PACIFIC JOURNAL OF MATHEMATICS
Vol. 160, No. 2, 1993

A NOTE ON MORTON'S CONJECTURE CONCERNING THE LOWEST DEGREE OF A 2-VARIABLE KNOT POLYNOMIAL

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This note is concerned with the behaviour of the 'HOMFLY' polynomial of oriented links, $P_L(v, z)$. In particular, we show that the gap between the two lowest powers of v can be made arbitrarily large. This casts doubt on whether Morton's conjecture on the least v-degree can be established in general by the kind of combinatorial approach that has been successfully applied to some special cases.

Introduction. The two-variable knot polynomial $P_L(v, z)$ of a link L, announced in **[FYHLMO]**, **[PT]**, can be written in the form

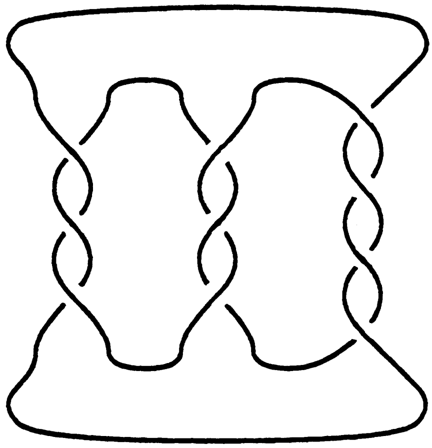
$$P_L(v, z) = \sum_{i=e}^{E} a_i(z)v^i$$

where $a_i(z)$ is a polynomial in z for each i, $a_e(z) \neq 0$, and $a_E(z) \neq 0$. Let $f(P_L)$ denote the least degree in v in the polynomial P_L. Say that $f(P_L)$ is the *first* degree of P_L. With the above formulation $f(P_L) = e$. Let $s(P_L)$ be the least $i > e$ such that $a_i(z) \neq 0$. Say that $s(P_L)$ is the *second* degree of P_L.

In **[Mo3]** H. Morton conjectured that

$$f(P_L) \leq 1 - \chi(L)$$

for all links L where $\chi(L)$ is the maximum Euler characteristic over all orientable surfaces spanning L. In **[Cr]** I showed that the conjecture is satisfied by the homogeneous links (a class containing the positive and alternating links as special cases). A computer search for counterexamples in other classes of links showed up an interesting phenomenon: sometimes polynomials were produced where $s(P_L) - f(P_L)$ was quite large and $a_e(z) = 1$. In these cases it was only the term v^e, isolated from the other non-zero terms in the polynomial, which saved the conjecture from being violated. This prompted the question of whether $s(P_L) - f(P_L)$ could be arbitrarily large. Here I provide examples to show that it can.



(3, −3, 4)

FIGURE 1

EXAMPLES. The simplest examples that I have found can be viewed as pretzel knots of the form $(3, -3, 2a)$ for any $a \in \mathbb{N}$ (see Figure 1). Writing the polynomial of this knot as $P(3, -3, 2a)$ we get

$$\begin{aligned} P(3, -3, 2a) &= v^2 P(3, -3, 2(a-1)) \\ &\quad + vzP \text{ (two component trivial link)} \\ &= v^2 P(3, -3, 2(a-1)) - v^2 + 1 \\ &= v^{2a}(P(3, -3, 0) - 1) + 1. \end{aligned}$$

Now $(3, -3, 0)$ is a square or reef knot—the connected sum of a trefoil and its mirror image. Its polynomial is

$$P(3, -3, 0) = (-2 - z^2)v^{-2} + (5 + 4z^2 + z^4) + (-2 - z^2)v^2.$$

Letting K denote the pretzel knot $(3, -3, 2a)$ we obtain

$$s(P_K) - f(P_K) = \begin{cases} 2, & 0 \le a \le 2, \\ 2(a-1), & 2 < a. \end{cases}$$

Thus $s(P_K) - f(P_K)$ can be made as large as we please.

Applying Seifert's algorithm to the standard diagram of the pretzel knot, K, shows that $1 - \chi(K) \le 6$. So whenever $a > 4$, we have $s(P_K) > 1 - \chi(K)$ and the constant term in P_K is the only term which validates the conjecture. The difference $s(P_K) - (1 - \chi(K))$ can also be made arbitrarily large. These examples suggest that it may be difficult to prove Morton's conjecture true in general using a combinatorial approach like that in **[Cr]**.

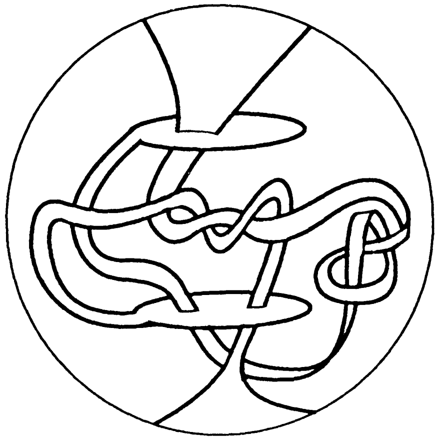


FIGURE 2

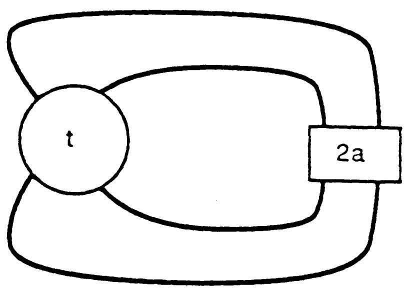


FIGURE 3

Many other examples are easily constructed. All that is required is a tangle t such that its numerator $N(t)$ is the trivial link with two components and its denominator $D(t)$ is a non-trivial knot (using Conway's notation for the closures of a tangle [**Co**]). Such tangles are easily constructed: take two discs embedded in the interior of a ball and connect each of them to the boundary of the ball by a ribbon. The ribbons may pass through the discs in ribbon singularities. An example is shown in Figure 2.

Inserting $2a$ positive half-twists into $D(t)$, as shown in Figure 3, produces the same behaviour in the polynomial as before. That is

$$P(D(t) \text{ with } 2a \text{ half-twists}) = v^{2a}(P(D(t)) - 1) + 1 .$$

Substituting $v = 1$ in this expression shows that all of the knots derived from $D(t)$ in this way have the same Conway polynomial. Thus the square knot, 8_{20}, and 10_{140} all have the same Conway polynomial since they are $(3, -3, 0)$, $(3, -3, 2)$ and $(3, -3, 4)$ respectively.

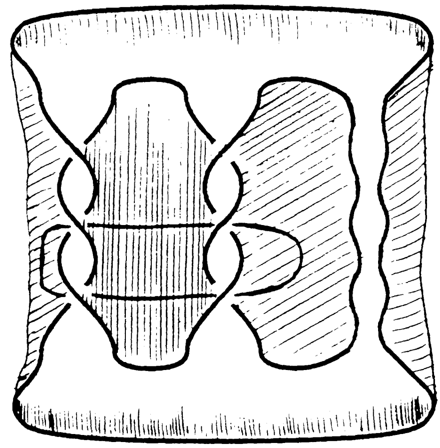


FIGURE 4

REMARKS. Morton observed that the connected sum of two trefoils is a fibred knot and that the insertion of twists described above can be achieved by $(1, n)$ Dehn surgery about an unknotted untwisted curve in the fibre surface. Such a curve is shown in Figure 4. Hence there is an infinite family of fibred knots all having the same Alexander polynomial but which can be distinguished by $P(v, z)$. This provides further counterexamples to the conjecture (made in [**Ne**]) that at most finitely many fibred knots could have the same Alexander polynomial. It was Morton who showed that the conjecture is false [**Mo1**], [**Mo2**].

The referee drew attention to a related result of Akio Kawauchi [**Ka**] who has shown that a gap in the z-degree can also be made as large as desired. More specifically, he constructed a family of knots whose polynomials have the form

$$P_L(v, z) = 1 + \sum_{i=m}^{M} b_i(v) z^i$$

(where $b_m(v)$ and $b_M(v)$ are non-zero polynomials in v). The value of m can be made arbitrarily large.

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Received July 1, 1991 and in revised form August 24, 1992. The author was supported by a S.E.R.C. post-doctoral fellowship.

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