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WHEN L^1 OF A VECTOR MEASURE IS AN AL-SPACE

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We consider the space of real functions which are integrable with respect to a countably additive vector measure with values in a Banach space. In a previous paper we showed that this space can be any order continuous Banach lattice with weak order unit. We study a priori conditions on the vector measure in order to guarantee that the resulting L^1 is order isomorphic to an AL-space. We prove that for separable measures with no atoms there exists a c_0-valued measure that generates the same space of integrable functions.

Introduction. Given a vector measure ν we consider the space of classes of real functions which are integrable with respect to ν in the sense of Lewis **[L-1]**, denoted by $L^1(\nu)$. In **[C**, Theorem 8] we showed that every order continuous Banach lattice with weak unit can be obtained as $L^1(\nu)$ for a suitable vector measure ν. In particular we have Hilbert spaces as L^1 of a vector measure. A natural question arises. Under what conditions on the vector measure, or on the Banach space in which the measure takes its values, is the resulting L^1 of the vector measure order isomorphic to an AL-space? Recall that a Banach lattice is an *AL-space* when the norm is additive for disjoint vectors. An *order isomorphism* is a linear isomorphism that preserves the lattice operations. So the question can be restated in the following way. When can $L^1(\nu)$ be equivalently renormed so that endowed with the new norm and the same order is a Banach lattice where the norm is additive for disjoint functions?

In §1 we fix notation and basic definitions.

In §2 we show the special role that the space $L^1(|\nu|)$ plays in the problem we are studying, where $|\nu|$ is the variation of ν. It is shown in Proposition 2 that $L^1(\nu)$ is an $\mathscr{L}_1$-space, in the sense of Lindenstrauss and Pelczynski **[L-P]**, if and only if it is order isomorphic to $L^1(|\nu|)$. From this it follows that bounded variation of the measure is a necessary condition. The conditions cannot be placed on the Banach space in which the measure takes its values. This is shown in Example 1, that also shows that neither bounded variation nor domination of the variation by the semivariation are sufficient conditions.

In §3 we study measures with values in $C(K)$ spaces. In Theorem 1 we show that if ν is a Banach space valued measure that is separable and has no atoms then there exists a c_0-valued measure μ such that $L^1(\nu)$ and $L^1(\mu)$ are order isomorphic and isometric. We also show that this is not the case in general for purely atomic measures. Theorem 2 gives a characterization of the purely atomic measures with values in $C(K)$ that generate an AL-space.

In §4 we study the dual of $L^1(\nu)$ in $L_\infty(|\nu|)$ via the Gelfand transform. This allows us to characterize the measures for which $L^1(\nu)$ is given by a finite number of spaces $L^1(|x^*\nu|)$ where x^* is in X^*, X being the Banach space in which ν takes its values (Theorem 3). We study the characterization of weak convergence in $L^1(\nu)$ by weak convergence of the integrals over arbitrary sets and show that when this does not happen $L^1(\nu)$ contains a complemented copy of l^1 (Theorem 4).

In §5 we exhibit examples of measures that separate the properties studied in the paper.

1. Let (Ω, Σ) be a measurable space, X a Banach space with unit ball B_X and dual space X^*, and $\nu: \Sigma \to X$ a countably additive vector measure. The semivariation of ν is given by $\|\nu\|(A) = \sup\{|x^*\nu|(A): x^* \in B_{X^*}\}$, where $|x^*\nu|$ is the variation of the scalar measure $x^*\nu$. A Rybakov control measure for ν is a measure $\lambda = |x_0^*\nu|$ such that $\lambda(A) = 0$ if and only if $\|\nu\|(A) = 0$ (see **[D-U**, Theorem IX.2.2**]**). Thus the concepts of $\|\nu\|$-almost everywhere and λ-almost everywhere are equivalent.

Following D. R. Lewis **[L-1]** we will say that a measurable function $f: \Omega \to \mathbb{R}$ is *integrable* with respect to ν if

(1) f is $x^*\nu$ integrable for every $x^* \in X^*$ (*scalarly integrable*), and

(2) for each $A \in \Sigma$ there exists an element of X denoted by $\int_A f\, d\nu$, such that

$$x^* \int_A f\, d\nu = \int_A f\, dx^*\nu \quad \text{for every } x^* \in X^*.$$

Identifying two functions if the set where they differ has null semivariation we obtain a linear space of classes of functions which when endowed with the norm

$$\|f\|_\nu = \sup\left\{\int_\Omega |f|\, d|x^*\nu|: x^* \in B_{X^*}\right\}$$

becomes a Banach space. We will denote it by $L^1(\nu)$. It is a Banach lattice for the natural λ-almost everywhere order. Simple functions are dense in $L^1(\nu)$ and the formal identity is a continuous inclusion of the space of λ-essentially bounded functions into $L^1(\nu)$. An equivalent norm for $L^1(\nu)$ is given by $\|f\| = \sup\{\|\int_A f\,d\nu\|_X : A \in \Sigma\}$.

Let λ be a Rybakov control measure for ν. Then $L^1(\nu)$ is an order continuous Banach lattice with weak order unit over $(\Omega, \Sigma, \lambda)$ (see **[C**, Theorem 1]). Thus it can be regarded as a lattice ideal in $L^1(\lambda)$ and $L^1(\nu)^*$ can be identified with the set of functions g in $L^1(\lambda)$ such that fg is λ integrable for all f in $L^1(\nu)$. The action of such a g over $L^1(\nu)$ is given by $f \in L^1(\nu) \mapsto \langle g, f\rangle = \int fg\,d\lambda$ (see **[L-T**, vol. II, Theorem 1.b.14]). $L^1(\nu)^*$ is a lattice ideal in $L^1(\lambda)$, that is, if g is in $L^1(\nu)^*$, h is in $L^1(\lambda)$ and $|h| \le |g|$ holds but for a set of λ-measure zero, then h is in $L^1(\nu)^*$.

For general facts on vector measures we refer the reader to **[D-U]**. For Banach lattices see **[A-B]**, **[MN]** and **[S]**.

2. It is a general fact that if $|\nu|$ is the variation of ν then the formal identity is a continuous inclusion of the space $L^1(|\nu|)$ into $L^1(\nu)$ and $\|f\|_\nu \le \|f\|_1 = \int |f|\,d|\nu|$ **[L-2**, Theorem 4.1]. However it is important to notice that $|\nu|$ is not involved in the construction of the space $L^1(\nu)$. Thus, $L^1(|\nu|)$ is not relevant for the general theory of the space $L^1(\nu)$. Consider for example the following measures, defined on the Lebesgue measurable sets of $[0, 1]$, $A \in \mathscr{M} \mapsto \nu(A) = \chi_A \in L^p[0, 1]$ for $1 < p < +\infty$. Then $L^1(\nu)$ is order isomorphic and isometric to $L^p[0, 1]$, but $L^1(|\nu|) = \{0\}$ as $|\nu|$ is infinite in every nonnull set.

The following proposition characterizes the elements of $L^1(\nu)$ that belong to $L^1(|\nu|)$.

PROPOSITION 1 (**[L-2**, Theorem 4.2]). *Let f be in $L^1(\nu)$ and consider the measure ν_f with density f with respect to ν. Then f is in $L^1(|\nu|)$ if and only if the measure ν_f has bounded variation, and in this case*

$$|\nu_f|(A) = \int_A |f|\,d|\nu| \quad \text{for all } A \in \Sigma.$$

The role of the space $L^1(|\nu|)$ in the problem we are investigating is more important than at first it seems. The following proposition illustrates this point. For the theory of $\mathscr{L}_p$-spaces we refer the reader to **[L-P]**.

PROPOSITION 2. *Let $\nu\colon \Sigma \to X$ be a vector measure. The following conditions are equivalent, and imply that ν has bounded variation:*

(a) $L^1(\nu)$ *is an* $\mathscr{L}_1$*-space.*

(b) $L^1(\nu)$ *is order isomorphic to an AL-space.*

(c) *The natural inclusion is an order isomorphism from* $L^1(|\nu|)$ *into* $L^1(\nu)$.

Proof. As $L^1(\nu)$ is an order continuous Banach lattice the results of Abramovich and Wojtaszczyk on the uniqueness of order **[A-W]** show that (a) is equivalent to (b). Thus we just have to prove that (b) implies (c). In view of Proposition 1 we have to prove that for every $f \in L^1(\nu)$ the measure with density f with respect to ν has bounded variation. We can assume that f is positive. Let T be an order isomorphism from $L^1(\nu)$ into an AL-space. Consider a finite measurable partition (A_n). Then:

$$\begin{aligned}\sum\left\|\int_{A_n} f\,d\nu\right\| &\le \sum \|f\cdot\chi_{A_n}\|_\nu \le \|T^{-1}\|\cdot\sum\|T(f\cdot\chi_{A_n})\| \\ &= \|T^{-1}\|\cdot\left\|\sum T(f\cdot\chi_{A_n})\right\| \\ &= \|T^{-1}\|\cdot\|Tf\| \le \|T^{-1}\|\cdot\|T\|\cdot\|f\|_\nu\end{aligned}$$

due to the fact that the $T(f\cdot\chi_{A_n})$ are positive and disjoint functions in an AL-space. Taking the supremum over all partitions we get the desired result. To see that the variation is bounded just consider $f = \chi_\Omega$. □

In view of the previous proposition we will assume from now on that the measure ν has bounded variation.

There is another condition equivalent to Proposition 2 that should be mentioned. The *integration map* $\nu\colon L^1(\nu) \to X$ is defined by $\nu(f) = \int f\,d\nu$. The condition is that the set $\nu^*(B_{X^*})$ is order bounded in $L^1(\nu)^*$. This implies, in particular, that if the measure takes its values in an AL-space an equivalent condition is that the integration map is *regular*, i.e. the difference of two positive maps (see [S, IV.1.2, IV.1.5]). This occurs, for example, when the AL-space valued measure is positive, or when it has a Hahn decomposition.

We seek conditions in order that $L^1(\nu)$ is order isomorphic to an AL-space. Our first example shows that these conditions cannot be placed on the Banach space X in which the measure takes its values. It also shows that neither the condition that ν has bounded variation nor the stronger condition:

(A) there exists $C > 0$ such that $|\nu|(A) \le C \cdot \|\nu\|(A)$ for every $A \in \Sigma$,

although both necessary by Proposition 2, are sufficient.

EXAMPLE 1. Let X be an infinite dimensional Banach space. Let x_n be a sequence in X such that the series $\sum x_n$ converges unconditionally but not absolutely. Let $\mathscr{P}(\mathbb{N})$ be the σ-algebra of all subsets of the natural numbers. Consider the measure $\nu: \mathscr{P}(\mathbb{N}) \to X$ given by

$$\nu(A) = \sum_{n \in A} \frac{x_n}{\|x_n\| \cdot 2^n} \quad \text{for } A \subset \mathbb{N}.$$

It is well defined and countably additive. Moreover it has bounded variation as the variation of the measure is given by $|\nu|(A) = \sum_{n \in A} 1/2^n$ for $A \subset \mathbb{N}$. A sequence $b = (b_n)$ belongs to $L^1(\nu)$ if the series $\sum (b_n / \|x_n\| \cdot 2^n) x_n$ converges unconditionally in X, and belongs to $L^1(|\nu|)$ if the same series converges absolutely, that is $\sum |b_n|/2^n$ is finite. Thus $b = (\|x_n\| \cdot 2^n)$ gives an element in $L^1(\nu)$ but not in $L^1(|\nu|)$, so from Proposition 2 we have that $L^1(\nu)$ is not an AL-space. Let $A \subset \mathbb{N}$, set $n_0 = \min\{n : n \in A\}$; then:

$$|\nu|(A) = \sum_{n \in A} \frac{1}{2^n} \le \sum_{n \ge n_0} \frac{1}{2^{n_0}} = \frac{2}{2^{n_0}}.$$

On the other hand: $\|\nu\|(A) \ge \sup\{\|\nu(B)\| : B \subset A\} \ge \|\nu(n_0)\| = 1/2^{n_0}$. Thus $|\nu|(A) \le 2\|\nu\|(A)$ for every $A \subset \mathbb{N}$. Thus condition (A) is satisfied.

3. We study in this section measures with values in a space $C(K)$ of continuous functions on a compact Hausdorff topological space K. This case includes measures with values in AM-spaces. Recall that a Banach lattice is called an *AM-space* if $\|x + y\| = \max(\|x\|, \|y\|)$ for disjoint elements x, y. By a theorem of Kakutani, every AM-space is lattice isomorphic and isometric to a sublattice of $C(K)$ for some compact Hausdorff topological space K. On the other hand the following theorem shows that for separable measures with no atoms we can restrict our attention to measures with values in c_0.

The measure ν is said to be *separable* if the Saks pseudometric space associated to a Rybakov control measure λ is separable. Clearly this is equivalent to $L^1(\nu)$ being separable. A measurable set is an atom for ν if and only if it is an atom for λ (this also being equivalent to χ_A being an atom in the lattice $L^1(\nu)$).

THEOREM 1. *Let $\nu: \Sigma \to X$ be separable and with no atoms. Then there exists a measure $\mu: \Sigma \to c_0$ such that $L^1(\mu)$ is order isomorphic and isometric to $L^1(\nu)$.*

Proof. Let λ be a Rybakov control measure for ν. $L^1(\nu)$ is separable and the simple functions are dense so we can find a sequence (f_n) of simple functions that is dense in $L^1(\nu)$. Recall that $L^1(\nu)^*$ can be regarded as a lattice ideal in $L^1(\lambda)$.

Step 1. There exists a sequence (h_n) of simple functions which is w*-dense in the unit ball of $L^1(\nu)^*$.

Let g_0 be in the unit ball of $L^1(\nu)^*$. There exists a sequence (g_n) of simple functions such that $|g_n| \leq |g_0|$ for all n, and g_n converges pointwise to g_0. This implies that $\int g_n f\, d\lambda = \langle g_n, f \rangle$ converges to $\int g_0 f\, d\lambda = \langle g_0, f \rangle$ for every f in $L^1(\nu)$. So (g_n) converges to g_0 in the w*-topology in $L^1(\nu)^*$. From $|g_n| \leq |g_0|$ it follows that $\|g_n\| \leq \|g_0\| \leq 1$. Thus the simple functions are w*-dense in the unit ball of $L^1(\nu)^*$. The claim follows from the separability of $L^1(\nu)$ which implies that the unit ball of $L^1(\nu)^*$ is w*-metrizable.

Step 2. There exists an increasing sequence of finite sub-σ-algebras (Σ_n) and a sequence (g_n) of simple functions such that the functions $f_1, \ldots, f_n$ are Σ_n-measurable, $|g_n| = |h_n|$ holds for all n and (g_n, Σ_n) is a martingale difference sequence, that is, the conditional expectation of g_n with respect to Σ_{n-1} is zero.

Set $g_1 = h_1$. Let us define g_n. Set

$$\Sigma_{n-1} = \sigma(f_1, \ldots, f_{n-1}, g_1, \ldots, g_{n-1})$$

the smallest σ-algebra that makes the functions $f_1, \ldots, f_{n-1}, g_1, \ldots, g_{n-1}$ measurable. It is finite as the functions are simple. Let A be a set of constancy of h_n where it has the value a and let B be an atom of Σ_{n-1}. By the nonatomic nature of λ we can find two disjoint sets of equal measure whose union is $A \cap B$. Define g_n to be a on one and $-a$ on the other. It is plain to see that g_n satisfies the requirements.

Step 3. Define the measure $\mu: \Sigma \to c_0$ by $\mu(A) = (\int_A g_n\, d\lambda)_0^\infty$, where g_0 is identically one. It is well defined, countably additive and if f is in $L^1(\nu)$ then f is in $L^1(\mu)$.

To see this let f be in $L^1(\nu)$. Given $\varepsilon > 0$ find n_0 such that $\|f - f_{n_0}\|_\nu < \varepsilon$. Then

$$\begin{aligned}\left|\int f g_n \, d\lambda\right| &\leq \left|\int (f - f_{n_0}) g_n \, d\lambda\right| + \left|\int f_{n_0} g_n \, d\lambda\right| \\ &= |\langle f - f_{n_0}, g_n\rangle_{L^1(\nu)}| + \left|\int f_{n_0} g_n \, d\lambda\right| \\ &\leq \varepsilon + \left|\int f_{n_0} g_n \, d\lambda\right|.\end{aligned}$$

For $n > n_0$ the function f_{n_0} is Σ_{n-1} measurable and thus the last integral is zero. This proves that the measure is well defined.

Let $x^* = (a_n)$ be in l^1. Then

$$x^*\mu(A) = \sum a_n \int_A g_n \, d\lambda = \int_A \left(\sum a_n g_n\right) d\lambda$$

as the g_n are bounded in $L^1(\nu)^*$ and so bounded in $L^1(\lambda)$. Thus the measure $x^*\mu$ is countably additive. So μ is weakly countably additive and by the Orlicz-Pettis theorem it is countably additive. The two previous equations show that if f is in $L^1(\nu)$ then f is in $L^1(\mu)$.

Step 4. The inclusion of $L^1(\nu)$ into $L^1(\mu)$ is onto and norm preserving.

Let f be in $L^1(\mu)$. Fix x^* in X^*. For every $n \in \mathbb{N}$ denote $f_n = f \cdot \chi_{A_n}$, where $A_n = \{\omega : |f(\omega)| \leq n\}$. As f_n is essentially bounded it belongs to $L^1(\nu)$. Set h_{x^*} for the Radon-Nikodym derivative of the measure $x^*\nu$ with respect to λ. It is in the unit ball of $L^1(\nu)^*$. Find a subsequence (h_{n_i}) that w*-converges to $|h_{x^*}|$. Then

$$\begin{aligned}\int |f_n| d|x^*\nu| &= \int |f_n| \, |h_{x^*}| \, d\lambda = \lim_i \int |f_n| h_{n_i} \, d\lambda \\ &\leq \lim_i \int |f_n| \, |h_{n_i}| \, d\lambda = \lim_i \int |f_n| \, |g_{n_i}| \, d\lambda \\ &= \lim_i \langle |f_n|, |g_{n_i}|\rangle_{L^1(\mu)} \leq \|f_n\|_\mu \leq \|f\|_\mu.\end{aligned}$$

To see this we have used that λ is a Rybakov control measure for μ and thus integration with respect to $g_n \, d\lambda$ defines a linear functional on $L^1(\mu)$ with norm not greater than one. Thus f is $|x^*\nu|$ integrable. So f is scalarly integrable with respect to ν and

$$\sup\left\{\int |f| \, d|x^*\nu| : x^* \in B_{X^*}\right\} \leq \|f\|_\mu.$$

To see that f is actually in $L^1(\nu)$ just apply the previous argument to $f_n - f_m$ to get $\|f_n - f_m\|_\nu \leq \|f_n - f_m\|_\mu$. So f_n is a Cauchy sequence in $L^1(\nu)$ that converges λ-a.e. to f. Hence f is in $L^1(\nu)$ and $\|f\|_\nu \leq \|f\|_\mu$.

On the other hand for $x^* = (a_n)$ in the unit ball of l^1, we have that $\sum a_n g_n$ is in $L^1(\nu)^*$ and has norm not greater than one, so

$$\int |f|\, d|x^*\mu| = \int |f| \cdot \left|\sum a_n g_n\right| d\lambda = \left\langle |f|, \left|\sum a_n g_n\right|\right\rangle_{L^1(\nu)} \leq \|f\|_\nu.$$

Hence $\|f\|_\mu \leq \|f\|_\nu$. □

The situation in the purely atomic case is quite different. Purely atomic countably additive vector measures can be realized over the σ-algebra $\mathscr{P}(\mathbb{N})$ of all subsets of the natural numbers. The spaces $L^1(\nu)$ generated by these measures are sequence spaces where the characteristic functions of singletons form an unconditional basis and the order is the coordinatewise order. Let $\nu: \mathscr{P}(\mathbb{N}) \to X$ be given by $\nu(\{n\}) = x_n$. Then a sequence (a_n) is in $L^1(\nu)$ if and only if the sequence in $(a_n x_n)$ is unconditionally summable in X and in this case its norm in $L^1(\nu)$ is

$$\|(a_n)\|_\nu = \sup\left\{\sum |a_n\langle x^*, x_n\rangle| : x^* \in B_{X^*}\right\}.$$

It follows that $\|\chi_{\{n\}}\|_\nu = \|x_n\|$. Conversely let E be a Banach space with an unconditional basis (y_n). E is a Banach lattice for the coordinatewise order and the equivalent norm $\|\!|\sum a_n y_n|\!\| = \sup\{\|\sum_A a_n y_n\| : A \subset \mathbb{N}\}$. Then E can be obtained (order isomorphically) as $L^1(\nu)$ for the measure $\nu: \mathscr{P}(\mathbb{N}) \to E$ defined by $\nu(\{n\}) = a_n y_n$ where $\sum a_n y_n$ is in E and $a_n > 0$ for all n (see [C, Theorem 8 and Lemma 2]).

The previous discussion shows that c_0 can be obtained from a c_0-valued measure. The space l^1 can be obtained from the measure $\nu: \mathscr{P}(\mathbb{N}) \to \mathbb{R} \subset c_0$ defined by $\nu(\{n\}) = a_n$, where (a_n) is in l^1 with $a_n > 0$ for all n.

On the other hand, consider a Banach space E with a normalized unconditional basis (y_n) that does not contain subspaces isomorphic to l^1 or c_0 (E reflexive). Then E cannot be obtained (order isomorphically) as $L^1(\nu)$ for a measure $\nu: \mathscr{P}(\mathbb{N}) \to c_0$. Assume by way of contradiction that this is not the case. Let $\nu: E \to c_0$ be the integration map associated to the measure ν. Set $\nu(y_n) = x_n \in c_0$. Then $\sum a_n y_n$ converges unconditionally in E if and only if $\sum a_n x_n$ converges unconditionally in c_0. As y_n is an atom in E, it follows that $\|y_n\| = \|x_n\|$. So (x_n) is normalized in c_0. As E does not contain subspaces isomorphic to l^1 the basis (y_n) is shrinking. So (y_n) is weakly null. The integration map ν is continuous, so (x_n) is weakly null in c_0, and normalized. By the Bessage-Pelczynski Selection Principle (see **[L-T**, vol. I, Proposition 1.a.12]) there exists a subsequence (x_{n_k}) which is a basic sequence and equivalent to a block basis of c_0. Thus (x_{n_k}) is an unconditional basic sequence and so the restriction of the operator ν to the closed linear span of (y_{n_k}) in E is an isomorphism onto the closed linear span of (x_{n_k}) in c_0. This last space is isomorphic to c_0 (see **[L-T**, vol. I, Proposition 2.a.1]), so we arrive at a contradiction.

For particular situations we have a complete characterization.

THEOREM 2. *Consider a measure $\nu:\mathscr{P}(\mathbb{N})\to C(K)$, for K a compact Hausdorff topological space. Set $f_n=\nu(\{n\})$ and let $|f_n|$ be the modulus of f_n in $C(K)$. Then the following are equivalent:*

(a) *$L^1(\nu)$ is order isomorphic to an AL-space.*

(b) *The measure ν satisfies the following condition:*

$$\text{(B)}\qquad 0\notin\overline{\text{co}}\left\{\frac{|f_n|}{\|f_n\|}:n\in\mathbb{N}\right\}.$$

Proof. In view of Proposition 2 we have to prove that $L^1(\nu)$ is isomorphic to $L^1(|\nu|)$. The simple functions are dense and a function and its modulus have the same norm. Thus, by using the equivalent norm in $L^1(\nu)$ mentioned in the introduction, it follows that (a) is equivalent to the existence of a positive constant $C>0$ such that for nonnegative a_n and $N\in\mathbb{N}$ we have

$$C\cdot\sum_1^N a_n\|f_n\|\le\sup\left\{\left\|\sum_{n\in B}a_nf_n\right\|:B\subset\{1,\dots,N\}\right\},$$

since $|\nu|(\{n\})=\|f_n\|$. Set $g_n=f_n/\|f_n\|$. By homogeneity the previous expression is equivalent to

$$C\le\sup\left\{\left\|\sum_{n\in B}\alpha_ng_n\right\|:B\subset\{1,\dots,N\}\right\}$$

where $\alpha_n\ge0$ and $\sum_1^N\alpha_n=1$. The above supremum is not greater than $\|\sum_1^N\alpha_n|g_n|\|_\infty$. Proving the equivalence of both expressions would finish the proof. For some $t_0\in K$ we have

$$\left\|\sum_1^N\alpha_n|g_n|\right\|_\infty=\sum_1^N\alpha_n|g_n(t_0)|=\sum_{B_1}\alpha_ng_n(t_0)-\sum_{B_2}\alpha_ng_n(t_0)$$

where $B_1 = \{1 \leq n \leq N\colon g_n(t_0) \geq 0\}$ and B_2 is its complement in $\{1, \ldots, N\}$; thus

$$\left\|\sum_1^N \alpha_n |g_n|\right\|_\infty \leq \left\|\sum_{B_1} \alpha_n g_n\right\| + \left\|\sum_{B_2} \alpha_n g_n\right\|$$
$$\leq 2 \cdot \sup\left\{\left\|\sum_{n\in B} \alpha_n g_n\right\| : B \subset \{1, \ldots, N\}\right\}. \qquad \square$$

4. If ν has finite variation the natural inclusion of $L^1(|\nu|)$ into $L^1(\nu)$ has dense range, since the simple functions are dense in $L^1(\nu)$, and so $L^1(\nu)^*$ can be identified with a linear subspace of the Banach algebra $L_\infty(|\nu|)$. Moreover, $L^1(\nu)^*$ is an *ideal* in $L_\infty(|\nu|)$, in both the lattice and the algebraic sense (which coincide in $L_\infty(|\nu|)$). To see this consider a Rybakov control measure λ. $L^1(\nu)^*$ is a lattice ideal in $L^1(\lambda)$. On the other hand λ and $|\nu|$ have the same null sets and $L^1(\nu)^* \subset L_\infty(|\nu|) \subset L^1(\lambda)$.

Let us consider the space $C(\Delta)$ of continuous functions over the compact totally disconnected topological space Δ of continuous, linear and multiplicative functionals (characters) over the space $L_\infty(|\nu|)$. Then $L_\infty(|\nu|)$ and $C(\Delta)$ are order and algebraically isomorphic and isometric under the Gelfand transform:

$$f \in L_\infty(|\nu|) \mapsto f^\frown \in C(\Delta) \quad \text{where for } s \in \Delta \;\; f^\frown(s) = s(f).$$

We define the following sets in Δ. Let H be the set of all *characters that are null over* $L^1(\nu)^*$, i.e. the set of zeros of the image of $L^1(\nu)^*$ by the Gelfand transform in $C(\Delta)$.

Given $x^* \in X^*$, let h_{x^*} be the Radon-Nikodym derivative of the scalar measure $x^*\nu$ with respect to $|\nu|$. The map $f \in L^1(\nu) \mapsto \int f\, dx^*\nu$ defines a bounded linear functional on $L^1(\nu)$, so we have that h_{x^*} is in $L^1(\nu)^*$. We will denote by $\mathscr{I}$ the ideal generated by the set $\{h_{x^*}\colon x^* \in X^*\}$ in $L^1(\nu)^*$.

Let H^* be the set of all *characters that are null over* $\mathscr{I}$, i.e. the set of zeros of the image of $\mathscr{I}$ by the Gelfand transform in $C(\Delta)$. Obviously $H \subset H^*$.

The next proposition follows easily, by taking into account that the closure of a proper ideal is itself a proper ideal.

Proposition 3. *The following conditions are equivalent*:

(a) H *is empty.*

(b) $L^1(\nu)$ *is order isomorphic to an AL-space.*

The usefulness of the definition of H^* for the problem we are investigating is shown in the next proposition.

PROPOSITION 4. *The following conditions are equivalent:*

(a) H^* *is empty.*

(b) *There exists a finite partition* $(A_i)_1^n$ *of the measure space and* $x_1^*, \dots, x_n^*$ *in* X^* *such that* $L^1(\nu)$ *is order isomorphic, via the identity, to* $L^1(\mu)$ *where* $\mu = \sum_1^n \mu_i$ *and* μ_i *is the restriction of the measure* $|x_i^*\nu|$ *to the trace of* Σ *over* A_i.

Proof. (a) $\Rightarrow$ (b) If H^* is empty then H is empty so from Proposition 3 $L^1(\nu)$ is order isomorphic to an AL-space and from Proposition 2 it is order isomorphic to $L^1(|\nu|)$. On the other hand the ideal generated by $\{h_{x^*}: x^* \in X^*\}$ is dense; thus it is $L_\infty(|\nu|)$. So, there exist $x_1^*, \dots, x_n^*$ in X^* such that $\chi_\Omega \leq \sum_1^n |h_{x_i^*}|$. By multiplying, if necessary, the functionals x_k^* by positive constants, we can assume that there exists a disjoint partition $(A_i)_1^n$ such that $\chi_\Omega \leq \sum_1^n |h_{x_i^*}| \cdot \chi_{A_i}$. Integrating with respect to $|\nu|$ we have $|\nu|(A) \leq \sum_1^n |x_i^*\nu|(A \cap A_i)$ for all A in Σ. As we always have $\sum_1^n |x_i^*\nu|(A \cap A_i) \leq \max_i \|x_i^*\| \cdot |\nu|(A)$ we see that $L^1(|\nu|)$ is order isomorphic to $L^1(\mu)$. So (b) follows.

(b) $\Rightarrow$ (a) From Proposition 2 it follows that $L^1(|\nu|)$ is order isomorphic via the identity to $L^1(\mu)$. Thus there exists a constant $C > 0$ such that for every measurable set A we have

$$C \cdot |\nu|(A) < \sum_1^n |x_i^*\nu|(A \cap A_i).$$

By integrating with respect to $|\nu|$ we have

$$C \cdot \int_A \chi_\Omega \, d|\nu| = \int_A \left(\sum_1^n |h_{x_i^*}| \cdot \chi_{A_i} \right) d|\nu|$$

for every measurable set A. So $\chi_\Omega \leq C^{-1} \cdot \sum_1^n |h_{x_i^*}| \cdot \chi_{A_i}$ $|\nu|$-almost everywhere. Thus, the unit of $L_\infty(|\nu|)$ is in the ideal generated by the functions h_{x^*}. Hence H^* is empty. □

The following result is well known. It describes how the action of a character on a function can be computed.

LEMMA. *Let* s *be a character on* $L_\infty(\Omega, \Sigma, \mu)$. *Then there exists an ultrafilter* $\mathscr{U}$ *in* $\Sigma/\mu^{-1}(0)$ *such that*

$$s(f) = \lim_{A\in\mathscr{U}} \frac{\int_A f\,d\mu}{\mu(A)}$$

for every f *in* $L_\infty(\Omega, \Sigma, \mu)$. *Conversely, for every ultrafilter as above, the previous expression defines a character over* $L_\infty(\Omega, \Sigma, \mu)$.

The previous propositions allow us to give an answer to the problem we are investigating. For a vector measure $\nu: \Sigma \to X$, consider the following condition:

$$\text{(C)} \qquad 0 \notin \overline{\text{co}}\left\{ \frac{\nu(A)}{|\nu|(A)} : |\nu|(A) \neq 0 : A \in \Sigma \right\}.$$

THEOREM 3. *The following conditions are equivalent*:

(a) $L^1(\nu)$ *is order isomorphic, via the identity, to* $L^1(\mu)$ *where the measure* μ *is as in Proposition* 4.

(b) *There exists a finite partition* $(B_j)_1^k$ *of the measure space such that the restriction of* ν *to each set* B_j *satisfies condition* (C).

Proof. (a) $\Rightarrow$ (b) From our hypothesis there exists a constant $C > 0$ such that $C \cdot |\nu|(A) \leq \sum_1^n |x_i^*\nu|(A \cap A_i)$ for disjoint A_i. For each i apply a Hahn decomposition to the measure $x_i^*\nu$ and decompose A_i into two disjoint sets A_{i1} and A_{i2} so that on the first $x_i^*\nu$ is nonnegative and on the second is nonpositive. Consider the restriction of ν to A_{ik} for $1 \leq i \leq n$ and $k = 1, 2$. We have that

$$\left| \frac{x_i^*\nu(A)}{|\nu|(A)} \right| \geq C \quad \text{for all } A \in \Sigma,\ A \subset A_{ik}$$

and this implies that ν satisfies condition (C) on each A_{ik}.

(b) $\Rightarrow$ (a) Let ν_j be the restriction of ν to B_j. As the ν_j are disjointly supported, $L^1(\nu)$ is order isomorphic to $(\bigoplus_1^k L^1(\nu_j))_1$. Thus we can assume that ν satisfies condition (C). By Proposition 4 we just have to verify that H^* is empty. Suppose not; then there exists $s \in \Delta$ such that $s(h_{x^*}) = 0$ for all x^* in X^*. The lemma tells as that there exists an ultrafilter $\mathscr{U}$ in $\Sigma/|\nu|^{-1}(0)$ such that for every x^* in X^*.

$$0 = \lim_{A\in\mathscr{U}} \frac{\int_A h_{x^*}\,d|\nu|}{|\nu|(A)} = \lim_{A\in\mathscr{U}} \frac{x^*\nu(A)}{|\nu|(A)}$$

that is, the net $\{\nu(A)/|\nu|(A) : A \in \mathscr{U}\}$ is weakly null, contradicting our hypothesis. □

The coincidence of the sets H and H^* is related to a natural characterization of the weak convergence in $L^1(\nu)$. This is shown in the next theorem. For this theorem no assumptions are made on the variation. Thus we will consider $L^1(\nu)^*$ as an ideal in $L^1(\lambda)$ for a control measure λ. For $x^* \in X^*$ and $A \in \Sigma$ we will denote by φ_{x^*A} the element of $L^1(\nu)^*$ defined by $f \in L^1(\nu) \mapsto \int_A f\,dx^*\nu \in \mathbb{R}$ (which corresponds in $L^1(\lambda)$ to the Radon-Nikodym derivative with respect to λ of the measure $x^*\nu$ restricted to the set A). Let $\mathcal{J}$ denote the ideal generated in $L^1(\nu)^*$ by the set $\{\varphi_{x^*A}\colon\ x^* \in X^*,\ A \in \Sigma\}$.

THEOREM 4. *Consider the following conditions*:

(a) $L^1(\nu)$ *does not contain a complemented subspace isomorphic to* l^1.

(b) *The ideal* $\mathcal{J}$ *is dense in* $L^1(\nu)^*$.

(c) *In* $L^1(\nu)$ *weak convergence of bounded nets is characterized by weak convergence* (*in* X) *of the integrals over arbitrary sets, that is, if* $\sup_\alpha \|f_\alpha\| < +\infty$ *then*

$$\text{(D)}\quad f_\alpha \overset{w}{\to} f \text{ in } L^1(\nu) \Leftrightarrow \int_A f_\alpha\,d\nu \overset{w}{\to} \int_A f\,d\nu \quad \text{in } X \text{ for every } A \in \Sigma.$$

Then (a) *implies* (b) *and* (b) *implies* (c). *If* ν *has finite variation, then* (c) *implies*

(d) $H = H^*$.

Proof. (a) $\Rightarrow$ (b) Assume $L^1(\nu)$ does not contain a complemented subspace isomorphic to l^1. It follows from results of Bessaga and Pelczynski that $L^1(\nu)^*$ does not contain a subspace isomorphic to l_∞. As $L^1(\nu)^*$ is an order complete Banach lattice, we deduce that $L^1(\nu)^*$ is order continuous (see **[A-B**, Theorem 14.9]). In order continuous Banach lattices every closed ideal is a band (see **[MN**, Corollary 2.4.4]). As $L^1(\nu)$ is order continuous it follows that every band in $L^1(\nu)^*$ is w*-closed (see **[MN**, Corollary 2.4.7]). Thus, the closure of the ideal $\mathcal{J}$ (which is itself an ideal) is w*-closed in $L^1(\nu)^*$.

On the other hand, the set $\{\varphi_{x^*A}\colon x^* \in X^*,\ A \in \Sigma\}$ is total. To see this let $f \in L^1(\nu)$ be such that $\langle \varphi_{x^*A}, f\rangle = 0$ for every $x^* \in X^*$ and every $A \in \Sigma$. Then $\int_A f\,dx^*\nu = 0$ for every $x^* \in X^*$ and every $A \in \Sigma$, so $f \equiv 0$. This fact implies that the closure of the ideal $\mathcal{J}$ is w*-dense in $L^1(\nu)$. Hence $\mathcal{J}$ is dense in $L^1(\nu)^*$.

(b) $\Rightarrow$ (c) We have to prove that condition (D) holds. In fact in (D) we just have to prove sufficiency as necessity follows from the continuity of the integration map. Let (f_α) be a net such that

$\int_A f_\alpha \, dx^*\nu$ goes to zero for every x^* in X^* and every $A \in \Sigma$. For fixed x^*, by considering the Hahn decomposition of the measure $x^*\nu$, we deduce that (f_α) is a weakly null net in $L^1(|x^*\nu|)$. Given h in the ideal generated by $\{\varphi_{x^*A} : x^* \in X^*,\ A \in \Sigma\}$ there exist $x_1^*, \dots, x_n^*$ and $A_1, \dots, A_n$ such that $|h| \le \sum_1^n |\varphi_{x_i^* A_i}|$. Set μ for the measure with density $|h|$ with respect to λ. The space $(\bigoplus_1^n L^1(|x_i^*\nu|))_1$ is continuously included in $L^1(\mu)$ and thus (f_α) is weakly null in $L^1(\mu)$, and this implies that $\int_A f_\alpha |h| \, d\lambda$ goes to zero, for every measurable set A. By decomposing h into its positive and negative parts we conclude that $\int f_\alpha h \, d\lambda = \langle h, f_\alpha \rangle$ goes to zero. From (b) and the boundedness of the net (f_α), this result holds for every h in $L^1(\nu)^*$. Thus (f_α) is weakly null.

(c) $\Rightarrow$ (d) Assume now that $|\nu|$ is finite. Let $s \in H^*$. It is given by an ultrafilter $\mathscr{U}$ in $\Sigma/|\nu|^{-1}(0)$. Consider the net $\{f_A = \chi_A/|\nu|(A) : A \in \mathscr{U}\}$ in $L^1(\nu)$. It is bounded as $\|f_A\| = \|\nu\|(A)/|\nu|(A) \le 1$. Let $B \in \mathscr{U}$; then

$$\begin{aligned}
\lim_{A \in \mathscr{U}} \int_B f_A \, dx^*\nu &= \lim_{A \in \mathscr{U}} \frac{\int_{A\cap B} h_{x^*} \, d|\nu|}{|\nu|(A)} \\
&= \lim_{A \in \mathscr{U},\, A \subset B} \frac{\int_A h_{x^*} \, d|\nu|}{|\nu|(A)} \\
&= \lim_{A \in \mathscr{U}} \frac{\int_A h_{x^*} \, d|\nu|}{|\nu|(A)} = s(h_{x^*}) = 0
\end{aligned}$$

since $s \in H^*$. The same holds for $B \notin \mathscr{U}$. So the net (f_A) has the property that the integrals over arbitrary sets go weakly to zero in X. Thus by (c) (f_A) goes weakly to zero in $L^1(\nu)$, so for every h in $L^1(\nu)^*$

$$s(h) = \lim_{A \in \mathscr{U}} \frac{\int_A h \, d|\nu|}{|\nu|(A)} = \lim_{A \in \mathscr{U}} \langle h, f_A \rangle = 0.$$

Hence $s(h) = 0$, so s is in H. □

It should be noticed that, as $L^1(\nu)$ is an order continuous Banach lattice, it follows from results of Tzafriri [T, Theorem 16] that whenever it contains a subspace isomorphic to l^1 in fact it contains a complemented subspace isomorphic to l^1. Condition (d) does not imply (a), in general. To see this consider the following measure defined on the Lebesgue measurable sets of $[0, 1]$. Let $\nu : \mathscr{M} \to c_0$ be defined as $\nu(A) = (\int_A r_n(t) \, dm(t))$ where the r_n are the Rademacher functions. It was shown in [C] that $L^1(\nu)$ is order isometric to $L^1[0, 1]$. On the other hand, as $r_1 \equiv 1$ and as the variation of ν is the Lebesgue measure m, it is easy to see that condition (C) is satisfied and so $H^* = H = \varnothing$. S. Okada has proved independently, using a different technique, the implication (a) $\Rightarrow$ (b) for sequences of functions, **[O]**. From the Nikodym Boundedness Theorem (see **[D-U**, Theorem I.3.1]) it follows that a sequence in $L^1(\nu)$ satisfying that the integrals over arbitrary sets are weakly convergent, is norm bounded.

5. We exhibit several measures in order to show that the conditions studied in this paper are indeed different.

EXAMPLE 2. In every infinite dimensional Banach space there exists a countably additive vector measure of bounded variation not satisfying condition (A).

To see this let (x_n) be a sequence in X such that the series $\sum x_n$ converges unconditionally but not absolutely. Suppose that we can find a sequence (α_n), $0 \leq \alpha_n \leq 1$, for all n, such that

(a) $\sum \alpha_n \|x_n\| < +\infty$, and

(b) $(\sum_{i \geq n} \alpha_i \|x_i\|) \cdot (\sup\{\sum_{i \geq n} |x^* x_i| : x^* \in B_{X^*}\})^{-1} \to +\infty$.

Consider the measure $A \in \mathscr{P}(\mathbb{N}) \mapsto \nu(A) = \sum_{n \in A} \alpha_n x_n$. It has bounded variation by (a). By considering $A_n = \{n, n+1, \dots\}$, we deduce from (b) and from the fact that $0 \leq \alpha_n \leq 1$, that $|\nu(A_n)|/\|\nu\|(A_n)$ goes to infinity, so ν does not satisfy condition (A).

The existence of the sequence (α_n) follows from the next claim by setting $\beta_n = \|x_n\|$, $\gamma_n = \sup\{\sum_{i \geq n} |x^* x_i| : x^* \in B_{X^*}\}$ and $\alpha_n = (a_n - a_{n+1})/\beta_n$.

Claim. Let sequences (β_n) and (γ_n) be given with γ_n decreasing to zero and $\sum \beta_n = +\infty$. Then there exists a sequence (α_n) decreasing to zero such that

(a) $\beta_n \geq a_n - a_{n+1}$, and

(b) $a_n \gamma_n^{-1} \to \infty$.

To prove this notice that the sequence $(\sqrt{\gamma_n})$ decreases to zero and the series $\sum \beta_n$ diverges. Using these two facts we can find, via an inductive process, a strictly increasing sequence of positive integers (n_k) and a sequence of sets J_k, such that $\bigcup J_k = \mathbb{N}$ and $n < m$ for every $n \in J_k$ and $m \in J_{k+1}$, satisfying

$$A_k = \sum_{n \in J_k} \beta_n > \frac{1}{2^k}, \quad \text{and} \quad \frac{1}{2^{n_k}} \geq \sqrt{\gamma_n} > \frac{1}{2^{n_{k+1}}} \quad \text{for } n \in J_k.$$

Set $a_1 = 1$ and $a_{n+1} = a_n - \beta_n/(2^k A_k)$ for $n \in J_k$. This is the sequence we were looking for.

EXAMPLE 3. We now exhibit a measure for which $L^1(\nu)$ is an AL-space but it is not given by a finite number of spaces $L^1(|x^*\nu|)$. That is $\varnothing = H \neq H^*$. It follows that condition (D) does not hold in $L^1(\nu)$. Suppose that we can find a sequence of functions (f_n) in $C[0, 1]$ that satisfy the following conditions: $\|f_n\| = 1$, zero is a weak cluster point of (f_n) and there exists an $\varepsilon > 0$ and a norm one functional $\mu \in C[0, 1]^*$ such that $\mu(|f_n|) \geq \varepsilon > 0$ for all n. Consider the measure

$$A \in \mathscr{P}(\mathbb{N}) \mapsto \nu(A) = \sum_{n \in A} \frac{1}{2^n} f_n \in C[0, 1].$$

It is countably additive and has bounded variation. $L^1(\nu)$ is order isomorphic to an AL-space by Theorem 2 as the measure satisfies condition (B): for a convex combination of $|\nu(n)|/\|\nu\|(n)$ we have

$$\left\|\sum \alpha_n |f_n|\right\| \geq \mu\left(\sum \alpha_n |f_n|\right) \geq \varepsilon.$$

On the other hand, as zero is a weak cluster point of (f_n), the measure ν cannot be decomposed into a finite number of disjointly supported measures each one of them satisfying condition (C). So, from Theorem 3 and Proposition 4, it follows that H^* is not empty.

The existence of a sequence satisfying the required conditions follows from the separability of $C[0, 1]$ and the fact that the map $f \in C[0, 1] \mapsto |f| \in C[0, 1]$ is not weak-to-weak continuous at zero. To see this it suffices to prove, by using Liapunov's and Lusin's Theorems, that for any $\mu_1, \mu_2, \ldots, \mu_k$ in $C[0, 1]^*$ and $\varepsilon > 0$ there exists a continuous function g such that $|\int g\, d\mu_i| < \varepsilon$ for $1 \leq i \leq k$, and $\int |g|\, dm \geq 1/2$, where m is the Lebesgue measure in $[0, 1]$.

EXAMPLE 4. Let $1 < p < +\infty$ and $\alpha_n > 0$ with $\sum \alpha_n < +\infty$. Define $\mu\colon \mathscr{P}(\mathbb{N}) \to l^p$ by $\mu(A) = \sum_A \alpha_n e_n$ where (e_n) is the canonical basis in l^p. For this measure $\varnothing \not\subseteq H = H^*$. Consider the disjoint sum of μ and the measure ν of the previous example. The resulting measure satisfies $\varnothing \not\subseteq H \not\subseteq H^*$.

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