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MAPS BETWEEN SEIFERT FIBERED SPACES OF INFINITE π_1

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A theorem of A. Edmonds says that any nonzero degree map between closed surfaces is homotopic to a composition of a pinch map and a branched covering. Here we consider the analogous problem in dimension three. We prove that any nonzero degree map between P^2-irreducible Seifert fibered spaces of infinite π_1 is homotopic to a composition of "vertical pinches" and a fiber preserving branched covering, except for a few cases which we describe completely. In particular, any such degree one map is homotopic to a composition of vertical pinches.

0. Introduction. In this paper we study nonzero degree maps between closed P^2-irreducible Seifert fibered spaces of infinite π_1, or equivalently, closed aspherical Seifert fibered spaces. We prove that any such map is homotopic to a composition of vertical pinches (defined in §1) and a fiber preserving branched covering, except for certain cases which can be completely understood (Theorem 3.2). As a corollary, any degree one map between such spaces is homotopic to a composition of vertical pinches.

The analogous theorem for surfaces was proved by A. Edmonds [**1**]. Later R. Skora gave a simplified proof using the notion of geometric degree [**6**]. Our proof uses similar ideas as theirs. Some extra work must be done to adjust the map so that it is nice with respect to the Seifert fibrations of the manifolds.

In §1 we establish terminology. Pinches and squeezes are defined by analogy with those definitions in dimension two given by A. Edmonds [**1**]. In §2 we show our map can be homotoped into an equivariant fiber preserving map, but possibly followed by a covering between Euclidean manifolds (those which have the geometry of E^3). In §3 we give an inductive proof of our main theorem.

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1. Notations and terminology. For a Seifert fibered space M, h denotes either a regular fiber or its homotopy class. Tori and annuli are often regarded as Seifert fibered without singular fibers, and h has a similar meaning in these cases. For two or more spaces which are Seifert fibered, we use the same letter h to denote the regular fibers in them if this does not cause confusion. For an element x in a group, $\langle x\rangle$ denotes the subgroup generated by x. For a submanifold S, $N(S)$ denotes the regular neighborhood of S. I always denotes a closed interval, T denotes a torus, V a solid torus, and A an annulus. The abbreviation S.F.S. stands for Seifert fibered space. The geometric degree of a proper map $f\colon M^n \to N^n$, denoted by $G(f)$, is the least number d such that for some map g properly homotopic to f, and some disk in $\mathring{N}$, $g^{-1}(D)$ consists of exactly d disks and g maps each such disk homeomorphically onto D.

We now define pinches. Let M be a closed 3-manifold, F be a 2-sided closed surface which separates M into a union of M_1 and M_2. If there is a map q from M_2 onto a handlebody H such that $q|\partial$ is a homeomorphism, then we have a degree one map f $(= \mathrm{id} \cup q)$ from M to $N = M_1 \cup_F H$. We call such a map a 1-*pinch*. When F is a 2-sphere, a 1-pinch is a pinch in the usual sense, which we may call a "0-pinch". The following lemma says that a 1-pinch can always be homotoped so that it "pinches" M_2 onto a 1-dimensional complex. The proof is simple and is omitted.

LEMMA 1.1. *If $f\colon (W, \partial W) \to (H, \partial H)$ is a map from a 3-manifold onto a handlebody such that $f|\partial$ is a homeomorphism, then f can be homotoped* $\mathrm{rel}\,\partial$ *such that f sends a collar $\partial W \times [0, 1)$ of ∂W homeomorphically onto $H - c$, and sends $W - \partial W \times [0, 1)$ onto c, where c is a core of H.*

Proof. Let $s\colon \partial W \times [0, 1] \to [0, 1]$ be the projection. Let $r_t\colon H \to H$ be a deformation retract of H so that $r_0 = \mathrm{id}$, and $r_1(H) \subset c$. Let $f_t\colon W \to H$ be defined by $f_t(x) = r_{ts(x)} \circ f(x)$. It is easy to verify f_t is the desired homotopy. □

The next lemma tells us when a 1-pinch can occur:

LEMMA 1.2. *Let W be a compact 3-manifold with $\partial W \cong F$, a connected orientable surface. Then there exists a map $f\colon W \to H$ with $f|\partial$ is a homeomorphism if and only if there are g* $(= \mathrm{genus}(F))$ *disjoint simple closed curves on F which cut F into a $2g$-punctured sphere and bound disjoint surfaces in W.*

Proof. Let $D = \bigcup D_i$ be a system of meridian disks in H. If there is such a map f, homotop $f\,\mathrm{rel}\,\partial$ so that f is transverse to D.

Then $f^{-1}(D)$ is a surface whose boundary is a collection of s.c.c. on F satisfying the conclusion.

If there is such a collection of s.c.c. $\{l_1, \dots, l_g\}$ on F such that $l_i = \partial F_i$, and F_i's are disjoint, let H be the handlebody obtained by adding 2-handles to $F \times I$ along each l_i, and then a 3-handle to cap off the 2-sphere. There is a map from $N(\bigcup F_i)$ onto the union of the 2-handles by pinching. The map can then be extended over the remaining part onto the 3-handle by the Tietze extension theorem. $\square$

A manifold W satisfying the above lemma is called *pinchable.*

If M is a Seifert fibered space and T is a separating vertical torus in M such that one side of T is pinchable, then the resulting manifold N after the pinching is again a Seifert fibered space with an induced Seifert fibration from M. This is true because in M_2, the fiber h is not null-homologous. Such a 1-pinch is called a *vertical 1-pinch* or simply a *vertical pinch.*

Next we define squeezes. Let T be an incompressible torus in a 3-manifold M with a product neighborhood $T \times I$, l be an essential simple closed curve on T. Parameterize T by $T = S^1 \times S^1$ such that $l = S^1 \times \{p\}$. Let $X = M/\sim$, where $(x, y, t) \sim (x', y, t)$ for $(x, y, t), (x', y, t) \in S^1 \times S^1 \times I$. The quotient map $q: M \to X$ is called a *squeeze.* Topologically X is M cut open along T, union two solid tori along the boundary such that each meridian is identified with a copy of l, and then union an annulus connecting the cores of the two tori. If T is a vertical torus in a Seifert fibered space, a squeeze along T is called a *vertical squeeze.*

If a map $f: M \to N$ factors through a squeeze, then we say f admits a squeeze. If N is aspherical, then f admits a squeeze along a torus T iff f sends an essential s.c.c. on T onto a null-homotopic loop in N.

2. Equivariant fiber preserving maps. Let M be a Seifert fibered space. Fix a Seifert fibration of M and regard M as an S^1-bundle over its base orbifold O_M [**5**]. The transition group is SO(2) if the bundle is orientable and is O(2) otherwise. In any case, there is a well-defined local S^1-action on M. We denote the image of the action by tx for $t \in S^1$ and $x \in M$. Globally tx is well-defined up to changing t into t^{-1}.

Let a be a path in M connecting two fibers c_0 and c_1. An S^1-action tx on c_0 extends along a to an S^1-action $t_a x$ on c_1. If a' is another arc connecting c_0 and c_1, then $t_a x = t_{a'} x$ or $t_{a'}^{-1} x$

depending whether $w_1(a \cup a') = 0$ or 1, where w_1 is the first Stiefel-Whitney class of the bundle. In particular, the S^1-action is globally well-defined iff the bundle is orientable.

We say a map $f\colon M \to N$ is an *equivariant fiber preserving map* (EFPM) if f is a bundle homomorphism. By definition f is EFPM iff f is fiber preserving and $f(t(x)) = t^n f(x)$, where n is the integer for which $f_*(h) = h^n$. (n is globally well-defined when both M and N are oriented bundles, otherwise $|n|$ is well-defined and will be called the geometric fiber degree of f in §3.)

Similar definitions can be made for annuli, tori, and Klein bottles when they are regarded as Seifert fibered spaces.

We hope to show in this section that any nonzero degree map between aspherical S.F.S. M and N is homotopic to an EFPM for some Seifert fibrations of M and N. However, the following example shows that this is not true in general.

EXAMPLE. Let $M = F_g \times S^1$, where F_g is a closed orientable surface of genus > 1. Let $S = S^1 \times S^1 \times S^1$, N be the unique S.F.S. with orbifold $S^2(3, 3, 3)$ that is covered by S. Let $\alpha = p \times \mathrm{id}\colon M \to S$, where p is a (2-dimensional) pinch from F_g onto $S^1 \times S^1$. Hence α sends the fiber of M onto the last S^1-factor of S. Let $\beta\colon S \to N$ be a covering that sends the first S^1-factor onto the fiber of N. Now define $F = \beta \circ \alpha$. Under the unique Seifert fibrations of M and N, $f_*(h) \notin \langle h \rangle$. Hence f is not homotopic to a fiber preserving map.

The core of the above example is that S has two non-isotopic Seifert fibrations, so that α and β are both fiber preserving but under different Seifert fibrations of S. In other words, when a map factors through S, S switches its fibration. Since we restrict ourself to P^2-irreducible manifolds with infinite π_1, such S must be a Euclidean manifold [5]. There are only finitely many of such Seifert fibered spaces. We will show that this is the only way our map fails to be an EFPM (Lemma 2.1 and Proposition 2.4).

LEMMA 2.1. *Let $f\colon M \to N$ be a map of nonzero degree between aspherical S.F.S. of infinite π_1. Then, for any fixed Seifert fibration of M, either*

1. *there is a Seifert fibration of N such that $f_*(h) \in \langle h \rangle$, or*
2. *there is a covering $p\colon \widetilde{N} \to N$, a lifting $\tilde{f}\colon M \to \widetilde{N}$ such that $f = p \circ \tilde{f}$, and $\tilde{f}_*(h) \in \langle h \rangle$ for some fibration of $\widetilde{N}$. Furthermore, N and $\widetilde{N}$ are both Euclidean manifolds in this case.*

Proof. If $f_*(h) = 1$, $f_*(h) \in \langle h \rangle$. (In fact this can happen only when N is a solid torus since $\deg f \neq 0$.)

Now let us assume that $f_*(h) \neq 1$. If f_* is onto, then $\langle f_*(h) \rangle$ is an infinite cyclic normal subgroup of $\pi_1(N)$. By [2, VI.11(e)], $f_*(h) \in \langle h \rangle$ for some Seifert fibration of N.

If f_* is not onto, let $p \colon \widetilde{N} \to N$ be the covering corresponding to $f_*\pi_1(M)$. It must be a finite covering for $\deg f \neq 0$. Let $\tilde{f} \colon M \to \widetilde{N}$ be the lift of f. Since $\tilde{f}_*$ is onto, $\tilde{f}_*(h) \in \langle h \rangle$ for some Seifert fibration of $\widetilde{N}$. $\widetilde{N}$ also has a Seifert fibration induced from N by the map p so that p is fiber preserving covering. If the two Seifert fibrations of $\widetilde{N}$ agree, then $f_*(h) \in \langle h \rangle$, and we are in case 1. If they do not agree, then $\widetilde{N}$ has two Seifert fibrations that are not isotopic to each other. By [5], $\widetilde{N}$ is Euclidean, and therefore N is Euclidean since it is a quotient of $\widetilde{N}$. □

From now on, we will focus on maps of case 1 in the above lemma.

LEMMA 2.2. *Let $f \colon A \to N$ be a map from an annulus into an aspherical Seifert fibered space such that f sends ∂A into fibers and $f|\partial A$ is an EFPM. Then f can be homotoped* rel ∂ *such that f is an EFPM.*

Proof. Let $\partial A = a_0 \cup a_1$, a be a spanning arc of A. We consider 3 cases:

Case 1. Both $f(a_0)$ and $f(a_1)$ belong to regular fibers. Let $f(tx) = t^n f(x)$ on a_0. Define f_1 on A by $f_1(tx) = t^n f(x)$ for $t \in S^1$, $x \in a$. Then f_1 is an EFPM, and $f_1 = f$ on ∂A. Using $\pi_2(N) = \{0\}$, and $f_1 = f$ on a, we can easily see that $f_1 \simeq f \operatorname{rel} \partial$.

Case 2. $f(a_0)$ belongs to a regular fiber but not $f(a_1)$. Let c_1 be the singular fiber containing $f(a_1)$. The index p_1 of c_1 must divide $\deg\{f| \colon a_1 \to c\}$. Let a_1' be a parallel copy of a_1 in A, and $A_1 \subset A$ be the annulus between a_1' and a_1. We can homotop $f \operatorname{rel} \partial$ so that it is an EFPM on A_1, and $f(a_1')$ belongs to a regular fiber. By the previous case the result holds.

Case 3. $f(a_0)$ and $f(a_1)$ belong to singular fibers c_0 and c_1 respectively. Suppose that $f(a_i)$ covers c_i with degree n_i. Let O_N be the base orbifold of N and $\widetilde{O}_N$ be its universal covering. Let $\bar{c}_i \in \pi_1(O_N)$ be the coset of c_i in the quotient group $\pi_1(N)/\langle h \rangle \cong \pi_1(O_N)$. The map f gives a conjugacy relation $\bar{c}_0^{n_0} = \alpha \bar{c}_1^{n_1} \alpha^{-1}$ in $\pi_1(O_N)$. But $\bar{c}_i$ acts on $\widetilde{O}_N$ as a rotation around some point over the ith cone point. Hence the relation cannot hold unless either the two rotation angles are both multiples of 2π, or the two rotations have the same fixed point and α is also a rotation around this point. The first possibility implies that n_i is a multiple of the index of c_i, and the result then follows by a similar reason as in Case 2. For the second possibility, $c_0 = c_1$, and $f(a)$ is a loop whose homotopy class lies in $\langle c_0 \rangle$. Hence f is homotopic to the constant fiber c_0. □

Similar to Case 1 above, the following lemma is proved using $\pi_3(N) = \{0\}$:

LEMMA 2.3. *Let $f\colon V \to N$ be a map from a Seifert fibered torus of type $(1, 0)$ into an aspherical Seifert fibered space N such that $f|\partial V$ is an EFPM. Then $f \simeq f_1 \operatorname{rel} \partial$, where f_1 is an EFPM.*

PROPOSITION 2.4. *Let $f\colon (M, \partial M) \to (N, \partial N)$ be a nonzero degree map between aspherical S.F.S. of infinite π_1 such that $1 \neq f_*(h) \in \langle h \rangle$. Then f can be homotoped into an EFPM. If $f|\partial$ is already an EFPM, then the homotopy can be chosen to be fixed on the boundary.*

Proof. Consider first the case when M and N are closed. We can write M as a union $M = H_0 \cup H_1 \cup H_2$, where $H_0 = N(h \cup c_i)$, h is a regular fiber, $\bigcup c_i$ is the union of all the singular fibers, $H_1 = N(\bigcup A_j)$, where A_j's are disjoint vertical annuli such that $N(\partial A_j) \subset \partial H_0$, and $H_2 = N(h_1)$, where h_1 is a regular fiber and $\partial H_2 \subset \partial(H_0 \cup H_1)$. (This is called a round-handle decomposition of M [3].)

Since $f_*(h) \in \langle h \rangle$, $f_*(c_i)$ is a multiple of a fiber for each i [2, VI.11(f)]. Hence we can homotop f so that $f|H_0$ is EFPM. Then we apply Lemma 2.2 and Lemma 2.3 to finish the proof.

For manifolds with boundary, we first use the fact $1 \neq f_*(h) \in \langle h \rangle$ to homotop $f|\partial$ so that $f|\partial$ is an EFPM. The rest is similar as before. □

Let Z_n denote the cyclic subgroup of order n in S^1. Let $\sim_n$ be the equivalence relation on the S.F.S. M defined by $x \sim_n t(x)$ for all $x \in M$ and all $t \in Z_n$. Let $M_n = M/\sim_n$. Then M_n is a Seifert fibered space, and the quotient map is a fiber preserving branched covering. An EFPM f with $f_*(h) = h^n$ satisfies $f(t(x)) = t^{\pm n}(f(x))$. Thus $f(t(x)) = f(x)$ for all $t \in Z_n$. It follows that f factors through M_n:

COROLLARY 2.5. *Under the same hypothesis as in Proposition 2.4, $f \simeq f_1 \circ p$, where $p\colon M \to M_1$ is a fiber preserving branched covering, and $f_{1*}(h) = h$.*

3. Main theorem. We prove our main theorem in this section. First we give some definitions. Let $f\colon (M, \partial M) \to (N, \partial N)$ be a map of Seifert fibered spaces which is homotopic to a fiber preserving map. The *geometric orbifold degree* of f, denoted by $G_{ob}(f)$, is the minimum number of regular fibers in $f_1^{-1}(h)$, where f_1 ranges in all the maps which are properly homotopic to f, fiber preserving, and transverse to h. The *geometric fiber degree* of f, denoted by $G_h(f)$, is $|n|$ where $f_*(h) = h^n$. Notice that these definitions depend on the choice of Seifert fibrations of the spaces.

Denote by f_∂ the restriction of f on the boundary. We have

$$G(f) \le G_h(f)G_{ob}(f) \le G(f_\partial).$$

We say a fiber preserving map f is *allowable* if f_∂ is a covering of degree $G_{ob}(f)G_h(f)$. If $\partial M = \partial N = \varnothing$, then any fiber preserving map is allowable. From the definition, the following observation is easy to see and will be used later in our proof:

REMARK 1. If $f\colon M \to N$ is allowable, X is a vertical set (vertical annulus, vertical solid torus, etc.) in N, and $f_1 = f|\colon M - f^{-1}(N(X)) \to N - N(X)$, then f_1 is allowable if either

$f|f^{-1}(X)$ is a covering of degree $G_{ob}(f)G_h(f)$, or
$f|f^{-1}(X)$ is a covering of degree $G(f)$. In the later case, $G(f) = G_{ob}(f)G_h(f) = G(f_\partial)$.

PROPOSITION 3.1. *Let f be an allowable map between aspherical S.F.S. of infinite π_1 with $G(f) \neq 0$. Then $F \simeq g\pi$, where π is a composition of finitely many of vertical pinches, and g is a fiber preserving branched covering branched over fibers.*

This proposition together with Lemma 2.1 and Proposition 2.4 imply our main theorem:

THEOREM 3.2. *Let $f\colon M \to N$ be a nonzero degree map between closed P^2-irreducible S.F.S. of infinite π_1. Then $f \simeq p \circ g \circ \pi$, where π is a composition of finitely many vertical pinches, g is a fiber preserving branched covering, and p is a covering.*

REMARK 2. The covering p can be chosen to be the identity map unless N is an Euclidean manifold. There are only 10 such manifolds N [5]. For each such N, the covering space $\widetilde{N}$ must have two different Seifert fibrations and is Euclidean. There are only two such manifolds $\widetilde{N}$, the three torus and the double of the twisted I-bundle of the Klein bottle. Hence the possibilities of p are very limited.

When $G(f) = 1$, we have

COROLLARY 3.3. *Any degree one map between closed P^2-irreducible Seifert fibered spaces of infinite π_1 is a composition of finitely many vertical pinches.*

Before we prove Proposition 3.1, we first prove a lemma which serves the initial step of the induction.

LEMMA 3.4. *Let $f: M \to V$ be an allowable map, where M is a S.F.S. of infinite π_1, and V is a Seifert fibered solid torus. Assume that f does not admit a nontrivial vertical pinch, and $G_h(f) = 1$. Then f can be homotoped* rel ∂ *to a fiber preserving brc⁻ched covering.*

Proof. Since F has no nontrivial vertical pinch, the orbifold O_M of M must be a planar surface with at most one cone point. Let $a_1, \dots, a_k$ be proper arcs which cut O_M into a disk with possibly one cone point, let A_i be the annulus over a_i. Then $\bigcup A_i$ cut M into a Seifert fibered solid torus V_1. The map $f|A_i$ can be homotoped rel ∂ so that $f(A_i)$ is ∂-parallel. Now $f|V_1$ is a map between tori, hence can be homotoped rel ∂ such that off ∂V_1 it is a branched covering branched over the core c of V. It follows from Remark 1 that $f_1: M_1 \to N_1$ is allowable and $G_{ob}(f_1) = G_{ob}(f)$, where $N_1 = (N - N(c))^-$, $M_1 = f^{-1}(N_1)$, $f_1 = f|M_1$.

The Seifert fibered space N_1 is isomorphic to a product $A \times S^1$ where A is a horizontal annulus. Homotop f_1 rel ∂ such that $f_1^{-1}(A)$ is incompressible. It must be either vertical or horizontal by [7]. But it cannot be vertical since $f_{1*}(h) = h$ in $\pi_1(N_1)$. Hence $f_1^{-1}(A)$ must be a union of parallel horizontal surfaces $\bigcup F$, and M_1 is a fibered manifold with fiber F and a periodic gluing map. Using $G_h(f_1) = 1$, we conclude that there is only one copy of F and the gluing map must be the identity. The map $f_1|: f^{-1}(A) \to A$ is allowable as a map of surfaces (note that Edmonds and Skora both defined allowable maps, but the two definitions are in fact equivalent by [**6**]); also it does not admit a pinch since f does not admit a vertical pinch. By Edmonds theorem, it is homotopic rel ∂ to a branched covering g. This implies f_1 is homotopic rel ∂ to $g \times \mathrm{id}: F \times S^1 \to A \times S^1$, which is a fiber preserving branched covering. Hence f is homotopic rel ∂ to a fiber preserving branched covering. □

Proof of Proposition 3.1. After a finite number of vertical pinches we may assume that f has no vertical pinch. Next by Corollary 2.5, we can compose f as $f = f_1 \circ g$, where g is a fiber preserving branched covering branched over fibers and $G_h(f_1) = 1$. Note that f_1 does not admit a vertical pinch since a vertical pinch of f_1 would yield a vertical pinch of f. Therefore we may assume that $G_h(f) = 1$.

If $\partial M = \varnothing$, let h be a regular fiber in N such that f is transverse to h, and $f^{-1}(h)$ is a union of $G_{ob}(f)$ regular fibers. We then consider $f|M - N(f^{-1}(h))$, which is again an allowable map of the same geometric orbifold degree. Hence we may assume that $\partial M \neq \varnothing$.

From now on, we assume that $\partial M \neq \varnothing$, f has no vertical pinch, and $G_h(f) = 1$. Under these assumptions we prove that f is homotopic to a fiber preserving branched covering branched over fibers by an induction on the complexities of M and N.

The first step is the case when $N \cong V$. This has been done by Lemma 3.4.

Next is the inductive step:

Case 1. f admits a squeeze along an incompressible vertical torus T.

We take a maximum collection of disjoint non-parallel incompressible vertical tori along which f admits squeezes. Let $X = Q \cup A$ be the space obtained after these squeezes, where $Q = \bigcup Q_i$ is a union of S.F.S., $A = \bigcup A_j$ is a union of annuli such that ∂A is a disjoint union of (possibly singular) fibers in Q. After a homotopy, $f = (g \cup \alpha) \circ q$, where Q is a composition of squeezes, $g = \bigcup g_i$ is defined on Q, and α is defined on A. Each Q_i must be of infinite π_1 for otherwise $f_*(h)$ factors through an element of finite order in $\pi_1 Q_i$ which implies $f_*(h) = 1$ in $\pi_1(N)$, and thus $G(f) = 0$.

Claim. g is allowable and $G_{ob}(g) = G_{ob}(f)$, $G_h(g) = 1$.

Proof of Claim. Clearly, $G_h(g) = G_h(f) = 1$. Next, fix a regular fiber h in N, homotop g_i so that g_i is fiber preserving and $g_i^{-1}(h)$ is a union of $G_{ob}(g_i)$ regular fibers in M. By Lemma 2.2 we can homotop $\alpha \operatorname{rel} \partial$ so that α is fiber preserving; hence we can perturb it so that $\alpha(A)$ misses h. So $f^{-1}(h) = g^{-1}(h)$. Hence $G_{ob}(f) \leq G_{ob}(g)$. It follows that $G(f_\partial) = G_{ob}(f)G_h(f) \leq G_{ob}(g)G_h(g) \leq G(g_\partial)$. Since $f_\partial = g_\partial$, all inequalities must be equalities. Hence

$G_{ob}(g)G_h(g) = G(g_\partial)$, and thus g is allowable. It also follows from the above equalities that $G_{ob}(g) = G_{ob}(f)$.

By the maximality of q, g does not admit any squeeze, thus no vertical pinch.

For each i such that Q_i is closed, by Lemma 3.5 below we can homotop g_i such that $g_i(Q_i)$ is contained in a fiber of N. Let B_1 be the union of such fibers in N.

For each i such that Q_i has boundary, $G_{ob}(g_i) \neq 0$ since g is allowable. By the inductive hypothesis, each g_i is homotopic to a fiber preserving branched covering branched over fibers. Let B_2 be the union of the branched fibers in N.

Now consider $\alpha: A \to N$. $\alpha(\partial A) = g(\partial A)$ must be fibers in N. By Lemma 2.2, we may assume that α is fiber preserving after a homotopy $\operatorname{rel}\partial$.

If a component of ∂A_j is a singular fiber c in Q of order n (> 1), $\alpha(A_j)$ must be contained in one singular fiber. Otherwise, by Lemma 2.2, α sends a fiber c' of A_j in $\operatorname{Int} A_j$ into a regular fiber in N. So $\alpha(c') \simeq h^k$ and thus $g(c) \simeq \alpha(c') \simeq h^k$. This implies that $g(h) \simeq g(c^n) \simeq h^{nk}$, contradicting the fact that $G_h(g) = G_h(f) = 1$.

If both components of ∂A_j are regular fibers in Q, we can perturb them so that they are mapped into different regular fibers in N under g. By Lemma 2.2, $\alpha(A_j)$ can be homotoped into an immersed vertical annulus. Thus we can homotop $\alpha \operatorname{rel}\partial$ so that $\alpha(A_j)$ is an embedded vertical annulus by sliding the double lines off $\alpha(\partial A_j)$.

It follows that in any case, $\alpha(A_j)$ is contained in a vertical solid torus after a homotopy $\operatorname{rel}\partial$. Let B_3 be the union of these vertical solid tori in N.

Let E be a vertical essential annulus in N missing $B_1 \cup B_2 \cup B_3$. Now $f^{-1}(E)$ covers E with degree $G_h(f)G_{ob}(f)$, thus consists of a collection of annuli. The map $f|: M - N(f^{-1}(E)) \to N - N(E)$ is again allowable, so it is homotopic $\operatorname{rel}\partial$ to a fiber preserving branched covering by the inductive hypothesis. Hence f is homotopic to a fiber preserving branched covering.

Case 2. f does not admit a squeeze along any incompressible vertical torus. Since $\partial N \neq \varnothing$, and $N \not\cong V$, there is an essential vertical annulus E in N. Homotop f such that $f^{-1}(E)$ is incompressible. It must be either vertical or horizontal [7]. If F is a horizontal component of $f^{-1}(E)$, then the subgroup A generated by $\pi_1(F)$ and h has finite index in $\pi_1(M)$. Let $B = \pi_1(E)$, which is an infinite index subgroup of $\pi_1(N)$. Since $f_*(A) \subset B$, we have

$$\begin{aligned}[][\pi_1 N : B] &\leq [\pi_1 N : f_*(A)] = [\pi_1 N : f_*\pi_1 M][f_*\pi_1 M : f_* A] \\ &\leq [\pi_1 N : f_*\pi_1 M][\pi_1 M : A] < \infty, \quad \text{a contradiction}.\end{aligned}$$

Hence $f^{-1}(E)$ must be a union of incompressible vertical tori or annuli. It must be a union of annuli because f does not admit a squeeze. Denote the annuli by $\{E_i\}$. After a homotopy of f, $f|E_i$ either covers E or misses a boundary component of E. But the later case cannot happen, for otherwise $f^{-1}(h)$ has less than $G_{ob}(f)$ components for a regular fiber h on E that is close to ∂E, a contradiction. It follows that $f|M - N(f^{-1}(E))$ is an allowable map onto $N - N(E)$. Therefore the inductive hypothesis implies it can be homotoped rel ∂ into a fiber preserving branched covering. Hence f is homotopic to a fiber preserving branched covering. □

LEMMA 3.5. *Let $g: M \to N$ be a map of aspherical Seifert fibered spaces of infinite π_1 such that $1 \neq f_*(h) \in \langle h \rangle$. Assume that M is closed and $\partial N \neq \varnothing$. Then either f admits a squeeze along an incompressible vertical torus or $f \simeq f_1$, where $f_1(M) \subset$ a fiber of N.*

Proof. Let A be a union of essential vertical annuli in N which cut N into a union of solid tori. Homotop f so that $f^{-1}(A)$ is incompressible in M.

Case 1. If a component $F \subset f^{-1}(A)$ is horizontal, the group $G = \langle \pi_1(F), h \rangle$ is of finite index in $\pi_1(M)$. But $f_*(G)$ is cyclic since it is contained in an annulus group. Hence $f_*(\pi_1 M)$ contains a cyclic group of finite index. Therefore if M contains an incompressible vertical torus then f admits a squeeze along this torus. If M does not contain any incompressible vertical torus then M is a Seifert fibered space whose base orbifold is S^2 with three exceptional fibers. In this case $G = \pi_1(M)$. Thus $f_*(\pi_1 M) = f_*(G)$ is a cyclic group with a generator x. Since this group contains $f_*(h)$, and $f_*(h) \in \langle h \rangle$, x is represented by a power of fiber c in N [**2**, VI.11(f)]. Hence $f_*(\pi_1 M) \subset \langle c \rangle$, and so $f \simeq f_1$ where $f_1(M) \subset c$.

Case 2. If $f^{-1}(A)$ is a nonempty union of incompressible vertical tori, then f admits squeezes along these tori.

Case 3. If $f^{-1}(A) = \varnothing$, then $f(M)$ is contained in one of the tori of N cut along A. Hence $f(M)$ can be homotoped into the core of this torus, which is a fiber of N. □

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