

ON FACTOR REPRESENTATIONS OF DISCRETE RATIONAL NILPOTENT GROUPS AND THE PLANCHEREL FORMULA

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The purpose of this paper is to extend the Kirillov orbit picture of representation theory for nilpotent Lie groups to discrete groups $G_{\mathbf{Q}}$ defined over the rationals \mathbf{Q} , following a program begun by Roger Howe. Let Ad^* be the coadjoint action of $G_{\mathbf{Q}}$ on the Pontryagin dual $\widehat{\mathfrak{g}_{\mathbf{Q}}}$ of the Lie algebra of $G_{\mathbf{Q}}$. It is shown that each coadjoint orbit closure is a coset of the annihilator of an ideal of $\mathfrak{g}_{\mathbf{Q}}$, that a certain induced representation canonically associated with an orbit closure is a traceable factor, and that there is an orbital integral formula which gives the trace. Finally, a Plancherel formula is proved.

1. Introduction. In what follows, let $\mathfrak{g}_{\mathbf{Q}}$ be a nilpotent Lie algebra with rational structure constants; i.e., $\mathfrak{g}_{\mathbf{Q}}$ has a basis $\{X_1, \dots, X_n\}$ of vectors such that

$$[X_i, X_j] = \sum_{k=1}^n \alpha_{i,j,k} X_k,$$

with all $\alpha_{i,j,k} \in \mathbf{Q}$. Then we form a nilpotent group $G_{\mathbf{Q}}$ with base set $\mathfrak{g}_{\mathbf{Q}}$, using the Campbell-Baker-Hausdorff formula to define a polynomial group multiplication, and we have a map $\exp: \mathfrak{g}_{\mathbf{Q}} \rightarrow G_{\mathbf{Q}}$, $\exp \equiv \text{Id}$ on the set $\mathfrak{g}_{\mathbf{Q}}$. We will think of $\mathfrak{g}_{\mathbf{Q}}$ and $G_{\mathbf{Q}}$ as having the discrete topology. $G_{\mathbf{Q}}$ has no normal abelian group of finite index; therefore $G_{\mathbf{Q}}$ is not of Type I ([Tho]). The purpose of this paper is to develop an extension of the Kirillov orbit picture of representation theory for nilpotent Lie groups to discrete groups $G_{\mathbf{Q}}$ defined over the rationals, following a program begun by Roger Howe ([How1], [How2], [How3]). In particular, in [How2], he constructs an extension of the Kirillov theory to finitely generated discrete nilpotent groups without torsion; this work was a chief source of inspiration to us.

Although it is in general difficult to work with non-type-I groups (and in particular discrete groups lack some of the structure that non-type-I connected Lie groups G possess: see [Puk], [Gdt]), it is possible to say a great deal in the present case. Let $\widehat{\mathfrak{g}_{\mathbf{Q}}}$ denote the Pontryagin dual of the Lie algebra $\mathfrak{g}_{\mathbf{Q}}$; we examine the structure of closures of

orbits under the coadjoint action of $G_{\mathbf{Q}}$ on $\widehat{\mathfrak{g}_{\mathbf{Q}}}$. We find that the following holds:

1. If $\tilde{\lambda} \in \widehat{\mathfrak{g}_{\mathbf{Q}}}$, then the coadjoint orbit closure containing $\tilde{\lambda}$ is of the form $\tilde{\lambda} + \mathfrak{i}_{\lambda}^{\perp}$, where $\mathfrak{i}_{\lambda} \subseteq \mathfrak{g}_{\mathbf{Q}}$ is the largest ideal contained in the subalgebra $\mathfrak{r}_{\lambda} = \{X \in \mathfrak{g}_{\mathbf{Q}} \mid \tilde{\lambda}([X, Y]) = 1, \text{ all } Y \in \mathfrak{g}_{\mathbf{Q}}\}$. We note that since \mathfrak{i}_{λ} is an ideal in \mathfrak{r}_{λ} , the radical of the character $\tilde{\lambda}$, we have $\tilde{\lambda}(X^*Y) = \tilde{\lambda}(X + Y)$ for $X, Y \in \mathfrak{i}_{\lambda}$; therefore we may regard λ , the restriction of $\tilde{\lambda}$ to \mathfrak{i}_{λ} , as a character on the normal subgroup I_{λ} of $G_{\mathbf{Q}}$. Let K_{λ} be the kernel in I_{λ} of λ ; then $(G_{\mathbf{Q}}, I_{\lambda}) \subseteq K_{\lambda}$.

2. The representation $\tau_{\lambda} = \text{Ind}(\lambda, I_{\lambda} \uparrow G_{\mathbf{Q}})$ is a type II traceable factor representation which is canonically associated with the coadjoint orbit closure containing $\tilde{\lambda}$; there is a simple formula for the trace of τ_{λ} as an integral over the orbit closure.

3. A concrete Plancherel formula is proven for $G_{\mathbf{Q}}$; Plancherel measure is normalized Haar measure on $\widehat{Z(G_{\mathbf{Q}})}$, the Pontryagin dual of the center of $\mathfrak{g}_{\mathbf{Q}}$. (This is in contrast with other countable discrete groups, for example the free group on two generators, where Plancherel measure is atomic measure on one point; see [Mac].)

We construct the Pontryagin dual of $\mathfrak{g}_{\mathbf{Q}}$ as follows. Let \mathbf{A} denote the ring of adeles over the rationals (a standard reference for harmonic analysis on the ring of adeles is Tate’s thesis, in [CF]); the rationals embed diagonally as a cocompact, discrete additive subgroup. Then $\widehat{\mathbf{Q}} \cong \mathbf{A}/\mathbf{Q}$, when \mathbf{Q} is regarded as a discrete countable abelian group. If $\{X_1, \dots, X_n\}$ is a basis for $\mathfrak{g}_{\mathbf{Q}}$ over \mathbf{Q} , then we have the isomorphism

$$\begin{aligned} (\mathbf{A}/\mathbf{Q})^n &\rightarrow \widehat{\mathfrak{g}_{\mathbf{Q}}}, \\ (\alpha_1, \dots, \alpha_n) &\mapsto \tilde{\lambda} \end{aligned}$$

where $\tilde{\lambda}(q_1X_1 + \dots + q_nX_n) = \chi(\alpha_1q_1 + \dots + \alpha_nq_n)$ for some standard character $\chi \in \widehat{\mathbf{A}}$, usually taken to be the product of the standard characters on each component \mathbf{Q}_p (i.e., those characters which map \mathbf{Z}_p to 1).

In proving the assertion about the nature of the orbit closures $\widehat{\mathfrak{g}_{\mathbf{Q}}}$, we use the following result from [CP].

THEOREM. *Let $\alpha_1, \dots, \alpha_n$ be non-rational elements of \mathbf{A} . Then the set*

$$\{(\alpha_1x, \alpha_2x^2, \dots, \alpha_nx^n), x \in \mathbf{Q}\}$$

is dense in $(\mathbf{A}/\mathbf{Q})^n$. More generally, let $p_1(x), \dots, p_n(x)$ be polynomials with coefficients in \mathbf{A}/\mathbf{Q} , having no constant terms, which are lin-

early independent over \mathbf{Q} . Then the set $\{(p_1(x), \dots, p_n(x)) : x \in \mathbf{Q}\}$ is dense in $(\mathbf{A}/\mathbf{Q})^n$.

2. Traceable factor representations associated with coadjoint orbit closures. Recall that the representation τ_λ is a factor representation if $CR(\tau_\lambda) = \tau_\lambda(G_{\mathbf{Q}})' \cap \tau_\lambda(G_{\mathbf{Q}})'' = CI$ (\mathcal{A}' denotes the commutator of the set \mathcal{A}).

LEMMA 1. Let $U \in \tau_\lambda(G_{\mathbf{Q}})'$, and let H_λ be the Mackey space for the representation τ_λ . Then U is entirely determined by its value on the function $\delta_1 \in H_\lambda$ defined by

$$\delta_1(z) = \begin{cases} 0, & z \notin I_\lambda, \\ \lambda(z), & z \in I_\lambda. \end{cases}$$

Proof. We have that

$$H_\lambda := \{f : G_{\mathbf{Q}} \rightarrow \mathbf{C}\},$$

where

$$\int_{G_{\mathbf{Q}}/I_\lambda} |f|^2 < \infty, \\ f(kz) = \lambda(k)f(z), \quad k \in I_\lambda, z \in G_{\mathbf{Q}}.$$

Let $H'_\lambda = \{f \in H_\lambda : \int_{G_{\mathbf{Q}} \setminus I_\lambda} |f| < \infty\}$. Then H'_λ is a convolution algebra, with

$$f * g(z) = \int_{G_{\mathbf{Q}} \setminus I_\lambda} f(zx^{-1})g(x) dx$$

well-defined and in H'_λ . To see this, we must check that the integrand is constant on cosets of I_λ . Let $x, z \in G_{\mathbf{Q}}, t \in I_\lambda$. Then

$$f(x(tz)^{-1})g(tz) = f(xz^{-1}t^{-1})g(tz) = f(t^{-1}(txz^{-1}t^{-1}))g(tz).$$

We have that $t \in I_\lambda$, so $txz^{-1}t^{-1} = rxz^{-1}$, with $r = txz^{-1}E^1(xz^{-1})^{-1}$. Since r is in the commutator of $G_{\mathbf{Q}}$ with I_λ , $r \in K_\lambda$. Therefore

$$f(xz^{-1}t^{-1})g(tz) = f(t^{-1}rxz^{-1})g(tz) \\ = \lambda(t^{-1})\lambda(r)\lambda(t)f(x^{-1}z)g(z) = f(x^{-1}z)g(z).$$

We note also that the convolution of an element in H_λ with an element in H'_λ is again in H_λ .

We have that $\delta_1 * g = g$ for $g \in H'_\lambda$. If U commutes with τ_λ , then

$$\begin{aligned} U(g) &= U(\delta_1 * g) \\ &= U\left(\int_{G_{\mathbf{Q}} \setminus I_\lambda} \delta_1(zx^{-1})g(x) dx\right) \\ &= \int_{G_{\mathbf{Q}} \setminus I_\lambda} U(\delta_1(zx^{-1}))g(x) dx \\ &= \int_{G_{\mathbf{Q}} \setminus I_\lambda} U(\tau_\lambda(x^{-1})\delta_1(z))g(x) dx \\ &= \int_{G_{\mathbf{Q}} \setminus I_\lambda} \tau_\lambda(x^{-1})U(\delta_1(z))g(x) dx \\ &= \int_{G_{\mathbf{Q}} \setminus I_\lambda} U(\delta_1)(zx^{-1})g(x) dx = (U(\delta_1) * g)(z). \end{aligned}$$

So U restricted to H'_λ is really a convolution operator by the element $U(\delta_1) \in H_\lambda$. Since H'_λ is dense in H_λ , $U(\delta_1)$ determines U on all of H_λ . This completes the proof of Lemma 1.

LEMMA 2. *If $U \in CR(\tau_\lambda)$, then U is convolution by an element of H_λ which is constant on conjugacy classes. Furthermore, if $f \in H_\lambda$ is constant on conjugacy classes of $G_{\mathbf{Q}}$, and if convolution by f is a bounded operator on H_λ , then convolution by f is in $CR(\tau_\lambda)$.*

Proof. We show the first part of Lemma 2 by evaluating $U(\delta_1)$ for $U \in CR(\tau_\lambda)$; if λ_x, ρ_x are respectively the left and right regular representations of $G_{\mathbf{Q}}$ on H_λ , then both commute with $U \in CR(\tau_\lambda)$. Thus we have that

$$U(\delta_1)(z) = U(\lambda_x \rho_x \delta_1)(z) = \lambda_x \rho_x U \delta_1(z) = U(\delta_1(xzx^{-1})).$$

The second part of Lemma 2 follows when we note that τ'_λ is the strong operator closure of left translation operators on H_λ . So $U \in \tau'_\lambda$ if and only if $\lambda(g)U = U\lambda(g)$ for all $g \in G_{\mathbf{Q}}$. Therefore, convolution by $f \in H_\lambda$ satisfying the boundedness condition is in $CR(\tau_\lambda)$ if and only if it commutes with both left and right translation, and we have seen that this holds if f is constant on conjugacy classes. This completes Lemma 2.

LEMMA 3. *Let $I_\lambda^{(2)}$ be the inverse image in $G_{\mathbf{Q}}$ of the center of $G_{\mathbf{Q}} \setminus I_\lambda$. Then τ_λ is a factor representation if and only if for all $\gamma \in$*

$I_\lambda^{(2)} - I_\lambda$, the conjugacy class $C(\gamma) \subseteq G_Q$ is not contained in the set γK_λ .

Proof. Suppose $C(\gamma) \subseteq \gamma K_\lambda$. Then the function

$$\delta_\gamma(x) = \begin{cases} \lambda(z) & \text{if } x = z\gamma, z \in I_\lambda, \\ 0 & \text{otherwise} \end{cases}$$

is constant on conjugacy classes and is an element of H'_λ . Thus, by Lemma 2, it is in $CR(\tau_\lambda)$, but it is not a multiple of the identity, and so τ_λ is not a factor representation.

Conversely suppose $C(\gamma) \not\subseteq \gamma K_\lambda$, for all $\gamma \in I_\lambda^{(2)} - I_\lambda$. Then elements of H_λ cannot be constant on $C(\gamma)$. If $x \in G_Q - I_\lambda^{(2)}$, then $C(x)$ intersects infinitely many cosets of I_λ . To see this, note that G_Q acts on $G_Q \setminus I_\lambda$ by inner automorphisms; consider the orbit of xI_λ . Since $x \notin I_\lambda^{(2)}$, $yx y^{-1} \notin xI_\lambda$ for some $y \in G_Q$; therefore the isotropy group of x under this action is a proper subgroup of G_Q . Since G_Q contains no finite index subgroups, the orbit of xI_λ is infinite, and so the conjugacy class of x intersects infinitely many cosets of I_λ . Therefore, a function constant on $C(x)$ could not be in H_λ .

Thus, a function in H_λ constant on conjugacy classes could be nonzero only on I_λ , and so would have to be a multiple of $\delta_1 \in H_\lambda$. This completes the proof of Lemma 3.

The Pontryagin dual $\widehat{\mathfrak{g}}_Q$ of the discrete Lie algebra \mathfrak{g}_Q is isomorphic to $(\mathbf{A}/\mathbf{Q})^n$, where n is the dimension of \mathfrak{g}_Q as a vector space. The coadjoint orbits of G_Q in $\widehat{\mathfrak{g}}_Q$ are given by polynomials P_i , $i = 1, \dots, n$, where $P_i: G_Q \rightarrow \mathbf{A}/\mathbf{Q}$ are polynomials in \mathbf{Q}^n with coefficients in \mathbf{A}/\mathbf{Q} . In what follows, we rely heavily upon the technical results in [CP].

THEOREM 1. *Suppose $\tilde{\lambda} \in \widehat{\mathfrak{g}}_Q$. Then the closure of $\text{Ad}^*(G_Q)\tilde{\lambda}$ is “flat”; i.e., there exists a subgroup $V \cong \mathbf{A}/\mathbf{Q}^k$ of $\widehat{\mathfrak{g}}_Q$ such that the closure of the orbit is $\tilde{\lambda}V$.*

Define $P(\mathbf{Q}^k, \mathbf{A}/\mathbf{Q}^n)$ to be the space of n -tuples of polynomials in k rational variables, with coefficients in \mathbf{A}/\mathbf{Q} . We have that $\widehat{\mathbf{Q}}^k \cong (\mathbf{A}/\mathbf{Q})^k$. Conversely, the Pontryagin dual of $(\mathbf{A}/\mathbf{Q})^k$ is \mathbf{Q}^k .

LEMMA 4. *Suppose $\vec{p} = (p_1(x), \dots, p_n(x)) \in P(\mathbf{Q}, (\mathbf{A}/\mathbf{Q})^k)$, where the p_i have no constant terms. Then $\vec{p}(\mathbf{Q})$ is dense in some subgroup $V \subseteq (\mathbf{A}/\mathbf{Q})^n$, with $V \cong (\mathbf{A}/\mathbf{Q})^k$.*

Proof. First suppose that the p_i are linearly independent over \mathbf{Q} . Then the closure of the image of \mathbf{Q} under \vec{p} is $(\mathbf{A}/\mathbf{Q})^n$, since the conclusion of Theorem 2 in [CP] holds as well if the polynomial vector (p_1, \dots, p_n) consists of polynomials which are linearly independent over \mathbf{Q} , so that no \mathbf{Q} -linear combination of the p_i is 0.

Now suppose that $\vec{p} = (p_1, \dots, p_n)$ has \mathbf{Q} -linearly dependent entries. Then the map

$$T_{\vec{p}}: \mathbf{Q}^n \rightarrow P(\mathbf{Q}, \mathbf{A}/\mathbf{Q}),$$

$$\vec{q} \mapsto \vec{q} \cdot \vec{p}$$

has a nontrivial kernel K ; it is a vector subspace of \mathbf{Q}^n . Then $\mathbf{Q}^n \setminus K$ is the dual of a subgroup $K^\perp \subseteq (\mathbf{A}/\mathbf{Q})^n$ having annihilator K ; since K is \mathbf{Q}^s for some s , $K^\perp \cong (\mathbf{A}/\mathbf{Q})^{n-s}$. We see easily that $\vec{p}(\mathbf{Q}) \subseteq K^\perp$. Take a preimage for a basis of $\mathbf{Q}^n \setminus K$ in \mathbf{Q}^n , $\{\vec{q}_i\}_{i=1}^{n-s}$, and define the map

$$\pi: K^\perp \rightarrow (\mathbf{A}/\mathbf{Q})^{n-s},$$

$$\vec{\alpha} = (\alpha_1, \dots, \alpha_n) \mapsto (\vec{q}_1 \cdot \vec{\alpha}, \dots, \vec{q}_s \cdot \vec{\alpha}).$$

Now consider $\pi \circ \vec{p}: \mathbf{Q} \rightarrow (\mathbf{A}/\mathbf{Q})^{n-s}$. The components of $\pi \circ \vec{p}$ form a linearly independent set of polynomials in $(\mathbf{A}/\mathbf{Q})^{n-s}$ without constant term, so $\pi \circ \vec{p}(\mathbf{Q})$ is dense in $(\mathbf{A}/\mathbf{Q})^{n-s}$. We can show that in fact π is a homeomorphism of K^\perp onto $(\mathbf{A}/\mathbf{Q})^{n-s}$. It is injective, since if $\vec{q}_i \cdot \vec{\alpha} \equiv \mathbf{Q}$ for all $i = 1, \dots, n-s$, we have that $\vec{q} \cdot \vec{\alpha} \equiv \mathbf{Q}$ for all $\vec{q} \in \mathbf{Q}^n$, so that $\vec{\alpha} \equiv 0$ in $(\mathbf{A}/\mathbf{Q})^n$. It is clearly a continuous map, and is onto; the image of K^\perp contains a dense set in $(\mathbf{A}/\mathbf{Q})^{n-s}$ and is closed, since K^\perp is compact. Thus, the image of \mathbf{Q} under \vec{p} is dense in K^\perp .

This completes the proof of Lemma 4; we now consider the case where the polynomials under consideration are of several rational variables.

LEMMA 5. *Let $\{p_i\}_{i=1}^n \subseteq P(\mathbf{Q}^k, \mathbf{A}/\mathbf{Q})$ be a linearly independent set of polynomials in k rational variables without constant terms; then the image of \mathbf{Q}^k under $\vec{p} = (p_1, \dots, p_n)$ is dense in $(\mathbf{A}/\mathbf{Q})^n$.*

Proof. We will show that there is a map $\phi: \mathbf{Q} \rightarrow \mathbf{Q}^k$, $\phi(q) = (q^{N_1}, \dots, q^{N_k})$, such that the $p_i \circ \phi$ form a linearly independent set in $P(\mathbf{Q}, \mathbf{A}/\mathbf{Q})$. The result follows immediately.

\mathbf{N} will denote the natural numbers, including 0. We define an order on \mathbf{N}^k as follows: let $i \in 1, \dots, k$ be the greatest integer satisfying $m_i \neq m'_i$; then $(m_1, \dots, m_k) > (m'_1, \dots, m'_k)$ if $m_i > m'_i$.

We will define the degree of a monomial $x_1^{m_1} \cdots x_k^{m_k}$ to be (m_1, \dots, m_k) , and the degree of a polynomial to be the degree of its highest-valued monomial. Then we may assume without loss of generality, by interchanging p_i and p_j and by adding rational multiples of p_i to p_j , that the $\{p_i\}$ satisfy

1. $\deg p_i \geq \deg p_j$ if $i < j$;
2. if $\deg p_i = \deg p_{i+1} = \cdots = \deg p_j$, then the leading coefficients (i.e., the coefficients of the monomials of highest degree in each polynomial) form a linearly independent set of adeles over \mathbf{Q} .

A simple induction argument shows the following:

3. For each finite set $\{M_i\} \in \mathbf{N}^k$, there exists an element $N = (N_1, \dots, N_k) \in \mathbf{N}^k$ such that $N \cdot M_i > N \cdot M_j$ if $M_i > M_j$ in the order on \mathbf{N}^k ($M \cdot N$ denotes the usual dot product).

Now let $M_i = \deg p_i$, and let N be as in statement 3. For $x \in \mathbf{Q}$, let $\phi(x) = (x^{N_1}, \dots, x^{N_k})$. Then $\{p_i \circ \phi\}$ is a set of polynomials in one variable, with $\deg p_i \circ \phi > \deg p_j \circ \phi$ if and only if $\deg p_i > \deg p_j$; furthermore, the leading coefficients of the polynomials $p_i \circ \phi$ of a given degree form a linearly independent set over \mathbf{Q} in \mathbf{A}/\mathbf{Q} . Thus the polynomials $p_i \circ \phi$ form a linearly independent set in $P(\mathbf{Q}, \mathbf{A}/\mathbf{Q})$.

LEMMA 6. *Let $\vec{p} = (p_1, \dots, p_n) \in P(\mathbf{Q}^k, (\mathbf{A}/\mathbf{Q})^n)$, where the p_i are polynomials without constant term. Then the closure in $(\mathbf{A}/\mathbf{Q})^n$ of the image of \mathbf{Q}^n under this polynomial is a subgroup $V \cong (\mathbf{A}/\mathbf{Q})^s$ for some $s \leq k$. Furthermore, V is the subgroup of $(\mathbf{A}/\mathbf{Q})^k$ annihilated by $K = \ker T_{\vec{p}}$ where*

$$T_{\vec{p}}: \mathbf{Q}^n \rightarrow P(\mathbf{Q}^k, \mathbf{A}/\mathbf{Q}),$$

$$\vec{q} \mapsto \vec{q} \cdot \vec{p}.$$

Proof. Lemma 5 gives this result for the case when K is trivial. In general, we see easily that if $K = \ker T_{\vec{p}}$, then $\vec{p}(\mathbf{Q}^k) \subseteq K^\perp$. Define $\pi: K^\perp \rightarrow (\mathbf{A}/\mathbf{Q})^{n-s}$ as in the proof of Lemma 4; then the components of $\pi \circ \vec{p}$ form a linearly independent set of polynomials in $(\mathbf{A}/\mathbf{Q})^{n-s}$. We now use Lemma 5 to show that the image of $\pi \circ \vec{p}(\mathbf{Q}^k)$ is dense in $(\mathbf{A}/\mathbf{Q})^{n-s}$, and proceed as in the proof of Lemma 4 to show that $\vec{p}(\mathbf{Q}^k)$ is dense in K^\perp . This completes the proof of Lemma 6.

To complete the proof of Theorem 1, we take a strong Malcev basis $\{X_1, \dots, X_n\}$ of $\mathfrak{g}_{\mathbf{Q}}$ and write the elements of $\mathfrak{g}_{\mathbf{Q}}$ with respect to this basis as (x_1, \dots, x_n) . Recall that the base set of $G_{\mathbf{Q}}$ is $\mathfrak{g}_{\mathbf{Q}}$ and that the exponential map with respect to our realization of $G_{\mathbf{Q}}$ is the

identity on $\mathfrak{g}_{\mathbf{Q}}$. Writing the elements of $\widehat{\mathfrak{g}_{\mathbf{Q}}} \cong \mathbf{A}/\mathbf{Q}^n$ additively, we have that

$$\begin{aligned} \text{Ad}^*(x_1, \dots, x_n)\tilde{\lambda} &= \tilde{\lambda} + \sum_{k=1}^n \frac{\text{ad}^{*(k)}(x_1, \dots, x_n)\tilde{\lambda}}{k!} \\ &= \tilde{\lambda} + \sum_{k=1}^n \vec{p}_k(x_1, \dots, x_n) \end{aligned}$$

where the \vec{p}_k are polynomial vectors in $P(\mathbf{Q}^n, \mathbf{A}/\mathbf{Q}^n)$ without constant terms. Lemma 6 now shows that the closure of the orbit $\text{Ad}^*(G_{\mathbf{Q}})\tilde{\lambda}$, again written additively, is $\tilde{\lambda} + V$ for some subgroup $V \cong \mathbf{A}/\mathbf{Q}^s$.

This completes the proof of Theorem 1.

Let $I \subset \mathfrak{g}_{\mathbf{Q}}$ be an ideal of $\mathfrak{g}_{\mathbf{Q}}$. Let I^\perp denote that subgroup of $\widehat{\mathfrak{g}_{\mathbf{Q}}}$ which annihilates I . We will now prove

THEOREM 2. $\tau_\lambda = \text{Ind}(I_\lambda \uparrow G_{\mathbf{Q}}, \lambda)$ is a factor representation.

We first prove the following lemma.

LEMMA 7. The closure in $\widehat{\mathfrak{g}_{\mathbf{Q}}}$ of the coadjoint orbit of $\tilde{\lambda}$ is $\tilde{\lambda} + \mathfrak{i}_\lambda^\perp$.

Proof. From Theorem 1, we know that the closure of the orbit is a “flat”, $\tilde{\lambda} + W_0$. Suppose that $W \in \mathfrak{i}_\lambda$ and $Y \in \mathfrak{g}_{\mathbf{Q}}$. Then

$$\begin{aligned} \text{Ad}^*(\exp(-Y))\tilde{\lambda}(W) &= \tilde{\lambda}(\text{Ad}(\exp(Y)W)) \\ &= \tilde{\lambda}\left(W + [Y, W] + \frac{[Y, [Y, W]]}{2} + \dots\right). \end{aligned}$$

However, $[Y, W], [Y, [Y, W]],$ etc., are in $[\mathfrak{i}_\lambda, \mathfrak{g}_{\mathbf{Q}}]$, and so the above is just $\tilde{\lambda}(W)$. So $\text{Ad}^*(G_{\mathbf{Q}})\tilde{\lambda} \in \tilde{\lambda} + \mathfrak{i}_\lambda^\perp$.

Conversely, consider the group $V_0 \subseteq \widehat{\mathfrak{g}_{\mathbf{Q}}}$ generated by the elements $\text{Ad}^*(G_{\mathbf{Q}})\tilde{\lambda} - \tilde{\lambda}$. We have $V_0 \subseteq W_0$, since the coadjoint orbit closure is a flat. A typical element is

$$\text{ad}^*(Y)\tilde{\lambda} + \frac{\text{ad}^{*(2)}(Y)\tilde{\lambda}}{2} + \dots,$$

where the sum is finite. By taking linear combinations of these elements with $Y = X, 2X, \dots$ one can see that the individual terms are in V_0 . So for each $Y \in \mathfrak{g}_{\mathbf{Q}}$, $\text{ad}^*(rY)\tilde{\lambda} \in V_0$ for all $r \in \mathbf{Q}$. Note also that terms of the form $\text{Ad}^*(x)\text{Ad}^*(y)\tilde{\lambda} - \text{Ad}^*(y)\tilde{\lambda}$ are in V_0 , since $\text{Ad}^*(xy)\tilde{\lambda} - \tilde{\lambda}$ and $\text{Ad}^*(y)\tilde{\lambda} - \tilde{\lambda}$ are in V_0 .

The closure of V_0 is $V_0^{\perp\perp}$, so it suffices to show that $V_0^\perp = \mathfrak{i}_\lambda$. We have seen that $\mathfrak{i}_\lambda \subseteq V_0^\perp$. If $X \in V_0^\perp$, then $\text{ad}^*(Y)\tilde{\lambda}(X) = 1$ for all $Y \in \mathfrak{g}_\mathbb{Q}$, so $\tilde{\lambda}([X, Y]) = 1$, and so $X \in \mathfrak{r}_\lambda$. The same holds if we replace $\tilde{\lambda}$ with $\text{Ad}^*(y)\tilde{\lambda}$, for $y \in G_\mathbb{Q}$, so $X \in \mathfrak{r}_{\text{Ad}^*(y)\tilde{\lambda}} = \text{Ad}(y)\mathfrak{r}_\lambda$. That is, $X \in \bigcap_{y \in G_\mathbb{Q}} \text{Ad}(y)\mathfrak{r}_\lambda = \mathfrak{i}_\lambda$, the largest ideal of $\mathfrak{g}_\mathbb{Q}$ contained in \mathfrak{r}_λ . This completes the proof of Lemma 7.

We now prove Theorem 2. Let $\gamma = \exp(\Gamma)$ be in $I_\lambda^{(2)} - I_\lambda$. Then since $\text{Ad}^*(G_\mathbb{Q})\tilde{\lambda} - \tilde{\lambda} = \mathfrak{i}_\lambda^\perp$, the function

$$x \mapsto (\text{Ad}^*(x)\lambda - \lambda)(\Gamma)$$

is not identically 1 as a function of $x \in G_\mathbb{Q}$. We may write this as

$$\begin{aligned} & \text{Ad}^*(\exp(X))\tilde{\lambda} - \tilde{\lambda}(\Gamma) \\ &= \tilde{\lambda}([X, \Gamma] + [X, [X, \Gamma]] + \dots + [X, \dots [X, \Gamma], \dots]). \end{aligned}$$

However, since $[X, \Gamma] \in \mathfrak{i}_\lambda^{(2)}$, all the brackets after the first are in the kernel of λ ; therefore

$$\text{Ad}^*(x)\tilde{\lambda}(\Gamma) = \lambda([X, \Gamma]).$$

This shows that $\text{ad}(\mathfrak{g}_\mathbb{Q})\Gamma \not\subseteq \mathfrak{k}_\lambda$. Therefore the conjugacy class of γ is not contained in γK_λ . By Lemma 3, τ_λ must be a factor. This proves Theorem 2.

Note that for any element χ in the closure of $\text{Ad}^*(G_\mathbb{Q})\tilde{\lambda}$, the restriction of χ to \mathfrak{i}_λ is λ . Thus we see that the factor representation τ_λ is canonically associated with the closure of the orbit $\text{Ad}^*(G_\mathbb{Q})\tilde{\lambda}$ for each $\tilde{\lambda} \in \widehat{\mathfrak{g}_\mathbb{Q}}$.

3. The trace on the factor τ_λ and the Plancherel formula. In this section, we demonstrate that the factor representation τ_λ of the previous section is traceable, and develop an orbital trace formula and Plancherel formula.

Let $\tilde{\lambda} \in \widehat{\mathfrak{g}_\mathbb{Q}}$; there is then an ideal \mathfrak{i}_λ of $\mathfrak{g}_\mathbb{Q}$ such that the closure in $(\mathbf{A}/\mathbf{Q})^n$ of $\text{Ad}^*(G_\mathbb{Q})\tilde{\lambda}$ is $\tilde{\lambda} + \mathfrak{i}_\lambda^\perp$. Note that on \mathfrak{i}_λ , $\text{Ad}^*(x)\tilde{\lambda} = \tilde{\lambda}$ for all $x \in G_\mathbb{Q}$.

Let H_λ be the Mackey space for τ_λ as defined in the previous section. We define $\overline{G}_\mathbb{Q} = G_\mathbb{Q} \backslash I_\lambda$. Let $\alpha: \overline{G}_\mathbb{Q} \rightarrow G_\mathbb{Q}$ give a cross section of the cosets of $\overline{G}_\mathbb{Q}$ satisfying $\alpha(\overline{e}) = e$ (where e denotes the identity element of $G_\mathbb{Q}$) and $\alpha(\overline{x}^{-1}) = \alpha(\overline{x})^{-1}$. Then a basis for H_λ is given by the functions $\delta_{\alpha(\overline{x})}$, where

$$\delta_{\alpha(\overline{x})}(y) = \begin{cases} \tilde{\lambda}(z), & \text{if } y = \alpha(\overline{x})z, \quad z \in I_\lambda, \\ 0, & \text{if } y \notin \alpha(\overline{x}). \end{cases}$$

Let $J_{\bar{x}}: \mathbf{C} \rightarrow H_\lambda$ map 1 to $\delta_{\alpha(\bar{x})}$. Then $J_{\bar{x}}^*(f) = f(\alpha(\bar{x}))$ for $f \in H_\lambda$. So if $A \in L(H_\lambda)$, the space of bounded linear operators on H_λ , and we write $A_{\bar{x}, \bar{y}} = J_{\bar{x}}^* A J_{\bar{y}}$, then we have

$$\begin{aligned} A\delta_{\alpha(\bar{y})}(\alpha(\bar{x})) &= (A\delta_{\alpha(\bar{y})}, \delta_{\alpha(\bar{x})}) = (AJ_{\bar{y}}(1), J_{\bar{x}}(1)) \\ &= (J_{\bar{x}}^* A J_{\bar{y}}(1), 1) = A_{\bar{x}, \bar{y}} \in \mathbf{C}. \end{aligned}$$

We define $c: \overline{G_{\mathbf{Q}}} \times G_{\mathbf{Q}} \rightarrow I_\lambda$ by

$$\alpha(\bar{x})y = c(\bar{x}, y)\alpha(\bar{x}y).$$

We have the cocycle identity

$$c(\bar{x}, u)c(\bar{x}u, v) = c(\bar{x}, uv).$$

Note that if $u \in I_\lambda$, then $\bar{x} = \bar{x}u$; therefore $c(\bar{x}, u) = \alpha(\bar{x})u\alpha(\bar{x})^{-1}$, and we have

$$\alpha(\bar{x})u\alpha(\bar{x})^{-1}c(\bar{x}, v) = c(\bar{x}, uv)$$

for $u \in I_\lambda$. We define $\xi(\bar{x}, y) = \lambda(c(\bar{x}, y))$; we then have, for $u \in I_\lambda$,

$$\xi(\bar{x}, u) = \lambda(\alpha(\bar{x})u\alpha(\bar{x})^{-1}) = \text{Ad}^*(\alpha(\bar{x}))(u) = \lambda(u),$$

since $\text{Ad}^*(G_{\mathbf{Q}})\lambda \subseteq \lambda + \mathfrak{i}_\lambda^\perp$. It follows that

$$\xi(\bar{x}, v)\lambda(u) = \xi(\bar{x}, uv) = \xi(\bar{x}, vu)$$

if $u \in I_\lambda$.

We now compute $\tau_\lambda(g)_{\bar{x}, \bar{y}}$ for $g \in G_{\mathbf{Q}}$. Since $\tau_\lambda(ug) = \lambda(u)\tau_\lambda(g)$ if $u \in I_\lambda$, it suffices to consider the case where $g = \alpha(\bar{g})$. We then have

$$\tau_\lambda(g)_{\bar{x}, \bar{y}} = \tau_\lambda(g)\delta_{\alpha(\bar{y})}(\alpha(\bar{x})) = \delta_{\alpha(\bar{y})}(\alpha(\bar{x}\bar{g})) = \begin{cases} 0 & \text{if } \bar{y} \neq \bar{x}\bar{g}, \\ \xi(\bar{x}, g) & \text{if } \bar{y} = \bar{x}\bar{g}, \end{cases}$$

because if $\bar{y} = \bar{x}\bar{g}$, then $\alpha(\bar{x})g = c(\bar{x}, g)\alpha(\bar{x}\bar{g}) = c(\bar{x}, g)\alpha(\bar{y})$, and $\delta_{\alpha(\bar{y})}c(\bar{x}, g)\alpha(\bar{y}) = \lambda(c(\bar{x}, g)) = \xi(\bar{x}, g)$.

Since we have chosen $g = \alpha(\bar{g})$, it follows that

$$\tau_\lambda(g)_{\bar{x}, \bar{y}} = \begin{cases} \xi(\bar{x}, \alpha(\overline{x^{-1}y})) & \text{if } \bar{g} = \overline{x^{-1}y}, \\ 0 & \text{otherwise.} \end{cases}$$

This shows that $\tau_\lambda(g)_{\bar{e}, \bar{g}} = \xi(\bar{e}, \alpha(\bar{g}))$. However, $\alpha(\bar{e})\alpha(\bar{g}) = \alpha(\bar{g})$, so that $c(\bar{e}, \alpha(\bar{g})) = e$ and $\xi(\bar{e}, \alpha(\bar{g})) = 1$. We thus have

$$(1) \quad \tau_\lambda(g)_{\bar{x}, \bar{y}} = \xi(\bar{x}, \alpha(\overline{x^{-1}y}))\tau_\lambda(g)_{\bar{e}, \overline{x^{-1}y}}$$

if $g = \alpha(\bar{g})$.

This also holds, however, if $g = u\alpha(\bar{g})$, $u \in I_\lambda$; both sides are multiplied by a factor of $\lambda(u)$. It holds also if $x^{-1}y \neq \bar{g}$, since then both sides are 0. Therefore, (1) holds for all $\bar{x}, \bar{y} \in \overline{G_Q}$. Since (1) depends linearly upon $\tau_\lambda(g)$, we have also

$$(2) \quad \tau_\lambda(f)_{\bar{x}, \bar{y}} = \xi(\bar{x}, \alpha(\overline{x^{-1}y}))\tau_\lambda(f)_{\bar{e}, x^{-1}y}$$

for all $f \in C_c(G_Q)$. This relation is preserved under the taking of weak limits in $\tau_\lambda(f)$, so that (2) holds if $\tau_\lambda(f)$ is replaced by an element of the Von Neumann algebra generated by the $\tau_\lambda(f)$.

We now follow Dixmier ([Dx], I.4.2 and I.9.2). Let \mathcal{A} be the Von Neumann algebra generated by the $\tau_\lambda(f)$, for $f \in C_c(G_Q)$. We define $\phi: \mathcal{A} \rightarrow \mathbf{C}$ by $\phi(A) = A_{\bar{e}, \bar{e}}$. Then the following hold:

1. ϕ is linear.
2. $\phi(A^*A) = \phi(AA^*) \geq 0$, with equality only if $A = 0$.

Proof of Claim 2. We have $A\delta_{\alpha(\bar{e})} = \sum_{\bar{x} \in \overline{G_Q}} A_{\bar{x}, \bar{e}}\delta_{\alpha(\bar{x})}$, so that

$$(A^*A)_{\bar{e}, \bar{e}} = (A\delta_{\alpha(\bar{e})}, A\delta_{\alpha(\bar{e})}) = \sum_{\bar{x} \in \overline{G_Q}} |A_{\bar{x}, \bar{e}}|^2.$$

Similarly,

$$(AA^*)_{\bar{e}, \bar{e}} = \sum_{\bar{x} \in \overline{G_Q}} |A_{\bar{x}, \bar{e}}^*|^2.$$

To see that these sums are equal, we note that $A_{\bar{x}, \bar{y}}^* = \overline{A_{\bar{y}, \bar{x}}}$, so that in particular $A_{\bar{x}, \bar{e}}^* = \overline{A_{\bar{e}, \bar{x}}}$; also, from (2), $A_{\bar{x}, \bar{e}} = \xi(\bar{x}, \alpha(\overline{x^{-1}}))A_{\bar{e}, \bar{x}^{-1}}$. Therefore, $|A_{\bar{x}, \bar{e}}^*| = |A_{\bar{e}, \bar{x}}|$ and $|A_{\bar{x}, \bar{e}}| = |A_{\bar{e}, \bar{x}^{-1}}|$. Thus the sums over $\overline{G_Q}$ which determine $\phi(A^*A)$ and $\phi(AA^*)$ are the same.

For the second assertion, suppose $\phi(AA^*) = 0$. Then $A_{\bar{x}, \bar{e}} = 0$ for all $\bar{x} \in \overline{G_Q}$. By (2), we see that then $A_{\bar{x}, \bar{y}} = 0$ for all $\bar{x}, \bar{y} \in \overline{G_Q}$. Hence $A = 0$.

3. ϕ is normal, since $\phi(A) = (A\delta_{\alpha(\bar{e})}, \delta_{\alpha(\bar{e})})$; see ([Dx], Theorem 1, I.4.2). Therefore, ϕ is (up to positive multiples) the unique trace on the factor \mathcal{A} .

To compute ϕ on $L^1(G_Q)$, we compute it on $C_c(G_Q)$ and then use the L^1 norm-continuity of ϕ to extend to $L^1(G_Q)$. Since $\tau_\lambda(C_c(G_Q))$ consists of finite linear combinations of elements of $\tau_\lambda(G_Q)$, we compute ϕ on $\tau_\lambda(G_Q)$. We see that if $g \notin I_\lambda$, then $\phi(\tau_\lambda(g)) = 0$, and if $g \in I_\lambda$, then $\phi(\tau_\lambda(g)) = \lambda(g)$. Therefore we have

$$(3) \quad \phi(f) = \sum_{u \in I_\lambda} \lambda(u)f(u).$$

By linearity and continuity, (3) holds also for $f \in L^1(G_{\mathbf{Q}})$.

Now suppose that $F \in L^1(\mathfrak{g}_{\mathbf{Q}})$, and that $f = F \circ \log \in L^1(G_{\mathbf{Q}})$ (recall that, as we have defined it, log on the set level is just the identity). Let $\tilde{F}(X) = \tilde{\lambda}(X)F(X)$. Then for $\chi \in \mathfrak{i}_{\lambda}^{\perp}$,

$$\tilde{F}^{\wedge}(\chi) = \sum_{X \in \mathfrak{g}_{\mathbf{Q}}} \tilde{F}(X)\chi(X) = \sum_{X \in \mathfrak{g}_{\mathbf{Q}}} F(X)(\chi + \tilde{\lambda})(X) = F^{\wedge}(\chi + \tilde{\lambda}).$$

Therefore,

$$\begin{aligned} \phi(f) &= \sum_{u \in I_{\lambda}} \lambda(u)f(u) = \sum_{Y \in \mathfrak{i}_{\lambda}} \lambda(Y)F(Y) = \sum_{Y \in \mathfrak{i}_{\lambda}} \tilde{F}(Y) \\ &= \int_{\mathfrak{i}_{\lambda}^{\perp}} \tilde{F}^{\wedge}(\chi) d\chi = \int_{\mathfrak{i}_{\lambda}^{\perp}} F^{\wedge}(\chi + \tilde{\lambda}) d\chi = \int_{\overline{\mathcal{O}}_{\lambda}} F^{\wedge}(\chi) d\chi. \end{aligned}$$

The last set of equations follows from Poisson summation on \mathfrak{i}_{λ} . We note that the measure on the closure of \mathcal{O}_{λ} is the lift of normalized Haar measure on $\mathfrak{i}_{\lambda}^{\perp}$. This gives the orbital trace formula for τ_{λ} .

THEOREM 3. *Let ϕ be the unique trace on the factor \mathcal{A} . Then for $f \in L^1(G_{\mathbf{Q}})$, $\phi(f)$ is given by the orbital trace formula*

$$\phi(f) = \int_{\overline{\mathcal{O}}_{\lambda}} F^{\wedge}(\chi) d\chi,$$

where $F = f \circ \exp \in L^1(\mathfrak{g}_{\mathbf{Q}})$.

We finish by developing a Plancherel formula for $G_{\mathbf{Q}}$. For $\tilde{\lambda} \in \widehat{\mathfrak{g}_{\mathbf{Q}}}$, we call the closure of the $\text{Ad}^*(G_{\mathbf{Q}})$ -orbit $\overline{\mathcal{O}}_{\lambda}$ of $\tilde{\lambda}$ generic if $\overline{\mathcal{O}}_{\lambda} = \tilde{\lambda} + \mathfrak{z}(\mathfrak{g}_{\mathbf{Q}})^{\perp}$, or equivalently if $\mathfrak{i}_{\lambda} = \mathfrak{z}(\mathfrak{g}_{\mathbf{Q}})$. We will refer to $\tilde{\lambda}$ as generic if the closure of its $G_{\mathbf{Q}}$ -orbit is generic.

We define the rising central series of ideals of $\mathfrak{g}_{\mathbf{Q}}$ as follows: let $\mathfrak{g}_0 = (\mathbf{0})$, and let π_0 be the identity map on $\mathfrak{g}_{\mathbf{Q}}$. We then let $\pi_k: \mathfrak{g}_{\mathbf{Q}} \rightarrow \mathfrak{g}_{\mathbf{Q}} \setminus \mathfrak{g}_{k-1}$ be the natural map, and let \mathfrak{g}_k be the inverse image in $\mathfrak{g}_{\mathbf{Q}}$ of the center of $\mathfrak{g}_{\mathbf{Q}} \setminus \mathfrak{g}_{k-1}$. Then since $\mathfrak{g}_{\mathbf{Q}}$ is nilpotent, the rising central series terminates at some $\mathfrak{g}_n = \mathfrak{g}_{\mathbf{Q}}$.

LEMMA 8. *Suppose $\tilde{\lambda} \in \widehat{\mathfrak{g}_{\mathbf{Q}}}$. Then $\tilde{\lambda}$ is generic if and only if there does not exist $X \in \mathfrak{g}_2$ such that $\text{ad}(X)\mathfrak{g}_{\mathbf{Q}} \subseteq \ker(\tilde{\lambda})$.*

Proof. If \mathfrak{i}_{λ} properly contains $\mathfrak{z}(\mathfrak{g}_{\mathbf{Q}})$, then take $X \in \mathfrak{i}_{\lambda} \cap \mathfrak{g}_2$ such that $X \notin \mathfrak{z}(\mathfrak{g}_{\mathbf{Q}})$. Then $\tilde{\lambda}([X, \mathfrak{g}_{\mathbf{Q}}]) = 1$ by definition of \mathfrak{i}_{λ} , so $\text{ad}(X)\mathfrak{g}_{\mathbf{Q}} \subseteq \ker(\tilde{\lambda})$. Conversely, suppose there is an $X \in \mathfrak{g}_2$ such that $\text{ad}(X)\mathfrak{g}_{\mathbf{Q}} \subseteq$

$\ker(\tilde{\lambda})$. Then $\mathbf{Q}X \oplus \mathbf{z}(\mathbf{g}_{\mathbf{Q}}) \subseteq \mathfrak{r}_{\lambda}$ is an ideal of $\mathbf{g}_{\mathbf{Q}}$; hence it is in \mathfrak{i}_{λ} , and \mathfrak{i}_{λ} properly contains $\mathbf{z}(\mathbf{g}_{\mathbf{Q}})$. Thus $\tilde{\lambda}$ is not generic.

Note that the second condition of Lemma 7 implies that whether $\tilde{\lambda}$ is generic depends upon its restriction to $\mathbf{z}(\mathbf{g}_{\mathbf{Q}})$; thus we will (by abuse of notation) refer to $\lambda \in \hat{\mathbf{z}}(\mathbf{g}_{\mathbf{Q}})$ as generic if any (equivalently all) of its extensions to $\mathbf{g}_{\mathbf{Q}}$ are generic. We will show that for a set of $\lambda \in \hat{\mathbf{z}}(\mathbf{g}_{\mathbf{Q}})$ of full Haar measure, the second condition of Lemma 7 holds; in fact, we will show something stronger than that.

LEMMA 9. *The $\lambda \in \widehat{\mathbf{z}(\mathbf{g}_{\mathbf{Q}})}$ with $\ker \lambda = 0$ are of full Haar measure in $\widehat{\mathbf{z}(\mathbf{g}_{\mathbf{Q}})}$.*

Proof. Suppose $\mathbf{z}(\mathbf{g}_{\mathbf{Q}}) \cong \mathbf{Q}^k$, and let $\{Z_1, \dots, Z_k\}$ be a basis over \mathbf{Q} for $\mathbf{z}(\mathbf{g}_{\mathbf{Q}})$. If $\lambda \in \widehat{\mathbf{z}(\mathbf{g}_{\mathbf{Q}})}$, then we may identify λ with an element $\vec{\alpha} \in (\mathbf{A}/\mathbf{Q})^k$ by

$$\lambda(q_1 Z_1 + \dots + q_k Z_k) = \chi(\alpha_1 q_1 + \dots + \alpha_k q_k),$$

where $\vec{\alpha} = (\alpha_1, \dots, \alpha_k)$ and χ is the standard character on \mathbf{A}/\mathbf{Q} . This gives an isomorphism of $\widehat{\mathbf{z}(\mathbf{g}_{\mathbf{Q}})}$ with $(\mathbf{A}/\mathbf{Q})^k$. We define, for $\vec{\alpha} \in (\mathbf{A}/\mathbf{Q})^k$ and $\vec{q} \in \mathbf{Q}^k$,

$$\vec{\alpha} \cdot \vec{q} = \alpha_1 q_1 + \dots + \alpha_k q_k \in \mathbf{A}/\mathbf{Q}.$$

For an element $\vec{q} \in \mathbf{Q}^k$, we define the linear map

$$\begin{aligned} T_{\vec{q}}: (\mathbf{A}/\mathbf{Q})^k &\rightarrow \mathbf{A}/\mathbf{Q}, \\ \vec{\alpha} &\mapsto \vec{\alpha} \cdot \vec{q}. \end{aligned}$$

The set of λ with nontrivial kernel corresponds to the union of the kernels of $T_{\vec{q}}$, $\vec{q} \neq 0$. Since \mathbf{Q}^k is countable, we need only show that each kernel is of Haar measure 0 in $(\mathbf{A}/\mathbf{Q})^k$.

Suppose now that $\vec{\alpha} \in \ker T_{\vec{q}}$. Then we have $\vec{\alpha} \cdot \vec{q} \equiv 0$ in \mathbf{A}/\mathbf{Q} , and therefore we may take representatives α_i for each coordinate of $\vec{\alpha}$ in the adeles, so that

$$\alpha_1 q_1 + \dots + \alpha_k q_k = 0.$$

This means that in the \mathbf{Q}_p th coordinate,

$$(\alpha_1 q_1 + \dots + \alpha_k q_k)_p = 0.$$

This condition is satisfied by elements $\vec{\alpha} \in (\mathbf{A}/\mathbf{Q})^k$ such that each $(\vec{\alpha})_p = ((\alpha_1)_p, \dots, (\alpha_k)_p)$ lies in the hyperplane

$$x_1 q_1 + \dots + x_k q_k = 0,$$

where $(x_1, \dots, x_k) \in (\mathbf{Q}_p)^k$. Therefore, if $\vec{q} \neq \vec{0}$, the set of choices in each \mathbf{Q}_p^n has Haar measure zero, and so the preimage of $\ker T_{\vec{q}}$ in the adèles has measure zero. Therefore the Haar measure of $\ker T_{\vec{q}}$ is zero. This completes the proof of Lemma 8.

Now the Plancherel formula is nearly trivial. If $F \in C_c(\mathfrak{g}_{\mathbf{Q}})$, then $f = F \circ \log \in C_c(G_{\mathbf{Q}})$. Let $\{Z_1, \dots, Z_k, X_{k+1}, \dots, X_n\}$ be an extension of the basis given above to a basis of $\mathfrak{g}_{\mathbf{Q}}$. We then have an inclusion $\mathfrak{z}(\widehat{\mathfrak{g}_{\mathbf{Q}}}) \rightarrow \widehat{\mathfrak{g}_{\mathbf{Q}}}$ as follows; $\lambda \mapsto \tilde{\lambda}$ if

$$\tilde{\lambda}(q_1 Z_1 + \dots + q_k Z_k + q_{k+1} X_{k+1} + \dots + q_n Z_n) = \lambda(q_1 Z_1 + \dots + q_k Z_k)$$

for all $\vec{q} \in \mathbf{Q}^k$. Then

$$f(0) = F(0) = \int_{\widehat{\mathfrak{g}_{\mathbf{Q}}}} F^\wedge(\xi) d\xi = \int_{\mathfrak{z}(\widehat{\mathfrak{g}_{\mathbf{Q}}})} \int_{\mathfrak{z}(\mathfrak{g}_{\mathbf{Q}})^\perp} F^\wedge(\lambda + \chi) d\chi d\lambda,$$

where all Haar measures are normalized so that their (compact) supports have measure 1. We let $d_\lambda(\chi)$ be the lift of normalized Haar measure on $\mathfrak{z}(\mathfrak{g}_{\mathbf{Q}})^\perp$ to $\lambda + \mathfrak{z}(\mathfrak{g}_{\mathbf{Q}})^\perp$; if λ is generic, this is $\overline{\mathcal{O}}_\lambda$. For a.e. $\lambda \in \mathfrak{z}(\widehat{\mathfrak{g}_{\mathbf{Q}}})$, $d_\lambda(\chi)$ is the orbital measure on $\overline{\mathcal{O}}_\lambda$ in the trace formula, so the above integral becomes

$$\begin{aligned} & \int_{\mathfrak{z}(\widehat{\mathfrak{g}_{\mathbf{Q}}})} \left\{ \int_{\mathfrak{z}(\mathfrak{g}_{\mathbf{Q}})^\perp} F^\wedge(\lambda + \chi) d\chi \right\} d\lambda \\ &= \int_{\mathfrak{z}(\widehat{\mathfrak{g}_{\mathbf{Q}}})} \left\{ \int_{\overline{\mathcal{O}}_\lambda} F^\wedge(\chi) d_\lambda(\chi) \right\} d\lambda = \int_{\mathfrak{z}(\widehat{\mathfrak{g}_{\mathbf{Q}}})} \phi_\lambda(f) d\lambda. \end{aligned}$$

Thus we see that Plancherel measure for $G_{\mathbf{Q}}$ is normalized Haar measure on $\mathfrak{z}(\widehat{\mathfrak{g}_{\mathbf{Q}}})$. We have proved

THEOREM 4 (Plancherel Formula). *Suppose $f \in C_c(\mathfrak{g}_{\mathbf{Q}})$, and that for generic $\lambda \in \mathfrak{z}(\widehat{\mathfrak{g}_{\mathbf{Q}}})$, ϕ_λ is the trace associated with the factor representation τ_λ induced from the character $\lambda \in \mathfrak{z}(\widehat{\mathfrak{g}_{\mathbf{Q}}})$. Then $\lambda \mapsto \phi_\lambda(f)$ is defined for a.a. λ , and is integrable on $\mathfrak{z}(\widehat{\mathfrak{g}_{\mathbf{Q}}})$, and we have*

$$f(0) = \int_{\mathfrak{z}(\widehat{\mathfrak{g}_{\mathbf{Q}}})} \phi_\lambda(f) d\mu(\lambda),$$

where μ is Haar measure on $\mathfrak{z}(\widehat{\mathfrak{g}_{\mathbf{Q}}})$, normalized so that $\mu(\mathfrak{z}(\mathfrak{g}_{\mathbf{Q}})) = 1$.

Editor’s note. The mathematical community was deeply saddened by the death of Larry Corwin on March 19, 1992. Larry had made

fundamental contributions to the harmonic analysis of p -adic reductive groups, nilpotent Lie groups and, as this paper illustrates, discrete groups. His friends also recall him as an effective administrator and innovative teacher, and remember his incredible wit, his playful and spontaneous sense of humor, and his remarkable energy and zest for life.

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