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ON THE EXISTENCE OF CONVEX CLASSICAL SOLUTIONS TO MULTILAYER FLUID PROBLEMS IN ARBITRARY SPACE DIMENSIONS

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We study certain multilayer free-boundary problems, in which the layer interfaces constitute a nested family of convex closed surfaces, each characterized by a Bernoulli joining condition between the potentials in the neighboring layers. In this context, we develop convex variational methods based on a family of convexity-preserving free-boundary perturbation operators, and we apply these methods in the study of the existence of convex solutions.

1. Introduction. The main purpose of this paper is to apply convex variational techniques to study the question of the existence of convex classical solutions to certain multiple-free-boundary problems arising in fluid dynamics, called multilayer fluid problems.

1.1. *Problem.* In $\mathbb{R}^m$, $m \geq 2$, let an annular domain Ω of the form $\Omega = D^+ \backslash \mathrm{Cl}(D^-)$ be given, where $D^{\pm}$ are fixed, bounded, simply-connected, nested C^1-domains. Given $n \in \mathbb{N}$ and continuous functions $\lambda_i(x)\colon \mathrm{Cl}(D^+) \to \mathbb{R}$, $i = 1, 2, \ldots, n$, we seek a nested family of C^1-domains $D_1, D_2, \ldots, D_n$ (with boundaries $\Gamma_i = \partial D_i$) such that $\mathrm{Cl}(D_i) \subset D_{i+1}$ for $i = 0, \ldots, n$, (where we set $D_0 = D^-$ and $D_{n+1} = D^+$) and such that

$$|\nabla U_i|^2 = |\nabla U_{i+1}|^2 + \lambda_i(x) \quad \text{on } \Gamma_i \tag{1.1}$$

for $i = 1, \ldots, n$, where $U(x)$ solves the boundary value problem

$$\Delta U = 0 \quad \text{in } \Omega \backslash (\Gamma_1 \cup \cdots \cup \Gamma_n), \qquad U(\Gamma_i) = i \quad \text{for } i = 0, 1, \ldots, n+1, \tag{1.2}$$

and where, for each i, U_i denotes the restriction of U to the closure of the annular domain $\Omega_i := D_i \backslash \mathrm{Cl}(D_{i-1})$ with boundary $\partial \Omega_i = \Gamma_i \cup \Gamma_{i-1}$.

1.2. *Problem.* This denotes the modified version of Problem 1.1 in which Γ_0 becomes a free boundary characterized by the requirement that

$$|\nabla U_1| = a_0(x) \quad \text{on } \Gamma_0. \tag{1.3}$$

Here, $a_0(x)\colon \mathrm{Cl}(D^+) \to \mathbb{R}$ is a given, continuous, weakly-positive function.

For Problem 1.1, we will show (see Theorem 7.2) that if the given domains $D^\pm$ are convex, D^- is a C^2-domain, the given functions $\lambda_i(x)\colon \mathrm{Cl}(D^+) \to \mathbb{R}$ are all strictly positive, and the related functions $b_i(x) := [\lambda_i(x)]^{-1/2}$ are all concave in D^+, then there exists a classical solution $D = (D_1, D_2, \ldots, D_n)$ such that the domains D_i, $i = 1, \ldots, n$, are all convex. We obtain essentially the same result for Problem 1.2 (under the additional assumption that the function $a_0(x)$ is concave in the convex set $\{a_0(x) > 0\}$), except that an additional assumption is needed to prevent degeneracy (see Theorem 6.3). It is reasonable to assume that the functions $\lambda_i(x)$ are all strictly positive, since the author has given an example of Problem 1.1 (see **[10]**) in which $n = 1$, $m = 2$, the given domains $D^\pm$ are both convex, $\lambda_1(x)$ is a *negative* constant, and *no* convex solution exists. The present convexity result generalizes a portion of the author's work in **[11]**, where a similar conclusion was reached under considerably stronger assumptions. In particular, the present convexity results hold in arbitrary space dimensions, whereas the previous convexity results for Problem 1.1 (in **[11]**, §6]) were restricted to 2 or 3 space dimensions $(m \le 3)$. It was previously assumed essentially that $0 \in D^-$ and that the functions $t^2\lambda_i(tv)$ (for $i = 1, \ldots, n$ and $v \in \mathbb{R}^m$) were all weakly increasing in $t > 0$. This latter assumption guaranteed that the solution of Problem 1.1 would be unique and continuously dependent on the data (properties which played an important role in the proof). In contrast to this, the present existence results hold in the absence of any knowledge concerning uniqueness.

We also mention the work of Laurence and Stedulinsky **[21]**, who proved in two space dimensions that Problem 1.1 and a modified version of Problem 1.2 both have convex solutions under convex conditions, provided that the functions $a_0(x)$ and $\lambda_i(x)$, $i = 1, \ldots, n$, are all positive constants. Laurence and Stredulinsky have also shown (this time in arbitrary dimensions; see **[22]**) that solutions of certain nonlinear PDEs can be approximated by convex solutions of Problems 1.1 in the limit as $n \to \infty$, so that our convexity results for Problems 1.1 have direct consequences regarding the existence of solutions with convex level surfaces for these nonlinear PDEs.

The author's results on convexity, both here and in **[11]**, are based on a certain one-parameter family of free-boundary perturbation operators T_ε, $0 < \varepsilon < 1$, which preserve the geometric convexity of the free-boundary surfaces under suitable conditions (see §§2.8, 2.9). In fact our assumption that the functions $a_0(x)$ and $b_i(x) := [\lambda_i(x)]^{-1/2}$, $i = 1, \dots, n$, all be concave arises as a natural requirement for the convexity-preserving property of these operators. However, our present treatment is otherwise entirely different from **[11]**, where the convexity results were obtained essentially as consequences of the maximum-norm convergence of a successive approximation scheme based on the (convexity-preserving) operators T_ε, and thus required the additional assumption stated above, which was crucial to the convergence of the successive approximations.

In our present approach, which is patterned after the author's first papers on the existence of convex free boundaries (see **[3]**, **[5]**) and work of Caffarelli and Spruck **[16**, §§4,5] the "operator method" is studied in the context of convex functional minimization (i.e. convex variational inequalities). This approach permits us to obtain existence results in the absence of any knowledge regarding uniqueness of solutions. We will now briefly outline this method in the context of Problem 1.2: One begins with the standard integral functional associated with Problem 1.2 via the method of variational inequalities, and one minimizes this functional among those functions having only convex level surfaces. Now, it turns out that convex minimizers can exist under quite general circumstances, including many cases where no convex solution exists for the corresponding free boundary problem (see **[10**, Remark 2]). Therefore, the crucial step in our method is the proof, under suitable assumptions, that U will satisfy the Euler equations for the original (non-convex) variatonal problem, namely (1.1), (1.2), and (1.3). A convex minimizer U will be harmonic except on the free boundaries (i.e. the surfaces corresponding to integer values of U), as follows from results in the literature on the convexity of level surfaces of the capacity potential (see **[15]**, **[16**, §2], **[17]**, **[18]**, **[19]**). Therefore, it remains to prove the joining conditions (1.1) and (1.3). The main tool for verifying the joining conditions is the examination of convex variations in the free boundaries of the convex minimizer, which cannot decrease the functional. The author's method for this, called the "operator method" (introduced in **[3]**), consists of defining one specific global convex variation in each free boundary, chosen in such a way that the functional will be diminished (to first order in the variation parameter) unless the joining condition on the free boundary is satisfied at least in some weak sense. We will show (in §3) that the convex variation needed to establish the joining conditions (1.1) is accomplished by precisely the one-parameter operator family T_ε already used in **[11]**, while (1.3) follows by applying the operator family defined in **[3]**.

An alternative perspective on convex-free-boundary problems has been studied by Laurence and Stredulinsky in the work previously cited (see **[21]**). After establishing the existence of a suitable convex minimizer (in arbitrary space dimensions), they study the joining conditions (1.1) by the "method of flat places". This method (which had already been studied in several other contexts by the author in **[6]**) is based on the observation that local convex variations establish the joining conditions on the free boundaries of a convex minimizer, except at the "flat places" in these surfaces, which require a separate analysis based on maximum principles and non-local convex variations. The main drawback in this method is the fact that the verification of the joining conditions on the flat places becomes increasingly difficult as the number of dimensions increases. In fact the method has never been applied in more than two space dimensions. By contrast, the operator method, although it was originally introduced in a 2-dimensional context, is actually insensitive to dimension in all essential aspects, perhaps because it circumvents difficult questions related to details of surface geometry. However, the method of flat places, when applicable, appears to lead to slightly more general assumptions for the existence of convex solutions to Problems 1.1 and 1.2. For example, the method of flat places would lead to the requirement that $1/a_0(x)$ be convex in $\{a_0(x) > 0\}$, which is slightly more general than our assumption that the function $a_0(x)$ be concave (see **[6]**).

We remark that neither a proof nor a counterexample has been found for the existence of convex solutions under convex conditions for Problem 1.2 in the case where the functions $a_0(x)$ and $\lambda_i(x)$, $i = 1, \dots, n$, are all constants, but at least one of the λ_i is negative. (A closely related problem, in which all the λ_i are negative, was proposed by Laurence and Stredulinsky in **[20]**, but not resolved.) However, we obtain a counterexample in §7.6 applicable to the case where $m = 2$, $n = 1$, $\lambda_1(x) = -\alpha^2 < 0$, and $a_0(x)$ is concave in the convex set $\{a_0(x) > 0\}$. The operator method does not show that the convex minimizer solves Problem 1.2 in this particular case because T_ε does not preserve convexity when applied to $\Gamma_1 := \partial D_1$.

A simple, but powerful observation in the study of the multilayer fluid problem (and multiple free boundary problems in general) is the following: If a nested family of free boundaries $(\Gamma_1, \Gamma_2, \dots, \Gamma_n)$

solves Problem 1.1 in the case of $(n+1)$ layers, then each surface Γ_i is the solution of a two-layer-version of Problem 1.1 relative to its own immediate neighbors, Γ_{i-1} and Γ_{i+1}. This principle allows many questions pertaining to multilayer problems to be resolved in the 2-layer case. It is the plan of this paper to make maximum use of this principle by first carefully studying the convex version of Problem 1.1 in the 2-layer case (in §§2, 3, 4, 5), and then solving the convex version of Problem 1.2 by multiple application of these results (see §6). Finally, in §7, we will solve the convex version of Problem 1.1 essentially by regarding it as a limiting case of Problem 1.2.

2. A variational approach to the 2-layer problem in the convex case.

2.1. *Problem.* In $\mathbb{R}^m$, $m \geq 2$, let be given an annular domain Ω in the form $\Omega = D^+ \backslash \mathrm{Cl}(D^-)$, where $D^\pm$ are given bounded, convex, nested C^1-domains with boundaries $\Gamma^\pm = \partial D^\pm$, and let $a(x)\colon \mathrm{Cl}(\Omega) \to \mathbb{R}$ denote a strictly-positive, continuous function such that the related function $b(x) := (1/a(x))$ is concave on Ω. (This means that $a(x)(\partial^2 a(x)/\partial \nu^2) \geq 2(\partial a(x)/\partial \nu)^2$ at any point $x \in \Omega$ and in any direction ν, provided that $a(x)$ is sufficiently differentiable.) We seek a convex domain D (or its boundary $\Gamma = \partial D$) such that $\mathrm{Cl}(D^-) \subset D \subset \mathrm{Cl}(D) \subset D^+$ and such that

$$|\nabla U^-|^2 = |\nabla U^+|^2 + a^2(x) \quad \text{on } \Gamma, \tag{2.1}$$

where $\Omega^\pm$ denotes the annular domain whose boundary is $\partial\Omega^\pm = \Gamma \cup \Gamma^\pm$, and where the functions $U^\pm(x)$ solve the boundary value problems

$$\Delta U^\pm = 0 \quad \text{in } \Omega^\pm, \qquad U^\pm(\Gamma) = 0, \qquad U^\pm(\Gamma^\pm) = 1. \tag{2.2}$$

2.2. *Problem.* In the context of Problem 2.1, let $\mathbb{X}_c$ denote the family of all closed, convex $(m-1)$-surfaces Γ of the form $\Gamma = \partial D$, where D denotes a convex domain such that $\mathrm{Cl}(D^-) \subset D \subset \mathrm{Cl}(D) \subset D^+$. For any surface $\Gamma \in \mathbb{X}_c$, we define the functions $U^\pm(\Gamma; x)\colon \mathrm{Cl}(\Omega^\pm(\Gamma)) \to \mathbb{R}$ to be the solutions of the boundary value problem (2.2), where $\Omega^\pm := \Omega^\pm(\Gamma)$ denotes the annular domain bounded by $\Gamma \cup \Gamma^\pm$. We seek to minimize the functional $I(\Gamma)\colon \mathbb{X}_c \to \mathbb{R}$ defined by

$$I(\Gamma) := K^+(\Gamma) + K^-(\Gamma) + \|\Omega^-(\Gamma)\|. \tag{2.3}$$

Here $K^\pm(\Gamma)$ denotes the capacity of $\Omega^\pm(\Gamma)$, i.e.

$$K^\pm(\Gamma) = \int_{\Omega^\pm(\Gamma)} |\nabla U^\pm(\Gamma; x)|^2 \, dx.$$

Also, we define $\|M\| = \int_M a^2(x)\,dx$ for any measurable subset M of $\mathrm{Cl}(\Omega)$.

2.3. Remarks. (a) In the notation of Problem 2.2, a solution of Problem 2.1 is a convex surface $\widetilde{\Gamma} \in \mathbb{X}_c$ such that

$$|\nabla \widetilde{U}^-(x)|^2 = |\nabla \widetilde{U}^+(x)|^2 + a^2(x) \quad \text{on } \widetilde{\Gamma},$$

where we define $\widetilde{U}^\pm(x) := U^\pm(\widetilde{\Gamma}; x)$.

(b) Observe that the definition of a solution $\widetilde{\Gamma}$ of Problem 2.2 requires that $\widetilde{\Gamma} \cap \Gamma^- = \varnothing$.

2.4. Definition. Assume $\widetilde{\Gamma}$ is a C^1-surface. Then the function $\widetilde{U}^\pm(x) \colon \mathrm{Cl}(\widetilde{\Omega}^\pm) \to \mathbb{R}$ is differentiable at a point $x_0 \in \widetilde{\Gamma}$ if there exists a value $\lambda^\pm = \lambda^\pm(x_0) \in \mathbb{R}$ such that

$$\widetilde{U}^\pm(x) = \pm\lambda^\pm \nu(x_0) \cdot (x - x_0) + o(|x - x_0|)$$

as $x \to x_0$ in $\mathrm{Cl}(\widetilde{\Omega}^\pm)$, where $\nu(x_0)$ denotes the exterior normal vector to $\widetilde{\Gamma}$ at $x_0 \in \widetilde{\Gamma}$. In this case, we define $\nabla \widetilde{U}^\pm(x_0) := \pm\lambda^\pm \nu(x_0)$.

2.5. Theorem. (a) *Assume in Problem* 2.2 *that* $m = 2$, *or that* $m = 3$ *and* Γ^- *is a* C^2-*surface. Then there exists at least one solution* $\widetilde{\Gamma} \in \mathbb{X}_c$. (b) *Assume (for arbitrary* $m \in \mathbb{N}$*) that* $\widetilde{\Gamma}$ *is a solution of Problem* 2.2. *Then*: (i) $\widetilde{\Gamma}$ *is a uniformly* C^1-*surface, and there exist positive constants* $0 < C_1 < C_2$ *such that* $C_1 \le |\nabla \widetilde{U}^\pm| \le C_2$ *near* $\widetilde{\Gamma}$ *in* $\widetilde{\Omega}^\pm(\Gamma)$. (ii) *The derivatives* $\nabla \widetilde{U}^\pm(x_0)$ *both exist at each point* $x_0 \in \widetilde{\Gamma}$, *in a sense given in Definition* 2.4, *and they satisfy* $C_1 \le |\nabla \widetilde{U}^\pm(x_0)| \le C_2$. (iii) *We have that* $\nabla \widetilde{U}^\pm(x) \to \nabla \widetilde{U}^\pm(x_0)$ *as* $x \to x_0$ *in* $\widetilde{\Omega}^\pm \cap \{\pm\nu(x_0) \cdot (x - x_0) > \alpha|x - x_0|\}$ *for any fixed* $0 < \alpha < 1$.

Proof. Concerning part (a), see **[21**, §1]. Part (b) was proved in **[16**, §4.2.7].

2.6. Theorem. *Let* $\widetilde{\Gamma}$ *be a solution of Problem* 2.2 *(which certainly exists if* $m \le 3$ *and* Γ^- *is a* C^2-*surface). Then* $\widetilde{\Gamma}$ *is a (convex, classical) solution of Problem* 2.1.

Proof. The proof of Theorem 2.6 is the main object of §§3, 4, 5 (and, in particular, Theorems 4.1 and 5.1). The proof will be based on the operator method, which we first briefly outline.

2.7. Lemma. *Let* $U(x)\colon \mathrm{Cl}(\Omega) \to \mathbb{R}$ *denote the capacity potential in a bounded annular domain* $\Omega \subset \mathbb{R}^m$ *whose boundary components* Γ^* *and* Γ *are convex surfaces (i.e.* $\Delta U = 0$ *in* Ω, $U(\Gamma) = 0$, $U(\Gamma^*) = 1$). *Then*: (a) *all level surfaces of* U *are convex,* (b) $|\nabla U|$ *is weakly increasing on curves of steepest ascent of* U *in the direction toward the interior boundary component,* (c) $|\nabla U|$ *is subharmonic, and* (d) $\ln(|\nabla U|)$ *is superharmonic.*

Proof sketch. Regarding the proof of (a), see **[15]**, **[16**, §2], **[17]**, **[18]**, and **[19]**. Part (b) follows immediately from Part (a). Part (c) is obvious, and part (d) is proved in **[8**, §3].

2.8. *Operator method.* We continue in the context of Problems 2.1 and 2.2. For $\Gamma_1, \Gamma_2 \in \mathbb{X}_c$, we say $\Gamma_1 \le \Gamma_2$ (resp. $\Gamma_1 < \Gamma_2$) if $D_1 \subset D_2$ (resp. $\mathrm{Cl}(D_1) \subset D_2$), where D_1, D_2 denote the corresponding interior complements. We define the operators $\Phi_\varepsilon^\pm(\Gamma)\colon \mathbb{X}_c \to \mathbb{X}_c$, $0 < \varepsilon < 1$, such that

$$\Phi_\varepsilon^\pm(\Gamma) = \{x \in \Omega^\pm(\Gamma)\colon U^\pm(\Gamma; x) = \varepsilon\}.$$

For any $\varepsilon > 0$ and given $(m-1)$-surfaces $\Gamma_1, \Gamma_2 \in \mathbb{X}_c$ satisfying $\Gamma_1 < \Gamma_2$, we define the $(m-1)$-dimensional surface

$$\Psi_\varepsilon(\Gamma_1, \Gamma_2) = \{x \in \omega\colon (\varepsilon^2/d^2(x, \Gamma_1)) - (\varepsilon^2/d^2(x, \Gamma_2)) = a^2(x)\} \in \mathbb{X}_c,$$

where ω denotes the annular domain between Γ_1 and Γ_2, and where $d(x, \Gamma) = \min\{|x - y|\colon y \in \Gamma\}$. Finally, we define the family of operators $T_\varepsilon(\Gamma)\colon \mathbb{X}_c \to \mathbb{X}_c$, $0 < \varepsilon < 1$, such that

$$T_\varepsilon(\Gamma) = \Psi_\varepsilon(\Phi_\varepsilon^-(\Gamma), \Phi_\varepsilon^+(\Gamma)).$$

2.9. Theorem. *In the context of Problem* 2.1 *and* 2.2, *we have* $\Phi_\varepsilon^\pm\colon \mathbb{X}_c \to \mathbb{X}_c$ *for* $0 < \varepsilon < 1$. *Also* $\Psi_\varepsilon(\Gamma_1, \Gamma_2) \in \mathbb{X}_c$ *for any* $\varepsilon > 0$ *and surfaces* $\Gamma_1, \Gamma_2 \in \mathbb{X}_c$ *satisfying* $\Gamma_1 < \Gamma_2$. *Therefore,* $T_\varepsilon\colon \mathbb{X}_c \to \mathbb{X}_c$ *for any* $0 < \varepsilon < 1$.

Proof. The operators $\Phi_\varepsilon^\pm$ preserve convexity due to Lemma 2.7(a). Then the proof that $\Psi_\varepsilon(\Gamma_1, \Gamma_2)$ is convex whenever the surfaces $\Gamma_1 < \Gamma_2$ are convex follows from maximum principles, properties of the distance function, and the assumed concavity of the function $b(x)$ in Ω. The details are given in **[9**, §4] and **[11**, §5].

3. Infinitesimal convex variations induced by the operators $T_\varepsilon(\Gamma): \mathbb{X}_c \to \mathbb{X}_c$.

3.1. NOTATION. Given a solution $\widetilde{\Gamma}$ of Problem 2.2, we define the functions $A(x)$, $B(x)$, $\alpha(x)$, $\beta(x): \widetilde{\Gamma} \to \mathbb{R}^+$ such that $A(x) = |\nabla \widetilde{U}^-|$, $B(x) = |\nabla \widetilde{U}^+|$, $\alpha(x) = 1/A(x)$, and $\beta(x) = 1/B(x)$. Clearly, these functions are all bounded and measurable relative to the surface-area measure on $\widetilde{\Gamma}$.

3.2. THEOREM. *Let $\widetilde{\Gamma}$ denote a (fixed) solution of Problem* 2.2. *For small $\varepsilon > 0$, define the function $h_\varepsilon(x): \widetilde{\Gamma} \to \mathbb{R}$ such that*

$$x_\varepsilon := x + h_\varepsilon(x)\nu(x) \in \widetilde{\Gamma}_\varepsilon := T_\varepsilon(\widetilde{\Gamma}) \tag{3.1}$$

for each $x \in \widetilde{\Gamma}$, where $\nu(x)$ is the exterior unit normal to $\widetilde{\Gamma}$ at $x \in \widetilde{\Gamma}$ and $|h_\varepsilon(x)|$ is minimum subject to (3.1). *Then for each $x \in \widetilde{\Gamma}$, we have*

$$\lim_{\varepsilon \to 0+} (h_\varepsilon(x)/\varepsilon) = h(x), \tag{3.2}$$

where $h(x)$ denotes the unique solution (in the interval $(-\alpha(x), \beta(x))$) of the equation

$$(\alpha(x) + h(x))^{-2} - (\beta(x) - h(x))^{-2} = a^2(x). \tag{3.3}$$

Proof. Let the continuous, strictly-positive functions $h_\varepsilon^\pm(x): \widetilde{\Gamma} \to \mathbb{R}$ be defined such that

$$x_\varepsilon^\pm := x \pm h_\varepsilon^\pm(x)\nu(x) \in \widetilde{\Gamma}_\varepsilon^\pm := \Phi_\varepsilon^\pm(\widetilde{\Gamma}), \tag{3.4}$$

where $h_\varepsilon^\pm(x) > 0$ is minimum subject to (3.4). For fixed $x \in \widetilde{\Gamma}$, it follows from Theorem 2.5(b) that

$$d(x_\varepsilon, \widetilde{\Gamma}_\varepsilon^\pm) = |x_\varepsilon^\pm - x_\varepsilon|(1 + \zeta(\varepsilon)) = |\pm h_\varepsilon^\pm(x) - h_\varepsilon(x)|(1 + \zeta(\varepsilon)) \tag{3.5}$$

as $\varepsilon \to 0+$, where $\zeta(\varepsilon)$ denotes any function such that $\zeta(\varepsilon) \to 0$ as $\varepsilon \to 0+$. Since $x_\varepsilon \in T_\varepsilon(\widetilde{\Gamma}) = \Psi_\varepsilon(\widetilde{\Gamma}_\varepsilon^-, \widetilde{\Gamma}_\varepsilon^+)$, we conclude using (3.5) that

$$[\varepsilon/(h_\varepsilon^-(x) + h_\varepsilon(x))]^2 - [\varepsilon/(h_\varepsilon^+(x) - h_\varepsilon(x))]^2 = a^2(x) + \zeta(\varepsilon) \tag{3.6}$$

as $\varepsilon \to 0+$. For fixed $x \in \widetilde{\Gamma}$, the theorem of the mean implies that

$$\begin{aligned}\varepsilon = \widetilde{U}^\pm(x_\varepsilon^\pm) &= \nabla \widetilde{U}^\pm(x) \cdot (x_\varepsilon^\pm - x) + o(|x_\varepsilon^\pm - x|) \\ &= |\nabla \widetilde{U}^\pm(x)| h_\varepsilon^\pm(x) + o(h_\varepsilon^\pm(x))\end{aligned}$$

as $\varepsilon \to 0+$, from which it follows that

$$h_\varepsilon^\pm(x) = (\varepsilon/|\nabla \widetilde{U}^\pm(x)|) + o(\varepsilon) \tag{3.7}$$

as $\varepsilon \to 0+$. By substituting (3.7) into (3.6), we conclude (again for fixed $x \in \widetilde{\Gamma}$) that

$$(\alpha(x) + (h_\varepsilon(x)/\varepsilon) + \zeta(\varepsilon))^{-2} - (\beta(x) - (h_\varepsilon(x)/\varepsilon) + \zeta(\varepsilon))^{-2} = a^2(x) + \zeta(\varepsilon)$$

as $\varepsilon \to 0+$, from which (3.3) follows in the limit.

3.3. LEMMA. *Given a (fixed) solution $\widetilde{\Gamma}$ of Problem 2.2, let the function $h(x)\colon \widetilde{\Gamma} \to \mathbb{R}$ be defined as in Theorem 3.2. Then there exists a positive function $r(x)\colon \widetilde{\Gamma} \to \mathbb{R}$ such that*

$$0 < r(x) \leq 4 \cdot \max\{A^3(x), (a^2(x) + B^2(x))^{3/2}\},$$
$$h(x) = (A^2(x) - B^2(x) - a^2(x))/r(x),$$

both for all $x \in \widetilde{\Gamma}$.

Proof. Fix $x \in \widetilde{\Gamma}$ and let

$$g(t) = (\alpha + t)^{-2} - (\beta - t)^{-2} \quad \text{for } -\alpha < t < \beta,$$

with $\alpha = \alpha(x)$ and $\beta = \beta(x)$. Then $g(0) = A^2 - B^2$, and $g(h) = a^2$, with $A = A(x)$, $B = B(x)$, $a = a(x)$, $h = h(x)$, and $r = r(x)$. By the theorem of the mean, we have

$$A^2 - B^2 - a^2 = g(0) - g(h) = \phi(s)h = rh,$$

where s lies between 0 and h, and where

$$\phi(t) := |h'(t)| = 2((\beta - t)^{-3} + (\alpha + t)^{-3}).$$

It remains to determine an upper bound for $\phi(s)$. Clearly the function $\phi(t)\colon (-\alpha, \beta) \to \mathbb{R}$ takes its global minimum at its center point $t_0 = (\beta - \alpha)/2$, and is decreasing (increasing) to the left (right) of the center point. Therefore, if $0 \leq t_0 \leq s \leq h < \beta$, then

$$\phi(s) \leq \phi(h) = 2([(\alpha + h)^{-2} - a^{-2}]^{3/2} + (\alpha + h)^{-3}) \leq 4A^3,$$

where we used the fact that $g(h) = a^2$. On the other hand, if $0 \leq s \leq t_0$, then $\phi(s) \leq \phi(0) = 2(A^3 + B^3) \leq 4A^3$. If $-\alpha < h \leq s \leq t_0 \leq 0$, then

$$\phi(s) \leq \phi(h) = 2((\beta - h)^{-3} + [a^2 + (\beta - h)^{-2}]^{3/2}) \leq 4(a^2 + B^2)^{3/2}.$$

Finally, if $t_0 \leq s \leq 0$, then $\phi(s) \leq \phi(0) = 2(A^3 + B^3) \leq 4B^3$.

3.4. LEMMA. *Let $\widetilde{\Gamma} \in \mathbb{X}_c$ denote a uniformly C^1-surface, and let Γ_ε, $0 < \varepsilon < \varepsilon_0$, denote a family of (convex) surfaces in $\mathbb{X}_c$ such that*

$\Gamma_\varepsilon \subset N_{\lambda\varepsilon}(\widetilde{\Gamma})$ (= *the* $(\lambda\varepsilon)$*-neighborhood of* $\widetilde{\Gamma}$) *for each* ε, *where* λ *is some positive constant. For each* $x \in \widetilde{\Gamma}$ *and* $0 < \varepsilon < \varepsilon_0$, *let* $\nu(x) =$ *the exterior unit normal vector to* $\widetilde{\Gamma}$ *at* $x \in \widetilde{\Gamma}$, *let* $y(x, \varepsilon)$ *be the point closest to* x *in* $\Gamma_\varepsilon \cap L(x)$ (*here* $L(x) = \{x + \alpha\nu(x) : \alpha \in \mathbb{R}\}$), *and let* $\nu(x, \varepsilon)$ *denote a unit vector such that* $\nu(x, \varepsilon) \cdot (z - y(x, \varepsilon)) \le 0$ *for all* $z \in \Gamma_\varepsilon$. *Then* $\phi(\varepsilon) := \sup\{|\nu(x, \varepsilon) - \nu(x)| : x \in \widetilde{\Gamma}\} \to 0$ *as* $\varepsilon \to 0$.

Proof. Choose a point $x^* \in D(\widetilde{\Gamma})$, and define $\widetilde{\Gamma}_\varepsilon = r_\varepsilon\widetilde{\Gamma} := \{x^* + r_\varepsilon(x - x^*) : x \in \widetilde{\Gamma}\}$, where the value $r_\varepsilon > 0$ is maximum subject to the requirement that $\widetilde{\Gamma}_\varepsilon \le \Gamma_\varepsilon$. Clearly $r_\varepsilon = 1 + O(\varepsilon)$, and $\nu(x, \varepsilon) \cdot (z - y(x, \varepsilon)) \le 0$ for all $x \in \widetilde{\Gamma}$, $0 < \varepsilon < \varepsilon_0$, and $z \in \widetilde{\Gamma}_\varepsilon$. Assume the assertion of the lemma is not true. Then there exist a value $\rho_0 > 0$, a positive null-sequence (ε_n), a vector sequence $(x_n) \subset \widetilde{\Gamma}$, and a sequence of unit vectors $(\tilde{\nu}_n)$ such that $|\tilde{\nu}_n - \nu_n| \ge \rho_0$ and $\tilde{\nu}_n \cdot (z - y_n) \le 0$ for all n and for all $z \in \widetilde{\Gamma}_n := \widetilde{\Gamma}_{\varepsilon_n}$, where $y_n = y(x_n, \varepsilon_n)$ and $\nu_n = \nu(x_n)$. In fact we can assume that $\tilde{\nu}_n$ maximizes $|\nu - \nu_n|$ subject to the requirement that $|\nu| = 1$ and $\nu \cdot (z - y_n) \le 0$ for all $z \in \widetilde{\Gamma}_n$. It follows that $\tilde{\nu}_n \cdot (z_n - y_n) = 0$, for some point $z_n \in \widetilde{\Gamma}_n$, where $\tilde{\nu}_n$ is the exterior normal to the surface $\widetilde{\Gamma}_n$ at z_n. By passing to a subsequence if necessary, we can assume that $x_n, y_n \to x \in \widetilde{\Gamma}$, $\nu_n \to \nu(x)$, $z_n \to z \in \widetilde{\Gamma}$, and $\tilde{\nu}_n \to \nu(z)$, all as $n \to \infty$, where $|\nu(z) - \nu(x)| \ge \rho_0$ and $\nu(z)\cdot(z - x) = 0$. However, the second property (that $\nu(z)\cdot(z-x) = 0$) implies that $\nu(x) = \nu(z)$, contradicting the first property.

4. Variational formulas.

4.1. THEOREM. *Let* $\widetilde{\Gamma} \in \mathbb{X}_c$ *be a solution of Problem* 2.2 (*with* $m \ge 2$). *Then*:

(a) *We have*

$$K^\pm(T_\varepsilon(\widetilde{\Gamma})) - \widetilde{K}^\pm \le \pm\varepsilon \int_{\widetilde{\Gamma}} |\nabla \widetilde{U}^\pm|^2 h(x)\, ds + \varepsilon\zeta(\varepsilon), \tag{4.1}$$

$$\|\Omega^-(T_\varepsilon(\widetilde{\Gamma}))\| - \|\widetilde{\Omega}^-\| = \varepsilon \int_{\widetilde{\Gamma}} a^2(x) h(x)\, ds + \varepsilon\zeta(\varepsilon) \tag{4.2}$$

for $\varepsilon \to 0+$, *where* $\widetilde{U}^\pm(x) = U^\pm(\widetilde{\Gamma}; x)$, *etc., and the function* $h(x)$: $\widetilde{\Gamma} \to \mathbb{R}$ *was defined in Theorem* 3.2.

(b) *We have*

$$I(T_\varepsilon(\widetilde{\Gamma})) \le I(\widetilde{\Gamma}) - \varepsilon \int_{\widetilde{\Gamma}} ([A^2(x) - B^2(x) - a^2(x)]^2 / r(x))\, ds + \varepsilon\zeta(\varepsilon) \tag{4.3}$$

as $\varepsilon \to 0+$, *where* $A(x) = |\nabla \widetilde{U}^-|$, $B(x) = |\nabla \widetilde{U}^+|$, *and the functional* $I(\Gamma)\colon \mathbb{X}_c \to \mathbb{R}$ *was defined by* (2.3).

(c) $\widetilde{\Gamma}$ *solves Problem* 2.1 *in the weak sense that* $A^2(x) = B^2(x) + a^2(x)$ *almost everywhere on* $\widetilde{\Gamma}$ *relative to* $(m-1)$*-dimensional Euclidean area.*

4.2. *Proof of parts* (b) *and* (c). By adding the three estimates given in (4.1) and (4.2), one obtains the inequality

$$(4.4)\quad I(T_\varepsilon(\widetilde{\Gamma})) \le I(\widetilde{\Gamma}) - \varepsilon \int_{\widetilde{\Gamma}} (B^2(x) + a^2(x) - A^2(x))h(x)\,ds + \varepsilon\zeta(\varepsilon).$$

Then the estimate (4.3) follows from (4.4) by applying Lemma 3.3. Since $\widetilde{\Gamma}$ minimizes the functional $I(\Gamma)\colon \mathbb{X}_c \to \mathbb{R}$, and since $T_\varepsilon\colon \mathbb{X}_c \to \mathbb{X}_c$, we conclude that $I(T_\varepsilon(\widetilde{\Gamma})) \ge I(\widetilde{\Gamma})$. It then follows from (4.3) that $A^2(x) = B^2(x) + a^2(x)$ almost everywhere on $\widetilde{\Gamma}$.

4.3. *Heuristic argument for Theorem* 4.1, *parts* (b) *and* (c). In the case of sufficient regularity of $\widetilde{\Gamma}$, the variation δI in the functional $I(\Gamma)$ caused by application of the operator T_ε to $\widetilde{\Gamma}$ is given approximately (i.e. to first order) by

$$\delta I \approx \int_{\widetilde{\Gamma}} [B^2(x) + a^2(x) - A^2(x)]\delta\nu(x)\,ds$$

(by the Poincaré variational formula for capacity), where $\delta\nu(x)$ denotes the exterior normal variation in $\widetilde{\Gamma}$ at x which is induced by T_ε. However, Theorem 3.2 and Lemma 3.3 imply that

$$\delta\nu(x) = h_\varepsilon(x) \approx h(x)\cdot\varepsilon = ([A^2(x) - B^2(x) - a^2(x)]/r(x))\cdot\varepsilon,$$

so that the assertion follows by substitution. Then $A^2(x) = B^2(x) + a^2(x)$ on $\widetilde{\Gamma}$ by the proof of Theorem 4.2 given above (see [**9**, §5] and [**11**, Remark 5.6]).

4.4. Lemma. *Let* $\widetilde{\Gamma}$ *solve Problem* 2.2. *Then*: (a) *Each point* $x_0 \in \widetilde{\Gamma}$ *is the endpoint of at least one maximal curve of steepest ascent* $\tilde{\gamma}^\pm$ *of the function* $\widetilde{U}^\pm$ (*here* $\tilde{\gamma}^\pm \subset \mathrm{Cl}(\widetilde{\Omega}^\pm)$). (b) *We have that* $\nabla\widetilde{U}^\pm(x) \to \nabla\widetilde{U}^\pm(x_0)$ *and* $\pm(|\nabla\widetilde{U}^\pm(x)| - |\nabla\widetilde{U}^\pm(x_0)|) \uparrow 0$ *as* $x \to x_0$ *monotonically on* $\tilde{\gamma}^\pm$, *where* $\nabla\widetilde{U}^\pm(x_0)$ *is defined in Definition* 2.4. (c) *For each* $x_0 \in \widetilde{\Gamma}$, *the curve* $\tilde{\gamma}^\pm$ *of steepest ascent is uniquely determined.* (d) *The curve of steepest ascent depends continuously on the endpoint* $x_0 \in \widetilde{\Gamma}$. (e) *Let* U *denote one of the functions* $\widetilde{U}^\pm(x)\colon \mathrm{Cl}(\widetilde{\Omega}^\pm) \to \mathbb{R}$, *let* Γ_ε

denote the level surface of U *at altitude* $0 < \varepsilon < 1$, *and let* ds *denote Euclidean differential surface area on* $\widetilde{\Gamma}$ *or* Γ_ε. *Consider the function* $y = \pi_\varepsilon(x) \colon \Gamma_\varepsilon \to \widetilde{\Gamma}$ *such that* x *lies on the curve of steepest ascent of* U *beginning at* y. *Then* $y = \pi_\varepsilon(x)$ *is continuous on* Γ_ε, *and we have*

$$\int_{\Gamma_\varepsilon} \phi(\pi_\varepsilon(x))|\nabla U(x)|\,ds = \int_{\widetilde{\Gamma}} \phi(x)|\nabla U(x)|\,ds$$

for any function ϕ *which is in* $L^1(\widetilde{\Gamma}, \mathbb{R})$ *relative to the surface area.*

Proof. See the appendix.

4.5. *Notation for the Proof of Theorem* 4.1(a). We will devote the remainder of this section to the proof of Theorem 4.1(a). Actually, we will prove the estimate (4.1) only in the "$+$" case, since the proof in the "$-$" case is nearly identical. The much more elementary proof of (4.2) will be omitted. Throughout the remainder of this section, $\widetilde{\Gamma}$ denotes a specific convex minimizer of the functional $I(\Gamma) \colon \mathbb{X}_c \to \mathbb{R}$. For small $0 < \varepsilon < 1$, we define $\widetilde{\Gamma}_\varepsilon = T_\varepsilon(\widetilde{\Gamma})$, $\overline{\Gamma}_\varepsilon = \{U^+(\widetilde{\Gamma}_\varepsilon; x) = \varepsilon^2\}$, and $\Gamma_\varepsilon = \Phi^+_{\lambda\varepsilon}(\widetilde{\Gamma}) = \{U^+(\widetilde{\Gamma}; x) = \lambda\varepsilon\}$, where the constant $\lambda > 0$ is chosen such that $\{U^+(\widetilde{\Gamma}; x) = \lambda\varepsilon/2\} > \overline{\Gamma}_\varepsilon$ for all sufficiently small $\varepsilon > 0$. Since we plan to explicitly prove (4.1) only in the "$+$" case, we simplify the notation by omitting the superscript "$+$". Thus, Γ^+ becomes Γ^*, and we use $\widetilde{\Omega}_\varepsilon$, $\widetilde{U}_\varepsilon$, $\widetilde{K}_\varepsilon$, $\overline{\Omega}_\varepsilon$, $\overline{U}_\varepsilon$, $\overline{K}_\varepsilon$, Ω_ε, U_ε, K_ε, to denote the annular domains $\Omega^+(\widetilde{\Gamma}_\varepsilon)$, $\Omega^+(\overline{\Gamma}_\varepsilon)$, $\Omega^+(\Gamma_\varepsilon)$, their respective capacity potentials, and their respective capacities. For small $\varepsilon > 0$, and for each $x \in \widetilde{\Gamma}$, x_ε denotes the point in Γ_ε which is joined to x by a curve of steepest ascent of $\widetilde{U}$, and $\overline{x}_\varepsilon$ denotes the point in $\overline{\Gamma}_\varepsilon$ which is joined to x_ε by a curve of steepest ascent of $\overline{U}_\varepsilon$. The three variables x, x_ε, $\overline{x}_\varepsilon$ are related to each other in a bijective, continuous way (see Lemma 4.4). $C_\varepsilon(x)$ denotes the curve of steepest ascent of $\widetilde{U}$ joining x to x_ε, and $\overline{C}_\varepsilon(x)$ denotes the curve of steepest ascent of $\overline{U}_\varepsilon$ joining $\overline{x}_\varepsilon$ to x_ε. On $\widetilde{\Gamma}$, we let $f(x) = g(x) = |\nabla\widetilde{U}(x)|$, $f_\varepsilon(x) = |\nabla\widetilde{U}(x_\varepsilon)|$, $g_\varepsilon(x) = |\nabla\overline{U}_\varepsilon(x_\varepsilon)|$, $Q_\varepsilon(x) = (|\nabla\widetilde{U}(x)|/|\nabla\widetilde{U}(x_\varepsilon)|) = f(x)/f_\varepsilon(x)$, $\overline{h}_\varepsilon(x) = (|\overline{C}_\varepsilon(x)|/\varepsilon)$, and $\overline{h}(x) = (\lambda/|\nabla\widetilde{U}(x)|) - h(x)$, where $|\overline{C}_\varepsilon(x)|$ refers to the arc-length of $\overline{C}_\varepsilon(x)$, and where $h(x)$ is defined in Theorem 3.2. Observe that the functions f, g, h, $\overline{h}$, $Q_\varepsilon \colon \widetilde{\Gamma} \to \mathbb{R}$ are bounded and measurable (in terms of the $(m-1)$-dimensional area measure on $\widetilde{\Gamma}$), whereas the functions f_ε, g_ε, $\overline{h}_\varepsilon \colon \widetilde{\Gamma} \to \mathbb{R}$ are continuous.

4.6. LEMMA. *Let $\widetilde{\Gamma}$ solve Problem* 2.2. *Then*

$$(\widetilde{K}-\widetilde{K}_\varepsilon)/\varepsilon \ge \int_{\widetilde{\Gamma}} f(x)g_\varepsilon(x)\overline{h}_\varepsilon(x)\,ds - \lambda\widetilde{K} - \zeta(\varepsilon) \tag{4.5}$$

as $\varepsilon \to 0+$.

Proof. We have

$$\widetilde{K}-\widetilde{K}_\varepsilon = (\widetilde{K}-K_\varepsilon)+(K_\varepsilon-\overline{K}_\varepsilon)+(\overline{K}_\varepsilon-\widetilde{K}_\varepsilon).$$

Now $(1-\lambda\varepsilon)U_\varepsilon = \widetilde{U}-\lambda\varepsilon$ in Ω_ε, from which it follows that $(1-\lambda\varepsilon)|\nabla U_\varepsilon| = |\nabla\widetilde{U}|$ and $(1-\lambda\varepsilon)K_\varepsilon = \widetilde{K}$. Therefore $K_\varepsilon - \widetilde{K} = \lambda\widetilde{K}\varepsilon + O(\varepsilon^2)$ as $\varepsilon \to 0+$. A similar argument shows that $\overline{K}_\varepsilon - \widetilde{K}_\varepsilon = O(\varepsilon^2)$ as $\varepsilon \to 0+$. By applying Green's second identity to the functions U_ε and $(U_\varepsilon - \overline{U}_\varepsilon)$ in the domain Ω_ε, one easily sees that

(4.6)

$$\begin{aligned} K_\varepsilon - \overline{K}_\varepsilon &= \int_{\Gamma^*} \frac{\partial}{\partial\nu}(U_\varepsilon - \overline{U}_\varepsilon)\,ds = \int_{\Gamma_\varepsilon} \overline{U}_\varepsilon \frac{\partial}{\partial\nu} U_\varepsilon\,ds \\ &= \frac{1}{1-\lambda\varepsilon}\int_{\Gamma_\varepsilon} |\nabla\widetilde{U}|\overline{U}_\varepsilon(x)\,ds \ge \int_{\Gamma_\varepsilon} |\nabla\widetilde{U}|\overline{U}_\varepsilon(x)\,ds - \varepsilon\zeta(\varepsilon). \end{aligned}$$

But for $x_\varepsilon \in \Gamma_\varepsilon$ (corresponding to $x \in \widetilde{\Gamma}$), we have

$$\overline{U}_\varepsilon(x_\varepsilon) = \int_{\overline{C}_\varepsilon(x)} |\nabla\overline{U}_\varepsilon(y)||dy| \ge |\nabla\overline{U}_\varepsilon(x_\varepsilon)||\overline{C}_\varepsilon(x)|,$$

due to the monotonicity of $|\nabla\overline{U}_\varepsilon|$ on the curve $\overline{C}_\varepsilon(x)$ (see Lemma 2.7(b)). Also, the differential areas on the surfaces Γ_ε and $\widetilde{\Gamma}$ are related by $ds_\varepsilon = Q_\varepsilon(x)\,ds$ (see Lemma 4.4(e)). Therefore, (4.6) implies that

$$K_\varepsilon - \overline{K}_\varepsilon \ge \int_{\widetilde{\Gamma}} |\nabla\widetilde{U}(x)||\nabla\overline{U}_\varepsilon(x_\varepsilon)||\overline{C}_\varepsilon(x)|\,ds - \varepsilon\zeta(\varepsilon)$$

as $\varepsilon \to 0+$, from which (4.5) follows.

4.7. LEMMA. *Given* $\varepsilon_0 > 0$, *let* $\mathbb{Y}_c(\varepsilon_0)$ *denote the set of all convex* C^2-*surfaces* Γ *and that* $N_{\varepsilon_0}(D(\Gamma)) \subset D^* := \operatorname{int}(\Gamma^*)$ *and* $B_{\varepsilon_0}(x_0) \subset D(\Gamma)$ *for some* $x_0 \in D^*$. *Then the integral* $\int_\Gamma |\nabla U(x)|^2\,ds$ *is uniformly bounded over all* $\Gamma \in \mathbb{Y}_c(\varepsilon_0)$.

Proof. Given a surface $\Gamma \in \mathbb{Y}_c(\varepsilon_0)$ (and a corresponding point x_0 with $B_{\varepsilon_0}(x_0) \subset D(\Gamma)$), let $\Gamma_\delta = \{x \in \Omega(\Gamma) \colon d(x,\Gamma) = \delta\}$ for sufficiently small $\delta > 0$. Also let the value $r(\delta) > 1$ be minimum subject to the requirement that $r(\delta)\Gamma \ge \Gamma_\delta$, where we define

$\alpha\Gamma = \{x_0 + \alpha(x - x_0) \colon x \in \Gamma\}$. By the Poincaré variational formula for capacity, we have $\int_\Gamma |\nabla U(x)|^2\, ds = \text{limit}_{\delta \to 0+}[(K_\delta - K)/\delta]$. Also

$$\begin{aligned} K_\delta &:= K(\Gamma_\delta, \Gamma^*) \le K(r(\delta)\Gamma, \Gamma^*) \\ &= (r(\delta))^{2-m} K(\Gamma, (1/r(\delta))\Gamma^*) = K(\Gamma, \Gamma^*) + O(\delta), \end{aligned}$$

where (for this proof only) the notation $K(\gamma, \gamma^*)$ denotes the capacity of an annular domain with boundary components γ and γ^*. The fact that $K(\Gamma, (1/r(\delta))\Gamma^*) = K(\Gamma, \Gamma^*) + O(\delta)$ follows from the Lipschitz continuity of U near Γ^*, which is uniform relative to variations in $\Gamma \in \mathbb{Y}_c(\varepsilon_0)$. (See [**2**, §7] for an analogous argument.)

4.8. Lemma. *The integral* $\int_{\widetilde{\Gamma}} g_\varepsilon^2(x)\, ds = \int_{\widetilde{\Gamma}} |\nabla \overline{U}_\varepsilon(x_\varepsilon)|^2\, ds$ *is uniformly bounded as* $\varepsilon \to 0+$.

Proof. For small $\varepsilon > 0$, the differential surface areas of the surfaces Γ_ε and $\overline{\Gamma}_\varepsilon$ are related by the equation

$$|\nabla \overline{U}_\varepsilon(x_\varepsilon)|\, ds_\varepsilon = \alpha(\varepsilon, x_\varepsilon)|\nabla \overline{U}_\varepsilon(\overline{x}_\varepsilon)|\, d\overline{s}_\varepsilon,$$

where $\alpha(\varepsilon, x_\varepsilon) = [1/\cos(\theta(\varepsilon, x_\varepsilon))]$ and $\theta(\varepsilon, x_\varepsilon)$ denotes the angle between the vectors $\nabla \overline{U}_\varepsilon(x_\varepsilon)$ and $\nabla \widetilde{U}(x_\varepsilon)$. Since $\Gamma_\varepsilon, \overline{\Gamma}_\varepsilon \to \widetilde{\Gamma}$ as $\varepsilon \to 0+$ (in the polar coordinate maximum norm relative to a point in the interior complement of $\widetilde{\Gamma}$), we conclude from Lemma 3.4 that $\theta(\varepsilon, x_\varepsilon) \to 0$ and $\alpha(\varepsilon, x_\varepsilon) \to 1$, both uniformly over $x \in \widetilde{\Gamma}$, as $\varepsilon \to 0+$. Therefore,

$$\begin{aligned} (4.7)\quad \int_{\Gamma_\varepsilon} |\nabla \overline{U}_\varepsilon(x_\varepsilon)|^2\, ds_\varepsilon &= \int_{\overline{\Gamma}_\varepsilon} |\nabla \overline{U}_\varepsilon(x_\varepsilon)||\nabla \overline{U}_\varepsilon(\overline{x}_\varepsilon)|\alpha(\varepsilon, x_\varepsilon)\, d\overline{s}_\varepsilon \\ &\le \int_{\overline{\Gamma}_\varepsilon} |\nabla \overline{U}_\varepsilon(\overline{x}_\varepsilon)|^2 \alpha(\varepsilon, x_\varepsilon)\, d\overline{s}_\varepsilon \le M_\varepsilon \int_{\overline{\Gamma}_\varepsilon} |\nabla \overline{U}_\varepsilon(\overline{x}_\varepsilon)|^2\, d\overline{s}_\varepsilon, \end{aligned}$$

where $M_\varepsilon = \sup\{\alpha(\varepsilon, y) \colon y \in \Gamma_\varepsilon\}$. Both M_ε and the integral

$$\int_{\overline{\Gamma}_\varepsilon} |\nabla \overline{U}_\varepsilon(x)|^2\, d\overline{s}_\varepsilon$$

are bounded as $\varepsilon \to 0+$, as follows from Lemmas 3.4 and 4.7. Therefore, the first integral in (4.7) is uniformly bounded as $\varepsilon \to 0+$, and the assertion follows from the fact that the differential areas on the surfaces Γ_ε and $\widetilde{\Gamma}$ are related by $ds_\varepsilon = Q_\varepsilon(x)\, ds$ (see Lemma 4.4(e)), where $(1/Q_\varepsilon(x))$ is uniformly bounded from above as $\varepsilon \to 0+$ and x varies in $\widetilde{\Gamma}$.

4.9. LEMMA. *Let $f \in L^2(\widetilde{\Gamma}, \mathbb{R}^+)$, and let E denote a family of positive L^2-functions $\phi(x)\colon \widetilde{\Gamma} \to \mathbb{R}^+$ such that $\|f - \phi\|$ is arbitrarily small for a suitable choice of $\phi \in E$ (where $\|\cdot\|$ denotes the L^2-norm on $\widetilde{\Gamma}$). Let $g_\varepsilon(x)$, $0 < \varepsilon < \varepsilon_0$, denote a family of functions in $L^2(\widetilde{\Gamma}, \mathbb{R})$ such that $\|g_\varepsilon\|$ is uniformly bounded as $\varepsilon \to 0+$ and* $\liminf_{\varepsilon\to 0+} \int_{\widetilde{\Gamma}} \phi g_\varepsilon \, ds \geq 0$ *for each* $\phi \in E$. *Then* $\liminf_{\varepsilon\to 0+} \int_{\widetilde{\Gamma}} f g_\varepsilon \, ds \geq 0$.

Proof. $\int_{\widetilde{\Gamma}} f g_\varepsilon \, ds = \int_{\widetilde{\Gamma}} \phi g_\varepsilon \, ds + \int_{\widetilde{\Gamma}} (f - \phi) g_\varepsilon \, ds \geq \int_{\widetilde{\Gamma}} \phi g_\varepsilon \, ds - \|f - \phi\| \|g_\varepsilon\|$.

4.10. LEMMA. *For $\varepsilon \to 0+$, we have*

$$\int_{\widetilde{\Gamma}} f(x)\overline{h}(x)[g(x) - g_\varepsilon(x)]\, ds \leq \zeta(\varepsilon). \tag{4.8}$$

Proof. Let S denote a simply connected $(m-1)$-dimensional subsurface of $\widetilde{\Gamma}$ whose boundary relative to $\widetilde{\Gamma}$ is a smooth, closed curve. Given small $\varepsilon > 0$, let S_ε denote the $(m-1)$-dimensional surface of points $x \in \widetilde{\Omega}$ such that $\widetilde{U}(x) = \lambda\varepsilon$ and such that x is joined to S by a curve of steepest ascent of $\widetilde{U}$. For small $\delta > \varepsilon$, let $\omega_{\varepsilon,\delta}$ denote the set of all points $x \in \widetilde{\Omega}$ such that $\lambda\varepsilon < \widetilde{U}(x) < \lambda\delta$ and such that x is joined to S_ε by a curve of steepest ascent of $\overline{U}_\varepsilon$. Also let $S_{\delta,\varepsilon} = (\partial\omega_{\delta,\varepsilon}) \cap \Gamma_\delta$ and $\sigma_{\delta,\varepsilon} = (\partial\omega_{\delta,\varepsilon}) \cap \{\lambda\varepsilon < \widetilde{U} < \lambda\delta\}$. We have $\overline{U}_\varepsilon(x) \leq \zeta(d(x, \overline{\Gamma}_\varepsilon))$ in $\overline{\Omega}_\varepsilon$, uniformly for small $\varepsilon > 0$, as follows from a domain comparison argument using the convexity of $\overline{\Gamma}_\varepsilon$. Thus $|\overline{U}_\varepsilon - \widetilde{U}| \leq \zeta(\varepsilon)$ on $\partial(\widetilde{\Omega} \cap \overline{\Omega}_\varepsilon)$, and it follows by the maximum principle that $|\overline{U}_\varepsilon - \widetilde{U}| \leq \zeta(\varepsilon)$ uniformly in $\widetilde{\Omega} \cap \overline{\Omega}_\varepsilon$ as $\varepsilon \to 0+$. Therefore, the standard estimate for derivatives of harmonic functions shows that $|\nabla(\overline{U}_\varepsilon - \widetilde{U})| \leq \zeta_0(\varepsilon)/\delta$ on $S_{\delta,\varepsilon}$, uniformly over small $\varepsilon, \delta > 0$ with $\delta \geq 2\varepsilon$. Here, $\zeta_0(\varepsilon)$ denotes a specific function such that $\zeta_0(\varepsilon) \to 0$ as $\varepsilon \to 0+$ (we assume w.l.o.g. that $(\zeta_0(\varepsilon)/\varepsilon) \to \infty$ as $\varepsilon \to 0+$). We choose $\delta = [\zeta_0(\varepsilon)]^{1/2} > \varepsilon$, so that $\int_{S_{\delta,\varepsilon}} |\nabla(\overline{U}_\varepsilon - \widetilde{U})|\, ds \leq [\zeta_0(\varepsilon)]^{1/2}$ as $\varepsilon \to 0+$. Since $\nabla\widetilde{U}$ is bounded and the vector $\nabla\overline{U}_\varepsilon$ is always tangent to the surface $\sigma_{\delta,\varepsilon}$, it follows by applying the divergence theorem to $\nabla(\overline{U}_\varepsilon - \widetilde{U})$ in $\omega_{\delta,\varepsilon}$ that

$$\int_{S_\varepsilon} (|\nabla\overline{U}_\varepsilon| - |\nabla\widetilde{U}|)\, ds \geq \int_{S_\varepsilon} \frac{\partial}{\partial\nu}(\overline{U}_\varepsilon - \widetilde{U})\, ds \geq -\zeta(\varepsilon) \tag{4.9}$$

as $\varepsilon \to 0+$. Since the differential surface areas on the surfaces Γ_ε and $\widetilde{\Gamma}$ are related by $ds_\varepsilon = Q_\varepsilon(x)\, ds$ (see Lemma 4.4(e)), it follows from (4.9) that $\int_S (g - g_\varepsilon Q_\varepsilon)\,ds \le \zeta(\varepsilon)$ as $\varepsilon \to 0+$. Therefore

$$\begin{aligned}\int_S (g - g_\varepsilon)\,ds &= \int_S g_\varepsilon(Q_\varepsilon - 1)\,ds + \int_S (g - g_\varepsilon Q_\varepsilon)\,ds \\ &\le \left(\int_S g_\varepsilon^2(x)\,ds\right)^{1/2} \left(\int_S (Q_\varepsilon - 1)^2\,ds\right)^{1/2} + \zeta(\varepsilon) = \zeta(\varepsilon),\end{aligned}$$

where we have used the Schwartz inequality, Lemma 4.8, and the fact that the functions $Q_\varepsilon(x)\colon \widetilde{\Gamma} \to \mathbb{R}$ are uniformly bounded independent of small $\varepsilon > 0$ and converge pointwise to unity as $\varepsilon \to 0+$ (due to Theorem 2.5(b)). The assertion now follows by applying Lemma 4.9, since $f(x)\overline{h}(x)\colon \widetilde{\Gamma} \to \mathbb{R}^+$ is a bounded, measurable function which can be approximated in the L^2-norm by piecewise constant functions $\phi(x)\colon \widetilde{\Gamma} \to \mathbb{R}^+$, chosen such that each domain of constant ϕ is a subsurface $\widetilde{S}$ with the properties assumed above.

4.11. Lemma. *We have that* $\int_{\widetilde{\Gamma}} (\overline{h}_\varepsilon - \overline{h})^2\,ds \to 0$ *as* $\varepsilon \to 0+$, *in fact* $\overline{h}_\varepsilon(x) \to \overline{h}(x)$ *pointwise on* $\widetilde{\Gamma}$ *as* $\varepsilon \to 0+$.

Proof. The functions $\overline{h}(x)$, $\overline{h}_\varepsilon(x)\colon \widetilde{\Gamma} \to \mathbb{R}$ are measurable and uniformly bounded independent of small $\varepsilon > 0$. Therfore, it suffices to prove the pointwise convergence. For fixed $x \in \widetilde{\Gamma}$, it is easily seen using Lemma 3.4 that $|\nu(y) - \nu(x)| \le \zeta(\varepsilon)$ uniformly over all $y \in C_\varepsilon(x)$ and $|\overline{\nu}_\varepsilon(y) - \nu(x)| \le \zeta(\varepsilon)$ uniformly over all $y \in \overline{C}_\varepsilon(x)$, where $\nu(y) = \nabla\widetilde{U}(y)/|\nabla\widetilde{U}(y)|$ and $\overline{\nu}_\varepsilon(y) = \nabla\overline{U}_\varepsilon(y)/|\nabla\overline{U}_\varepsilon(y)|$. It follows that $(x_\varepsilon - x)\cdot\nu(x) = [(\lambda/|\nabla\widetilde{U}(x)|) + \zeta(\varepsilon)]\varepsilon$ (using Theorem 2.5(b)), $(\overline{x}_\varepsilon - x)\cdot\nu(x) = (h(x) + \zeta(\varepsilon))\varepsilon$ (using Theorem 3.2), and $(x_\varepsilon - \overline{x}_\varepsilon)\cdot\nu(x) = |\overline{C}_\varepsilon(x)|(1 + \zeta(\varepsilon)) = (\overline{h}_\varepsilon(x) + \zeta(\varepsilon))\varepsilon$, all as $\varepsilon \to 0+$. The assertion follows from the definition: $\overline{h}(x) = (\lambda/|\nabla\widetilde{U}(x)|) - h(x)$ by comparing these equations.

4.12. *Proof of* (4.1) *in the* " + " *case.* By Lemma 4.6, we have

$$(\widetilde{K} - \widetilde{K}_\varepsilon)/\varepsilon \ge \int_{\widetilde{\Gamma}} f(x)g_\varepsilon(x)\overline{h}_\varepsilon(x)\,ds - \lambda\widetilde{K} - \zeta(\varepsilon)$$

as $\varepsilon \to 0+$. Also, we have

$$\begin{aligned}\int_{\widetilde{\Gamma}} f(g_\varepsilon\overline{h}_\varepsilon - g\overline{h})\,ds &= \int_{\widetilde{\Gamma}} f\overline{h}(g_\varepsilon - g)\,ds + \int_{\widetilde{\Gamma}} fg_\varepsilon(\overline{h}_\varepsilon - \overline{h})\,ds \\ &\ge \int_{\widetilde{\Gamma}} f\overline{h}(g_\varepsilon - g)\,ds - \left(\int_{\widetilde{\Gamma}} (fg_\varepsilon)^2\,ds\right)^{1/2} \left(\int_{\widetilde{\Gamma}} (\overline{h}_\varepsilon - \overline{h})^2\,ds\right)^{1/2} \\ &\ge -\zeta(\varepsilon),\end{aligned}$$

where we have used Lemmas 4.8, 4.10, and 4.11. Therefore,

$$\begin{aligned}(\widetilde{K}-\widetilde{K}_\varepsilon)/\varepsilon &\geq \int_{\widetilde{\Gamma}} f(x)g(x)\overline{h}(x)\,ds - \lambda\widetilde{K} - \zeta(\varepsilon)\\ &= \int_{\widetilde{\Gamma}} |\nabla\widetilde{U}(x)|^2(\overline{h}(x) - (\lambda/|\nabla\widetilde{U}(x)|))\,ds - \zeta(\varepsilon)\\ &= -\int_{\widetilde{\Gamma}} |\nabla\widetilde{U}(x)|^2 h(x)\,ds - \zeta(\varepsilon),\end{aligned}$$

completing the proof of (4.1).

4.13. REMARK. The estimate (4.3) does not apply only to minimizers. It applies to any surface $\widetilde{\Gamma} \in \mathbb{X}_c$ having the properties asserted in Theorem 2.5(b). It is hoped that it could be extended in a meaningfull way to all $\Gamma \in \mathbb{X}_c$ (see [**3**, Theorem 3]). This could provide the basis for a successive approximation scheme for solutions of Problem 2.1 (as well as the generalizations in §§6, 7) which is valid in the absence of uniqueness (see [**3**, §5]).

5. Regularity of the free boundary.

5.1. THEOREM. *Let $\widetilde{\Gamma} \in \mathbb{X}_c$ be a solution of Problem 2.2 such that $A^2(x) = B^2(x) + a^2(x)$ almost everywhere on $\widetilde{\Gamma}$ (in terms of $(m-1)$-dimensional surface measure), where $A(x) = |\nabla\widetilde{U}^-|$ and $B(x) = |\nabla\widetilde{U}^+|$. Then in fact we have $A^2(x) = B^2(x) + a^2(x)$ at every point $x \in \widetilde{\Gamma}$. Moreover, the functions $A(x), B(x)\colon \widetilde{\Gamma} \to \mathbb{R}^+$ are continuous.*

5.2. LEMMA. *Let $\widetilde{\Gamma}$ solve Problem 2.2. Then the function $A(x)\colon \widetilde{\Gamma} \to \mathbb{R}^+$ is upper semicontinuous ($A(x_0) \geq \limsup_{x\to x_0} A(x)$) and the function $B(x)\colon \widetilde{\Gamma} \to \mathbb{R}^+$ is lower semicontinuous ($B(x_0) \leq \liminf_{x\to x_0} B(x)$).*

Proof. We prove the second assertion. Let γ denote the unique curve of steepest ascent of the function $\widetilde{U}^+$ beginning at the point $x_0 \in \widetilde{\Gamma}$. Then $|\nabla\widetilde{U}^+|$ is continuous and decreasing with increasing $\widetilde{U}^+$ on γ, and $|\nabla\widetilde{U}^+(x)| \to B(x_0)$ as $x \to x_0$ in γ. Given $\varepsilon > 0$, choose $x_\varepsilon \in \gamma$ such that $|\nabla\widetilde{U}^+(x_\varepsilon)| > B(x_0) - (\varepsilon/2)$. Then choose $\delta > 0$ such that $|\nabla\widetilde{U}^+| > B(x_0) - \varepsilon$ in the ball $B_\delta(x_\varepsilon)$. Let S denote the set of all points in $\widetilde{\Gamma}$ which are joined to $B_\delta(x_\varepsilon)$ by curves of steepest ascent of $\widetilde{U}^+$. Then $B(x) > B(x_0) - \varepsilon$ for all $x \in S$. Moreover, S contains a neighborhood of x_0 relative to $\widetilde{\Gamma}$, as follows from Lemma 4.4(d). A similar argument applies to the first assertion. We remark that the preceding argument has been previously used in [**4**, §5, part 2], [**8**, §5.7], and [**21**, Proposition 2.3].

5.3. DEFINITION. Given a solution $\widetilde{\Gamma}$ of Problem 2.2 and a point $x_0 \in \widetilde{\Gamma}$, we define the blow-up functions

$$\widetilde{U}_n^{\pm}(x) := 2^n \widetilde{U}^{\pm}(x_0 + 2^{-n}(x - x_0)), \qquad n \in \mathbb{N},$$

in the blow-up domains $\widetilde{\Omega}_n^{\pm} := \{x_0 + 2^n(x - x_0) \colon x \in \widetilde{\Omega}^{\pm}\}$, with common boundary $\widetilde{\Gamma}_n := \{x_0 + 2^n(x - x_0) \colon x \in \widetilde{\Gamma}\}$ (see [**16**, §4]). Observe that the differentiability of the functions $\widetilde{U}^{\pm}(x)$ at the point $x_0 \in \widetilde{\Gamma}$ (see Definition 2.4) is equivalent to the property that

$$\widetilde{U}_n^{\pm}(x) = \pm\lambda^{\pm}\nu(x_0) \cdot (x - x_0) + 2^n o(2^{-n}|x - x_0|) \tag{5.1}$$

relative to the set $\mathrm{Cl}(\widetilde{\Omega}_n^{\pm})$.

5.4. LEMMA. *Assume at a point $x_0 \in \widetilde{\Gamma}$ that $|\nabla \widetilde{U}^{\pm}| = \lambda^{\pm}$ for values $\lambda^{\pm} > 0$. Then for any given value $\eta > 0$, we have*

$$\pm 2^n \int_{\tilde{\gamma}_n} (|\nabla \widetilde{U}^{\pm}|^2 - (\lambda^{\pm})^2)\, ds = \pm \int_{C_n} (|\nabla \widetilde{U}_n^{\pm}|^2 - (\lambda^{\pm})^2)\, ds < \eta \tag{5.2}$$

for all sufficiently large $n \in \mathbb{N}$, where $\tilde{\gamma}_n := \{x \in \widetilde{\Gamma} \colon |x - x_0| < 2^{-n}\}$, $C_n = \{x \in \widetilde{\Gamma}_n \colon |x - x_0| < 1\}$, and where $\widetilde{U}_n^{\pm}$ and $\widetilde{\Gamma}_n$ were defined in §5.3.

Proof. We will prove the assertion in the " + " case in detail and then remark briefly on the proof of the " − " case. The proof is expressed in the blow-up notation of Definition 5.3. Since the entire proof concerns a fixed solution surface $\widetilde{\Gamma}$ of Problem 2.2, and is restricted to " + " case, we simplify the notation by deleting the tilde and the plus sign, so that $\widetilde{U}_n^{+}(x)$, $\widetilde{\Omega}_n^{+}$, $\widetilde{\Gamma}_n$, and λ^{+} become $U_n(x)$, Ω_n, Γ_n, and λ. We also choose Cartesian coordinates such that $x_0 = 0$, $\nu(0) = (0, \dots, 0, 1)$, and $x = (y, z) = (y_1, \dots, y_{m-1}, z)$. Let $Q(y)$ denote a convex, radially symmetric, C^2-function of y such that $Q(0) = -2\lambda$, $\nabla_y Q(0) = 0$, and $Q(y) = 0$ for $|y| = 1$. Our proof is based on Green's second identity, in the form

$$\begin{aligned} &\int_{\Omega_{\delta,\varepsilon,n}} (\psi_n \Delta\phi_n - \phi_n \Delta\psi_n)\, dx \\ &\qquad = \int_{\partial\Omega_{\delta,\varepsilon,n}} \left(\psi_n \frac{\partial}{\partial\nu}\phi_n - \phi_n \frac{\partial}{\partial\nu}\psi_n\right) ds, \end{aligned} \tag{5.3}$$

where we define $\phi_n := |\nabla U_n|^2 - \lambda^2$, $\psi_n := (U_n + Q)$, and

$$\Omega_{\delta,\varepsilon,n} := \{x \in \Omega_n : U_n > \delta, \ z < \varepsilon, \ U_n + Q < 0\}$$

for all small $\delta > 0$, $\varepsilon > 0$, and large $n \in \mathbb{N}$. For (δ/ε) sufficiently small, a partition of $\partial\Omega_{\delta,\varepsilon,n}$ into disjoint surfaces is given by $\partial\Omega_{\delta,\varepsilon,n} = \Gamma_{\delta,n} \cup L_{\varepsilon,n} \cup \Sigma_{\delta,\varepsilon,n}$, where $\Gamma_{\delta,n} = \{U_n = \delta, \ U_n + Q \le 0\}$, $L_{\varepsilon,n} = \{z = \varepsilon, \ U_n + Q \le 0\}$, and $\Sigma_{\delta,\varepsilon,n} = \{U_n > \delta, \ z < \varepsilon, \ U_n + Q = 0\}$. Now $\Delta\phi_n \ge 0$ and $\psi_n \le 0$ in $\Omega_{\delta,\varepsilon,n}$, whence $\psi_n \Delta\phi_n \le 0$ in $\Omega_{\delta,\varepsilon,n}$. Also, $\sup\{|\phi_n(x)\Delta\psi_n| : x \in \Omega_{\delta,\varepsilon,n}\} \le M$, uniformly for all small $\delta, \varepsilon > 0$ and large $n \in \mathbb{N}$, because $\Delta\psi_n = \Delta Q$ and because $\phi_n := (|\nabla U_n|^2 - \lambda^2)$ is uniformly bounded by Theorem 2.5(b) and the identity: $\nabla U_n(x) = \nabla U(x_0 + 2^{-n}(x - x_0))$ (where $U = \widetilde{U}^+$). Finally, we have $|\Omega_{\delta,\varepsilon,n}| \le O(\varepsilon) + \zeta(2^{-n})$ (independent of $\delta > 0$) because $\widetilde{\Gamma}$ is a uniformly C^1-surface, where $|\cdot|$ denotes Euclidean volume. Thus

$$\int_{\Omega_{\delta,\varepsilon,n}} (\psi_n \Delta\phi_n - \phi_n \Delta\psi_n)\, dx \le O(\varepsilon) + \zeta(2^{-n}) \tag{5.4}$$

as $\delta, \varepsilon \to 0+$ and $n \to \infty$. Also we have

$$\int_{L_{\varepsilon,n}} \left(\psi_n \frac{\partial}{\partial\nu}\phi_n - \phi_n \frac{\partial}{\partial\nu}\psi_n\right) ds = \zeta_\varepsilon(2^{-n}), \tag{5.5}$$

where for each $\varepsilon > 0$, $\zeta_\varepsilon(t)$ denotes a function such that $\zeta_\varepsilon(t) \to 0$ as $t \to 0+$. This is because $\max\{|\psi_n|, |\nabla\psi_n| : x \in L_{\varepsilon,n}\}$ is uniformly bounded for fixed $\varepsilon > 0$ as $n \to \infty$, while $\max\{|\phi_n(x)|, |\partial\phi_n(x)/\partial\nu| : x \in L_{\varepsilon,n}\} \to 0$ as $n \to \infty$ for fixed $\varepsilon > 0$, as is easily deduced from (5.1) and maximum principles and a standard derivative estimate. We also have

$$\int_{\Sigma_{\delta,\varepsilon,n}} \left(\psi_n \frac{\partial}{\partial\nu}\phi_n - \phi_n \frac{\partial}{\partial\nu}\psi_n\right) ds = -\int_{\Sigma_{\delta,\varepsilon,n}} \phi_n \frac{\partial}{\partial\nu}\psi_n\, ds = O(\varepsilon) \tag{5.6}$$

as $\delta, \varepsilon \to 0+$, independent of $n \in \mathbb{N}$, because $\psi_n = 0$ on $\Sigma_{\delta,\varepsilon,n}$, and because both ϕ_n and $\nabla\psi_n$ remain uniformly bounded in a uniform neighborhood of Γ_n as $n \to \infty$, while the surface area of $\Sigma_{\delta,\varepsilon,n}$ is bounded by $O(\varepsilon)$ (independent of $\delta > 0$ and large $n \in \mathbb{N}$) as $\varepsilon \to 0+$. By substituting (5.4), (5.5), and (5.6) into (5.3), one obtains

$$\int_{\Gamma_{\delta,n}} (\phi_n \nabla\psi_n - \psi_n \nabla\phi_n) \cdot \nu_n\, ds \le O(\varepsilon) + \zeta_\varepsilon(2^{-n}),$$

(with $\nu_n = \nabla U_n / |\nabla U_n|$ on $\Gamma_{\varepsilon,n}$). This is equivalent (using the definitions of ϕ_n and ψ_n) to

(5.7)
$$\int_{\Gamma_{\delta,n}} (|\nabla U_n|^2 - \lambda^2)(|\nabla U_n| + (\partial Q(y)/\partial \nu_n))\, ds \\ \leq 2 \int_{\Gamma_{\delta,n}} |\delta + Q||\nabla U_n||(\partial |\nabla U_n|/\partial \nu_n)|\, ds + O(\varepsilon) + \zeta_\varepsilon(2^{-n}).$$

Now $|\delta + Q| \leq M$ and $C_1 \leq |\nabla U_n| \leq C_2$ on $\Gamma_{\delta,n}$, both uniformly as $n \to \infty$ and $\delta, \varepsilon \to 0+$. Also, it follows from Lemma 3.4 that

$$\max\{|\nu_n(x) - \nu(0)| : x \in \Gamma_{\delta,n}\} \leq \zeta(2^{-n}) + \zeta(\delta) \tag{5.8}$$

as $n \to \infty$ and $\delta \to 0+$. Therefore $\max\{|\nabla Q(y) \cdot \nu_n(x)| : x \in \Gamma_{\delta,n}\} \leq \zeta(2^{-n}) + \zeta(\delta)$ as $n \to \infty$ and $\delta \to 0+$, since $\nabla Q(y)$ has no component in the $\nu(0)$ direction. Therefore, (5.7) implies

$$\int_{\Gamma_{\delta,n}} (|\nabla U_n|^2 - \lambda^2)\, ds \leq C \int_{\Gamma_{\delta,n}} (|\partial^2 U_n / \partial \nu_n^2| / |\nabla U_n|)\, ds \\ + \zeta(\delta) + O(\varepsilon) + \zeta_\varepsilon(2^{-n}). \tag{5.9}$$

For sufficiently large $n \in \mathbb{N}$ and sufficiently small $\delta > 0$, $\Gamma_{\delta,n}$ is the graph of a smooth function $z = \Gamma_{\delta,n}(y) : G_{\delta,n} \to \mathbb{R}$. In terms of this representation, we have

$$(\partial^2 U_n(x)/\partial \nu_n^2)/|\nabla U_n(x)| \\ = \nabla_y \cdot (\nabla_y \Gamma_{\delta,n}(y)/[1 + |\nabla_y \Gamma_{\delta,n}(y)|^2]^{1/2}), \tag{5.10}$$

where both sides represent $(m-1)$ times the mean curvature of the surface $\Gamma_{\delta,n}$ at $x = (y, z) = (y, \Gamma_{\delta,n}(y)) \in \Gamma_{\delta,n}$. By substituting (5.10) into the second integral of (5.9), estimating $ds/dy = (1 + |\nabla \Gamma_{\delta,n}(y)|^2)^{1/2}$ by a constant, and applying the divergence theorem, one obtains

$$\int_{\Gamma_{\delta,n}} (|\nabla U_n|^2 - \lambda^2)\, ds \leq C \int_{\partial G_{\delta,n}} |\nabla \Gamma_{\delta,n}(y)|\, ds \\ + \zeta(\delta) + O(\varepsilon) + \zeta_\varepsilon(2^{-n}), \tag{5.11}$$

where the integrand of the second integral is uniformly bounded by $\zeta(2^{-n}) + \zeta(\delta)$, due to (5.8), and where ds in the second integral refers to $(m-2)$-dimensional surface area. In the limit as $\delta \to 0+$, we obtain

$$\int_{C_n} (|\nabla U_n|^2 - \lambda^2)\, ds \leq O(\varepsilon) + \zeta_\varepsilon(2^{-n}),$$

where $C_n = \Gamma_n \cap \{x = (y, z) \colon |y| < 1\}$. This implies the assertion in the "+" case (despite the slightly different definition of C_n).

Finally, for the corresponding proof in the "−" case, one again simplifies the notation so that $\widetilde{U}_n^-(x)$, $\widetilde{\Omega}_n^-$, $\widetilde{\Gamma}_n$, and λ^- become $U_n(x)$, Ω_n, Γ_n, and λ. The proof again starts with Green's second identity (5.3), where this time $\phi_n = \ln(|\nabla U_n|/\lambda)$ (notice that $\Delta\phi_n \leq 0$ by Lemma 2.7(d)), $\psi_n = U_n + Q$ (here $Q = Q(y)$ has the same properties as before) and $\Omega_{\delta,\varepsilon,n} = \{x \in \Omega_n \colon U_n(x) > \delta,\ z > -\varepsilon,\ U_n(x) + Q(y) > 0\}$. Continuing as in the "+" case, one can show (in the "−" case) that

$$\int_{\Gamma_{\delta,n}} \ln(|\nabla U_n|/\lambda)\, ds \geq -M \int_{\Gamma_{\delta,n}} (|\partial^2 U_n/\partial \nu_n^2|/|\nabla U_n|)\, ds \\ - \zeta(\delta) - O(\varepsilon) - \zeta_\varepsilon(2^{-n}),$$

where $\Gamma_{\delta,n} = \{x \in \Omega_n \colon U_n(x) = \delta,\ U_n(x) + Q(y) > 0\}$. Then the assertion follows by the steps given above.

5.5. *Proof of Theorem* 5.1. Let $\widetilde{\Gamma}$ solve Problem 2.2. Then $A^2(x) = B^2(x) + a^2(x)$ almost everywhere on $\widetilde{\Gamma}$, by Theorem 4.1(c). Since the function $A(x) \colon \widetilde{\Gamma} \to \mathbb{R}^+$ is upper semicontinuous and the function $B(x) \colon \widetilde{\Gamma} \to \mathbb{R}^+$ is lower semicontinuous (Lemma 5.2), it immediately follows that $A^2(x) \geq B^2(x) + a^2(x)$ at every point $x \in \widetilde{\Gamma}$. Moreover, for each point $x_0 \in \widetilde{\Gamma}$ and $\eta > 0$, it follows from Theorem 4.1(c), Lemma 5.4, and the continuity of the function $a(x)$ that

$$2^n(A^2(x_0) - B^2(x_0))|\tilde{\gamma}_n| \leq 2^n \int_{\tilde{\gamma}_n} (A^2(x) - B^2(x))\, ds + 2\eta \\ = 2^n \int_{\tilde{\gamma}_n} a^2(x)\, ds + 2\eta \leq 2^n a^2(x_0)|\tilde{\gamma}_n| + 3\eta$$

for all sufficiently large $n \in \mathbb{N}$, where $\tilde{\gamma}_n := \{x \in \widetilde{\Gamma} \colon |x - x_0| < 2^{-n}\}$ and $|\tilde{\gamma}_n|$ refers to Euclidean $(m-1)$-area of $\tilde{\gamma}_n$. Thus $A^2(x_0) - B^2(x_0) \leq a^2(x_0)$, and we conclude that $A^2(x) = B^2(x) + a^2(x)$ at every point $x \in \widetilde{\Gamma}$. At this point, Lemma 5.2 implies that the functions $A(x), B(x) \colon \widetilde{\Gamma} \to \mathbb{R}^+$ are both continuous.

6. A modified multi-layer problem in the convex case.

6.1. *Problem.* In $\mathbb{R}^m$, $m \geq 2$, let be given a bounded, convex, C^1-domain D^* and a convex domain $P \subset D^*$. For a fixed integer

$n \geq 0$, let $a_i(x)\colon \mathrm{Cl}(D^*) \to \mathbb{R}$, $i = 0, 1, 2, \dots, n$, denote $n+1$ functions with the following properties: The function $a_0(x)$ is Lipschitz-continuous in $\mathrm{Cl}(D^*)$, vanishes in $D^*\backslash P$, and is strictly positive and concave in P (thus, the set $\{(x, z) \in P \times \mathbb{R}\colon 0 < z < a_0(x)\}$ is convex in $\mathbb{R}^{m+1}$). The remaining functions $a_i(x)$, $i = 1, \dots, n$, are strictly-positive and continuous in $\mathrm{Cl}(D^*)$, and are such that the related functions $b_i(x) := (1/a_i(x))$, $i = 1, \dots, n$, are all concave in D^* (thus, the sets $\{(x, z) \in D^* \times \mathbb{R}\colon 0 < z < b_i(x)\}$, $i = 1, \dots, n$, are convex in $\mathbb{R}^{m+1}$). We seek a nested family of convex C^1-domains $D_0, D_1, \dots, D_n$ (with boundaries $\Gamma_i = \partial D_i$) such that $\mathrm{Cl}(D_0) \subset P$ and $\mathrm{Cl}(D_i) \subset D_{i+1}$ for $i = 0, \dots, n$ (where we set $D_{n+1} = D^*$), and also such that

$$(6.1) \qquad |\nabla U_1| = a_0(x) \quad \text{on } \Gamma_0,$$

$$(6.2) \qquad |\nabla U_i|^2 = |\nabla U_{i+1}|^2 + a_i^2(x) \quad \text{on } \Gamma_i, \ i = 1, \dots, n,$$

where $U(x)$ solves the boundary value problem

(6.3)

$$\Delta U = 0 \quad \text{in } D^*\backslash(\Gamma_0 \cup \dots \cup \Gamma_n), \qquad U(\Gamma_i) = i \text{ for } i = 0, 1, \dots, n+1,$$

and where, for each $i = 1, 2, \dots, n+1$, U_i denotes the restriction of U to the closure of the annular domain $\Omega_i := D_i\backslash\mathrm{Cl}(D_{i-1})$ with boundary $\partial\Omega_i = \Gamma_i \cup \Gamma_{i-1}$.

6.2. *Problem.* In the context of Problem 6.1, we seek to minimize the functional $I(v)\colon \mathbb{X}_c \to \mathbb{R}$ defined by

$$(6.4) \quad I(v) := \int_{D^*} \left(|\nabla v|^2 + a_0^2(x)H(v(x)) + \sum_{i=1}^{n} a_i^2(x)H(i - v(x))\right) dx,$$

where $H(t)$ denotes the Heaviside function ($H(t) = 0$ for $t \leq 0$, $H(t) = 1$ for $t > 0$) and $\mathbb{X}_c$ denotes the set of all functions $v \in L^1(\mathrm{Cl}(D^*))$ such that $\nabla v \in L^2(D^*)$, $v = n+1$ on ∂D^*, and, up to a set of Lebesgue measure 0, the set $\{v(x) < t\}$ is convex for each $t \in \mathbb{R}$.

6.3. THEOREM. *Assume in Problem* 6.2 *that there exists a value* $\varepsilon_0 > 0$ *and a function* $v_0 \in \mathbb{X}_c$ *such that* $|\{v_0(x) \leq 0\}| > \varepsilon_0$ (*where* $|\cdot|$ *denotes Euclidean volume*) *and such that* $I(v) > I(v_0)$ *for any* $v \in \mathbb{X}_c$ *such that* $|\{v(x) \leq 0\}| \leq \varepsilon_0$. *Then*: (a) *Problem* 6.2 *has at least one continuous solution* $U \in \mathbb{X}_c$ *such that* $|\{U(x) \leq 0\}| > \varepsilon_0$. (b) *Given a continuous solution* $U \in \mathbb{X}_c$ *of Problem* 6.2 *such that* $|\{U(x) \leq 0\}| > \varepsilon_0$, *let* $D_0 = \mathrm{interior}\{U \leq 0\}$ *and* $D_i = \{U(x) < i\}$

for $i = 1, \ldots, n+1$. *Then* $\mathfrak{D} := (D_0, D_1, \ldots, D_n)$ *is a classical solution of Problem* 6.1.

Proof. We remark that a proof of part (a) has been given by Laurence and Stredulinsky **[21]** (for the case where the functions $a_i(x)$ are all constant). Turning to the proof of part (b), we let U be a continuous solution of Problem 6.2 such that $|\{U(x) \leq 0\}| > 0$, and let $\mathfrak{D} = (D_0, D_1, \ldots, D_n)$ be defined as in Theorem 6.3. Then the surface $\Gamma_0 := \partial D_0 \subset D_1$ is a minimizer of the functional $J(\Gamma) := K(\Gamma) + \|\Omega(\Gamma)\|_0$ in the family of all convex, closed $(m-1)$-surfaces $\Gamma \subset D_1$, where D_1 is fixed and convex, where $K(\Gamma)$ denotes the capacity of the annular domain $\Omega(\Gamma)$ bounded by $\Gamma \cup \Gamma_1$, and where $\|M\|_0 = \int_M a_0^2(x)\,dx$ for any measureable set $M \subset \mathrm{Cl}(D^*)$. Therefore $\Gamma_0 \subset P \subset D^*$, since otherwise it is easily seen that $J(\Gamma_{0,\delta}) < J(\Gamma_0)$ for sufficiently small $\delta > 0$, where we define $D_{0,\delta} = \{x \in P \cap D_0 \colon \mathrm{dist}(x, \partial P) > \delta\}$ and $\Gamma_{0,\delta} = \partial D_{0,\delta}$. It now follows from the results in **[3]**, **[16**, §5], or **[8**, §§4, 5] that the condition (6.1) is satisfied classically on Γ_0. For $i = 1, 2, \ldots, n$, the surface Γ_i is a solution of Problem 2.2 in the case where $\Gamma^- = \Gamma_{i-1}$, $\Gamma^+ = \Gamma_{i+1}$, and $a(x) = a_i(x)$. Since $\Gamma_i \cap \Gamma^- = \varnothing$, we conclude from Theorem 2.6 (see also Theorems 4.1 and 5.1) that Γ_i is a classical solution of Problem 2.1 in the new notation. Therefore Γ_i satisfies the joining condition (6.2) in the classical sense for each $i = 1, \ldots, n$, completing the proof of Theorem 6.3(b).

7. The multi-layer problem in the convex case.

7.1. *Problem.* In $\mathbb{R}^m$, $m \geq 2$, let an annular domain Ω of the form $\Omega = D^+ \backslash \mathrm{Cl}(D^-)$ be given, where $D^\pm$ are fixed, bounded, convex, nested domains. We assume that ∂D^- is a C^2 surface and ∂D^+ is a C^1 surface. Let be given $n \in \mathbb{N}$ and the strictly positive, continuous functions $a_i(x) \colon \mathrm{Cl}(D^+) \to \mathbb{R}$, $i = 1, 2, \ldots, n$, such that the related functions $b_i(x) := (1/a_i(x))$ are all concave in D^+. We seek a nested family of convex C^1-domains $D_1, D_2, \ldots, D_n$ (with boundaries $\Gamma_i = \partial D_i$) such that $\mathrm{Cl}(D_i) \subset D_{i+1}$ for $i = 1, \ldots, n$, (where we set $D_0 = D^-$ and $D_{n+1} = D^+$) and such that

$$(7.1) \qquad |\nabla U_i|^2 = |\nabla U_{i+1}|^2 + a_i^2(x) \quad \text{on } \Gamma_i$$

for $i = 1, \ldots, n$, where $U(x)$ solves the boundary value problem

(7.2)

$$\Delta U = 0 \quad \text{in } \Omega \backslash (\Gamma_1 \cup \cdots \cup \Gamma_n), \qquad U(\Gamma_i) = i \quad \text{for } i = 0, 1, \ldots, n+1,$$

and where, for each i, U_i denotes the restriction of U to the closure of the annular domain $\Omega_i := D_i \backslash \mathrm{Cl}(D_{i-1})$ with boundary $\partial\Omega_i = \Gamma_i \cup \Gamma_{i-1}$.

7.2. THEOREM. *Problem* 7.1 *has at least one solution* $D = (D_1, D_2, \dots, D_n)$.

7.3. LEMMA. *Given* $x_0 \in \mathbb{R}^m$ *and* $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n, \alpha_{n+1})$ *with* $0 < \alpha_0 < \alpha_1 < \cdots < \alpha_n < \alpha_{n+1}$, *let* $U(x_0, \alpha; x)$ *denote the solution of the boundary value problem* (7.2) *in the case where* $D_i = \{|x - x_0| < \alpha_i\}$ *for* $i = 0, 1, \dots, n, n+1$. *If* $m = 2$ *and* $\alpha_i = C\exp(i^2)$ *for* $i = 0, \dots, n+1$, *where* $C > 0$, *then* $U(x_0, \alpha; x)$ *satisfies the joining conditions* (7.1) *in the case where the functions* $a_i^2(x)$ *are replaced by the constants* $\lambda_i^2 := 8i/C^2(4i^2 - 1)\exp(2i^2)$, $i = 1, \dots, n$. *Alternately, assume that* $m \geq 3$ *and*

$$\alpha_i^{2-m} = C[\exp(n+2) - \exp(i)] \quad \textit{for } i = 0, \dots, n+1,$$

where $C > 0$. *Then* $U(x_0, \alpha; x)$ *satisfies* (7.1) *in the case where the functions* $a_i^2(x)$, $i = 1, \dots, n$, *are replaced by the constants* (*where* $e = \exp(1)$):

$$\lambda_i^2 := (m-2)^2[(e+1)/(e-1)]\exp(-2i)C^{2/(m-2)} \cdot [\exp(n+2) - \exp(i)]^{(2m-2)/(m-2)}.$$

Proof sketch. For $m = 2$, we have

$$U_i(x) = (i-1) + [\ln(r/\alpha_{i-1})/\ln(\alpha_i/\alpha_{i-1})]$$

in Ω_i, where $r = |x - x_0|$. For $m \geq 3$,

$$U_i(x) = (i-1) + [(r^{2-m} - \alpha_{i-1}^{2-m})/(\alpha_i^{2-m} - \alpha_{i-1}^{2-m})]$$

in Ω_i. Using these formulas, the constants λ_i^2 can be calculated explicitly.

7.4. LEMMA. (a) *In the context of Problem* 7.1, *there exist constants* $\delta, \varepsilon > 0$ *with the following property: Let* $\widehat{D}_0$ *denote any convex domain in* $\mathbb{R}^m$ *such that* $\widehat{D}_0 \subset N_\delta(D^-)$ *and* $D^- \subset N_\delta(\widehat{D}_0)$, *and let* $\widehat{D} = (\widehat{D}_1, \widehat{D}_2, \dots, \widehat{D}_n)$ *denote any classical solution of Problem* 7.1 *in the case where* D_0 *is replaced by* $\widehat{D}_0$. *Then* $\widehat{D}_1 \supset N_\varepsilon(\widehat{D}_0)$.

(b) *For any classical solution* $D = (D_1, D_2, \dots, D_n)$ *of Problem* 7.1, *we have* $0 < \delta_1 \leq \delta_2 \leq \cdots \leq \delta_{n+1}$, *where* $\delta_i = \mathrm{dist}(\Gamma_{i-1}, \Gamma_i)$.

Proof (*part* (a)). We apply Lemma 7.3 to construct barriers for solutions of Problem 7.1. By assumption, if $\rho > 0$ is sufficiently small (i.e. $0 < \rho < \rho_0$), then each point $x \in \Gamma^-$ is on the boundary of a ball $B_\rho(x_\rho)$, with center x_ρ and radius ρ, such that $B_\rho(x_\rho) \subset D^-$. For fixed $m \geq 2$ and for fixed, sufficiently small $\eta > 0$, one can choose the constant $C = C(\rho)$ in Lemma 7.3 such that $0 < \alpha_0 = \alpha_0(\rho) < (1-\eta)\rho$ and $\alpha_1 = \alpha_1(\rho) > (1+\eta)\rho$. One can then choose $\rho > 0$ so small that $\alpha_{n+1}(\rho) < \operatorname{dist}(\Gamma^-, \Gamma^+)$ and $\lambda_i^2 = \lambda_i^2(\rho) > a_i^2(x)$ throughout $\mathrm{Cl}(D^+)$ for each $i = 1, \dots, n$. Now, let $\delta = \varepsilon = \rho\eta/2$ in the assertion. Then for each $x \in \Gamma^-$, it follows from **[11**, §2], that the function $U(x_\rho, \alpha(\rho); x)$ is an upper barrier for the solution $\widehat{U}(x)$ of (7.2) corresponding to $\widehat{D} = (\widehat{D}_1, \widehat{D}_2, \dots, \widehat{D}_n)$, so that $B_{(1+\eta)\rho}(x_\rho) \subset \widehat{D}_1$. The assertion follows from this.

Proof (*Part* (b)). Choose $i \in \{1, \dots, n\}$ and collinear points $x \in \Gamma_{i-1}$, $y \in \Gamma_i$, $z \in \Gamma_{i+1}$ such that $|y - z| = \delta_{i+1}$. One easily shows using the maximum principle, the convexity of the domains D_{i-1}, D_i, D_{i+1}, and the joining condition (7.1) that

$$\delta_{i+1} \geq (1/|\nabla U_{i+1}(y)|) \geq (1/|\nabla U_i(y)|) \geq |x - y| \geq \delta_i.$$

7.5. *Proof of Theorem* 7.2. For each $k \in \mathbb{N}$, let the function $a_{0,k}(x)\colon \mathbb{R}^m \to \mathbb{R}$ be defined by

$$a_{0,k}(x) = \max\{0, k - k^2 \cdot \operatorname{dist}(x, D^-)\}. \tag{7.3}$$

Observe that for each $k \in \mathbb{N}$, the function $a_{0,k}(x)$ is concave inside the $(1/k)$-neighborhood of D^- (designated by P_k) and vanishes in $\mathbb{R}^m \backslash P_k$. For each sufficiently large $k \in \mathbb{N}$, let $U_k(x)\colon \mathrm{Cl}(D^+) \to \mathbb{R}$ and $\mathfrak{D}_k := (D_{0,k}, D_{1,k}, \dots, D_{n,k})$ denote corresponding solutions of Problems 6.2 and 6.1, respectively, in the case where $D^* = D^+$, $P = P_k$, and the function $a_0(x)\colon \mathrm{Cl}(D^+) \to \mathbb{R}$ is replaced by $a_{0,k}(x)$. These solutions exist for all sufficiently large $k \in \mathbb{N}$ (i.e. for $k > k_0$) by Theorem 6.3. Moreover, $I_k(U_k) \geq k^2|D^- \backslash D_{0,k}|$ for each $k > k_0$, where $|\cdot|$ denotes Euclidean volume and the functional $I_k\colon \mathbb{X}_c \to \mathbb{R}$ is defined by (6.4) (with $a_0(x)$ replaced by $a_{0,k}(x)$). Now there is a function $\phi \in \mathbb{X}_c$ (in Problem 6.2) such that $I_k(\phi)$ is uniformly bounded as $k \to \infty$. Thus $I_k(U_k)$ remains bounded as $k \to \infty$ (since $I_k(U_k) \leq I_k(\phi)$), implying that $|D^- \backslash D_{0,k}| \to 0$ as $k \to \infty$. One easily concludes (using the convexity of $D_{0,k}$ and the smoothness of Γ^-) that for any $\varepsilon > 0$, $D_{0,k}$ contains the ε-interior of D^- for all sufficiently large $k \in \mathbb{N}$. We also have $D_{0,k} \subset P_k$ for $k > k_0$ by Theorem 6.3 (and the definition of a solution of Problem 6.1). Therefore $\Gamma_{0,k} \to \Gamma^-$ as $k \to \infty$ (in the polar coordinate maximum norm relative to a point $x_0 \in D^-$), where $\Gamma_{0,k} = \partial D_{0,k}$. By Lemma 7.4, there exists a constant $\delta_0 > 0$ so small that $\operatorname{dist}(\Gamma_{i,k}, \Gamma_{i+1,k}) \geq \delta_0$, uniformly for $i = 0, 1, \dots, n$ and for all sufficiently large k (where $\Gamma_{n+1,k} = \Gamma^+$). Moreover, the convexity of the surfaces $\Gamma_{i,k}$ (for $i = 1, \dots, n$ and $k > k_0$) implies the equicontinuity of their polar coordinate representations relative to a point $x_0 \in D^-$ (see **[3**, §2**]** for a similar argument). By applying the theorem of Ascoli-Arzela, and passing to a subsequence (still indexed by k) if necessary, we conclude that there exists a nested family of convex domains $D_1, D_2, \dots, D_n$ (with boundaries $\Gamma_i = \partial D_i$) such that $\mathrm{Cl}(D_i) \subset D_{i+1}$ for $i = 0, 1, \dots, n$, (where we set $D_0 = D^-$ and $D_{n+1} = D^+$) and such that $\Gamma_{i,k} \to \Gamma_i$ as $k \to \infty$ (in the maximum norm in polar coordinates) for each $i = 0, \dots, n$. Now for each $i = 1, \dots, n$, and for $k > k_0$, $\Gamma_{i,k}$ is a solution of Problem 2.2 in the case where $\Gamma^- = \Gamma_{i-1,k}$, $\Gamma^+ = \Gamma_{i+1,k}$, and $a(x) = a_i(x)$. However, it is well known that the capacity of an annular domain between nested convex surfaces depends continuously on these surfaces as they undergo convex perturbations (while remaining uniformly separated). By using this, and the convergence of the surfaces $\Gamma_{i,k}$ to Γ_i as $k \to \infty$, one easily concludes that for each $i = 1, \dots, n$, Γ_i is a solution of Problem 2.2 in the case where $\Gamma^- = \Gamma_{i-1}$, $\Gamma^+ = \Gamma_{i+1}$, and $a(x) = a_i(x)$. Therefore, Γ_i solves Problem 2.1 in the same case, due to Theorem 2.6 (see also Theorems 4.1 and 5.1). Therefore, $D := (D_1, D_2, \dots, D_n)$ is a solution of Problem 7.1.

7.6. REMARK. Some of the ideas in the preceding proof suffice to extend the author's counterexample in **[10]** to Problem 1.2. Assume for $m = 2$, $n = 1$, $\lambda_1(x) = -\alpha^2 < 0$, and for a particular choice of convex nested domains $D^\pm$, that Problem 1.1 does not have a convex classical solution D_1 (we know this situation occurs due to **[10]**). For large $k \in \mathbb{N}$, the functional $I_k(\phi)\colon \mathbb{X}_c \to \mathbb{R}$ has a convex minimizer $U_k \in \mathbb{X}_c$, where

$$I_k(\phi) := \int_{D^*} [|\nabla\phi|^2 + a_{0,k}^2(x)H(\phi) + \alpha^2 H(\phi - 1)]\, dx,$$

the function $a_{0,k}(x)$ is defined by (7.3), and $\mathbb{X}_c$ is as defined in Problem 6.2, with $n = 1$ and $D^* = D^+$. We assume that $D_k = (D_{k,0}, D_{k,1})$ solves Problem 1.2 with $m = 2$, $n = 1$, $a_0(x) = a_{0,k}(x)$, $\lambda_1(x) = -\alpha^2$, and the same domain $D^* = D^+$. (Here $D_{k,0} = \operatorname{int}\{U_k \leq 0\}$ and $D_{k,1} = \{U_k < 1\}$.) After passing to a sub-sequence (still indexed by k), one concludes by arguments given in §7.5 that $D_{k,1} \to D_1$ as $k \to \infty$ (in the sense that $\partial D_{k,1} \to \partial D_1$ in a suitable polar coordinate maximum norm), where D_1 is a convex solution of Problem 1.1 in the case described above. This contradiction shows that even under convex conditions, the minimizer of the functional $I_k(\phi): \mathbb{X}_c \to \mathbb{R}$ is not always a (convex) classical solution of the corresponding formulation of Problem 1.2.

7.7. REMARKS. (a) Hopefully, Theorems 6.3 and 7.2 will generalize to cases where the Laplacian is replaced by more general elliptic operators in divergence form (such as the p-Laplacian) for which results analogous to Lemma 2.7(a) are true. (b) If, in Problem 7.1, the regions $D^{\pm}$ and the functions $a_i(x)$, $i = 1, 2, \ldots, n$, are all symmetric relative to each member of a given family of $(m-1)$-dimensional planes, then there exists a solution $D = (D_1, D_2, \ldots, D_n)$ such that all the (convex) domains D_i have the same symmetry properties. (Observe that the operators T_ε preserve these symmetries.)

7.8. REMARKS. (a) Consider the modified version of Problem 2.1 in which $a(x) = \alpha$ (a constant), and the given domains $D^{\pm}$ are no longer convex, but are assumed to be directionally convex relative to a given direction ν_0. It is natural to conjecture that there exists a solution D of (2.1) such that D is also directionally convex relative to ν_0. In fact this conjecture is false, as can be seen by slightly modifying the author's counterexample given in [7, Example 2 and Figure 2]. (b) Consider the Bernoulli free-boundary problem, which is Problem 2.1 in the limiting case where $D^+ = \mathbb{R}^m$ and $U^+ = 0$ (we set $D^* = D^-$, $\Omega = \Omega^-$, $U = U^-$, $K = K^-$). Assume that $a(x) = 1$ and that D^* is directionally convex relative to ν_0. Let $\widetilde{\Gamma}$ denote a minimizer of the functional $I(\Gamma) = K(\Gamma) + \|\Omega(\Gamma)\|$ subject to the requirement that the interior complement of Γ be directionally convex relative to ν_0. We conjecture that $|\nabla \widetilde{U}| = 1$ on $\widetilde{\Gamma}$. Observe that the operator method (which was applied to the convex Bernoulli free-boundary problem in [3], [4], [8]) is not helpful in this problem because the operators do not preserve direction convexity (due to [7, Example 2]). In [6], the author used the method of flat places to prove our conjecture in the case where $m = 2$ and the minimizer is sufficiently regular (in [6, Figure 2], the regions labeled Ω_+ and Ω'_- should be interchanged). We hope this proof will generalize to arbitrary space dimensions.

Appendix: The Proof of Lemma 4.4. We restrict the proof of the "+" case and use U, Ω and Γ to denote $\widetilde{U}^+$, $\widetilde{\Omega}^+$ and $\widetilde{\Gamma}$. Part (b) follows easily from Theorem 2.5(b), Lemma 2.7(b) and Lemma

3.4. Concerning Part (a), let (x_n) denote a sequence of points in Ω such that $x_n \to x_0 \in \Gamma$ as $n \to \infty$. For each n, let γ_n denote the maximal arc of steepest ascent of U through x_n, parametrized by a function $p_n(t)\colon [0, 1] \to \mathbb{R}^m$ such that $U(p_n(t)) = t$. Then $p_n(t_n) = x_n$ and $p_n'(t) = V(p_n(t))$ for $0 < t < 1$, where $t_n = U(x_n)$ and $V(x) = \nabla U(x)/|\nabla U(x)|^2$. Since $|p_n'(t)| \leq (1/C_1)$ for $0 < t < 1$ (by Theorem 2.5(b)), we conclude by passing to a subsequence, again indexed by $n \in \mathbb{N}$, (and using the Theorem of Ascoli-Arzela) that $p_n(t) \to p(t)$ uniformly in $[0, 1]$, where the function $p(t)\colon [0, 1] \to \mathbb{R}^m$ is Lipschitz-continuous. Clearly $U(p(t)) = \lim_{n\to\infty} U(p_n(t)) = t$ for $0 \leq t \leq 1$. It follows from $p_n(t) - p_n(s) = \int_s^t V(p_n(\tau))\,d\tau$ that $p(t)-p(s) = \int_s^t V(p(\tau))\,d\tau$ for all $0 \leq s,\ t \leq 1$, so that $p'(t) = V(p(t))$ for $0 < t < 1$. Also $|p(0) - x_0| \leq |p(0) - p(t_n)| + |p(t_n) - x_0| \to 0$ as $n \to \infty$. Therefore, $p(t)\colon [0, 1] \to \mathbb{R}^m$ parametrizes a curve γ of steepest ascent of U beginning at x_0. Turning the proof of Part (c), let $q(t)\colon [0, 1] \to \mathbb{R}^m$ denote the parametrization of a second curve of steepest ascent of U beginning at x_0 (we assume that $U(q(t)) = t$). Observe that $V(p(t)) \to V(x_0)$ as $t \to 0+$, due to Part (b). It follows that $p'(t) = V(x_0) + \zeta(t)$ and $p(t) = x_0 + V(x_0)t + t\zeta(t)$, both as $t \to 0+$. Since $q(t) = x_0 + V(x_0)t + t\zeta(t)$ as $t \to 0+$ by the same argument, we conclude that $\delta(t) := |p(t) - q(t)| = t\zeta(t)$ as $t \to 0+$. For $0 < t < 1$, we let γ_t denote the shortest curve in $\{U(x) = t\}$ joining $p(t)$ to $q(t)$, observing that $|\gamma_t| \leq \delta(t)(1 + \zeta(t))$ as $t \to 0+$, where $|\gamma_t|$ refers to arc length. For any point $x \in \gamma_t$ and unit vector $\tau \perp \nabla U(x)$, we have $\partial V(x)/\partial\tau = (\nabla - 2\partial/\partial\nu)(\partial U(x)/\partial\tau)$, where $\nu = \nabla U(x)/|\nabla U(x)|$. Since $\Delta(\partial U/\partial\tau) = 0$ in Ω (for fixed τ) and $|\partial U/\partial\tau| \leq \zeta(t)$ in the ball $B_{\mu t}(x)$ (for suitable $0 < \mu < 1$) by Lemma 3.4, we conclude that $|\partial V(x)/\partial\tau| \leq \zeta(t)/t$ for any $x \in \gamma_t$. Therefore, $\delta'(t) \leq |p'(t)-q'(t)| = |V(p(t))-V(q(t))| \leq (\zeta(t)/t)\delta(t)$ for $0 < t < 1$, which integrates to give $\ln(\delta(t)/\delta(\alpha)) \leq \zeta(t)\ln(t/\alpha)$ for $0 < \alpha < t < 1$. By combining results, we conclude that

$$\delta(t) \leq (t/\alpha)^{\zeta(t)}\delta(\alpha) \leq (t/\alpha)^{\zeta(t)}\alpha\zeta(\alpha)$$

for $0 < \alpha < t < 1$. Choose $t_0 > 0$ sufficiently small, so that $\zeta(t) \leq 1$ for $0 \leq t \leq t_0$ in the above inequality. By letting $\alpha \to 0+$ for each fixed $0 \leq t \leq t_0$, we conclude that $\delta(t) = 0$ for $0 \leq t \leq t_0$ (thus $\delta(t) = 0$ for $0 \leq t \leq 1$, since $p(t)$ and $q(t)$ satisfy the same ordinary differential equation: $p'(t) = V(p(t))$ for $0 < t < 1$). Concerning Part (d), if the assertion is false, then there exists a value $\varepsilon_0 > 0$ and a sequence of points (x_n) in Γ such that $x_n \to x_0$ as

$n \to \infty$, but $|p_n(t_n) - p(t_n)| \geq \varepsilon_0$ for all $n \in \mathbb{N}$, where $0 < t_n \leq 1$ and where $p_n(t)\colon [0, 1] \to \mathbb{R}^m$ is the parametrization of the curve of steepest ascent of U beginning at x_n (such that $U(p_n(t)) = t$). By passing to a subsequence (again indexed by n) and repeating the procedure in the proof of Part (a), one easily concludes that $p_n(t) \to q(t)$ uniformly in $[0, 1]$, where the function $q(t)\colon [0, 1] \to \mathbb{R}^m$ (satisfying $U(q(t)) = t$) parametrizes an arc of steepest ascent of U beginning at x_0 which is distinct from $\gamma := \{p(t)\colon 0 \leq t \leq 1\}$. Concerning Part (e), the function $y = \pi_\varepsilon(x)$ is the inverse of a one-to-one, continuous function (by Parts (c) and (d)), and is therefore continuous. To prove the integral identity, it suffices to show that $J(\varepsilon) := \int_{S_\varepsilon} |\nabla U(x)|\, ds_\varepsilon = J := \int_S |\nabla U(x)|\, ds$ for $0 < \varepsilon < 1$, where $S \subset \Gamma$ is the graph of a smooth mapping $z = \Gamma(y)$ of a closed C^1-domain $F \subset \mathbb{R}^{m-1}$ into $\mathbb{R}$, and where $S_\varepsilon \subset \Gamma_\varepsilon$ is chosen such that $\pi_\varepsilon(S_\varepsilon) = S$. An application of the divergence theorem shows that $J(\varepsilon)$ is a constant for $0 < \varepsilon < 1$. Let $\widehat{J}(\varepsilon) = \int_{\widehat{S}_\varepsilon} |\nabla U(x)|\, ds$ for sufficiently small $\varepsilon > 0$. Here, $\widehat{S}_\varepsilon$ is the graph of the function $z = \Gamma_\varepsilon(y)\colon F \to \mathbb{R}$, which is a smooth local representation of Γ_ε in the previously used coordinates. Then $J(\varepsilon) = \widehat{J}(\varepsilon) + \zeta(\varepsilon)$ as $\varepsilon \to 0+$, as follows from Lemma 3.4 and the boundedness of $|\nabla U|$. However, the Lebesgue dominated convergence theorem implies that

$$\begin{aligned}\widehat{J}(\varepsilon) &= \int_F (1 + |\nabla_y \Gamma_\varepsilon(y)|^2)^{1/2} |\nabla U(y, \Gamma_\varepsilon(y))|\, dy \to J \\ &= \int_F (1 + |\nabla_y \Gamma(y)|^2)^{1/2} |\nabla U(y, \Gamma(y))|\, dy,\end{aligned}$$

because the integrands are uniformly bounded (independent of small $\varepsilon > 0$) and because the left integrand converges pointwise to the right integrand as $\varepsilon \to 0+$, by Theorem 2.5(b). It follows that $J(\varepsilon) = J$ for all $0 < \varepsilon < 1$. At this point, the integral identity in Part (e) follows by approximating the function $\phi(x)\colon \widetilde{\Gamma} \to \mathbb{R}$ by suitable piecewise constant functions.

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