PACIFIC JOURNAL OF MATHEMATICS
Vol. 162, No. 1, 1994

SPIN MODELS FOR LINK POLYNOMIALS, STRONGLY REGULAR GRAPHS AND JAEGER'S HIGMAN-SIMS MODEL

PIERRE DE LA HARPE

We recall first some known facts on Jones and Kauffman polynomials for links, and on state models for link invariants. We give next an exposition of a recent spin model due to F. Jaeger and which involves the Higman-Sims graph. The associated invariant assigns to an oriented link the evaluation for $a = -\tau^5$ and $z = 1$ of its Kauffman polynomial in the Dubrovnik form, where τ denotes the golden ratio.

1. Introduction. A *knot* is a simple closed curve in $\mathbb{R}^3$ and a *link* is a finite union of disjoint knots. We denote by $\vec{L}$ a link L together with an *orientation* on each of its components. Two oriented links $\vec{L}$, $\vec{L}'$ are *isotopic*, and we write $\vec{L}' \approx \vec{L}$, if there exists a family $(\phi_t)_{0\leq t\leq 1}$ of homeomorphisms of $\mathbb{R}^3$ such that the map $[0, 1] \to \mathbb{R}^3$ sending t to $\phi_t(x)$ is continuous for each $x \in \mathbb{R}^3$ and such that $\phi_0 = \mathrm{id}$, $\phi_1(\vec{L}) = \vec{L}'$, where the last equation indicates that orientations correspond via ϕ. Links considered here are always assumed to be *tame*, namely isotopic to links made of smoothly embedded curves. A Ω-*valued invariant* for oriented links is a map $\vec{L} \mapsto I(\vec{L})$ which associates to each oriented link $\vec{L}$ in $\mathbb{R}^3$ an element $I(\vec{L})$ of some ring Ω, for example $\mathbb{C}$ or a ring of Laurent polynomials, in such a way that $I(\vec{L}') = I(\vec{L})$ whenever $\vec{L}' \approx \vec{L}$.

Classically, one of the most studied example of link invariant is the *Alexander-Conway polynomial* $\Delta(L) \in \mathbb{Z}[t^{\pm 1}]$ defined by J. W. Alexander in 1928 [**Ale**], with a normalization made precise by J. H. Conway in 1969 [**Con**]; the notation (L rather than $\vec{L}$) indicates that, at least for knots, $\Delta(L)$ does not depend on the choice of an orientation on the knot. The polynomial invariant $L \to \Delta(L)$ is well understood in terms of *standard algebraic topology* (homology of "the" infinite cyclic covering of the complement of L in $\mathbb{R}^3$); see e.g. [**Rha**], [**Rol**] or [**BuZ**].

The subject entered a new era in 1984 [**Jo1**] with the discovery of the *Jones polynomial* $V(\vec{L}) \in \mathbb{Z}[t^{\pm 1/2}]$. This was the starting point of several other invariants, including the *Kauffman polynomial* reviewed below **[Ka2]**. (See also **[F+]**, **[Jo2]** and the survey in **[Li1]**; for 3-manifolds, see e.g. **[Kup]**, **[Tu2]**.) Let us mention three striking features of these new invariants:

(i) They have been used to solve old problems, including a conjecture of Tait on "alternating links" going back to last century (see **[Ka1]**, **[Mur]**, **[Thi]** as well as expositions in **[HKW]**, **[Ka3]**, **[Li1]**, **[Tu1]**).

(ii) They remain quite mysterious. For example, we do not know whether there exist nontrivial examples with $V(\vec{L}) = 1$, and we do not know when $f \in \mathbb{Z}[t^{\pm 1/2}]$ is of the form $V(\vec{L})$. (Compare with Δ: Seifert **[Sei]** has constructed nontrivial examples of L such that $\Delta(L) = 1$; he has also shown that $f \in \mathbb{Z}[t^{\pm 1}]$ is Δ of a knot if and only if $f(t^{-1}) = f(t)$ and $f(1) = 1$.)

(iii) They are related to an amazingly wide variety of subjects such as von Neumann algebras (**[Jo5]**, **[Wen]**, **[HJ1]**), representations of semi-simple Lie algebras, finite groups, and more generally of quantum groups **[ReT]**, statistical mechanics (**[Jo3]**, **[Ka1]**, **[HJ2]**), topological field theory **[Ati]**, and so on.

In this report, we shall focus on *combinatorics*, and indicate the connection between link polynomials and statistical mechanics going via *state models* (**[Bax]**, **[Ka1]**, **[Jo3]**), in particular via the spin models defined below. More precisely, we will explain how F. Jaeger **[Jae]** has found new models for evaluations of the Kauffman polynomial, using very special association schemes and strongly regular graphs such as the Higman-Sims graph.

In §2, we recall the definition of the Kauffman polynomial $F_{+1}(\vec{L})$ of an oriented link $\vec{L}$ in terms of a diagram $\vec{D}$ which represents $\vec{L}$ (we distinguish F_{+1} from its Dubrovnik variant F_{-1}). In §3, we define spin models for oriented links using signed graphs associated to diagrams. The simplest nontrivial examples appear as a family of Potts' models providing values of the Jones polynomial, as exposed in §4. The main ingredient of a spin model is the matrix R_+ of its so-called Boltzmann weights. As a Potts' model is characterized by these weights having two different values, the next step is to look for models with three values. We show how such a model is associated with a graph S which has to be *strongly regular*; this is explained in §5, which contains also examples with S having four or five vertices. The remarkable spin model related to the *Higman-Sims graph* **[HiS]** and which has been discovered by Jaeger **[Jae]** is exposed in §6. It is conceivable that the "pentagonal model" of §5 and the Jaeger model of §6 are members of a larger family on which we speculate in §7.

The results up to §5 are standard; those of §§6 and 7 are in **[Jae]**, but our exposition is slightly different. More precisely, we unfold as much as possible the consequences for spin models of the Reidemeister moves of type III. Following **[Jo3]**, we show how this gives rise to a braid relation for the Boltzmann weights (Equation (12) and Proposition 1). The strength of this braid relation allows us not to introduce any of the association scheme machinery of **[Jae]**. In §6, we also proceed to a geometric discussion of the Higman-Sims graph which is more detailed than in **[Jae]**, and we describe Jaeger's model independently of the general considerations of §7; indeed the reader interested first by this example could go quickly through §§1, 3 and 5.1 before focusing on our exposition in §6.

I am most grateful to R. Bacher, F. Jaeger and V. Jones for many useful conversations, as well as to J. Seidel and to the referee for their comments on a first draft of this paper. It is also a pleasure to thank for its hospitality the MSRI at Berkeley, where part of this work was done in September 1991.

2. Reidemeister moves for link diagrams and Kauffman polynomial. Consider $\mathbb{R}^3$ as an oriented Euclidean space. Given an *oriented* plane E in $\mathbb{R}^3$ we denote by Π_E the orthogonal projection of $\mathbb{R}^3$ onto E. We identify E with $\mathbb{R}^2$ and $\mathbb{R}^3$ with $E \times \mathbb{R}$. As $\mathbb{R}^3$ and E are oriented, it makes sense to say that a point $(x, z') \in E \times \mathbb{R}$ is *above* $(x, z) \in E \times \mathbb{R}$ if $z' > z$.

Let $\vec{L}$ be an oriented smooth link in $\mathbb{R}^3$. An oriented plane E is *generic* for $\vec{L}$ if the plane curve $\Pi_E(\vec{L})$ is smooth up to double points with transverse tangents. The corresponding *oriented link diagram* $\vec{D}$ is then the projection $\Pi_E(\vec{L})$ together with some indication showing at each double point which part is above the other.

Let D be a link diagram in $\mathbb{R}^2$. (The notation indicates that we forget the orientation for a while.) One may always colour the connected components of $\mathbb{R}^2 - D$ in black and white in such a way that (i) the unbounded component is white, (ii) two components which have a common boundary of strictly positive length are of different colours. By definition, the *signed graph* X associated to D has one vertex by black region, and edges between two given vertices x, y of X are in bijection with the double points of D in the intersection of the closures of the two corresponding components. Moreover, each edge of S has a sign encoding the type of the corresponding crossing, as in Figure 1. (It can be checked that D_1 and D_2 represent isotopic knots—right trefoil—though X_1 and X_2 are *not* isomorphic!)

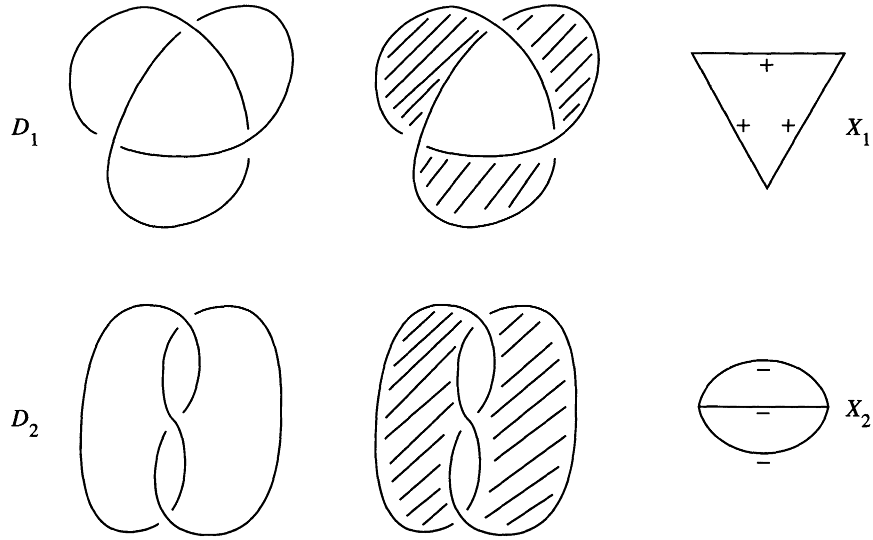


FIGURE 1. Diagrams and signed graphs.

A classical result **[Rei]** shows that two oriented diagrams $\vec{D}$, $\vec{D}'$ represent isotopic links if and only if one can change $\vec{D}$ into $\vec{D}'$ by a finite number of diagram isotopies and of so-called *Reidemeister moves*, as indicated in Figure 2. In this figure, two related pictures represent portions of diagrams of which the portions not represented are identical; it is understood that each of these moves holds for any pair of corresponding orientations (two pairs for each move of type I, four for type II, and eight for type III). The corresponding pictures for signed graphs are shown in Figure 3 on p. 62.

Each oriented crossing has also a *sign* as indicated in Figure 4 on p. 63. (This is completely independent of the colours around the crossing.) The *Tait number* $\mathrm{Tait}(\vec{D})$ of an oriented diagram $\vec{D}$ is the sum of these signs.

Kauffman's generalization of Jones polynomial is characterized by the next theorem for which we refer to **[Li1]**. Recall that two unoriented link diagrams D, D' are *regular isotopic* if one can change D into D' by a finite number of diagram isotopies and of Reidemeister moves of types II and III.

THEOREM. *Choose* $\varepsilon \in \{1, -1\}$. *There exists a function*

$$\Lambda_\varepsilon : \{\textit{unoriented link diagrams in } \mathbb{R}^2\} \to \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$$

that is defined uniquely by

(i) $\Lambda_\varepsilon(U) = 1$ *if* U *is the diagram with one component and no crossing,*

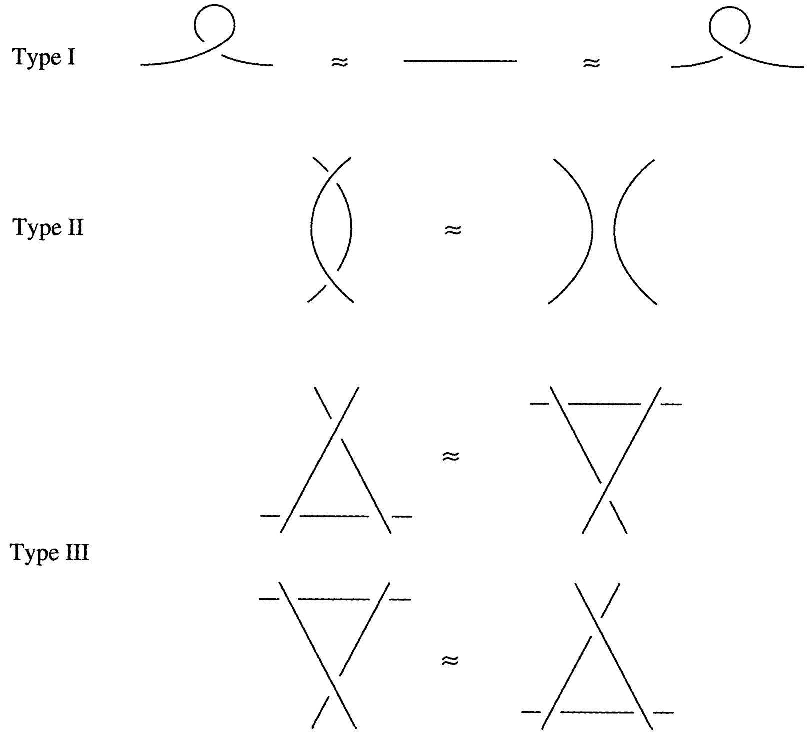


FIGURE 2. Reidemeister moves for diagrams.

(ii) $\Lambda_\varepsilon(D) = \Lambda_\varepsilon(D')$ *if* D, D' *are regular isotopic,*

(iii) $\Lambda_\varepsilon(\boxed{D}\!\!\times\!\!\bigcirc) = a\Lambda_\varepsilon(D)$ *and* $\Lambda_\varepsilon(\boxed{D}\!\!\times\!\!\bigcirc) = a^{-1}\Lambda_\varepsilon(D)$,

(iv) $\Lambda_\varepsilon(D_+)+\varepsilon\Lambda_\varepsilon(D_-) = z\{\Lambda_\varepsilon(D_0)+\varepsilon\Lambda_\varepsilon(D_\infty)\}$ *whenever* D_+, D_-, D_0, D_∞ *are the same except near one point where they are as shown in Figure* 5 (*see p.* 63).

If $\vec{D}$ *is an oriented diagram* (*with underlying unoriented diagram* D) *representing a link* $\vec{L}$, *then* $F_\varepsilon(\vec{L}) = a^{-\mathrm{Tait}(\vec{D})}\Lambda_\varepsilon(D)$ *depends only on the isotopy class of* $\vec{L}$. *Moreover*

$$F_{-1}(\vec{L})(a, z) = (-1)^{c(\vec{L})-1}F_{+1}(\vec{L})(ia, -iz)$$

where $c(\vec{L})$ *denotes the number of connected components of* $\vec{L}$, *and*

$$V(\vec{L})(t) = F_{+1}(\vec{L})(-t^{-3/4}, t^{-1/4} + t^{1/4})$$

is the original Jones polynomial, normalized as in **[Jo2]**.

The invariant F_{+1} is often denoted by F, and F_{-1} by F^*; the latter is the so-called *Dubrovnik polynomial*. For the equations relating F_{+1} to F_{-1} and V to F_{+1}, see **[Li2]**. The known proofs of this

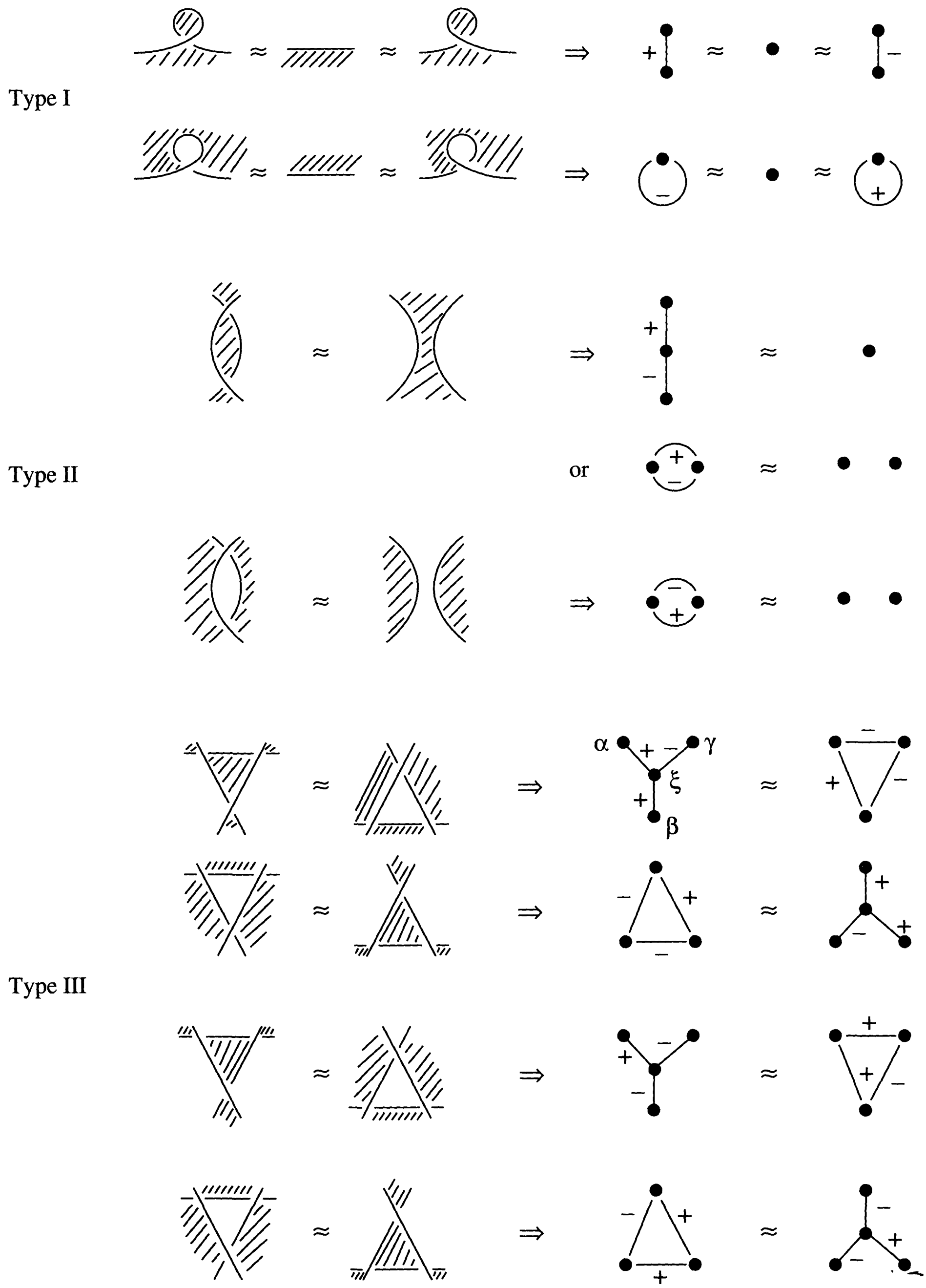


FIGURE 3. Reidemeister moves for signed graphs.

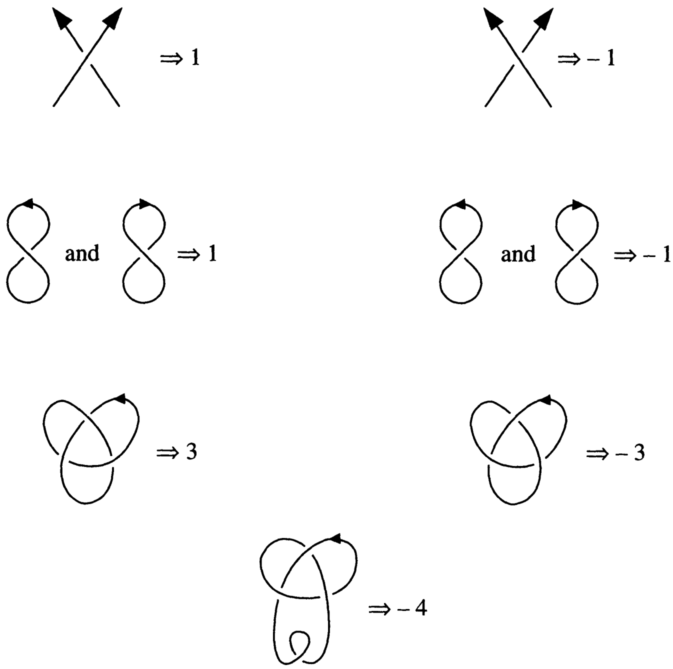


FIGURE 4. Tait numbers for oriented diagrams.

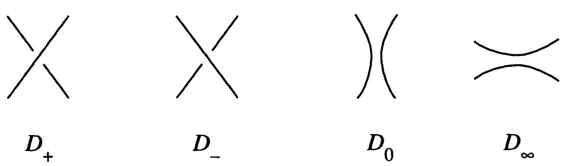


FIGURE 5. The local unoriented diagrams D_+, D_-, D_0, D_∞.

theorem, say when $\varepsilon = 1$, are much longer ([**Ka2**], [**TuR**]) than the simplest proofs (such as that of [**Ka1**]) for the particular case of the polynomial V.

Given $a, z \in \mathbb{C}^*$, the complex-valued function $D \mapsto \Lambda_\varepsilon(D)(a, z)$ and the related $\vec{L} \mapsto F_\varepsilon(\vec{L})(a, z)$ may also be uniquely characterized by (i) to (iv).

About complex-valued invariants related to Λ_ε or to F_ε and involving $z = 0$, let us first recall that $z^{c(\vec{L})-1}F_\varepsilon(\vec{L})(a, z) \in \mathbb{Z}[a^{\pm 1}, z]$ has positive powers of z only for any oriented link $\vec{L}$ (see Proposition 2.5 and 4.7 of [**Li1**]). Let us also mention that there are *several* complex-valued invariants satisfying (i) to (iv) for $\varepsilon = -1$, $z = 0$

and $a \in \{1, -1\}$: for any $d \in \mathbb{C}^*$ the complex-valued invariant $D \mapsto d^{c(L)-1}a^{\mathrm{Tait}(D)}$ satisfies (i) to (iv). (See the discussion of the so-called degenerate examples in §7 below.)

3. Reidemeister conditions and spin models. Graphs of interest here may have loops and multiple edges. A *signed graph* X has a set of vertices denoted by X^0 and a set of edges $X^1 = X^1_+ \amalg X^1_-$ partitioned in two subsets; for each edge $e \in X^1$ we denote arbitrarily one of its ends by e' and the other by e'' (and $e'' = e'$ iff e is a loop). A *spin model for signed graphs* is a quintuplet $M = (S, w_+, w_-, \Omega, d)$ where S is a finite set, where Ω is a ring given together with an invertible element $d \in \Omega$, and where w_+, w_- are symmetric maps $S \times S \to \Omega$ called the *Boltzmann weights*. Given such a model M and a signed graph X, the corresponding *partition function* is here (as in **[HaJ]**, except that $d = 1$ there)

$$Z_X^M = d^{-|X^0|} \sum_{\sigma : X^0 \to S} \prod_{e \in X^1} w_e(\sigma(e'), \sigma(e'')) \in \Omega$$

where $|X^0|$ is the number of vertices of X; we write w_e for w_+ when $e \in X^1_+$ and for w_- when $e \in X^1_-$.

Here are two examples for ordinary graphs; in this case $w_+ = w_-$ is simply denoted by w. One has $S = \{1, 2, \ldots, n\}$, $\Omega = \mathbb{C}$ and $d = 1$ for the two examples.

EXAMPLE 1. Set $w(\alpha, \beta) = 0$ if $a = \beta \in S$ and $w(\alpha, \beta) = 1$ otherwise. Then Z_X^M is the number $\chi_X(n)$ of so-called *proper* colourings of X namely of maps $\sigma: X^0 \to S$ such that $\sigma(e') \neq \sigma(e'')$ for all $e \in X^1$. In other words Z_X^M is the evaluation at n of the *chromatic polynomial* χ_X of X, as studied by Birkhoff **[Bir]** and Whitney **[Whi]**. This example is a prototype for many other "chromatic invariants", as discussed in **[HaJ]**.

EXAMPLE 2. Choose a constant $L \in \mathbb{C}$; set $w(\alpha, \beta) = \exp(\frac{L}{kT})$ if $\alpha = \beta$ and $w(\alpha, \beta) = \exp(\frac{-L}{kT})$ if $\alpha \neq \beta$, where k and T hold for the Boltzmann constant and the temperature. Set also $\varepsilon(\alpha, \beta) = -1$ if $\alpha = \beta$ and $\varepsilon(\alpha, \beta) = 1$ if $\alpha \neq \beta$. Then

$$Z_X^M = \sum_{\sigma : X^0 \to S} \exp\left(\frac{-1}{kT} E(X, \sigma)\right)$$

where each state σ has an energy

$$E(X, \sigma) = \sum_{e \in X^1} L\varepsilon(\sigma(e'), \sigma(e'')).$$

In statistical physics, Z_X^M is viewed as a function of T and is known as the partition function of X for the *Potts' model* of ferromagnetism (here without external field): X represents a crystal, vertices of X its atoms, and edges of X its interacting pairs of atoms. The quantity

$$\frac{1}{Z_X^M} \exp\left(-\frac{1}{kT} E(X, \sigma)\right)$$

is interpreted as a probability for the state σ among all possible states, and these probabilities define the so-called Gibbs measure on the (here finite) set of states. The particular case $n = 2$ is that of the (zero field) *Ising model*, first studied in the early 1920's **[Bru]**.

It could seem natural to ask whether there exist examples of models $M = (S, w_+, w_-, \Omega, d)$ such that $Z_X^M = Z_{X'}^M$ whenever the signed graphs X, X' correspond to diagrams D, D' representing links L, L' which are isotopic. But one has to modify slightly the question, in two ways.

First, consider the two following examples. (i) A trivial knot $\vec{U}$ represented by a diagram which is a circle, hence by a signed graph reduced to one point. (ii) A trivial link $\vec{UU}$ represented by a diagram made of two concentric circles, hence by a signed graph which is again reduced to *one* point. It would not be appropriate to study invariants whose values on $\vec{U}$ and $\vec{UU}$ are always the same! For this reason, we agree from now on with the following: whenever a link L is represented by a diagram D which has several connected components $D_1, \dots, D_m$, we represent D by a signed graph X which is the disjoint union of the corresponding signed graphs $X_1, \dots, X_m$. In particular $\vec{U}$ and $\vec{UU}$ as above are respectively represented by one and *two* points.

Second, a closer look taking orientations into account shows that the correct condition on $M = (S, w_+, w_-, \Omega, d)$ is:

> there exists an invertible $a \in \Omega$ such that $a^{-\mathrm{Tait}(\vec{D})} Z_X^M = a^{-\mathrm{Tait}(\vec{D}')} Z_{X'}^M$ whenever the signed graphs X, X' correspond to diagrams $\vec{D}$, $\vec{D}'$ representing oriented links $\vec{L}$, $\vec{L}'$ which are isotopic.

Figure 3 suggests three types of natural sufficient conditions for this. The conditions for moves of type II are

$$(1) \qquad \frac{1}{d^2} \sum_{\xi \in S} w_+(\alpha, \xi) w_-(\xi, \beta) = \delta(\alpha, \beta),$$

$$w_+(\alpha, \beta)w_-(\alpha, \beta) = 1 \tag{2}$$

for all $\alpha, \beta \in S$. (The first illustration for Type II in Figure 3 shows firstly the case where the number of connected components of the corresponding diagram does not change, and secondly the case where this number does change by 1.) The conditions for moves of type III are

$$\begin{aligned}\frac{1}{d}\sum_{\xi \in S} w_+(\alpha, \xi)w_+(\beta, \xi)w_-(\gamma, \xi) \\ = w_+(\alpha, \beta)w_-(\beta, \gamma)w_-(\gamma, \alpha),\end{aligned} \tag{3}$$

$$\begin{aligned}\frac{1}{d}\sum_{\xi \in S} w_-(\alpha, \xi)w_-(\beta, \xi)w_+(\gamma, \xi) \\ = w_-(\alpha, \beta)w_+(\beta, \gamma)w_+(\gamma, \alpha)\end{aligned} \tag{4}$$

for all $\alpha, \beta, \gamma \in S$. If (2) holds, (3) and (4) with $\alpha = \gamma$ imply first that there exists some invertible element $a \in \Omega$ such that

$$\begin{cases} w_+(\alpha, \alpha) = a, \\ w_-(\alpha, \alpha) = a^{-1} \end{cases} \tag{5}$$

for all $\alpha \in S$ and second that

$$\begin{cases} \dfrac{1}{d}\displaystyle\sum_{\xi \in S} w_-(\beta, \xi) = a, \\ \dfrac{1}{d}\displaystyle\sum_{\xi \in S} w_+(\beta, \xi) = a^{-1} \end{cases} \tag{6}$$

for all $\beta \in S$.

Observe that (5) and (6) are the natural sufficient conditions on M associated to moves of type I, which are thus consequences of the conditions (1) to (4) associated to moves of types II and III. It may also be shown that, if (1) and (2) hold, each of (3), (4) is a consequence of the other (see Figure 6).

Observe also that (1) and (2) imply

$$d^2 = n. \tag{7}$$

Let us now introduce the free module $V = \Omega^S$ together with its canonical basis $(v_\alpha)_{\alpha \in S}$. The matrices $w_+(\alpha, \beta)_{\alpha, \beta \in S}$ and $w_-(\alpha, \beta)_{\alpha, \beta \in S}$ correspond to endomorphisms of V. Traditionally, they are denoted respectively by R_+ and R_-, and we write also

$$M = (S, R_+, R_-, \Omega, d)$$

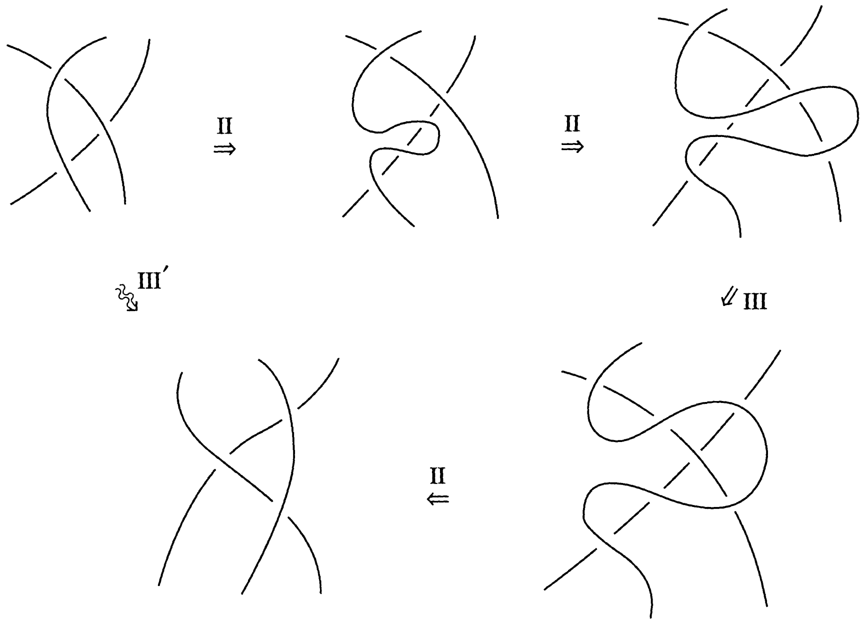


FIGURE 6. Two kinds of type III moves.

for (S, w_+, w_-, Ω, d). The matrix $\delta(\alpha, \beta)_{\alpha,\beta\in S}$ corresponds to the identity $I \in \mathrm{End}(V)$ and the matrix with all coefficients 1 to an endomorphism J whose image is the scalar multiples of $U = \sum_{\alpha\in S} v_\alpha$. We denote by $A \circ B$ the *Hadamard product* of two matrices $A, B \in \mathrm{End}(V)$, defined by $(A \circ B)(\alpha, \beta) = A(\alpha, \beta)B(\alpha, \beta)$. If all entries of A are invertible, $A^{-!}$ denotes the Hadamard inverse of A, so that $A \circ A^{-!} = A^{-!} \circ A = J$.

Equations (1) and (2) can respectively be written as

$$R_+R_- = d^2 I \quad \text{or} \quad R_- = d^2 R_+^{-1}, \tag{8}$$

$$R_+ \circ R_- = J \quad \text{or} \quad R_- = R_+^{-!} \tag{9}$$

while (5) and (6) can be written as

$$I \circ R_+ = aI, \quad I \circ R_- = a^{-1} I, \tag{10}$$

$$JR_- = daJ, \quad JR_+ = da^{-1}J. \tag{11}$$

We define moreover $R_1, R_2 \in \mathrm{End}(V \otimes V)$ by

$$R_1(v_\alpha \otimes v_\beta) = \sum_{\xi\in S} w_+(\alpha, \xi) v_\xi \otimes v_\beta,$$
$$R_2(v_\alpha \otimes v_\beta) = dw_-(\alpha, \beta) v_\alpha \otimes v_\beta$$

for all $\alpha, \beta \in S$. It is straightforward to check that (3) can be written as

$$R_1 R_2 R_1 = R_2 R_1 R_2. \tag{12}$$

As R_1, R_2 are invertible (because of (8) and (9)), one may multiply (12) to the left by R_2^{-1} and to the right by R_1^{-1} to obtain

$$R_2^{-1} R_1 R_2 = R_1 R_2 R_1^{-1}, \tag{12'}$$

and it is again straightforward to check the latter is a rewriting of (4). One has also the relations

$$R_1 = (R_1 R_2)^{-1} R_2 (R_1 R_2), \tag{12+}$$

$$R_1^{-1} = (R_1 R_2)^{-1} R_2^{-1} (R_1 R_2) \tag{12−}$$

each of which is equivalent (when (8) and (9) hold) to (12).

DEFINITION. A *spin model for oriented links* is a spin model for signed graphs $M = (S, R_+, R_-, \Omega, d)$ such that equations (8) to (12) hold for some invertible element $a \in \Omega$ called the *modulus* of the model. The number d is called the *loop variable* of the model; recall from (7) that it is a square root of the cardinal n of S.

We may sum up the discussion above as follows.

PROPOSITION 1. *Let $M = (S, R_+, R_-, \Omega, d)$ be a spin model for oriented links with modulus a. Given an oriented link $\vec{L}$ represented by a diagram $\vec{D}$ with corresponding signed graph X, the element $a^{-\operatorname{Tait}(\vec{D})} Z_X^M \in \Omega$ depends only on the isotopy class of $\vec{L}$. In other words, the assignment*

$$\vec{L} \mapsto a^{-\operatorname{Tait}(\vec{D})} Z_X^M$$

is a well defined Ω-valued link invariant.

As an exercise, the reader may check that $a^{-\operatorname{Tait}(\vec{D})} Z_X^M = d$ for various pairs $(\vec{D}, X)$ representing the trivial knot.

4. Spin models for the Jones polynomial. Let

$$M = (S, R_+, R_-, \Omega, d)$$

be a spin model for oriented links such that $n = |S| \geq 2$. Denote by m the number of *distinct* values of the nondiagonal entries of R_+. One may introduce symmetric S-times-S matrices $A_0 = I$, $A_1, \dots, A_m$ with entries in $\{0, 1\}$ such that $A_i \neq 0$ and

$$A_0 + A_1 + \cdots + A_m = J,$$

as well as pairwise distinct elements $t_0, t_1, \dots, t_m \in \Omega$ such that

$$R_+ = \sum_{0 \le i \le m} t_i A_i.$$

Equation (10) shows that t_0 is the modulus of M. It follows from (9) that the t_i's are invertible and that $R_- = \sum_{0 \le i \le m} t_i^{-1} A_i$.

In this section, we consider the case $m = 1$, and we write

$$R_+ = aI + b(J - I), \quad R_- = a^{-1}I + b^{-1}(J - I).$$

Equations (8), (11) and (12) read now respectively

$$ab^{-1} + a^{-1}b = 2 - n, \tag{13}$$

$$\begin{cases} a^{-1} + (n-1)b^{-1} = da, \\ a + (n-1)b = da^{-1}, \end{cases} \tag{14}$$

$$\begin{cases} a^2b^{-1} + a^{-1}b^2 + (n-2)b = dab^{-2}, \\ a + (n-1)b = da^{-1}, \\ 2a + a^{-1} + a^{-1}b^2 + (n-3)b = db^{-1}. \end{cases} \tag{15}$$

(Write $(R_1R_2R_1 - R_2R_1R_2)(v_\alpha \otimes v_\beta) = \sum_\eta C_{\alpha,\beta,\eta} v_\eta \otimes v_\beta = 0$. Because of (14), it is enough to consider the case $\alpha \neq \beta$. The three equations in (15) correspond then respectively to $C_{\alpha,\beta,\alpha} = 0$, $C_{\alpha,\beta,\beta} = 0$, and $C_{\alpha,\beta,\eta} = 0$ with $\eta \notin \{\alpha, \beta\}$; the last of these comes only when $n \geq 3$.)

Viewing first (14) as a linear system in a and a^{-1}, we obtain

$$a = b + db^{-1}, \qquad a^{-1} = b^{-1} + db.$$

This implies $d = -b^2 - b^{-2}$ by elimination of a, and also $a = b + db^{-1} = -b^{-3}$ (recall that $d^2 = n$ by (7)). One obtains in this way the following proposition, which appears already as Example 2.17 in **[Jo3]**, and again (with A for b^{-1}) as the last example of **[HJ2]**.

PROPOSITION 2. *Consider an integer $n \geq 2$ and a complex number b such that*

$$(b^2 + b^{-2})^2 = n.$$

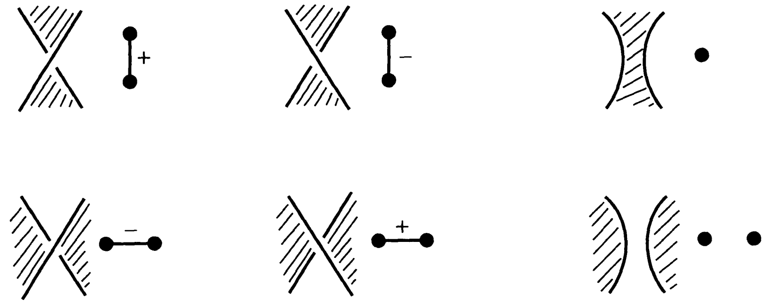


FIGURE 7. On the skein relations.

Define $M = (S, R_+, R_-, \mathbb{C}, d)$ *by* $S = \{1, 2, \dots, n\}$ *and*

$$R_+ = -b^{-3}I + b(J - I), \qquad R_- = -b^3 I + b^{-1}(J - I),$$
$$d = -b^2 - b^{-2}.$$

Then M *is a spin model for oriented links.*

Moreover, if $\vec{L}$, $\vec{D}$ *and* X *are as in Proposition* 1, *one has*

$$\frac{1}{d}(-b^{-3})^{-\operatorname{Tait}(\vec{D})} Z_X^M = V(\vec{L})(b^4)$$

where the right-hand term is the evaluation at b^4 *of the Jones polynomial of* $\vec{L}$.

Proof. The first claim has been proved above. For the second claim, we know already from §3 that $\frac{1}{d}(-b^{-3})^{-\operatorname{Tait}(\vec{D})} Z_X^M$ provides a link invariant, say $V_*(\vec{L}) \in \mathbb{C}$, giving the value 1 to the trivial knot. To finish the proof, it is thus enough to check that V_* satisfies the exchange property (see e.g. **[Li1]**, or indeed almost any reference on the Jones polynomial).

Consider a skein related triple $(\vec{D}_+, \vec{D}_-, \vec{D}_0)$ of oriented link diagrams. Around the distinguished crossing, the black and white colourings and the associated graphs look as in one of the two situations represented in Figure 7.

The exchange property

$$t^{-1}V(\vec{L}_+)(t) - tV(\vec{L}_-)(t) + (t^{-1/2} - t^{1/2})V(\vec{L}_0)(t) = 0$$

follows from the identities

$$\frac{1}{d}b^{-4}(-b^{-3})^{-1}w_+(\alpha, \beta) - \frac{1}{d}b^4(-b^{-3})w_-(\alpha, \beta) + (b^{-2} - b^2)\delta_{\alpha,\beta} = 0,$$
$$b^{-4}(-b^{-3})^{-1}w_-(\alpha, \beta) - b^4(-b^{-3})w_+(\alpha, \beta) + b^{-2} - b^2 = 0$$

for all $\alpha, \beta \in \{1, \dots, n\}$, which are both easy to check. □

The models of Proposition 2 are called *Potts' models*, because the values of $w_+(\alpha, \beta)$ and $w_-(\alpha, \beta)$ depend only on α and β being equal or not, as in Example 2 of §3.

5. Reidemeister conditions and strongly regular graphs. We keep the notations of §4, with the exception that S is now denoted by S^0. We consider a spin model $M = (S^0, R_+, R_-, \mathbb{C}, d)$ such that there are exactly $m = 2$ distinct nondiagonal entries of R_+ (or of R_-). We introduce a simple graph S with vertex set S^0 and with adjacency matrix A_1: a pair $(\alpha, \beta) \in S^0 \times S^0$ defines an edge of S if and only if $A_1(\alpha, \beta) = 1$.

5.1. *Strong regularity of the graph* S. As R_+ is now given by

$$R_+ = t_0 I + t_1 A_1 + t_2 A_2 = t_0 I + t_1 A_1 + t_2(J - I - A_1),$$

equation (11) shows that JA_1 is a scalar multiple of J and equation (8) shows then that A_1 satisfies a relation of the form

$$c_0 A_1^2 + c_1 A_1 + c_2 I = c_3 J \tag{16}$$

for some complex constants $c_0 \neq 0$, c_1, c_2, c_3. This implies the following.

(i) There exists a number, say k, such that each vertex of S has exactly k neighbours; in other words, the graph S is *k-regular*. (Indeed $k = c_0^{-1}(c_3 - c_2)$ follows from the equality of the diagonal entries in (16).)

(ii) There exists a number, say λ, such that two vertices $\alpha, \beta \in S^0$ joined by an edge in S have exactly λ common neighbours. (Consider the entries (α, β) in (16) such that $A_1(\alpha, \beta) = 1$.)

(iii) There exists a number, say μ, such that two distinct vertices $\alpha, \beta \in S^0$ not joined by an edge have exactly μ common neighbours. (Consider the entries (α, β) in (16) such that $\alpha \neq \beta$ and $A_1(\alpha, \beta) = 0$.) In other words, one has the following.

PROPOSITION 3. *With the notations above, S is a strongly regular graph with parameters (n, k, λ, μ) and one has*

$$A_1 J = J A_1 = kJ, \tag{17}$$

$$A_1^2 + (\mu - \lambda) A_1 + (\mu - k) I = \mu J. \tag{18}$$

Observe that one has necessarily

$$n \geq 4. \tag{19}$$

Indeed, for a regular graph with $n = 2$ or $n = 3$, one would have either $A_1 = 0$ or $A_2 = J - I - A_1 = 0$ and this would imply $m = 1$, in contradiction with our hypothesis $m = 2$.

5.2. *First conditions on the weights.* Equations (8) to (12) impose strong conditions on the weights t_0, t_1, t_2 appearing in R_+. Equation (9) can be seen as a definition of $R_- = R_+^{-!}$, and equation (12) is often complicated to deal with (see Propositions 6 and 7 below). We reformulate now (8), (10) and (11).

PROPOSITION 4. *Let* $R_+ = t_0 + t_1 A_1 + t_2(J - I - A_1)$ *be the* (+)*-matrix of weights of a spin model for oriented links, associated as above to a strongly regular graph of parameters* (n, k, λ, μ). *Then one has*

$$\frac{t_0}{t_2} + \frac{t_2}{t_0} + (k - \mu)\left(\frac{t_1}{t_2} + \frac{t_2}{t_1}\right) + n + 2(\mu - k - 1) = 0, \tag{20}$$

$$\frac{t_0}{t_1} + \frac{t_1}{t_0} - \left(\frac{t_0}{t_2} + \frac{t_2}{t_0}\right) - (\lambda - \mu + 1)\left(\frac{t_1}{t_2} + \frac{t_2}{t_1}\right) + 2(\lambda - \mu + 1) = 0, \tag{21}$$

$$t_0 = a \ (\textit{the modulus of the model}), \tag{22}$$

$$t_0^{-1} + k t_1^{-1} + (n - k - 1) t_2^{-1} = da, \tag{23}$$

$$t_0 + k t_1 + (n - k - 1) t_2 = da^{-1}. \tag{24}$$

Proof. Equations (22) to (24) are nothing but ways to write (10) and (11) in the present case. Next, a straightforward computation using (18) shows that

$$\begin{aligned} R_+ R_- &= [(t_0 - t_2)I + (t_1 - t_2)A_1 + t_2 J][(t_0^{-1} - t_2^{-1})I \\ &\qquad + (t_1^{-1} - t_2^{-1})A_1 + t_2^{-1} J] \\ &= -\left[\frac{t_0}{t_2} + \frac{t_2}{t_0} + (k - \mu)\left(\frac{t_1}{t_2} + \frac{t_2}{t_1}\right) + 2(\mu - k - 1)\right] I \\ &\quad + \left[\frac{t_0}{t_1} + \frac{t_1}{t_0} - \left(\frac{t_0}{t_2} + \frac{t_2}{t_0}\right) - (\lambda - \mu + 1)\left(\frac{t_1}{t_2} + \frac{t_2}{t_1}\right) \right. \\ &\qquad \left. + 2(\lambda - \mu + 1)\right] A_1 \\ &\quad + \left[\frac{t_0}{t_2} + \frac{t_2}{t_0} + (k - \mu)\left(\frac{t_1}{t_2} + \frac{t_2}{t_1}\right) + n + 2(\mu - k - 1)\right] J \end{aligned}$$

and it follows that (8) is equivalent to (20) and (21). □

EXAMPLE. *The case* $n = 4$. (This is alluded to in **[Jo3]**, just before Example 2.17.) Here, one must have $k \in \{1, 2\}$. Indeed, $k = 0$ would imply $A_1 = 0$ and $k = 3$ would imply $A_2 = 0$, and these are incompatible with $m = 2$. Upon exchanging A_1 and A_2, we may assume that $k = 2$, namely that the underlying graph is a square. Up to renumeration of the four vertices of S, one has then

$$R_+ = \begin{pmatrix} t_0 & t_1 & t_2 & t_1 \\ t_1 & t_0 & t_1 & t_2 \\ t_2 & t_1 & t_0 & t_1 \\ t_1 & t_2 & t_1 & t_0 \end{pmatrix}$$

for some $t_0, t_1, t_2 \in \mathbb{C}^*$.

As the parameters of the square are given by $(n, k, \lambda, \mu) = (4, 2, 0, 2)$, equation (20) reduces to $t_0t_2^{-1} + t_2t_0^{-1} + 2 = 0$, namely to $(t_0 + t_2)^2 = 0$, so that $t_2 = -t_0$, and this in turn implies that (21) holds. Now (23) and (24) read $2t_1^{-1} = dt_0$ and $2t_1 = dt_0^{-1}$. By (7) there exists $\varepsilon \in \{1, -1\}$ such that $d = -2\varepsilon$; then $t_1 = -\varepsilon t_0^{-1}$. A tedious but straightforward computation shows that (12) holds, so that we have shown the first half of the following.

PROPOSITION 5. *Given any* $a \in \mathbb{C}^*$ *and* $\varepsilon \in \{1, -1\}$, *set*

$$M_{a,\varepsilon} = (\{1, 2, 3, 4\}, R_+, R_+^{-!}, \mathbb{C}, -2\varepsilon)$$

where

$$R_+ = \begin{pmatrix} a & -\varepsilon a^{-1} & -a & -\varepsilon a^{-1} \\ -\varepsilon a^{-1} & a & -\varepsilon a^{-1} & -a \\ -a & -\varepsilon a^{-1} & a & -\varepsilon a^{-1} \\ -\varepsilon a^{-1} & -a & -\varepsilon a^{-1} & a \end{pmatrix}.$$

Then $M_{a,\varepsilon}$ *is a spin model for oriented links with module* a *and loop variable* $d = -2\varepsilon$.

Moreover, if $\vec{L}$, $\vec{D}$ *and* X *are as in Proposition* 1, *we have*

$$\frac{1}{d}a^{-\operatorname{Tait}(\vec{D})}Z_X^{M_{a,\varepsilon}} = F_\varepsilon(\vec{L})(a, -\varepsilon a - a^{-1}).$$

Proof. Set $z = -\varepsilon a - a^{-1}$ and $R_- = R_+^{-!}$. From the definition of R_+ a straightforward computation shows that

$$R_+ + \varepsilon R_- = z(dI + \varepsilon J).$$

It follows from the Theorem of §2 that one has, with the notations of this Theorem:

$$\frac{1}{d} Z_X^{M_{a,\varepsilon}} = \Lambda_\varepsilon(D)(a, z)$$

and consequently $\frac{1}{d} a^{-\mathrm{Tait}(\vec{D})} Z_X^{M_{a,\varepsilon}} = F_\varepsilon(\vec{L})(a, z)$. □

Recall that the invariant of Proposition 5 has a topological interpretation **[LiM]**:

$$F_+(\vec{L})(-q, q+q^{-1}) = \frac{1}{2}(-1)^{c(L)-1} \sum_{\vec{X} \subset \vec{L}} q^{-4\mathrm{lk}(\vec{X}, \vec{L}-\vec{X})}$$

$$F_-(\vec{L})(x^{-1}, x^{-1}-x) = \frac{1}{2} \sum_{\vec{X} \subset \vec{L}} x^{4\mathrm{lk}(\vec{X}, \vec{L}-\vec{X})}$$

for all $q, x \in C^*$, where the summations are over all pairs of components of $\vec{L}$ (including $\vec{X} = \varnothing$ and $\vec{X} = \vec{L}$) and where lk denotes a linking number (with $\mathrm{lk}(\varnothing, \vec{L}) = 0$); in particular

$$F_+(\vec{L})(-q, q+q^{-1}) = F_-(\vec{L})(x^{-1}, x^{-1}-x) = 1$$

for all $q, x \in C^*$ in case $\vec{L}$ is an actual knot.

Example: *a pentagonal model.* (See **[Jo4]**.) The pentagon is a strongly regular graph with parameters

$$(n, k, \lambda, \mu) = (5, 2, 0, 1).$$

Let A_1 denote the adjacency matrix of this graph (with vertices numbered in a cyclic order) and set

$$\begin{aligned} \omega &= \exp\left(\frac{2i\pi}{5}\right), \\ R_+ &= -iI - i\omega A_1 - i\omega^{-1}(J - I - A_1) \\ &= -i \begin{pmatrix} 1 & \omega & \omega^{-1} & \omega^{-1} & \omega \\ \omega & 1 & \omega & \omega^{-1} & \omega^{-1} \\ \omega^{-1} & \omega & 1 & \omega & \omega^{-1} \\ \omega^{-1} & \omega^{-1} & \omega & 1 & \omega \\ \omega & \omega^{-1} & \omega^{-1} & \omega & 1 \end{pmatrix}. \end{aligned}$$

Proposition 6. *Let $S^0 = \{0, 1, 2, 3, 4\}$ denote the set of vertices of a pentagon*; *with R_+ as above,*

$$M_5 = (S^0, R_+, R_+^{-!}, \mathbb{C}, -\sqrt{5})$$

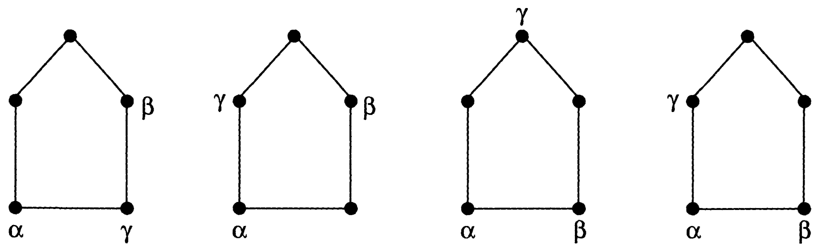


FIGURE 8. See the proof of Proposition 6.

is a spin model for oriented links with

$$\textit{modulus} \quad a = -i,$$
$$\textit{loop variable} \quad d = -\sqrt{5}.$$

Moreover, if $\vec{L}$, $\vec{D}$ *and* X *are as in Proposition* 1, *we have*

$$\begin{aligned}\frac{1}{d} a^{-\operatorname{Tait}(\vec{D})} Z_X^{M_5} &= F_{-1}(\vec{L})(-i, 2i\cos(2\pi/5)) \\ &= (-1)^{c(\vec{L})-1} F_{+1}(\vec{L})(1, 2\cos(2\pi/5)).\end{aligned}$$

Proof. The following proof is a warming up exercise for that of Proposition 7.

Let us first show that M_5 is a spin model for oriented links, namely that equations (8) to (12) are verified. For (8) to (11), it is enough by Proposition 4 to check (20) to (24), and this is straightforward (recall that $\operatorname{Re}(\omega) = \cos(2\pi/5) = (\sqrt{5}-1)/4$. The proof of the claim is now reduced to checking (12), or equivalently (3) for all $\alpha, \beta, \gamma \in S^0$.

We know already that (3) holds when $\alpha = \gamma$ or $\beta = \gamma$, because we know that (9) and (11) hold. If $\alpha = \beta$, equation (3) reduces to

$$\sum_{0 \le \xi \le 4} w_+(\alpha, \xi)^2 w_-(\gamma, \xi) = d a w_-(\alpha, \gamma)^2$$

and one may assume $\alpha \neq \gamma$. Thus, one has to check

$$t_0^2 t_1^{-1} + t_1^2 t_0^{-1} + t_2^2 t_1^{-1} + t_2^2 t_2^{-1} + t_1^2 t_2^{-1} \stackrel{?}{=} d a t_1^{-2}$$

if α and γ are connected by an edge in the pentagon, and

$$t_0^2 t_2^{-1} + t_1^2 t_1^{-1} + t_2^2 t_0^{-1} + t_2^2 t_1^{-1} + t_1^2 t_2^{-1} \stackrel{?}{=} d a t_2^{-2}$$

if not. These two identities $\stackrel{?}{=}$ are straightforward to check when t_0, t_1, t_2 are replaced respectively by $-i$, $-i\omega$, $-i\omega^{-1}$.

We may now assume that α, β, γ are all distinct: $5 \times 4 \times 3 = 60$ cases left. But one can use the symmetries of the pentagon, and it is enough to check (3) for the four cases of Figure 8.

In the first case, (3) reduces to

$$t_0t_2t_1^{-1} + t_1t_1t_0^{-1} + t_2t_0t_1^{-1} + t_2t_1t_2^{-1} + t_1t_2t_2^{-1} \stackrel{?}{=} dt_2t_1^{-1}t_1^{-1}$$

which is again straightforward to check when $(t_0, t_1, t_2) = (-i, -i\omega, -i\omega^{-1})$. The three other cases are similar, and the first claim is proved.

Set now $z = -t_1 + t_1^{-1} = 2i\cos(2\pi/5)$. An easy computation shows that

$$R_+ - R_+^{-!} = z(dI - J).$$

It follows from the theorem in §2 that one has

$$\frac{1}{d}Z_X^{M_5} = \Lambda_{-1}(D)(-i, 2i\cos(2\pi/5))$$

and consequently

$$\begin{aligned}\frac{1}{d}a^{-\operatorname{Tait}(\vec{D})}Z_X^{M_5} &= F_{-1}(\vec{L})(-i, 2i\cos(2\pi/5)) \\ &= (-1)^{c(\vec{L})-1}F_{+1}(\vec{L})(1, 2\cos(2\pi/5)). \qquad \square\end{aligned}$$

The invariant of Proposition 6 has again a topological interpretation. We refer to **[GoJ]** and **[Jo4]** for a complete description; but let us recall here that, for an actual knot K, one has

$$|F_{+1}(K)(1, 2\cos(2\pi/5))| = |F_{-1}(K)(-i, 2i\cos(2\pi/5))| = (\sqrt{5})^r$$

where r is the rank of the first homology with coefficients $\mathbb{Z}/5\mathbb{Z}$ of the 2-fold branched cover of $\mathbb{S}^3$ branched over K.

5.3. *A digression on products.* Consider a finite sequence $M_1, \dots, M_k$ of spin models for oriented links, where

$$M_j = (S_j, R_{j,+}, R_{j,-}, \Omega, d_j)$$

has modulus a_j for $j \in \{1, \dots, k\}$. The *product* $M = (S, R_+, R_-, \Omega, d)$ of these models is defined by

$$S = \prod_{1\le j\le k} S_j,$$

$$w_\pm((\alpha_1, \dots, \alpha_k), (\beta_1, \dots, \beta_k)) = \prod_{1\le j\le k} w_{j,\pm}(\alpha_j, \beta_j),$$

$$d = \prod_{1\le j\le k} d_j.$$

The following are straightforward to check:

M is again a spin model for oriented links;
its modulus is given by $a = \prod_{1 \le j \le k} a_j$;
one has $Z_X^M = \prod_{1 \le j \le k} Z_X^{M_j}$ for any signed graph X;
the link invariant $\vec{L} \mapsto a^{-\mathrm{Tait}(\vec{D})} Z_X^M$ associated to M as in Proposition 1 is the product of the invariants associated to the M_j's.

For a specific example, consider integers $n \ge 2$, $k \ge 1$, and let $M_1, \dots, M_k$ be k Potts' models with n spins, with the model M_j corresponding to a root b_j of the equation $(b^2 + b^{-2})^2 = n$, as in Proposition 2. The entries of the matrix R_+ of M are

$$\begin{array}{cl} -b_1^{-3},\, b_1 & \text{if } k = 1, \\ b_1^{-3} b_2^{-3},\, -b_1^{-3} b_2,\, -b_1 b_2^{-3},\, b_1 b_2 & \text{if } k = 2, \end{array}$$

and so on. Suppose moreover that $n = 2$, so that each b_j is in

$$\{e^{i\pi/8}, e^{-i\pi/8}, e^{3i\pi/8}, e^{-3i\pi/8}\}.$$

If $b_1 = b_2 = e^{i\pi/8}$, the entries of $M = M_1 \times M_2$ are

$$\begin{aligned} b^{-6} &= e^{-3i\pi/4} = a \quad \text{the modulus of } M, \\ -b^{-2} &= e^{3i\pi/4} = a^{-1}, \\ b^2 &= e^{i\pi/4} = -a \end{aligned}$$

which is *one* of the cases covered by Proposition 5. Observe that such products cover finitely many cases of Proposition 5, so that the latter goes really beyond this product's construction. On the other hand, the product defined by $b_1 = e^{i\pi/8}$ and $b_2 = e^{-i\pi/8}$ is a model for which R_+ has *three* distinct off-diagonal entries, namely i, $-i$, and 1 (which is also the diagonal entry), and consequently which is *not* covered by Proposition 5.

6. The Higman-Sims graph and the Jaeger model.

6.1. *The graph.* We are interested here in the graph HS discovered in the late 1960's by D. Higman and C. Sims **[HiS]**. For details of what follows, we refer to the exposition of Biggs and White **[BiW]**; for some more geometry in the projective plane $PG(2, 4)$ over the field of order 4, one may also see **[Edg]**. For other definitions of HS, see **[GoS]** and **[CoS]**.

Let $\mathscr{P}$ and $\mathscr{L}$ denote respectively the sets of points and lines in $PG(2, 4)$. One has

$$|\mathscr{P}| = |\mathscr{L}| = 4^2 + 4^1 + 4^0 = 21.$$

A *hexad* (or hyperoval) in $PG(2, 4)$ is a subset of $\mathscr{P}$ consisting of 6 points such that 3 of them are never collinear. It is known that the set $\mathscr{H}_{\mathrm{tot}}$ of all these hexads has 168 elements, and that the natural action of the group $\mathrm{PSL}_3(4)$ [respectively $\mathrm{PGl}_3(4)$] on $\mathscr{H}_{\mathrm{tot}}$ has 3 orbits of 56 elements each [resp. is transitive]. Choose one of these 3 orbits and denote it by $\mathscr{H}$; for distinct $H, K \in \mathscr{H}$, it is also known that $H \cap K$ consists of either 0 or 2 points.

The graph HS has two distinguished vertices denoted here by 0 and ∞, and one more vertex for each element of $\mathscr{P}$, $\mathscr{L}$ and $\mathscr{H}$. In particular

$$|HS^0| = 1 + 1 + 21 + 21 + 56 = 100.$$

The edges of HS are the following:

one edge with ends 0 and ∞,
one edge with ends 0 and p for all $p \in \mathscr{P}$,
one edge with ends ∞ and l for all $l \in \mathscr{L}$,
one edge with ends p and l for all $p \in \mathscr{P}$, $l \in \mathscr{L}$ such that $p \in l$,
one edge with ends p and H for all $p \in \mathscr{P}$, $H \in \mathscr{H}$ such that $p \in H$,
one edge with ends l and H for all $l \in \mathscr{L}$, $H \in \mathscr{H}$ such that $l \cap H = \varnothing$,
one edge with ends H and K for all $H \in \mathscr{H}$, $K \in \mathscr{H}$ such that $H \cap K = \varnothing$,

and Figure 9 should aid memory.

The numbers on Figure 9 should be read as follows: each $p \in \mathscr{P}$ defines a vertex in HS^0 which is adjacent to the vertex 0, to 5 vertices from $\mathscr{L}$ and to 16 from $\mathscr{H}$; each $H \in \mathscr{H}$ defines a vertex adjacent to 6 vertices from $\mathscr{P}$, to 6 from $\mathscr{L}$ and to 10 other vertices from $\mathscr{H}$; and so on.

We denote as in §5 by A_1 the adjacency matrix of HS and by $A_2 = J - I - A_1$ that of the complementary graph.

It is known that the group of all automorphisms of HS has a subgroup Γ of index 2 with the following properties (more precisely $\Gamma = \mathrm{Aut}(HS) \cap \mathscr{A}_{100}$ if $\mathscr{A}_{100}$ is the group of all even permutations of HS^0).

(a) Γ acts transitively on each of:

the set HS^0 of vertices of HS,
the set $\{(\alpha, \beta) \in HS^0 \times HS^0 : A_1(\alpha, \beta) = 1\}$ of its oriented edges,

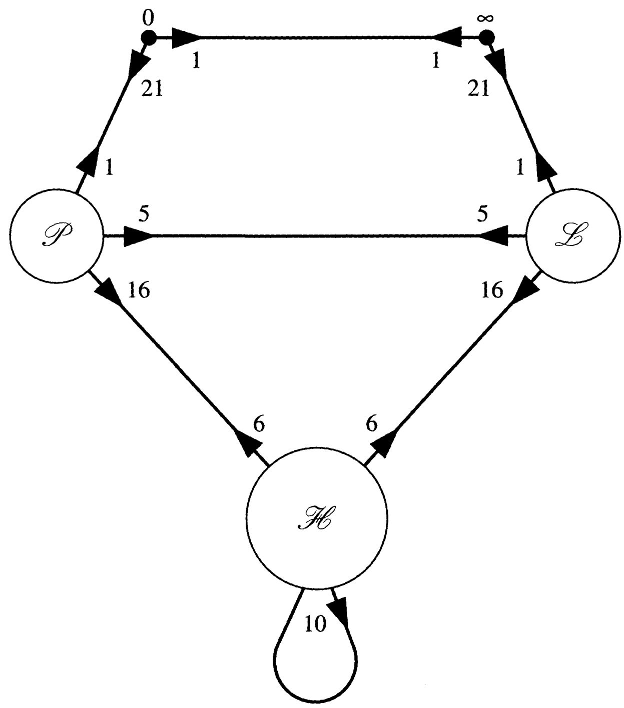


FIGURE 9. On the Higman-Sims graph.

the set $\{(\alpha, \beta) \in HS^0 \times HS^0 : A_2(\alpha, \beta) = 1\}$ of its oriented nonedges.

(b) For each $\alpha \in HS^0$, the isotropy group Γ_α acts transitively on each of the 10 following sets:

$$\{\beta \in HS^0 \colon A_j(\alpha, \beta) = 1\} \quad \text{for } j \in \{1, 2\},$$

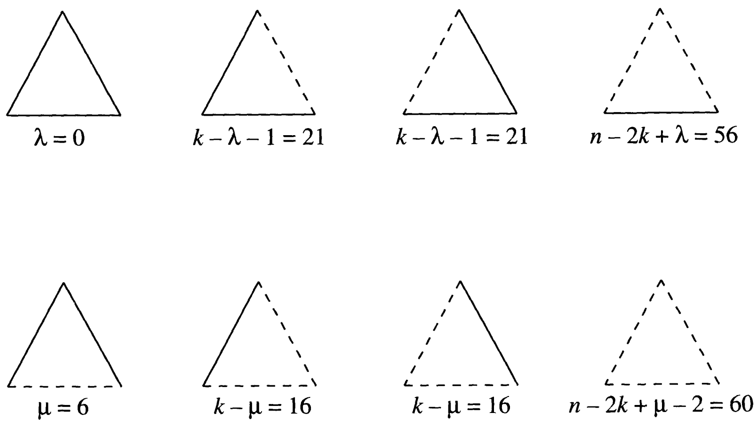
$$\{(\beta, \gamma) \in HS^0 \times HS^0 \colon A_j(\alpha, \beta) = 1, A_k(\alpha, \gamma) = 1, A_l(\beta, \gamma) = 1\} \qquad \text{for } j, k, l \in \{1, 2\}.$$

(c) Γ is a "sporadic" simple group of order $44 \cdot 352 \cdot 000$.
Claim (c) is useless here, but nice to know. Claims (a) and (b) have strong consequences on the geometry of HS. (In claim (b), the set with $j = k = l = 1$ is empty because one has $\lambda = 0$ for the graph HS.)

Claim (a) implies that HS is a strongly regular graph. One may compute its parameters

$$(n, k, \lambda, \mu) = (100, 22, 0, 6)$$

TABLE I. On the Higman-Sims Graph.



(see Proposition 3) and the eigenvalues of the adjacency matrix A_1 :

$$\begin{array}{rl} 22 & \text{with multiplicity } 1, \\ 2 & \text{with multiplicity } 77, \\ -8 & \text{with multiplicity } 22. \end{array}$$

The strong regularity implies that, given two distinct $\alpha_1, \alpha_2 \in HS^0$ and given $\delta_1, \delta_2 \in \{0, 1\}$, the cardinality of

$$\{\beta \in HS^0 : \beta \neq \alpha_k \text{ and } A_1(\alpha_k, \beta) = \delta_k \text{ for } k = 1, 2\}$$

depends only on $A_1(\alpha_1, \alpha_2)$. These cardinalities are shown in Table I where two vertices are joined by a line if they define an edge and by a dotted line otherwise.

Claim (b) implies that, given three distinct $\alpha_1, \alpha_2, \alpha_3 \in HS^0$ and given $\delta_1, \delta_2, \delta_3 \in \{0, 1\}$, the cardinality of

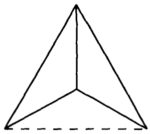
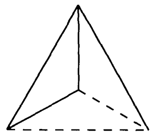
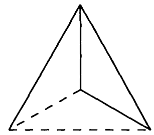
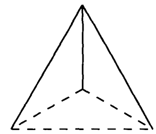
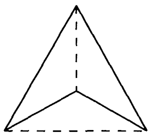
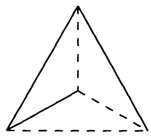
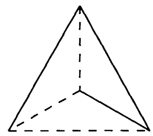
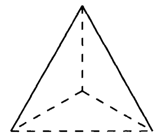
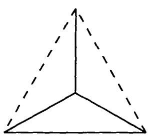
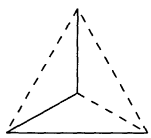
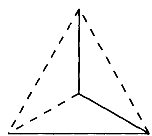
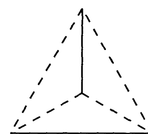
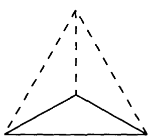
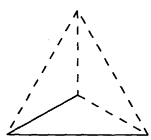
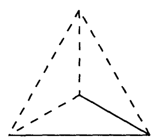
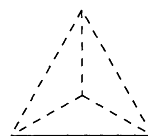
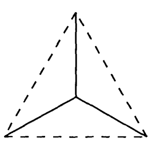
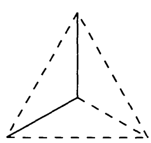
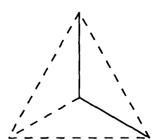
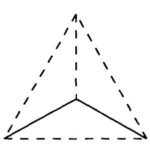
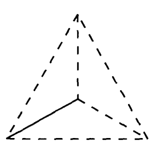
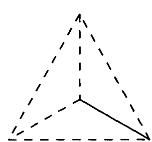
$$\{\beta \in HS^0 : \beta \neq \alpha_k \text{ and } A_1(\alpha_k, \beta) = \delta_k \text{ for } k = 1, 2, 3\}$$

depends only on $A_1(\alpha_1, \alpha_2)$, $A_1(\alpha_2, \alpha_3)$, $A_1(\alpha_3, \alpha_1)$. These cardinalities are shown in Table II. (The eight cardinalities corresponding to the situation $A_1(\alpha_1, \alpha_2) = A_1(\alpha_2, \alpha_3) = A_1(\alpha_3, \alpha_1) = 1$ are 0, because $\lambda = 0$.)

Here are some indications for Table II. For the first octet, choose $p \in \mathscr{P}$ and observe that (Figure 10 on p. 82)

$$|\{q \in \mathscr{P} : q \neq p\}| = 20, \quad |\{l \in \mathscr{L} : l \ni p\}| = 5,$$
$$|\{l \in \mathscr{L} : l \not\ni p\}| = 16, \quad 97 - (20 + 5 + 2 \times 16) = 40.$$

TABLE II. On the Higman-Sims graph.

			
0	0	0	20
			
5	16	16	40
			
0	6	6	10
			
0	15	15	45
			
2	4	4	12
			
4	12	12	47

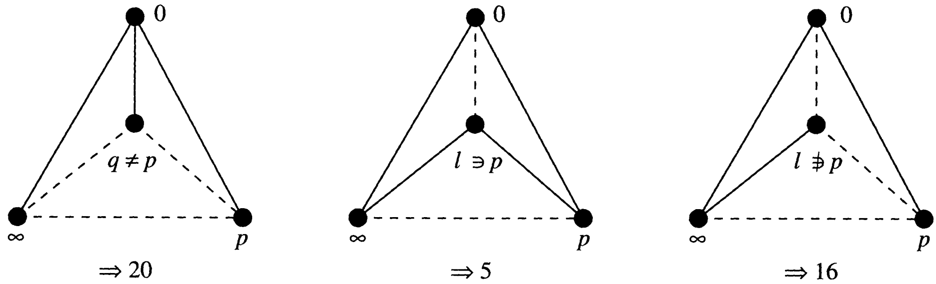


FIGURE 10

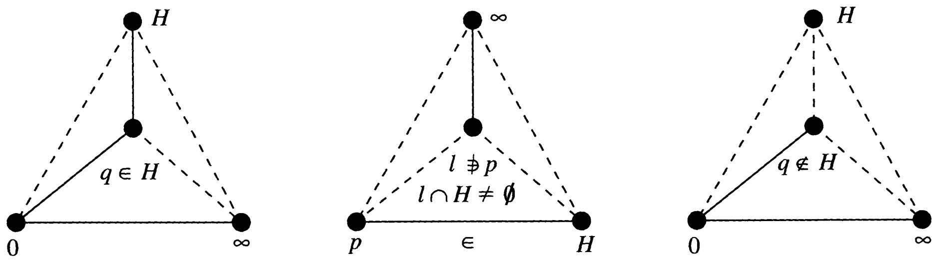


FIGURE 11

For the second octet, choose $H \in \mathscr{H}$, $p \in H$, and observe that (Figure 11)

$$|\{q \in \mathscr{P} : q \in H\}| = 6, \qquad |\{q \in \mathscr{P} : q \notin H\}| = 15,$$
$$|\{l \in \mathscr{L} : l \cap H \neq \varnothing \text{ and } l \not\ni p\}| = 10,$$
$$97 - (2 \times 6 + 10 + 2 \times 15) = 45.$$

For the three appropriate cases of the third octet let x [respectively y, z] denote the cardinality which has to be shown equal to 2 [resp. 4, 12]. The numbers of ordered 4-tuples $(\alpha_1, \alpha_2, \alpha_3)$ of distinct vertices of HS^0 providing the configurations of Figure 12.i are the same, so that

$$100 \times 77 \times 6 \times 20 = 100 \times 77 \times 60 \times x \Rightarrow x = 2.$$

Similarly for those of Figures 12.ii and 12.iii, so that

$$100 \times 77 \times 6 \times 40 = 100 \times 77 \times 60 \times y \Rightarrow y = 4,$$
$$100 \times 22 \times 56 \times 45 = 100 \times 77 \times 60 \times z \Rightarrow z = 12.$$

The last cardinality is of course $97 - (2 + 3 \times 4 + 3 \times 12) = 47$.

Given any vertex $\alpha \in HS^0$, the graph spanned by its neighbours has 22 vertices (because $k_{HS} = 22$) and no edge (because $\lambda_{HS} = 0$). The

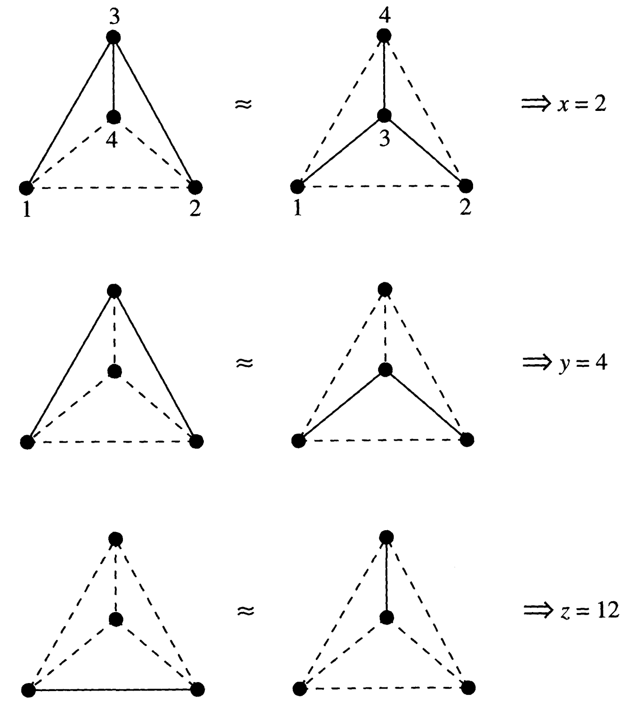


FIGURE 12

graph spanned by the vertices of HS at distance 2 from α is again strongly regular (because of Claim (b) above), say with parameters (n', k', λ', μ') given by

$$\begin{aligned} n' &= n_{HS} - k_{HS} - 1 = 77, \\ k' &= k_{HS} - \mu_{HS} = 16, \\ \lambda' &= \lambda_{HS} = 0, \\ \mu' &= \frac{k'(k' - \lambda' - 1)}{n' - k' - 1} = 4. \end{aligned}$$

It is known that there exists a unique strongly regular graph with parameters $(77, 16, 0, 4)$; see the remarks following Theorem 13.1.1 in [**BCN**]. It is also known that there exists an unique strongly regular graph with parameters $(100, 22, 0, 6)$; see §9 in [**CGS**].

6.2. *The weights.* We may now define the *main example*, due to F. Jaeger, of the present paper. It is a spin model with two nondiagonal Boltzmann weights in the matrix R_+, as discussed in the beginning of §5. The relevant finite set is the set HS^0 of vertices of the Higman-Sims graph. The loop variable is $d = -10$ (observe that equation (7) holds). Let

$$\tau \doteq \frac{1+\sqrt{5}}{2}$$

be the golden ratio. Set

$$(25) \qquad \begin{cases} t_0 = -5\tau - 3 = -\tau^5, \\ t_1 = \varepsilon t = -\tau, \\ t_2 = t^{-1} = \tau - 1, \end{cases} \qquad \begin{cases} t_0^{-1} = -5\tau + 8, \\ t_1^{-1} = -\tau + 1, \\ t_2^{-1} = \tau, \end{cases}$$

and define

$$\begin{aligned} R_+ &= t_0 I + t_1 A_1 + t_2(J - I - A_1), \\ R_- &= t_0^{-1} I + t_1^{-1} A_1 + t_2^{-1}(J - I - A_1) = R_+^{-!} \end{aligned}$$

where A_1 denotes the adjacency matrix of the Higman-Sims graph. (Recall that t_0 is also denoted by a, and is the *modulus* of the model.)

PROPOSITION 7. *The Jaeger's model* $JM = (HS^0, R_+, R_-, \mathbb{C}, -10)$ *defined above is a spin model for oriented links. The notations being again as in the Theorem of* §2 *and the Proposition* 1, *one has*

$$\frac{-1}{10}(-\tau)^{-5\,\mathrm{Tait}(\vec{D})} Z_X^{JM} = F_{-1}(\vec{L})(-\tau^5, 1)$$

for any oriented link $\vec{L}$ *represented by a diagram* $\vec{D}$ *and the corresponding signed graph* X. (*Recall from* §2 *that* F_{-1} *is the Dubrovnik version of the Kauffman polynomial.*)

Proof. The steps are similar to those of the proof of Proposition 6.

To show that JM is a spin model for oriented links, one has first to check (20) to (24), which is straightforward; one has then to check (3) for all $\alpha, \beta, \gamma \in HS^0$, a priori 10^6 checks! We know again that (3) holds when $\alpha = \gamma$ or $\beta = \gamma$, because we know that (9) and (11) hold. If $\alpha = \beta$, equation (3) reads

$$\sum_{\xi \in HS^0} w_+(\alpha, \xi)^2 w_-(\gamma, \xi) = daw_-(\alpha, \gamma)^2$$

and it suffices to check this when $\alpha \neq \gamma$. Because of Claim (a) in 6.1, this reduces to two computations. The first one, for $A_1(\alpha, \gamma) = 1$, is

$$a^2 t_1^{-1} + t_1^2 a^{-1} + 21(t_1^2 t_2^{-1} + t_2^2 t_1^{-1}) + 56 t_2 \stackrel{?}{=} dat_1^{-2}$$

(the left-hand side has one term for $\xi = \alpha$, one for $\xi = \gamma$, and the others for the cases of multiplicities 21 and 56 in Table I) and the second one, for $A_2(\alpha, \gamma) = 1$, is

$$a^2 t_2^{-1} + t_2^2 a^{-1} + 6t_1 + 16(t_1^2 t_2^{-1} + t_2^2 t_1^{-1}) + 60 t_2 \stackrel{?}{=} dat_2^{-2}.$$

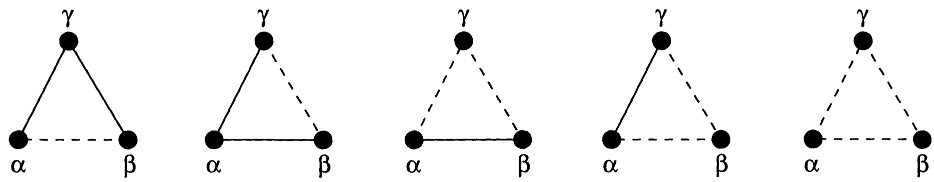


FIGURE 13

Both of these $\stackrel{?}{=}$ are identities easy to check when a, t_1, t_2 are replaced by the values of (25).

We may now assume that α, β, γ are all distincts: $970 \cdot 200$ cases left. Claims (a) and (b) in Subsection 6.1 show that these cases reduce to precisely 5 which are shown in Figure 13.

Reading Table II, we can write down equation (3) for (say) the first two cases of Figure 13 as

$$\begin{aligned} &t_0t_2t_1^{-1} + t_2t_0t_1^{-1} + t_1t_1t_0^{-1} + 20t_2t_2t_1^{-1} \\ &\qquad + 5t_1t_1t_2^{-1} + 16t_1t_2t_2^{-1} + 16t_2t_1t_2^{-1} + 40t_2t_2t_2^{-1} \\ &\stackrel{?}{=} -10t_2t_1^{-1}t_1^{-1} \end{aligned}$$

and

$$\begin{aligned} &t_0t_1t_1^{-1} + t_1t_0t_2^{-1} + t_1t_2t_0^{-1} + 20t_1t_2t_2^{-1} + \\ &5t_2t_1t_1^{-1} + 16t_2t_2t_1^{-1} + 16t_2t_1t_2^{-1} + 40t_2t_2t_2^{-1} \\ &\qquad \stackrel{?}{=} -10t_1t_2^{-1}t_1^{-1}. \end{aligned}$$

Both these $\stackrel{?}{=}$, as well as those corresponding to the three last cases of Figure 13, are again easy to check when t_0, t_1, t_2 are replaced by the values of (25).

The proof of the second claim is similar to that of Proposition 2. Consider four diagrams D_+, D_-, D_0, D_∞ as in §2. Around the distinguished crossing, the black and white colourings look as in one of the two situations represented in Figure 14 (next page).

The exchange property

$$Z_{D_+}^{JM} - Z_{D_-}^{JM} = Z_{D_0}^{JM} - Z_{D_\infty}^{JM}$$

holds because one has the identity

$$R_+ - R_- = dI - J.$$

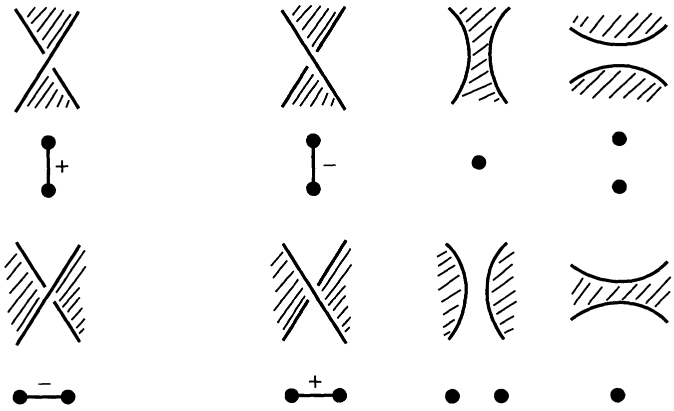


FIGURE 14

Indeed, the latter follows from the three equalities

$t_0 - t_0^{-1} = d - 1$ (matrix entries (α, β) such that $\alpha = \beta$),

$t_1 - t_1^{-1} = -1$ (matrix entries (α, β) such that $\{\alpha, \beta\} \in HS^1$),

$t_2 - t_2^{-1} = -1$ (matrix entries (α, β) such that $\{\alpha, \beta\} \in \overline{HS}^1$)

which are straightforward. The last claim of Proposition 7 follows now from the Theorem in §2. □

7. Looking for other models. It is tempting to see the pentagonal model of §5 and Jaeger's model as members of the same sequence. At the time of writing, it is an open problem to decide whether this sequence has any more terms. In the present section, we show what could be some of the properties of the corresponding graphs (if they exist).

Consider as in §5 a graph S with adjacency matrix A_1 and a spin model for oriented links

$$M = (S^0, R_+, R_-, \mathbb{C}, d)$$

such that the matrix

$$R_+ = t_0 I + t_1 A_1 + t_2 (J - I - A_1)$$

has exactly two distinct nondiagonal entries. We know from Proposition 3 that S is a strongly regular graph, say with parameters

(n, k, λ, μ). As the cases with $n \leq 4$ appear already in Proposition 5, we assume from now on that $n \geq 5$.

7.1. *Formal self-duality of* S. As $A_1 J = J A_1 = kJ$, the image of J is a one dimensional eigenspace of A_1 of eigenvalue k. By (18), the restriction to A_1 to $\operatorname{Ker}(J)$ has *two eigenvalues* denoted by r, s with multiplicities respectively denoted by m_1, $m_2 = n - m_1 - 1$. The numbers r, s are the two roots of the polynomial

$$x^2 + (\mu - \lambda)x + \mu - k$$

and the multiplicities can be computed from the relation $\operatorname{Trace}(A_1) = 0 = k + m_1 r + m_2 s$.

PROPOSITION 8. *With the notations above, one has*

$$n = (r-s)^2, \tag{26}$$

$$k \in \{r^2 + r - rs, s^2 + s - rs\}. \tag{27}$$

Proof. The eigenvalues and multiplicities of $R_+ = (t_0 - t_2)I + (t_1 - t_2)A_1 + t_2 J$ are

$$\begin{cases} t_0 + t_1 k + t_2(n-k-1), \\ \text{simple,} \end{cases} \quad \begin{cases} t_0 - t_2 + (t_1 - t_2)r, \\ \text{multiplicity } m_1, \end{cases}$$

$$\begin{cases} t_0 - t_2 + (t_1 - t_2)s, \\ \text{multiplicity } m_2, \end{cases}$$

and those of R_2 (as defined in §3) are

$$\begin{cases} dt_0^{-1}, \\ \text{multiplicity } n, \end{cases} \quad \begin{cases} dt_1^{-1}, \\ \text{multiplicity } nk, \end{cases} \quad \begin{cases} dt_2^{-1}, \\ \text{multiplicity } n(n-k-1). \end{cases}$$

As $R_1 = R_+ \otimes \mathrm{id}$ is conjugate to R_2 by (12), one has either $m_1 = n - k - 1$, and then $t_0 - t_2 + (t_1 - t_2)r = dt_2^{-1}$, or $m_1 = k$, and then $t_0 - t_2 + (t_1 - t_2)r = dt_1^{-1}$. We are going to discuss these cases one after the other; moreover we deal first with the generic situation $\mu \neq 0$, and second with the situation $\mu = 0$.

In this proof, we choose notations such that $r \geq 0$ and $s \leq -1$ (but we'll agree for another choice later! See (29) and (30)).

In the "generic" situation for which $\mu \neq 0$, one has $s + 1 \neq 0$, $\mu = k + rs$ and

$$n = \frac{(k-r)(k-s)}{k+rs}, \quad m_1 = \frac{k(k-s)(s+1)}{(k+rs)(s-r)} \tag{28}$$

(see e.g. Theorem 1.3.1 in **[BCN]**).

If $m_1 = n-k-1$, the values in (28) give a formula which simplifies to $k = r^2 + r - rs$. Then one has also the eigenvalue relations for $R_1 \sim R_2$ (where $\sim$ means "conjugate")

$$\left.\begin{array}{l} t_0 - t_2 + (t_1 - t_2)r = dt_2^{-1}, \\ t_0 - t_2 + (t_1 - t_2)s = dt_1^{-1} \end{array}\right\} \Rightarrow (t_1 - t_2)(r-s) = d(t_2^{-1} - t_1^{-1})$$

and similarly for $R_1^{-1} \sim R_2^{-1}$

$$\left.\begin{array}{l} t_0^{-1} - t_2^{-1} + (t_1^{-1} - t_2^{-1})r = dt_2, \\ t_0^{-1} - t_2^{-1} + (t_1^{-1} - t_2^{-1})s = dt_1 \end{array}\right\} \Rightarrow (t_1^{-1} - t_2^{-1})(r-s) = d(t_2 - t_1).$$

It follows that $(r-s)^2 = d^2$.

If $m_1 = k$, one obtains similarly $k = s^2 + s - rs$ and $(r-s)^2 = d^2$.

In the situation $\mu = 0$, one has moreover $r = k$ and $s = -1$; the graph S is a union of b cliques, and $b = \frac{n}{k+1}$ (see again Theorem 1.3.1 in **[BCN]**). One has $m_1 = b - 1$ and $m_2 = kb$. The eigenvalues of R_+ may be written as

$$\left\{\begin{array}{l} t_0 - t_2 + (t_1 - t_2)k + t_2 n, \\ \text{simple}, \end{array}\right. \qquad \left\{\begin{array}{l} t_0 - t_2 + (t_1 - t_2)k, \\ \text{multiplicity } m_1, \end{array}\right.$$

$$\left\{\begin{array}{l} t_0 - t_1, \\ \text{multiplicity } m_2. \end{array}\right.$$

If one had $m_1 = \frac{n}{k+1} - 1 = n - k - 1$, one would have $n = k + 1$ and S would just be a clique, which is ruled out $(t_1 \neq t_2)$. Thus $m_1 = \frac{n}{k+1} - 1 = k$, namely $n = (k+1)^2 = (r-s)^2$, and $s^2 + s - rs = 1 - 1 + r = k$, so that the proof is complete. □

A strongly regular graph S with parameters (n, k, λ, μ) and eigenvalues k, r, s is said to be *formally self-dual* if it fulfills the conditions of Proposition 8. The parameters of such a graph satisfy also the relations

$$\mu = r^2 + r, \qquad \lambda = r^2 + 2r + s$$

(because $\mu = k + rs$ and $\lambda = k + rs + r + s$ in any strongly regular graph).

The words "formally self-dual" come from a duality property of the Bose-Mesner algebra defined by such a graph. For the background behind this definition, see e.g. **[Neu]**, in particular Corollary 2 of Theorem 1. Let us only indicate here the following: a strongly regular graph which is formally self-dual has in particular its eigenvalues r, s with multiplicities m_1, m_2 satisfying

$$\{m_1, m_2\} = \{k, n-k-1\}.$$

For simplicity, we assume from now on that the graph S has parameter $\mu \neq 0$. As observed in the proof of Proposition 8, Equation (12) implies $t_0 + t_1 k + t_2(n-k-1) = dt_0^{-1}$ and

$$\{t_0 - t_2 + (t_1 - t_2)r,\ t_0 - t_2 + (t_1 - t_2)s\} = \{dt_1^{-1},\ dt_2^{-1}\}.$$

We choose to denote by r the eigenvalue of S such that $t_0 - t_2 + (t_1 - t_2)r = dt_2^{-1}$. This may imply $r \leq -1$ and $s \geq 0$ (unlike **[BCN]**). But this does imply

$$k = r^2 + r - rs \tag{29}$$

and

$$\begin{cases} t_0^\nu + t_1^\nu k + t_2^\nu(n-k-1) = dt_0^{-\nu} \\ t_0^\nu - t_2^\nu + (t_1^\nu - t_2^\nu)r = dt_2^{-\nu} \end{cases} \tag{30}$$

for $\nu \in \{-1, 1\}$. (Compare (29) with (27), and observe that the first equation in (30) just repeats (23) and (24).)

Observe the following. If the multiplicities m_1, m_2, of r, s are distinct, namely if S is not a so-called *conference graph*, then our choice of notations is simply defined as follows: r is of multiplicity $n-k-1$ and s of multiplicity k. If $m_1 = m_2 = n-k-1 = k$, I don't know a simple description of the appropriate choice, but this case hardly happens at all (Proposition 9.ii below).

7.2. *On the weights* t_0, t_1, t_2. Consider again a model M and the corresponding strongly regular graph S, satisfying the hypothesis above ($n \geq 5$, $\mu \neq 0$). From $d^2 = n$ (see (7)) and from $n = (r-s)^2$ (see Proposition 8), we know that there exists a sign ε such that

$$d = \varepsilon(r - s), \quad \varepsilon \in \{1, -1\}. \tag{31}$$

Our conventions on r and the proof of Proposition 8 show that $t_1 - t_2 = \varepsilon(t_2^{-1} - t_1^{-1})$. As $t_1 \neq t_2$, this implies

$$t_2 = \varepsilon t_1^{-1} \quad \text{namely} \quad \begin{cases} t_1 = \varepsilon t, \\ t_2 = t^{-1} \end{cases}$$

for some $t \in \mathbb{C}^*$.

Writing a for t_0, we have from (30)

$$\begin{cases} a + \varepsilon t k + t^{-1}(n-k-1) = da^{-1}, \\ a^{-1} + \varepsilon t^{-1} k + t(n-k-1) = da \end{cases}$$

as well as

$$\begin{cases} a - t^{-1} + (\varepsilon t - t^{-1})r = dt, \\ a^{-1} - t + (\varepsilon t^{-1} - t)r = dt^{-1}. \end{cases}$$

The first pair of equations implies

$$\varepsilon a - a^{-1} + (t - \varepsilon t^{-1})(2k - n - 1) = \varepsilon d(a^{-1} - \varepsilon a).$$

As

$$\begin{aligned}(1 + \varepsilon d)(1 + r + s) = (1 + r - s)(1 + r + s) &= 1 + 2r + r^2 - s^2 \\ &= 2k - n + 1\end{aligned}$$

by (29), (7) and (31), this simplifies to

$$\varepsilon a - a^{-1} + (t - \varepsilon t^{-1})(1 + r + s) = 0$$

(observe that $1 + \varepsilon d \neq 0$, because $d^2 = n \neq 1$). The second pair of equations implies

$$\varepsilon a + a^{-1} - \varepsilon t^{-1} - t = \varepsilon d(t + \varepsilon t^{-1})$$

or

$$\varepsilon a + a^{-1} - (t + \varepsilon t^{-1})(1 + r - s) = 0. \tag{32}$$

Solving for a and a^{-1} one obtains

$$\begin{cases} a = t^{-1}(1 + r) - \varepsilon t s, \\ a^{-1} = t(1 + r) - \varepsilon t^{-1} s. \end{cases}$$

The obvious compatibility condition $aa^{-1} = 1$ implies

$$1 = (1 + r)^2 + s^2 - \varepsilon(1 + r)s(t^2 + t^{-2}).$$

Given r and s, this equation is of degree 4 in t. (Indeed, $s \neq 0$, otherwise the equation above implies $(r + 1)^2 = 1$, hence $r = -2$, hence $n = (r - s)^2 = 4$, and we have assumed that $n \geq 5$; similarly $r \neq -1$, otherwise $s = 1$, and again $n = 4$.)

We have shown (i) of the following. For (ii), see the proof of Proposition 7 in **[Jae]**.

PROPOSITION 9. *Assume that there exists a spin model $M = (S, R_+, R_-, \mathbb{C}, d)$ for oriented links with associated strongly regular graph S such that the parameters (n, k, λ, μ) of S satisfy*

$$n \geq 5, \quad \mu \neq 0.$$

(i) *Suppose that $n \neq 2k + 1$. Let r [respectively s] denote the eigenvalue of S of multiplicity $n - k - 1$ [resp. k], and let ε be the sign such that $d = \varepsilon(r - s)$. Then the weights t_0, t_1, t_2 of the matrix $R_+ = aI + \varepsilon t A_1 + t_2 A_2$ satisfy the following equations*

$$s^2 + (r + 1)^2 - \varepsilon s(r + 1)(t^2 + t^{-2}) = 1, \tag{33}$$

$$a = (r+1)t^{-1} - \varepsilon ts, \tag{34}$$

$$t_0 = a, \quad t_1 = \varepsilon t, \quad t_2 = t^{-1}. \tag{35}$$

(ii) *Suppose that* $n = 2k+1$, *namely that* S *is a so-called conference graph. Then* S *is either a pentagon as in Proposition* 5, *or the lattice graph* $L_2(3)$ *with* 9 *vertices which is a Cartesian product of two triangles* (*see* §5.3).

In **[Jae]**, there are necessary *and sufficient* conditions on a strongly regular graphs S for the existence of a model M involving S. These conditions are the following:

(i) S is formally self-dual,
(ii) the subconstituents of S are strongly regular,
(iii) both S and its complement are connected (recall that $|S^0| \geq 5$).

(By definition, to each vertex $\alpha \in S^0$ correspond two subconstituents: the subgraph of S induced by the neighbours of α and the subgraph of S induced by the vertices $\beta \in S^0$ at distance 2 from α.) If S fulfills these conditions, the three equations of Proposition 9 are necessary and sufficient conditions on the weights a, εt and t_2 which enter the model.

Of course, a graph may satisfy (i) to (iii) above without giving a really new model: this is for example the case of the so-called lattice graphs $L_2(q)$, also called Hamming graphs of diameter 2 and denoted by $H(2, q)$ in **[BCN]**: the corresponding models are just squares (in the sense of the products of §5.3) of Potts' models. A graph S may also lead to a "degenerate model" (see below).

PROPOSITION 10. *Let the notations be as in Propositions* 8 *and* 9, *so that in particular*

$$R_+ = aI + \varepsilon t A_1 + t^{-1} A_2, \qquad R_- = a^{-1} I + \varepsilon t^{-1} A_1 + t A_2.$$

Set

$$z = t + \varepsilon t^{-1}.$$

Then

$$R_+ + \varepsilon R_- = (a + \varepsilon a^{-1}) I + (\varepsilon t + t^{-1})(A_1 + A_2) = z(dI + \varepsilon J).$$

Suppose moreover that $z \neq 0$. *The notations being also as in the Theorem of* §2 *and in Proposition* 1, *one has*

$$\frac{1}{d} a^{-\operatorname{Tait}(\vec{D})} Z_X^M = F_\varepsilon(\vec{L})(a, z)$$

for any link $\vec{L}$ represented by a diagram $\vec{D}$ and the corresponding signed graph X. In particular, the model M gives an evaluation of the Kauffman polynomial F_ε.

Proof. As

$$a+\varepsilon a^{-1}-\varepsilon t-t^{-1}=\varepsilon(t+\varepsilon t^{-1})(r-s)=zd$$

by (31) and (32), one has the formula for $R_+ + \varepsilon R_-$. The proof of the last statement follows, as that of the exchange property in the proof of Proposition 2. □

A degenerate example. Let M be a model as above, and assume now that the underlying graph has eigenvalues r, s such that $s(r+1)\neq 0$ and $r+s+1\in\{1,-1\}$. Then $s^2+(r+1)^2+2s(r+1)=1$ and comparison with (33) shows that $t^2=-\varepsilon$. This implies

$$z=t+\varepsilon t^{-1}=0.$$

By (34) one has $a=(r+s+1)t^{-1}$, and then $-\varepsilon a-a^{-1}=0$ because $t^2=-\varepsilon$.

Assume for simplicity that $\varepsilon=-1$, so that $R_+=R_-$ by Proposition 10, and $a=a^{-1}$. Consider an oriented link $\vec{L}$ with $c(L)$ components represented by a diagram $\vec{D}$ and a signed graph X; consider also a trivial link $\vec{L}_0$ with $c(L)$ components, represented by a diagram $\vec{D}_0$ made up of $c(L)$ disjoint circles and by the edgeless graph X_0 having $c(L)$ vertices. As $R_+=R_-$ one has

$$\frac{1}{d}a^{-\operatorname{Tait}(\vec{D})}Z_X^M=\frac{1}{d}a^{-\operatorname{Tait}(\vec{D}_0)}Z_{X_0}^M=\frac{1}{d^{c(L)+1}}n^{c(L)}=d^{c(L)-1}.$$

In particular $\frac{1}{d}a^{-\operatorname{Tait}(\vec{D})}Z_X^M=1$ whenever $\vec{L}$ is a knot, and the model is of little use for links. However such models may be of interest to graph theorists, and we describe now briefly an example.

Let C be the complement of the *Clebsch graph*: its vertices are subsets of $\{1,2,3,4,5\}$ of even cardinality, and two such are adjacent in C if their symmetric difference has cardinality 4. Standard computations show that C is strongly regular with parameters $(n,k,\lambda,\mu)=(16,5,0,2)$. Its eigenvalues are 5, 1, -3, respectively with multiplicities 1, 10, 5. Its constituents are on one hand graphs with 5 vertices and no edge, on the other hand Petersen graphs. Thus C satisfies conditions (i) to (iii) stated after Proposition 9. In our notations, r is of multiplicity $n-k-1=10$, so that $r=1$, $s=-3$ and $d=4\varepsilon$.

One has $t^2 = -\varepsilon$ by (33) and $a = -t^{-1}$ by (34). There are two possible models with $\varepsilon = 1$, $d = 4$ and weights

$$(a, t_1, t_2) = (\pm i, \pm i, \mp i)$$

and two other models with $\varepsilon = -1$, $d = -4$ and weights

$$(a, t_1, t_2) = (\pm 1, \pm 1, \mp 1).$$

In all cases one has $z = 0 = -\varepsilon a - a^{-1}$.

F. Jaeger has found other similar examples of models with underlying graphs having eigenvalues r, s such that $s(r+1) \neq 0$ and $r+s+1 \in \{1, -1\}$.

The reader should carefully distinguish the values of the Clebsch model described here from the following limit case of the Kauffman polynomial. For an oriented link $\vec{L}$, the values $F_{-1}(\vec{L})(a, a-a^{-1})$ are well understood [LiM]. In particular $F_{-1}(\vec{L})(a, a-a^{-1}) = 1$ for all $a \in \mathbb{C}^*$ such that $a \neq \pm 1$ in case $\vec{L}$ is an actual knot, and

$$\lim_{a \to 1} F_{-1}(\vec{L})(a, a-a^{-1}) = 2^{c(L)-1}$$

in all cases. This limit is clearly *not* the value $(-4)^{c(L)-1}$ given by the Clebsch model (see above the end of §2).

Variations. A model M with underlying graph S as above has various companion models. We use below the same notations as in Propositions 9 and 10.

One may describe a first variation of M in terms of the complement $\overline{S}$ of S. If S has parameters (n, k, λ, μ) and eigenvalues k, r, s, then $\overline{S}$ has parameters

$$(n, n-k-1, n-2k+\mu-2, n-2k+\lambda)$$

and eigenvalues $n-k-1$, $-s-1$, $-r-1$. This variation has the same parameters d, ε, a, z as M, but

$$t \text{ is replaced by } \varepsilon t^{-1}.$$

One may also keep S and change the sign and the weights according to

$$\varepsilon,\ a,\ t,\ z \Rightarrow -\varepsilon,\ -i\varepsilon a,\ i\varepsilon t,\ i\varepsilon z.$$

This is compatible with the relations

$$\Lambda_{+1}(D)(a, z) = \Lambda_{-1}(D)(-ia, iz),$$
$$F_{+1}(\vec{L})(a, z) = (-1)^{c(\vec{L})-1} F_{-1}(\vec{L})(-ia, iz)$$

of the theorem in §2. In §6, we have chosen the variant with $\varepsilon = -1$ to have $a = -\tau^5$, $t = \tau$ and $z = t + \varepsilon t^{-1}$ real. The same choice $\varepsilon = -1$ implies that a and z are imaginary in our pentagonal model.

Final questions. Let us finally review our favourite examples.

The *Potts' models* of Proposition 2 provide an infinite number of evaluations of the Kauffman polynomial $F_{+1}(\vec{L})(a, -t - t^{-1})$ on the curve of equation

$$a = t^3.$$

The *square models* of Proposition 5 provide evaluations of $F_\varepsilon(\vec{L})(a, -\varepsilon t - t^{-1})$ at all points of the curves

$$a = \varepsilon t^{-1}$$

(for $\varepsilon = 1$ and $\varepsilon = -1$).

The *pentagonal model* M_5 of Proposition 6, of which the underlying graph *is* a conference graph with eigenvalues $r = \tau - 1$ and $s = -\tau$ (both of multiplicity 2), provides the evaluation of $F_{-1}(\vec{L})(a, t - t^{-1})$ for

$$a = -i, \quad t = i \exp\left(\frac{2i\pi}{5}\right) \Rightarrow a = -t^5.$$

The *Clebsch model* discussed after Proposition 10, for which the parameters a, t satisfy

$$a = -t^{-1} = \varepsilon t^5.$$

The *Jaeger model* JM of §6 provides the evaluation of

$$F_{-1}(\vec{L})(a, t - t^{-1}) \quad \text{for } a = -\tau^5, t = \tau \Rightarrow a = -t^5.$$

One may thus make more precise the question asked in the beginning of §7:

Do there exist other models as above which provide evaluations of the Kauffman polynomial $F_\varepsilon(\vec{L})(a, -\varepsilon t - t^{-1})$ at other points of the curves

$$a = \varepsilon t^5?$$

Here is one more question in purely graph theoretical terms. Consider the class $\mathcal{S}$ of strongly regular graphs with the following properties:

(a) they are not lattice graphs (see the end of §7.1),

(b) they satisfy conditions (i) to (iii) stated after Proposition 9,

(c) they are "nondegenerate" in the sense that their eigenvalues r, s are such that $r + s + 1 \notin \{1, -1\}$;

does $\mathcal{S}$ contain any graph with $n > 100$ vertices?

References

[Ale] J. W. Alexander, *Topological invariants of knots and links*, Trans. Amer. Math. Soc., **30** (1928), 275–306.

[Ati] M. Atiyah, *The Geometry and Physics of Knots*, Cambridge Univ. Press, 1990.

[Bax] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, London, 1982.

[BiW] N. L. Biggs and A. T. White, *Permutation Groups and Combinatorial Structures*, London Math. Soc. Lecture Notes 33, Cambridge Univ. Press, 1979.

[Bir] G. D. Birkhoff, *A determinant formula for coloring a map*, Ann. Math., **14** (1912), 42–46.

[BCN] A. E. Brouwer, A. M. Cohen and A. Neumaier, *Distance Regular Graphs*, Springer, Berlin, 1989.

[Bru] S. G. Brush, *History of the Lenz-Ising model*, Rev. Mod. Physics, **39** (1967), 883–893.

[BuZ] G. B. Burde and H. Ziechang, *Knots*, de Gruyter, 1985.

[CGS] P. J. Cameron, J. M. Goethals and J. J. Seidel, *Strongly regular graphs having strongly regular subconstituants*, J. Algebra, **55** (1978), 257–280.

[Con] J. H. Conway, *An enumeration of knots and links*, in *Computational problems in abstract algebra*, (J. Leech, ed.), Pergamon Press (1969), 329–358.

[CoS] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, Springer, 1988.

[Edg] W. L. Edge, *Some implications of the geometry of the* 21*-point plane*, Math. Zeit., **87** (1965), 348–362.

[F+] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett and A. Ocneanu, *A new polynomial invariant of knots and links*, Bull. Amer. Math. Soc., **12** (1985), 239–246.

[GoS] J. M. Goethals and J. J. Seidel, *Strongly regular graphs derived from combinatorial designs*, Canad. J. Math., **22** (1970), 597–614.

[GoJ] D. M. Goldschmidt and V. F. R. Jones, *Metaplectic link invariants*, Geometriae Ded., **31** (1989), 165–191.

[HaJ] P. de la Harpe and F. Jaeger, *Chromatic invariants for finite graphs: Theme and polynomial variations*, preprint, Genève and Grenoble (1992).

[HJ1] P. de la Harpe and V. F. R. Jones, *Paires de sous-algèbres semi-simples et graphes fortement réguliers*, C. R. Acad. Sci. Paris, Série I, **311** (1990), 147–150.

[HJ2] ——, *Graph invariants related to statistical mechanical models: Examples and problems*, J. Combin. Theory B, **57** (1993), 207–227.

[HKW] P. de la Harpe, M. Kervaire and C. Weber, *On the Jones polynomial*, Enseign. Math., **32** (1986), 271–335.

[HiS] D. Higman and C. Sims, *A simple group of order* $44 \cdot 352 \cdot 000$, Math. Z., **105** (1968), 110–113.

[Jae] F. Jaeger, *Strongly regular graphs and spin models for the Kauffman polynomial*, Geometriae Ded., **44** (1992), 23–52.

[Jo1] V. F. R. Jones, *A polynomial invariant for knots via von Neumann algebras*, Bull. Amer. Math. Soc., **12** (1985), 103–111.

[Jo2] ——, *Hecke algebra representations of Braid groups and link polynomials*, Ann. of Math., **126** (1987), 335–388.

[Jo3] ——, *On knot invariants related to some statistical mechanical models*, Pacific J. Math., **137** (1989), 311–334.

[Jo4] ——, *On a certain value of the Kauffman polynomial*, Comm. Math. Phys., **125** (1989), 459–467.

[Jo5] ——, *Subfactors and Knots* (CBMS 80), Amer. Math. Soc., 1981.

[Ka1] L. Kauffman, *State models and the Jones polynomial*, Topology, **26** (1987), 395–407.

[Ka2] ——, *An invariant of regular isotopy*, Trans. Amer. Math. Soc., **318** (1990), 417–471.

[Ka3] ——, *New invariants in the theory of knots*, Amer. Math. Monthly, **95** (1988), 195–242.

[Kup] G. Kuperberg, *Involutory Hopf algebras and 3-manifold invariants*, Internat. J. Math., **2** (1991), 41–66.

[Li1] W. B. R. Lickorish, *Polynomials for links*, Bull. London Math. Soc., **20** (1988), 558–588.

[Li2] ——, *Some link-polynomial relations*, Math. Proc. Camb. Phil. Soc., **105** (1989), 103–107.

[LiM] W. B. R. Lickorish and K. C. Millett, *An Evaluation of the F-Polynomial of a Link*, Springer Lecture Notes in Math., **1350** (1988), 104–108.

[Mur] K. Murasugi, *The Jones polynomial and classical conjectures in knot theory*, Topology, **26** (1987), 187–194 (see also *Jones polynomials and classical conjectures in knot theory* II, Math. Proc. Camb. Phil. Soc., **102** (1987), 317–318).

[Neu] A. Neumaier, *Duality in coherent configurations*, Combinatorica, **9** (1989), 59–67.

[Rei] K. Reidemeister, *Elementare Begründung der Knotentheorie*, Abh. Math. Sem. Univ. Hamburg, **5** (1926), 24–32.

[ReT] N. Y. Reshetikin and V. Turaev, *Invariants for 3-manifolds via link polynomials and quantum groups*, Invent. Math., **103** (1991), 547–597.

[Rha] G. de Rham, *Lectures on Introduction to Algebraic Topology*, Tata Inst. Fund. Res., Bombay, 1969.

[Rol] D. Rolfsen, *Knots and Links*, Publish or Perish, 1976.

[Sei] H. Seifert, *Über das Geschlecht von Knoten*, Math. Ann., **110** (1935), 571–592.

[Thi] M. B. Thistlethwaite, *A spanning tree expansion of the Jones polynomial*, Topology, **26** (1987), 297–309.

[Tu1] V. G. Turaev, *A simple proof of Murasugi and Kauffman theorem on alternating links*, Enseign. Math., **33** (1987), 203–225.

[Tu2] ——, *State Sum Models in Low-Dimensional Topology*, I. C. M. Kyoto 1990 I (Springer 1991), 689–698.

[Wen] H. Wenzl, *Quantum groups and subfactors of type B, C and D*, Comm. Math. Phys., **133** (1990), 383–432.

[Whi] H. Whitney, *A logical expansion in mathematics*, Bull. Amer. Math. Soc., **38** (1932), 572–579.

Received February 15, 1992 and in revised form May 15, 1992.

SECTION DE MATHÉMATIQUES
UNIVERSITY OF GENEVA
C. P. 240, CH-1211 GENEVA 24
SWITZERLAND
E-mail address: Laharpe@cgeuge11.bitnet