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PRODUCTIVE POLYNOMIALS

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The problem addressed is: When is a class B of polynomials in n non-commuting indeterminates closed under substitution into a given polynomial q ?

1. Introduction. Let $\mathbb{F}$ be a field and let $\mathbb{F}\langle x_1, \dots, x_n\rangle$ be the linear algebra of polynomials in the non-commuting indeterminates $x_1, \dots, x_n$. Let $q \in \mathbb{F}\langle x_1, \dots, x_n\rangle$. Let A be an associative[1] algebra over $\mathbb{F}$. q defines a mapping $\hat{q}$ of $A \times \cdots \times A = A^n$ into A whose value $\hat{q}(a_1, \dots, a_n)$ at $(a_1, \dots, a_n)$ is the result of replacing each x_i in q by the corresponding a_i, and then carrying out the algebraic operations proper to A. A linear subspace B of the algebra A will be called q-closed if whenever $\mathbf{A} = (a_1, \dots, a_n) \in A^n$ then $\hat{q}(\mathbf{a}) \in B$. Let $q((B))$ be the smallest q-closed linear subspace containing B. We study mainly the case that A is $\mathbb{F}\langle x_1, \dots, x_n\rangle$ itself, and B is the linear subspace generated by $x_1, \dots, x_n$ and the unit 1. The q-closed set generated by $x_1, \dots, x_n$ and 1 will be denoted in this case simply by $((q))$.

We will usually use just P to stand for $\mathbb{F}\langle x_1, \dots, x_n\rangle$. $q \in P$ will be called *productive* if $((q)) = P$; and otherwise, *non-productive.*

Two questions interest us:

1.1. When is a given $q \in P$ productive,
and

1.2. If it is not, how to find elements p which are not in $((q))$?

A clear-cut answer to 1.1 is given by 3.9. An answer to 1.2 is given in §4, illustrated by an example 8.5. We regard q as an n-ary operation and prepare a suitable ideal theory.

2. Theorems establishing productivity. Consider $q = x_1x_2$. Then a linear subspace B is q-closed if it contains the product of any pair of members: B is a subalgebra.[2] Thus, if B is the linear subspace generated by $x_1, \dots, x_n$ and 1, then $((q))$ is the algebra generated by $x_1, \dots, x_n$ and 1. This being P, x_1x_2 is productive.

[1]In this paper, all algebras are supposed to be associative, and so we omit the term.

[2]Note that B is x_1x_2-closed if and only if it is x_ix_j-closed, where i, j are any two distinct indices.

We will usually write x, y for x_1, x_2. The main example of a productive element is xy.[3] As observed, $((xy))$ is the subalgebra generated by $x_1, \dots, x_n$ and the unit 1, and this is patently P.

For finding sufficient conditions for elements to be productive, we are aided by the following concept. Let $E = (e(1), \dots, e(n))$ be n non-negative integers. Let $p \in P$. Then p is *homogeneous of type* E if it is homogeneous of degree $e(i)$ in x_i, for each i.

2.1. DEFINITION. Given $q \in P$, its *homogeneous constituent* q_E shall be the sum of its monomial terms homogeneous of type E.

Obviously, q is the sum of its homogeneous constituents.

2.2. LEMMA. *If the field* $\mathbb{F}$ *is infinite, then* $q_E \in ((q))$ *for each homogeneity type* E.

Proof. Let $\lambda = (\lambda_1, \dots, \lambda_n)$, and define $q\lambda$ as $\hat{q}(\lambda_1 x_1, \dots, \lambda_n x_n)$. We have $q\lambda = \sum q_E \lambda = \sum \lambda^E q_E$. By making enough different choices for λ, we get a system of linear equations for the various q_E. These can then be expressed in terms of the left-hand sides. The left-hand sides belongs to $((q))$ no matter how the λ are selected. Therefore, so do the q_E.

2.3. THEOREM. *Let* $q \in P$. *If* $p \in ((q))$ *then* $((q))$ *is p-closed and* $((p))$ *is contained in* $((q))$.

This can be deduced from the following representation of $q((C_1))$ as the union of sets C_m where C_1 is a subset of P. We will define C_m for $m = 2, 3, \dots$. Suppose now that C_m has been defined. Then C_{m+1} shall consist of the elements of C_m, their linear combinations, and the values of $q(\mathbf{u})$ where $\mathbf{u}$ varies over the n-member subsets of C_m. Let C^∞ be the union of the sets C_m. Using induction, we see that each C_m lies in $q((C_1))$, so C^∞ is contained in $q((C_1))$. Since C^∞ is obviously q-closed, it must include $q((C_1))$. Thus

$$q((C_1)) = C^\infty. \tag{2.31}$$

Such a sequence of sets $\{C_m\}$ may be called a *q-system.* There is a q-system Q with $C_1 = B$, the span of 1 and $x_1, \dots, x_n$. Now suppose p lies in the set C_m and let $r_1, \dots, r_n$ lie in C_k. We want to show that $\hat{p}(r_1, \dots, r_n)$ lies in $((q))$.

In the system Q replace each x_i by r_i, giving Q'. This is a q-system $\{C'_m\}$ and it has $\hat{p}(r_1, \dots, r_n)$ in the mth set. C'_1 is

[3] See also 2.4.

contained in C_k. Consequently C'_m is contained in C_{k+m-1}, so $\hat{p}(r_1, \ldots, r_n)$ lies in $((q))$. This shows that $((q))$ is p-closed, whence the p-closure of B, namely $((p))$ is contained in $((q))$.

Thus 2.3 is proved. We mention an immediate generalization.

COROLLARY. *Let $q \in P$. Let C be a subset of P. Let $p \in q((C))$. Then $p((C))$ is contained in $q((C))$.*

2.4. COROLLARY. *Let $q \in P$. Then $P = ((q))$ if and only if $xy \in ((q))$.*

This follows from 2.3 with $p = xy$.

2.5. COROLLARY. *Let $q \in P$. If any q_E is productive*[4] *then q is productive.*

Proof. $((q_E))$ is contained in $((q))$, so if the former contains xy, so does the latter.

The next proposition can be as easily proved as was 2.4.

2.6. PROPOSITION. *Let $q \in P$. Select an index i. Define $q_i(x_1, \ldots, x_n)$ as the coefficient of t in $q(x_1, \ldots, x_i + t1, \ldots, x_n)$ where t ranges over $\mathbb{F}$ and 1 is the unit of $\mathbb{F}\langle x_1, \ldots, x_n \rangle$. Then q_i belongs to $((q))$.*

This can be used to show that $q = x^2y + xyx + xy^2$ is productive. First we get $x^2y + xyx \in ((q))$ by 2.2. Then 2.6 tells us that $2xy + yx + xy \in ((q))$, whence $3xy + yx \in ((q))$, and by substitution of x for y and y for x, $3yx + xy \in ((q))$. Linear combination gives us xy.

COROLLARY. *If f and g have positive degree, then $f(x)g(y)$ is productive.*

Proof. By differentiating an appropriate number[5] of times and applying 2.6, we can obtain $xy \in ((f(x)g(y)))$. By 2.3, $((f(x)g(y)))$ includes $((xy))$, $= \mathbb{F}\langle x_1, \ldots, x_n \rangle$.

The connection with Jordan algebras may be noted. (See **[BK]**.)

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[4]However, if all the q_E are non-productive, the sum k may still be productive. See 4.5.

[5]Using 2.3 after each differentiation.

REMARK. If q is a polynomial in one variable and has degree at least 2, then $((q)) = ((x^2))$, i.e., it is the special Jordan algebra $\mathbb{F}\langle x_1, \dots, x_n\rangle^+$.

Proof. Let q be a polynomial in x. Replace x by $1 + x$ and expand. One sees from 2.2 that $x^2 \in ((q))$. On the other hand, replace x by x^2 and see that $x^4, x^8, x^{16}, \dots \in ((x^2))$. Keep this up until the exponent exceeds the degree of q. Replacing x by $1+x$ and using 2.2, one sees that all the powers in q are in $((x^2))$. Forming linear combinations, one obtains $q \in ((x^2))$. Thus, by 2.3 applied twice, $((q)) = ((x^2))$.

3. Characterizing productivity. For $k = 1, 2, \dots$ let P_k stand for the set of homogeneous polynomials of degree k. For $p \in P$ let $p_{,k}$ be the $P_0 + \cdots + P_k$ component of p. We will use Quad(p) to denote the P_2 component of p.

We are going to be dealing with n-tuples $(p_1, \dots, p_n)$ of elements of P. We will abbreviate such an n-tuple by $\mathbf{p}$. We will write $\mathbf{p}_{,k}$ for $(p_{1,k}, \dots, p_{n,k})$.

We will also make use of the linear automorphism $p \to p^*$ of P defined inductively by $1^* = 1$, $(px_i)^* = x_i p^*$. An element u for which $u^* = u$ will be called symmetric, and one for which $u^* = -u$, skew.

We will let S_+ denote the linear subspace of P of those elements p for which Quad(p) is symmetric, and S_- for those for which it is skew. In fact p is in S_+ or S_- if and only if $p_{,2}$ is symmetric or skew, respectively.

PROPOSITION. *Let $q \in P$. Then one of the three sentences* 3.1, 3.2, *or* 3.3 *must be true.*

3.1. *If $f_1, \dots, f_n$ belong to $P_0 + P_1$, then* Quad $q(\mathbf{f})$ *is symmetric.*

3.2. *If $f_1, \dots, f_n$ belong to $P_0 + P_1$, then* Quad $q(\mathbf{f})$ *is skew.*

3.3. *Both of the following hold:*

3.31. *There are $f_1, \dots, f_n$ in $P_0 + P_1$, and* Quad $q(\mathbf{f})$ *is not symmetric.*

3.32. *There are $f_1, \dots, f_n$ in $P_0 + P_1$, and* Quad $q(\mathbf{f})$ *is not skew.*

We pass on to consider the consequences of each of these sentences or conditions.

LEMMA. *Condition* 3.1 *holds* $\Leftrightarrow$

3.4. Quad$(q(\lambda + \mathbf{x}))$ *is symmetric for all* $\lambda = (\lambda_1, \dots, \lambda_n)$.

Proof. $\Rightarrow$: Choose $f_i = \lambda_i + x_i$. By 3.1, $\mathrm{Quad}(q(\lambda + \mathbf{x}))$ is symmetric. $\Leftarrow$: Suppose $\mathrm{Quad}(q(\lambda+\mathbf{x})) = \sum \alpha_{ij}x_i x_j$. This forces α_{ij} to be symmetric as a matrix, which makes $\sum \alpha_{ij}X_i X_j$ symmetric when the X_i are any linear forms. So $\mathrm{Quad}(q(\lambda + \mathbf{x})) = \sum \alpha_{ij}X_i X_j$ is symmetric as 3.1 requires.

In the same way one can show

LEMMA. *Condition* 3.2 *holds* $\Leftrightarrow$
3.5. $\mathrm{Quad}(q(\lambda + \mathbf{x}))$ *is skew for all* $\lambda = (\lambda_1, \dots, \lambda_n)$.

3.6. LEMMA. *If* 3.1 *holds, then* $p_1, \dots, p_n \in S_+$ *implies that* $q(p_1, \dots, p_n) \in S_+$.

Proof. $\mathrm{Quad}\, q(\mathbf{p})$ is identical to $\mathrm{Quad}(q(\mathbf{p}_{,2}))$. Now $\mathrm{Quad}(q(\mathbf{p}_{,2})) = \mathrm{Quad}(q(\mathbf{p}_{,1}))$ plus a linear combination of terms $p_{i,2}$. Because $p_1, \dots, p_n \in S_+$, these extra terms are symmetric. By 3.1, $\mathrm{Quad}(q(\mathbf{p}_{,1}))$ is symmetric, whence $\mathrm{Quad}(q(\mathbf{p}))$ is symmetric, and $q(p_1, \dots, p_n) \in S_+$.

3.7. LEMMA. *If* 3.2 *holds, then* $p_1, \dots, p_n \in S_-$ *implies that* $q(p_1, \dots, p_n) \in S_-$.

A proof may be obtained by replacing "symmetric" by "skew".

3.71. LEMMA. *If* 3.31 *holds, then* $x_1x_2 - x_2x_1$ *belongs to* $((q))$.

Proof. From 2.2, we see that $\mathrm{Quad}(q(\mathbf{f}))$ belongs to $((q))$. Thus there is an element $p = \sum \alpha_{ij}x_i x_j$ in $((q))$ such that $p - p^* \neq 0$. This leads to an element $\sum \alpha_{ij}(x_i x_j - x_j x_i)$ in $((q))$ and not 0. By 2.2 again, some $x_i x_j - x_j x_i$ is in $((q))$. Permuting the x_k gives us 3.71.

3.72. LEMMA. *If* 3.32 *holds, then* $x_1x_2 + x_2x_1$ *belongs to* $((q))$.

A proof can be assembled from the preceding, except that we might arrive at the element $x_1x_1 + x_1x_1$. From here it is easy to get to $x_1x_2 + x_2x_1$ by polarization. We note an obvious consequence.

3.8. LEMMA. *If* 3.31 *and* 3.32 *both hold then* x_1x_2 *belongs to* $((q))$.

3.9. THEOREM. *Let $q \in P$. If 3.4 holds then $((q))$ is included in S_+ and q is nonproductive. If 3.5 holds then $((q))$ is included in S_- and q is non-productive. If neither of these hold, then q is productive.*

Proof. Suppose 3.4 holds. Therefore, 3.1 holds, so by 3.6, S_+ is invariant under q. Now $1, x_1, \ldots, x_n$ are symmetric and thus in S_+. Therefore, $((q))$ is included in S_+. Obviously S_+ is not all of P.

Supposing 3.5 holds, we proceed in an analogous fashion, and conclude that q is non-productive.

If neither 3.4 nor 3.5 holds then we obtain 3.31 and 3.32. We appeal to 3.9 and to 2.4, and conclude that q is productive.

4. Examples.

4.1. THEOREM. *Suppose either $\alpha_{ij} = \alpha_{ji}$ for all i, j or $\alpha_{ij} = -\alpha_{ji}$ for all i, j. Then, $\sum \alpha_{ij} x_i x_j$ is non-productive, and conversely.*

Proof. Suppose $\alpha_{ij} = \alpha_{ji}$ for all i, j. In $q = \sum \alpha_{ij} x_i x_j$ replace x_i by $\lambda_i + x_i$ where λ_i is a scalar. Obviously $\mathrm{Quad}(\lambda + \mathbf{x})$ is q itself, which is surely symmetric when $\alpha_{ij} = \alpha_{ji}$. The case $\alpha_{ij} = -\alpha_{ji}$ is treated in a like manner. By 3.9, q is nonproductive.

If neither $\alpha_{ij} = \alpha_{ji}$ for all i, j or $\alpha_{ij} = -\alpha_{ji}$ for all i, j then neither 3.4 nor 3.5 holds. This completes the proof of 4.1

4.2. THEOREM. *Let $q = x^i y^j x^k$, where the exponents are all positive. Here x is x_1 and y is x_2. Then q is non-productive precisely when $i = k$.*

Proof. Suppose 3.4 is true. Let the $\lambda_i = 1$. It is verifiable that

$$\mathrm{Quad}((1+x)^i(1+y)^j(1+x)^k) = x^2(ik + i' + k') + xyij + yxjk + y^2 j',$$

where i' means $i(i-1)/2$, etc. So $j = k$, and conversely, if $j = k$ then $\mathrm{Quad}(q(\lambda + \mathbf{x}))$ is symmetric. But we have not yet shown that $j = k$ implies $\mathrm{Quad}(q(\lambda + \mathbf{x}))$ is symmetric even when some λ_i is 0. In this case we have to examine $\mathrm{Quad}(x^i(\mu + y)^j x^k)$ which is either $\mu^j x^{i+k}$ or 0, in any case, symmetric.

This theorem implies that $x^i y^j x^i$ is non-productive. Observe that $x^i y^j x^i$ is symmetric. We indicate another way to see this.

4.3. THEOREM. *If q is symmetric, then every element of $((q))$ is symmetric and thus q is non-productive.*

Proof. If $f_1, \ldots, f_n$ are symmetric and q is symmetric, then $q(f_1, \ldots, f_n)$ is symmetric. Now 1, and the x_i are symmetric, so every element of $((q))$ is symmetric.

The converse is not true. Examples abound, by virtue of the next theorem.

4.4. THEOREM. *Let* $q = x^h y^i x^j y^k$. *Then* q *is non-productive precisely when*

$$hk + hi + ji = ij. \tag{4.41}$$

Proof. This condition is necessary for 3.4 and 3.5, as can be seen by putting $\lambda = (1, \ldots, 1)$. Then it can be shown sufficient for 3.4, in case $\lambda = (\lambda_1, \ldots, \lambda_n)$ when no λ_i is 0, and finally when some are 0.

4.5. EXAMPLE (see 2.5). One can have q productive but every q_E non-productive.

Let $q_1 = xyx$. This is non-productive by 4.2. In the proof we showed that it generates $xy + yx$. Let $q_2 = xy - yx$. This is non-productive by 4.1. Let q be their sum. From q we can get q_1 and q_2 back, by 2.2. Hence we can get $xy + yx$ and $xy - yx$, whence by 2.4, q is productive.

5. Remarks about the rest of this paper. We want to present some ideas which will enable us to assert, for example, that xyx does not belong to $((xy^2))$. See 8.5 below. To establish such propositions we apparently have to bring up the concept of *seminorms*, which is familiar, and that of *ideals*, which may not be familiar in this context.

6. Seminorms. From now on, $\mathbb{F}$ will be either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$.

DEFINITION. Suppose A is a linear algebra over $\mathbb{F}$. Suppose that a real-valued function S defined on A is a seminorm with respect to the linear space structure of A. A seminorm defines a topology in A, and thus also in $A \times \cdots \times A$. It makes sense to ask whether this makes $\hat{q}$ continuous at any selected point of $A \times \cdots \times A$. If so, we will say that $\hat{q}$ is *continuous in the topologies defined by* S, or that S renders $\hat{q}$ continuous[6] at that point.

[6]We will abbreviate this to " q is continuous in the topology defined by S ". If no particular point is mentioned, continuity at all points is implied.

6.1. LEMMA. *Suppose $q \in \mathbb{F}\langle x_1, \dots, x_n \rangle$. Let A be an algebra over $\mathbb{F}$. Let S be a seminorm on A. Suppose q is continuous in the topology defined by S. If $p \in ((q))$, the q-closed subset of $\mathbb{F}\langle x_1, \dots, x_n \rangle$ generated by $\{1, x_1, \dots, x_n\}$ then p is continuous in the topology defined by S.*

Proof. Construct the q-system $\{C_i\}$ as in 2.3, with C_1 being the set $\{1, x_1, \dots, x_n\}$. The union of the C_i is $((q))$. Every element of C_1 is certainly continuous in the topology defined by S. Assume it is true for every element of C_m. Each element of C_{m+1} is either a linear combination of these, or the value of $\hat{q}$ on n of them, and thus surely also continuous.

We will actually need a slight but immediate corollary of this.

6.2. COROLLARY. *Suppose $q \in \mathbb{F}\langle x_1, \dots, x_n \rangle$. Let A be an algebra over $\mathbb{F}$. Let S be a seminorm on A. Suppose q has 0 constant term and is continuous at $(0, \dots, 0)$ in the topology defined by S. Suppose $p \in ((q))$, and suppose that p has constant term 0. Then p is continuous at $(0, \dots, 0)$ in the topology defined by S.*

This result will enable us to exhibit some polynomials p and q where $p \notin ((q))$. We just have to find a seminorm such that $\hat{p}$ is not continuous. To do that we develop an appropriate ideal theory.

7. q-ideals.

DEFINITION. Let $q \in \mathbb{F}\langle x_1, \dots, x_n \rangle$. Let A be an algebra. Then a linear subspace J of A will be called a q-ideal if whenever $\mathbf{a} = (a_1, \dots, a_n) \in A^n$ and $\mathbf{j} \in J^n$, then $\hat{q}(\mathbf{a}+\mathbf{j}) - \hat{q}(\mathbf{a}) \in J$.

One example is $q = x_1 x_2$. Here a q-ideal is just an ordinary two-sided ideal of the algebra A. Another example is $q = (x_1)^2$. In this case a q-ideal is an ideal of the Jordan algebra A^+ (see **[BK]**).

LEMMA. *Let $q \in \mathbb{F}\langle x_1, \dots, x_n \rangle$. Let A be an algebra. Let J and K be q-ideals. Then $J \cap K$ and $J + K$ are q-ideals.*

Proof. The assertion about $J \cap J$ is elementary. As to the other, $\hat{q}(\mathbf{a}+\mathbf{j}+\mathbf{k}) - \hat{q}(\mathbf{a}) = [\hat{q}(\mathbf{a}+\mathbf{j}+\mathbf{k}) - \hat{q}(\mathbf{a}+\mathbf{j})] + [\hat{q}(\mathbf{a}+\mathbf{j}) - \hat{q}(\mathbf{a})]$. The first bracket belongs to K and the second to J. Thus the lemma is proved.

7.1. DEFINITION. The smallest q-ideal containing a given subset B of A is the q-ideal generated by B and will be denoted by $I_q(B)$.

The next lemma is helpful in discovering what elements belong to $I_q(B)$ in specific cases.

DEFINITION. Let q be an element of $\mathbb{F}\langle x_1, \dots, x_n\rangle$. Let J be a linear subspace of a linear algebra A. Let $(a_1, \dots, a_n) \in A^n$ and let $(j_1, \dots, j_n) \in J^n$. For each $i, j \leq 1 \leq n$, let

$$(7.2) \qquad q_i(a_1, \dots, a_n, j_1, \dots, j_n) = \frac{d}{dt}\hat{q}(a_1, \dots, a_i + tj_i, \dots, a_n)|_{t=0}.$$

LEMMA. *A linear subspace J is a q-ideal if and only if*

$$(7.3) \qquad q_i(a_1, \dots, q_n, j_1, \dots, j_n) \in J$$

whenever $a_1, \dots, a_n \in A$ and $j_1, \dots, j_n \in J$.

Proof. It seems adequate to us to give a proof only for a special example, say $q = xyx$. Here we write x for x_1 and y for x_2 and $n = 2$. Also let $a_1 = a$, $a_2 = b$, $j_1 = j$, and $j_2 = k$. In this case of $q = xyx$, conditions 7.3 say

$$(7.4) \qquad jba + abj \text{ and } aka \in J \text{ whenever } j, k \in J \text{ and } a, b \in A.$$

We have to show that $(a+j)(b+k)(a+j) - aba \in J$ whenever $j, k \in J$ and $a, b \in A$, if and only if 7.4 holds.

If $(a+sj)(b+tk)(a+sj) - aba$ lies in J for all s and t, we can deduce, by algebraic procedures, that the derivatives in 7.4, that is the derivatives 7.3, lie in J.

Conversely, does $(a+j)(b+k)(a+j) - aba$ lie in J if 7.4 holds? Actually, we undertake to prove that $(a+sj)(b+tk)(a+sj) - aba$ lies in J for all real s and t, if 7.4 holds. Let

$$f(s) = (a+sj)(b+tk)(a+sj) - aba.$$

Then $f(s) = \sum c_h s^h$, a sum of three terms. Conditions 7.4 imply that all derivatives of $f(s)$ are in J. Therefore, all the c_h are in J for $h > 0$. The constant of integration c_0 is also in J because $f(0) = atka \in J$. Hence $f(s) \in J$, as we promised to show.

PROPOSITION. *Consider the element $q = xyx$ of $\mathbb{F}\langle x, y\rangle$. Let $I_{xyx}(x)$ be the q-ideal generated by the single element x of $\mathbb{F}\langle x, y\rangle$. Then a basis for the elements of this ideal of degree not greater than* 3 *is*

$$(7.5) \qquad \{x, x^2, xy + yx, x^3, x^2y, xyx, yx^2, xy^2 + y^2x\}.$$

Proof. By repeated use of 4.52 one can show that any xyx-ideal which contains x, must contain x^2 and $xy+yx$. Also one can show that an ideal which contains these must contain the entire list 7.5. The list is clearly linearly independent. Then, one can show that on the other hand, that the span of 7.5 together with all polynomials of degree at least four, is an xyx-ideal. This sketch should suffice for a proof.

PROPOSITION. *Let* $q \in \mathbb{F}\langle x_1, \dots, x_n\rangle$. *Let* $A = \mathbb{F}\langle x_1, \dots, x_m\rangle$. *Let* k *be a positive integer. Let* X^k *be the set of polynomials whose constituent homogeneous summands are all of degree* k *or more. Then* X^k *is a* q*-ideal.*

Proof. Let $a_1, \dots, a_m$ be members of A and let $j_1, \dots, j_m$ be members of X^k. The expression $q(\dots, a_i+j_i, \dots)-q(\dots, a_i, \dots)$ is a sum of monomials in the a_i and j_i. After the obvious cancellation, every term must have at least one factor j_h in it, and so the sum must belong to X^k.

7.6. LEMMA. *Let* $q \in P \equiv \mathbb{F}\langle x_1, \dots, x_n\rangle$. *Let* m *be an integer not necessarily related to* n. *Let* $A = \mathbb{F}\langle x_1, \dots, x_m\rangle$. *Let* J *be a* q*-ideal in* A *and let* p *be an element of* A *which does not lie in* J. *Then there is a* q*-ideal* K *to which* p *does not belong, which has finite codimension in* A, *and contains* J.

Proof. Let k be greater than the degree of p. Let $K = J + X^k$. This is a q-ideal, and it has finite codimension, because X^k does. If the element p were to be in K, say $p = j + z$, then the homogeneous constituents of z could not appear in p because the degree of p is too small. Thus they would find and cancel their negatives in j, and p would lie in J.

8. Establishing non-producivity. For convenience, we introduce another term.

DEFINITION. Suppose A is a linear algebra over $\mathbb{F}$. Suppose that S is a seminorm on A, and μ is a real number satisfying $S(\hat{q}(a_1, \dots, a_n)) \le \mu$ whenever $S(a_i) \le 1$ for all i. Then μ is a *q-factor for* S, and S will be said to have a q-factor.

8.1. LEMMA. *Let* $q \in P \equiv \mathbb{F}\langle x_1, \dots, x_n\rangle$. *Let* A *be an algebra and let* J *be a* q*-ideal such that* A/J *is finite dimensional. Then there is a seminorm* S *on* A *that has a* q*-factor and has kernel* J.

In the following proof, we always intend summation over repeated indices. Also, any lower case j stands for a member of J.

Let A/J have dimension m. Let $e_1, \ldots, e_m$ be a basis for A/J. Now $\hat{q}(\sum \alpha_{1h}e_h, \ldots, \sum \alpha_{nh}e_h)$ is well defined because J is a q-ideal, which implies that evaluating $\hat{q}$ is well defined modulo J. Since q is a polynomial, $\hat{q}(\sum \alpha_{1h}e_h, \ldots, \sum \alpha_{nh}e_h) = P_i e_i$, where the P_i are polynomials in the α_{ih}.

Choose a norm $\|\cdot\|$ for A/J. Define the seminorm S on A by $S(a) = \|a + J\|$. Then $S(\hat{q}(\sum \alpha_{1h}e_h, \ldots, \sum \alpha_{nh}e_h)) = \|P_i e_i\|$.

The condition $\|a+J\| \leq 1$ defines a closed bounded set B in A/J. For $a_1, \ldots, a_n$ to have $S(a_i) \leq 1$ is the same as having the $a_i + J$ lie in B. This in turn, makes $\|P_i e_i\| \leq \mu$ for some real μ. We have shown that $S(a_i) \leq 1$ for $i = 1, \ldots, n$ implies $S(\hat{q}(a_1, \ldots, a_n)) \leq \mu$. Thus the seminorm has a q-factor.

The kernel of S is obviously J. This proves 8.1.

8.2. LEMMA. *Let $q \in \mathbb{F}\langle x_1, \ldots, x_n\rangle$. Let p be an element of $A = \mathbb{F}\langle x_1, \ldots, x_m\rangle$ vanishing at $(0, \ldots, 0)$ of $A \times \cdots \times A = A^m$. Suppose that p does not lie in the q-ideal $I_q(x_1, \ldots, x_m)$ generated by $x_1, \ldots, x_m$ in A. Then there exists a seminorm S on A that has q-factors, but does not render $\hat{p}$ continuous at $(0, \ldots, 0)$.*

Proof. By 7.6, there is a q-ideal K to which p does not belong, and such that A/K has finite dimension. By 8.1 there is a seminorm S having q-factors, and having K for its null space. So $S(p) > 0$ because p is not in K. However, $S(x_i) = 0$. Now suppose $\hat{p}$ were continuous at $(0, \ldots, 0)$. Its value there is of course the zero element of A. Hence for every $\varepsilon > 0$ there is a $\delta > 0$ such that for those $a_1, \ldots, a_m$, with $S(a_i) < \delta$ for all i, one will have $S(\hat{p}(a_1, \ldots, a_m)) < \varepsilon$. Take $\varepsilon = S(p)$ which is positive. Consider letting a_i have the value x_i. With that a_i, we surely have $S(a_i) < \delta$ for all i. So $S(\hat{p}(a_1, \ldots, a_m)) < \varepsilon$. But now $S(\hat{p}(a_1, \ldots, a_m)) = S(\hat{p}(x_1, \ldots, x_m))$, and a little reflection on the definition of the function $\hat{p}$ shows that $\hat{p}(x_1, \ldots, x_m)$ is indeed p. So $S(p) < S(p)$, is obviously a contradiction. This completes our proof of 8.2.

8.3. THEOREM. *Let $q \in \mathbb{F}\langle x_1, \ldots, x_n\rangle$. Let p be an element of $A = \mathbb{F}\langle x_1, \ldots, x_m, z_1, \ldots, z_r\rangle$ that is homogeneous in each of $z_1, \ldots, z_r$ and homogeneous of positive degree in x_1. Suppose p does not belong to the q-ideal generated by $z_1, \ldots, z_r$. Then there is* *a seminorm S on A which has q-factors but renders $\hat{p}$ discontinuous at $(0, \ldots, 0)$.*

Proof. As above, there is a q-ideal K including $z_1, \ldots, z_r$, but not p, and such that A/K is finite dimensional. Consequently there is a seminorm S such that $S(z_i) = 0$, but $S(p) > 0$. Say $S(p) = \varepsilon$. Suppose $\hat{p}$ were continuous at $(0, \ldots, 0)$. Then there is a $\delta > 0$ such that if $S(a_i) < \delta$ for all i and $S(b_j) < \delta$ for all j then $S(\hat{p}(a_1, \ldots, a_m, b_1, \ldots, b_r)) < \varepsilon$. Let $a_i = \alpha_i x_i$ and $b_j = \beta_j z_j$. If the α_i are sufficiently small, then $S(a_i) < \delta$ for all i and $S(b_j) < \delta$ for all j, whereupon $S(\hat{p}(a_1, \ldots, a_m, b_1, \ldots, b_r)) < \varepsilon$. Now $S(\hat{p}(a_1, \ldots, a_m, b_1, \ldots, b_r))$ is a product of α's and β's times $S(\hat{p}(x_1, \ldots, x_m, z_1, \ldots, z_r))$. After the α's have been chosen small enough, and $\beta_2, \ldots, \beta_n$ have been set equal to 1, the product of α's and β_1 can be made arbitrarily large by varying β_1. Thus $S(\hat{p}(x_1, \ldots, x_m, z_1, \ldots, z_r))$ is actually 0. But as mentioned at the end of the proof of 8.2, $S(\hat{p}(x_1, \ldots, x_m, z_1, \ldots, z_r))$ is $S(p)$. This contradiction proves 8.3.

8.4. THEOREM. *Let $q \in \mathbb{F}\langle x_1, \ldots, x_n\rangle$. Let p be an element of $\mathbb{F}\langle x_1, \ldots, x_m, z_1, \ldots, z_r\rangle$ that is homogeneous in each of $z_1, \ldots, z_r$ and homogeneous of positive degree in x_1. Suppose p does not belong to the q-ideal generated by $z_1, \ldots, z_r$. Then p does not belong to $((q))$.*

Proof. If $p \in ((q))$ then $\hat{p}$ is continuous by 6.1. But this is contradicted by 8.3.

In order to apply this more smoothly to a situation met earlier, we make a mere alphabetic variation:

8.4′. THEOREM. *Let $q \in \mathbb{F}\langle x_1, \ldots, x_n\rangle$. Let p be an element of $\mathbb{F}\langle y_1, \ldots, y_m, x_1, \ldots, x_r\rangle$ that is homogeneous in each of $x_1, \ldots, x_r$ and homogeneous of positive degree in y_1. Suppose p does not belong to the q-ideal generated by $x_1, \ldots, x_r$. Then p does not belong to $((q))$.*

8.5. THEOREM. *Consider the element $q = xyx$ of $\mathbb{F}\langle x, y\rangle$. Let p be an element of $\mathbb{F}\langle x, y\rangle$ that is homogeneous of positive degree in y, homogeneous in x, and homogeneous of the third degree in x, y. Suppose p is not a linear combination of the elements of the list 7.5. Then $p \notin ((xyx))$.*

Proof. By hypothesis, p does not belong to the ideal $I_{xyx}(x)$. We can appeal to 8.4′ and conclude that $p \notin ((xyx))$.

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