [14; p. 537, eq. (10.1)]. If one of z and n is a nonnegative integer, then

$$(2.4)\qquad {}_4F_3\left[\begin{matrix} a, b, -z, -n; 1 \\ u, v, w \end{matrix}\right] = \frac{\Gamma(v+z+n)\Gamma(w+z+n)\Gamma(v)\Gamma(w)}{\Gamma(v+z)\Gamma(v+n)\Gamma(w+n)\Gamma(w+z)} \times {}_4F_3\left[\begin{matrix} u-a, u-b, -z, -n; 1 \\ 1-v-z-n, 1-w-z-n, u \end{matrix}\right].$$

We note in passing that (2.4) specialized to $a=\alpha/2$, $b=(\alpha+1)/2$, $z=-\beta-\nu$, $n=\nu$, $v=(\beta+1)/2$, $w=(\beta+2)/2$, $u=\alpha$ yields

$$(2.5)\qquad {}_4F_3\left[\begin{matrix} \frac{\alpha}{2}, \frac{\alpha+1}{2}, \beta+\nu, -\nu; 1 \\ \frac{\beta+1}{2}, \frac{\beta+2}{2}, \alpha \end{matrix}\right] = \frac{\beta}{\beta+2\nu}\, {}_4F_3\left[\begin{matrix} \frac{\alpha-1}{2}, \frac{\alpha}{2}, \beta+\nu, -\nu; 1 \\ \frac{\beta+1}{2}, \frac{\beta}{2}, 1+(\alpha-1) \end{matrix}\right] = \frac{\beta}{\beta+2\nu}\,\frac{(\beta-\alpha+1)_\nu}{(\beta)_\nu} \quad \text{(by (2.2))}.$$

Thus (2.3) is an immediate consequence of (2.2) and (2.4).

LEMMA 1. *Let α be a positive integer, then*

$$(2.6)\qquad {}_4F_3\left[\begin{matrix} -\frac{\alpha}{2}, -\frac{\alpha-1}{2}, b+z, -z; 1 \\ \frac{b}{2}, \frac{b+1}{2}, 1-\alpha \end{matrix}\right] = \frac{(b+z)_\alpha}{(b)_\alpha} + \frac{(-z)_\alpha}{(b)_\alpha}.$$

Proof. We begin by considering (2.2) when $a = -\alpha$ a negative integer. The index of summation j runs from 0 to n (when $j > n$ all terms $= 0$). Furthermore, the terms with $\alpha/2 < j < \alpha$ are all identically zero. For $n \geq j \geq \alpha$ there are cancelling zeros in numerator and denominator.

Thus by (2.2)

$$\begin{aligned}\frac{(b+\alpha)_n}{(b)_n} &= \sum_{0\leq 2j\leq\alpha} \frac{\left(-\frac{\alpha}{2}\right)_j \left(\frac{1-\alpha}{2}\right)_j (b+n)_j(-n)_j}{j!\left(\frac{b}{2}\right)_j\left(\frac{b+1}{2}\right)_j(1-\alpha)_j} \\ &\quad + \sum_{j=\alpha}^{n} \frac{(-\alpha)(-\alpha+1)\cdots(-1)(1)(2)\cdots(-\alpha+2j-1)(b+n)_j(-n)_j}{j!(b)_{2j}(1-\alpha)(2-\alpha)\cdots(-1)\cdot(1)\cdot(2)\cdots(j-\alpha)} \\ &= S_1 + S_2.\end{aligned}$$

Now

$$\begin{aligned} S_2 &= \sum_{j=0}^{n-\alpha} \frac{(-1)^\alpha \alpha!(\alpha+2j-1)!(b+n)_{j+\alpha}(-n)_{j+\alpha}}{(j+\alpha)!(b)_{2j+2\alpha}(-1)^{\alpha-1}(\alpha-1)!j!} \\ &= -\frac{(b+n)_\alpha(-n)_\alpha}{(b)_{2\alpha}} \sum_{j=0}^{n-\alpha} \frac{(\alpha)_{2j}(b+n+\alpha)_j(-n+\alpha)_j}{j!(b+2\alpha)_{2j}(\alpha+1)_j} \\ &= -\frac{(b+n)_\alpha(-n)_\alpha}{(b)_{2\alpha}} {}_4F_3\left(\begin{matrix}\frac{\alpha}{2},\ \frac{\alpha+1}{2},\ b+2\alpha+(n-\alpha),\ -(n-\alpha);\ 1\\ \frac{b+2\alpha}{2},\ \frac{b+2\alpha+1}{2},\ 1+\alpha\end{matrix}\right) \\ &= -\frac{(b+n)_\alpha(-n)_\alpha}{(b)_{2\alpha}} \frac{(b+\alpha)_{n-\alpha}}{(b+2\alpha)_{n-\alpha}} \qquad \text{(by (2.2))} \\ &= -\frac{(-n)_\alpha}{(b)_\alpha}. \end{aligned}$$

Hence

$$\sum_{0\leq 2j\leq\alpha} \frac{\left(-\frac{\alpha}{2}\right)_j \left(\frac{1-\alpha}{2}\right)_j (b+n)_j(-n)_j}{j!\left(\frac{b}{2}\right)_j\left(\frac{b+1}{2}\right)_j(1-\alpha)_j} = \frac{(b+\alpha)_n}{(b)_n} + \frac{(-n)_\alpha}{(b)_\alpha} = \frac{(b+n)_\alpha}{(b)_\alpha} + \frac{(-n)_\alpha}{(b)_\alpha}, \tag{2.7}$$

which is precisely (2.6) when z is any positive integer, n. However, both sides of (2.6) are polynomials in z of degree at most α, and since they agree for all positive integral z we see that (2.6) holds for all real z. □

LEMMA 2. *Let α be a nonnegative integer, then*

$$(2.8)\qquad {}_4F_3\left[\begin{matrix}-\frac{\alpha}{2}, \frac{-\alpha+1}{2}, b+z, -z; 1\\ \frac{b+1}{2}, \frac{b+2}{2}, -\alpha\end{matrix}\right] = \frac{(b+z)_{\alpha+1}}{(b+2z)(b+1)_\alpha} - \frac{(-z)_{\alpha+1}}{(b+1)_\alpha(b+2z)}.$$

Proof. In parallel with Lemma 1, we begin by considering (2.3) with $a = -\alpha$ a negative integer. If j is the index of summation in the ${}_4F_3$, then the nonzero terms of the sum occur for $0 \le j \le \alpha/2$ and $\alpha < j \le n$. If we call the two resulting sums T_1 and T_2, then

$$(2.9)\qquad \frac{(b+\alpha+1)_n}{(b+1)_{n-1}(b+2n)} = T_1 + T_2.$$

Now

$$\begin{aligned}
T_2 &= \sum_{j=\alpha+1}^{n} \frac{(-\alpha)_\alpha(1)_{2j-\alpha-1}(b+n)_j(-n)_j}{j!(b+1)_{2j}(-\alpha)_\alpha(1)_{j-\alpha-1}}\\
&= \sum_{j=0}^{n-\alpha-1} \frac{(1)_{2j+\alpha+1}(b+n)_{j+\alpha+1}(-n)_{j+\alpha+1}}{(j+\alpha+1)!(b+1)_{2j+2\alpha+2}j!}\\
&= \frac{(\alpha+1)!(b+n)_{\alpha+1}(-n)_{\alpha+1}}{(\alpha+1)!(b+1)_{2\alpha+2}}\\
&\quad\times {}_4F_3\left[\begin{matrix}\frac{\alpha}{2}+1, \frac{\alpha}{2}+\frac{3}{2}, b+n+\alpha+1, -n+\alpha+1; 1\\ \frac{b+2\alpha+3}{2}, \frac{b+2\alpha+4}{2}, \alpha+2\end{matrix}\right]\\
&= \frac{(b+1)_{\alpha+n}(-n)_{\alpha+1}}{(b+1)_{n-1}(b+1)_{2\alpha+2}}\frac{(b+\alpha+1)_{n-\alpha-1}}{(b+2\alpha+3)_{n-\alpha-2}(b+2n)} \qquad \text{(by (2.3))}\\
&= \frac{(-n)_{\alpha+1}}{(b+1)_\alpha(b+2n)}.
\end{aligned}$$

Hence

$$\begin{aligned}
(2.10)\qquad {}_4F_3\left[\begin{matrix}-\frac{\alpha}{2}, \frac{-\alpha+1}{2}, b+n, -n; 1\\ \frac{b+1}{2}, \frac{b+2}{2}, -\alpha\end{matrix}\right]
&= T_1 = \frac{(b+\alpha+1)_n}{(b+1)_{n-1}(b+2n)} - \frac{(-n)_{\alpha+1}}{(b+1)_\alpha(b+2n)}\\
&= \frac{(b+n)_{\alpha+1}}{(b+1)_\alpha(b+2n)} - \frac{(-n)_{\alpha+1}}{(b+1)_\alpha(b+2n)},
\end{aligned}$$

which is (2.8) when z is any positive integer. Since (2.8) is an identity of rational functions in z, the result in full generality follows immediately. □

3. The main theorem. Our proof of Theorem 1 relies on the following binomial coefficient summations.

LEMMA 3. *For integers* $i, j \geq 0$

$$(3.1)\quad 2\sum_{k=0}^{i} \frac{(y-x)}{(y-x+i-k)}\binom{y-x+i-k}{2i-2k}\binom{k+j+x+y}{2k-j} = \binom{i+j+2x}{2i-j}+\binom{i+j+2y}{2i-j}.$$

$$(3.2)\quad \sum_{k=0}^{i} \frac{(2x-2y+1)}{(y-x+i-k)}\binom{y-x+i-k}{2i-2k+1}\binom{k+j+x+y}{2k-j} = \binom{i+j+2x+1}{2i-j+1}-\binom{i+j+2y}{2i-j+1}.$$

Proof. We note that the only nonzero terms on the left-hand side of (3.1) occur for $i \geq k \geq j/2$. Consequently, if $j > 2i$ then both sides of (3.1) are zero. Hence we may assume $2i \geq j$.

Therefore

$$(3.3)\quad 2\sum_{k=0}^{i} \frac{(y-x)}{(y-x+i-k)}\binom{y-x+i-k}{2i-2k}\binom{k+j+x+y}{2k-j}$$

$$= 2\sum_{k=0}^{i} \frac{(y-x)}{(y-x+k)}\binom{y-x+k}{2k}\binom{i+j-k+x+y}{2i-j-2k}$$

$$= 2\binom{i+j+x+y}{2i-j}\sum_{k=0}^{j} \frac{(y-x)_k(-y+x)_k(-2i+j)_{2k}}{(2k)!(-i-j-x-y)_k(2j-i+x+y+1)_k}$$

$$= 2\binom{i+j+x+y}{2i-j}\,{}_4F_3\left[\begin{matrix} y-x,\ -y+x,\ -i+\frac{j}{2},\ -i+\frac{j+1}{2};\ 1 \\ \frac{1}{2},\ -i-j-x-y,\ 2j-i+x+y+1 \end{matrix}\right]$$

$$= 2\binom{i+j+x+y}{2i-j}$$

$$\times \frac{\Gamma(i+j+x+y+1/2)\Gamma(2i-j)\Gamma(2j-i+x+y+1)\Gamma(1/2)}{\Gamma(3j/2+x+y+1)\Gamma(1/2+i-j/2)\Gamma(3j/2+x+y+1/2)\Gamma(i-j/2)}$$

$$\times\, {}_4F_3\left[\begin{matrix} -i-j-2y,\ -i-j-2x,\ -i+\frac{j}{2},\ -i+\frac{j+1}{2};\ 1 \\ -j-i-x-y+\frac{1}{2},\ -i-j-x-y,\ 1-2i+j \end{matrix}\right] \quad \text{(by (2.4))}$$

$$= \binom{i+j+x+y}{2i-j}\frac{\Gamma(i+j+x+y+1/2)\Gamma(2j-i+x+y+1)2^{2i+2j+2x+2y}}{\Gamma(1/2)\Gamma(3j+2x+2y+1)}$$

$$\times\left(\frac{(-i-j-2x)_{2i-j}}{(-2i-2j-2x-2y)_{2i-j}}+\frac{(-i-j-2y)_{2i-j}}{(-2i-2j-2x-2y)_{2i-j}}\right)$$

(by Lemma 1 and Gauss's duplication formula [5; p. 5, eq. (15)])

$$= \binom{i+j+2x}{2i-j}+\binom{i+j+2y}{2i-j}$$

as desired for (3.1).

For (3.2) again we obtain zero equals zero unless $2i \geq j$. Hence

$$(3.4)\quad \sum_{k=0}^{i} \frac{(2x-2y+1)}{(y-x+i-k)} \binom{y-x+i-k}{2i-2k+1} \binom{k+j+x+y}{2k-j}$$
$$= \sum_{k=0}^{i} \frac{(2x-2y+1)}{(y-x+k)} \binom{y-x+k}{2k+1} \binom{i-k+j+x+y}{2i-j-2k}$$
$$= (2x-2y+1) \binom{i+j+x+y}{2i-j}$$
$$\times {}_4F_3 \left[\begin{matrix} y-x,\ 1+x-y,\ -i+\frac{j}{2},\ -i+\frac{j}{2}+\frac{1}{2};\ 1 \\ \frac{3}{2},\ -i-j-x-y,\ x+y+2j-i+1 \end{matrix} \right]$$
$$= (2x-2y+1) \binom{i+j+x+y}{2i-j}$$
$$\times \frac{\Gamma(2i-j+1)\Gamma(i+j+x+y+1/2)\Gamma(3/2)\Gamma(2j-i+x+y+1)}{\Gamma(i-\frac{j}{2}+\frac{3}{2})\Gamma(i-\frac{j}{2}+1)\Gamma(\frac{3j}{2}+x+y+1)\Gamma(\frac{3j}{2}+x+y+\frac{1}{2})}$$
$$\times {}_4F_3 \left[\begin{matrix} -i-j-2y,\ -i-j-2x-1,\ -i+\frac{j}{2},\ -i+\frac{j}{2}+\frac{1}{2};\ 1 \\ -i-j-x-y,\ -2i+j,\ -i-j-x-y+\frac{1}{2} \end{matrix} \right] \quad \text{(by (2.4))}$$
$$= (2x-2y+1) \binom{i+j+x+y}{2i-j}$$
$$\times \frac{\Gamma(i+j+x+y+\frac{1}{2})2^{2i+2j+2x+2y}\Gamma(2j-i+x+y+1)}{\Gamma(\frac{1}{2})\Gamma(3j+2x+2y+1)(2i-j+1)}$$
$$\times \left(\frac{(-i-j-2x-1)_{2i-j+1}}{(2y-2x-1)(-2i-2j-2x-2y)_{2i-j}} - \frac{(-i-j-2y)_{2i-j+1}}{(2y-2x-1)(-2i-2j-2x-2y)_{2i-j}} \right)$$
$$= \binom{i+j+2x+1}{2i-j+1} - \binom{i+j+2y}{2i-j+1}. \qquad \square$$

Proof of Theorem 1. We define five matrices

$$(3.5)\qquad \mathbf{M}_n(x) = \left(\binom{i+j+x}{2i-j} \right)_{0 \leq i,j \leq n-1},$$

$$(3.6)\qquad \mu_n(x,y) = \left(\binom{i+j+x}{2i-j} + \binom{i+j+y}{2i-j} \right)_{0 \leq i,j \leq n-1},$$

$$(3.7)\qquad \nu_n(x,y) = \left(\frac{2}{x+1-y} \left(\binom{i+j+x+1}{2i-j+1} - \binom{i+j+y}{2i-j+1} \right) \right)_{0 \leq i,j \leq n-1},$$

$$(3.8)\qquad \tau_n(x,y) = \left(\frac{2(y-x)}{(y-x+i-j)} \binom{y-x+i-j}{2i-2j} \right)_{0 \leq i,j \leq n-1},$$

(3.9) $$\sigma_n(x,y) = \left(\frac{2}{(y-x+i-j)}\binom{y-x+i-j}{2i-2j+1}\right)_{0\le i,j\le n-1}.$$

Clearly

(3.10) $$m_n(x) = \det(\mathbf{M}_n(x)),$$

(3.11) $$M_n(x,y) = \det(\mu_n(x,y)),$$

(3.12) $$N_n(x,y) = \det(\nu_n(x,y)),$$

(3.13) $$\det(\tau_n(x,y)) = 2^n,$$

and

(3.14) $$\det(\sigma_n(x,y)) = 2^n.$$

Equations (3.10)–(3.12) are restatements of (1.1), (1.4) and (1.5), while (3.13) and (3.14) are obvious since each matrix in question is lower triangular.

Now

(3.15) $$\begin{aligned}&\tau_n(x,y)\mathbf{M}_n(x+y)\\ &= \left(\sum_{k=0}^{i}\frac{2(y-x)}{(y-x+i-k)}\binom{y-x+i-k}{2i-2k}\binom{k+j+x+y}{2k-j}\right)_{0\le i,j\le n-1}\\ &= \left(\binom{i+j+2x}{2i-j}+\binom{i+j+2y}{2i-j}\right)_{0\le i,j\le n-1} \qquad \text{(by Lemma 3, eq. (3.1))}\\ &= \mu_n(2x,2y).\end{aligned}$$

Hence by (3.15) and (1.1)

$$\begin{aligned}M_n(x,y) &= \det(\mu_n(x,y))\\ &= \det\left(\tau_n\left(\frac{x}{2},\frac{y}{2}\right)\mathbf{M}_n\left(\frac{x+y}{2}\right)\right)\\ &= 2^n\cdot\det\left(\mathbf{M}_n\left(\frac{x+y}{2}\right)\right)\\ &= \prod_{k=0}^{n-1}\Delta_{2k}(x+y),\end{aligned}$$

which proves the first part of (1.6).

Similarly

(3.16) $\sigma_n(x, y)\mathbf{M}_n(x+y)$

$$= \left(\sum_{k=0}^{i} \frac{2}{(y-x+i-k)}\binom{y-x+i-k}{2i-2k+1}\binom{k+j+x+y}{2k-j}\right)_{0\le i,j\le n-1}$$

$$= \left(\frac{2}{(2x-2y+1)}\left(\binom{i+j+2x+1}{2i-j+1}-\binom{i+j+2y}{2i-j+1}\right)\right)_{0\le i,j\le n-1}$$

(by Lemma 3, eq. (3.2))

$$= \nu_n(2x, 2y).$$

Therefore by (3.16) and (1.1)

$$\begin{aligned} N_n(x, y) &= \det(\nu_n(x, y)) \\ &= \det\left(\sigma_n\left(\frac{x}{2}, \frac{y}{2}\right)\mathbf{M}_n\left(\frac{x+y}{2}\right)\right) \\ &= 2^n \cdot \det\left(\mathbf{M}_n\left(\frac{x+y}{2}\right)\right) \\ &= \prod_{k=0}^{n-1} \Delta_{2k}(x+y). \end{aligned}$$ □

4. Applications to the T.S.S.C.P.P. conjecture. We shall consider in this section a few instances of the results we have obtained . We begin with a rather odd determinant for generalized harmonic numbers.

COROLLARY 1. *Let* $H_n(x) = \sum_{j=0}^{n} \frac{1}{x+j}$, *then*

$$\det\left(\binom{i+j+x}{2i-j+1}(H_{i+j}(x) - H_{2j-i-1}(x))\right)_{0\le i,j\le n-1} = \frac{1}{2^n}\prod_{k=0}^{n-1}\Delta_{2k}(2x-1). \tag{4.1}$$

Proof. From (1.5) and (1.6) with x replaced by $x-1$ we find

$$\det\left(\frac{1}{x-y}\left(\binom{i+j+x}{2i-j+1} - \binom{i+j+y}{2i-j+1}\right)\right)_{0\le i,j\le n-1} = 2^{-n}\prod_{k=0}^{n-1}\Delta_{2k}(x+y-1). \tag{4.2}$$

Now let $y \to x$, and we obtain

$$\det\left(\frac{d}{dx}\binom{i+j+x}{2i-j+1}\right)_{0\le i,j\le n-1} = 2^{-n}\prod_{k=0}^{n-1}\Delta_{2k}(2x-1). \tag{4.3}$$

Identity (4.1) is merely (4.3) after the differentiation has been completed. □

In [8], Mills, Robbins and Rumsey define "*a totally symmetric plane partition* of size n (to be) a plane partition whose three-dimensional Ferrers graph is contained in the box

$$X_n = [1, n] \times [1, n] \times [1, n]$$

and which is mapped to itself under all permtutations of the coordinate axes. The complement of the Ferrers graph of such a plane partition (that is, the set of lattice points in the box X_n that do not belong to the Ferrers graph) is again totally symmetric when viewed from the vantage point of the vertex $(n+1, n+1, n+1)$. A totally symmetric plane partition is self complementary if it is congruent (in the geometrical sense) to its complement. This cannot occur unless $n = 2m$ is even".

If we define A_n by the recurrence (4.6), then Mills et al. [8] conjecture that A_n is the total number of TSSCPP's in X_{2n}.

In [12], J. Stembridge essentially proved that the TSSCPP conjecture reduces to the following result (the details of Stembridge's result and its equivalence to the following are provided in [2; Sec. 2]). It should be noted that our proof of Corollary 2 in the final analysis relies on (1.1) and thus is quite different from the proof of the T.S.S.C.P.P. Conjecture given in [2].

COROLLARY 2.

$$\det(a_{ij})_{0\le i,j\le n-1} = A_n^2, \tag{4.4}$$

where

$$a_{ij} = \begin{cases} 1 & \text{if } i = j = 0, \\ 0 & \text{if } i = j > 0, \\ \displaystyle\sum_{s=2i-j+1}^{2j-i} \binom{i+j}{s} & \text{if } i < j, \\ -a_{ji} & \text{if } i > j, \end{cases} \tag{4.5}$$

and

$$A_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!} = \frac{(3n-2)!(n-1)!}{(2n-1)!(2n-2)!} A_{n-1}. \tag{4.6}$$

Proof. We define several new matrices:

$$w(n) = \left(\binom{i+j+1}{2j-i} + \binom{i+j}{2j-i-1}\right)_{0\le i,j\le n-1}, \tag{4.7}$$

$$u(n) = (\delta_{ij} - 2\delta_{i,j+1})_{0\le i,j\le n-1}, \tag{4.8}$$

$$v(n) = \left(\binom{i+j+2}{2j-i+1}\frac{(3i+1)(3j+1)(3j-3i)}{(i+j)(i+j+1)(i+j+2)}\right)_{0\le i,j\le n-1} \tag{4.9}$$

(where we define the $(0, 0)$-th entry of $v(n)$ to be 1),

$$\begin{aligned} u_1(n) &= u(n) + (2\delta_{(i-1)^2+j,0})_{0\le i,j\le n-1} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & -2 & 1 & 0 & \cdots \\ 0 & 0 & -2 & 1 & \cdots \end{pmatrix}_{n\times n}, \end{aligned} \tag{4.10}$$

$$st(n) = (a_{ij})_{0\le i,j\le n-1}. \tag{4.11}$$

The matrices u and u_1 are introduced to perform certain simple row and column operations. In particular, elementary algebra reveals that

$$u(n)w(n) = v(n) \tag{4.12}$$

and

$$u_1(n)st(n)(u_1(n))^T = v(n). \tag{4.13}$$

Finally, if we expand $M_{n+1}(-2, -1)$ along the top row we find

$$\begin{aligned} M_{n+1}(-2,-1) &= 2\ \det\left(\binom{i+j}{2i-j+1} + \binom{i+j+1}{2i-j+1}\right)_{0\le i,j\le n-1} \\ &= 2\ \det\left(\binom{i+j}{2j-i-1} + \binom{i+j+1}{2j-i}\right)_{0\le i,j\le n-1} \\ &\qquad \left(\text{since } \binom{A}{B} = \binom{A}{A-B}\right) \\ &= 2\ \det(w(n)). \end{aligned} \tag{4.14}$$

Consequently, since the determinants of u and u_1 are each 1, it follows from (1.6), (4.12), (4.13) and (4.14) that

$$\begin{aligned}\det(a_{ij})_{0\le i,j\le n-1} &= \det(st(n)) = \det(v(n)) \\ &= \det(w(n)) = \frac{1}{2}M_{n+1}(-2,-1) \\ &= \prod_{k=1}^{n} \Delta_{2k}(-3) = A_n^2\end{aligned} \tag{4.15}$$

because setting $u=-3$ in (1.2) we find

$$\Delta_{2k}(-3) = \left(\frac{(3k-2)!(k-1)!}{(2k-2)!(2k-1)!}\right)^2 . \tag{4.16}$$ □

5. A related identity and another proof of the T.S.S.C.P.P. conjecture. We have not found any related identities for two variables that are genuinely different from (1.6). However, there is one with one variable that merits mention. To this end we need the $\Delta_j(n)$ with odd subscript [**1**; p. 196]:

$$\Delta_{2j-1}(\mu) = \frac{(\mu+2j)_{j-1}(\frac{1}{2}\mu+2j+\frac{1}{2})_j}{(j)_j(\frac{1}{2}\mu+j+\frac{1}{2})_{j-1}}. \tag{5.1}$$

The next formula is implicit in the work of Mills-Robbins-Rumsey [**9**]. However, they do not state it so we record it here: Let

$$P_n(\mu) = \left(\binom{i+j+\mu}{2i-j} + 2\binom{i+j+\mu+2}{2i-j+1}\right)_{0\le i,j\le n-1}. \tag{5.2}$$

In the notation of [**9**; p. 50], the determinant of $P_n(\mu)$ is $R_n(1,\mu)$. This is easily seen by setting $x=1$ in their definition of $R_n(x,\mu)$ [**9**; p. 50] and applying the Chu-Vandermonde summation. Furthermore, from their Theorems 5 and 7, it is easy to see that

$$\begin{aligned}\det(P_n(\mu)) &= \frac{\det\left(\delta_{ij} + \binom{i+j+2\mu}{i}\right)_{0\le i,j\le 2n-1}}{\det\left(\binom{i+j+\mu}{2i-j}\right)_{0\le i,j\le n-1}} \\ &= \frac{2^n \prod_{k=0}^{2n-1} \Delta_k(2\mu)}{\prod_{k=0}^{n-1} \Delta_{2k}(2\mu)} = 2^n \prod_{k=1}^{n} \Delta_{2k-1}(2\mu).\end{aligned} \tag{5.3}$$

Our final result gives the determinant for

$$W_n(x) = \left(\binom{i+j+x+1}{2i-j} + \binom{i+j+x}{2i-j-1}\right)_{1\le i,j\le n}. \tag{5.4}$$

THEOREM 2.

$$\det(W_n(x)) = \prod_{k=1}^{n} \Delta_{2k-1}(2x+3). \tag{5.5}$$

Proof. We require an auxiliary matrix

$$\mathscr{S}_n(x) = \left(\frac{(-1/2)_{i-j}(-1)^{i-j}}{2^{2i-2j-1}(i-j)!} \right)_{0\le i,j\le n-1}. \tag{5.6}$$

Consequently,

$$\mathscr{S}_n(x) \cdot W_n(x) = P_n(x+3/2). \tag{5.7}$$

This is easily seen if we introduce

$$f(i,j,X) = \sum_{k=0}^{i} \frac{(-1/2)_{i-k}(-1)^{i-k}}{2^{2i-2k-1}(i-k)!} \binom{k+j+X}{2k-j}, \tag{5.8}$$

and note that as an immediate corollary of the Pfaff-Saalschultz summation [**4**; p. 9]:

$$f(i,j,X) = \binom{X+i+j-1/2}{2i-j} + \binom{X+i+j+1/2}{2i-j}. \tag{5.9}$$

Thus

$$\begin{aligned}
&\mathscr{S}_n(x)W_n(x) \\
&= \left(\sum_{k\ge 0} \frac{(-1/2)_{i-k}(-1)^{i-k}}{2^{2i-2k-1}(i-k!} \left\{ \binom{k+j+x+3}{2k-j+1} + \binom{k+j+x+2}{2k-j} \right\} \right)_{0\le i,j\le n-1} \\
&= (f(i,j-1,x+4) + f(i,j,x+2))_{0\le i,j\le n-1} \\
&= \left(\binom{x+i+j+5/2}{2i-j+1} + \binom{x+i+j+7/2}{2i-j+1} + \binom{x+i+j+3/2}{2i-j} + \binom{x+i+j+5/2}{2i-j} \right)_{0\le i,j\le n-1} \\
&= \left(2\binom{x+i+j+7/2}{2i-j+1} + \binom{x+i+j+3/2}{2i-j} \right)_{0\le i,j\le n-1} \\
&= P_n(x+3/2),
\end{aligned}$$

as asserted in (5.7). Consequently since $\det(\mathscr{S}_n(x)) = 2^n$, we see from (5.7) and (5.3) that

$$\det(W_n(x)) = \prod_{k=1}^{n} \Delta_{2k-1}(2x+3)$$

as asserted in Theorem 2. □

We note that Corollary 2 is also derivable from Theorem 2. This is because from (4.7), (5.4) and (5.5)

$$\det(w(n)) = \det(W_{n-1}(0)) = \prod_{k=1}^{n-1} \Delta_{2k-1}(3) = A_n^2,$$

since

$$\Delta_{2k-1}(3) = \frac{(3+2k)_{k-1}(2+2k)_k}{(k)_k(2+k)_{k-1}} = \left(\frac{(3k+1)!k!}{(2k)!(2k+1)!}\right)^2.$$

We also remark that other special values for $W_n(x)$ can be derived from Theorem 2 besides $\det(W_n(0)) = A_{n+1}^2$. Namely

$$\det(W_n(-1)) = H_{2n+1},$$
$$\det(W_n(-2)) = H_{2n},$$
$$\det(W_n(-3)) = A_{n-1}A_n,$$

where the sequence H_n is defined by $H_0 = 1$,

$$\frac{H_{2n+1}}{H_{2n}} = \binom{3n}{n} \Big/ \binom{2n}{n}, \quad \frac{H_{2n}}{H_{2n-1}} = \frac{4}{3}\frac{\binom{3n}{n}}{\binom{2n}{n}}.$$

The sequences A_n and H_n occur in a number of unsolved problems of Mills-Robbins-Rumsey (cf. **[11]** for a survey of the problems).

6. Conclusion. The problem of enumerating symmetry classes of plane partitions is considered extensively in **[12]**. Indeed the identity (1.1) which we rely on heavily throughout our work was used by Mills et al. to treat plane partitions in a different symmetry class **[9]**. G. Kuperberg **[7]** has recently prepared an appealing survey of this topic.

Also recently M. Ishikawa **[6]** has found a nice plane partition theoretic interpreation of $\det(v(n)) = A_n^2$ (see (4.9) and (4.15)).

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