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# CURVATURE CHARACTERIZATION OF CERTAIN BOUNDED DOMAINS OF HOLOMORPHY

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**In this note, we study the relation between the existence of a negatively curved complete hermitian metric on a complex manifold $M$ and the product structure of (or contained in) $M$. We introduce the concept of geometric ranks and give a curvature characterization of the rank one manifolds, which generalizes the previous results of P. Yang and N. Mok (see below). In the proof, we used the old techniques of Yau's Schwartz lemma and Cheng-Yau's result on the existence of Kähler-Einstein metrics.**

**1. Introduction and statement of results.** Let $M = M_1 \times M_2$ be the product of two complex manifolds. Then it is generally believed that $M$ does not admit any complete Kähler metric with bisectional curvature bounded between two negative constants. When $M$ is compact, this is certainly true since the cotangent bundle $T^*_M$ is not ample. In the noncompact case, the first result toward this direction was obtained by Paul Yang in 1976:

THEOREM ([Y]). *For any $n \geq 2$, there exists no complete Kähler metric on the polydisc $C^n$ with bisectional curvature bounded between two negative constants.*

In [M], as an application of his metric rigidity theory, Mok generalized the above to give an interesting curvature characterization of the rank one bounded symmetric domains:

THEOREM ([M]). *If $\Omega$ is a bounded symmetric domain of rank $\geq 2$, then there exists no complete hermitian metric on $\Omega$ with bounded torsion and with bisectional curvature bounded between two negative constants.*

Mok's proof is a constructive one. It used the existence of a uniform lattice $\Gamma$ on $\Omega$, as well as the integral formula on $\Omega/\Gamma$ discovered by Mok (cf. Proposition (3.2) in [M]). This proof is very interesting by itself. However, we noticed that Yang's approach can be used to give a more straightforward proof of Mok's result, and the conclusion holds for a larger class of manifolds (since one avoids the use of uniform lattice). Intuitively speaking, the reason for the non-existence of the above negatively curved metrics on $D^n$ or $\Omega$ is the product structure on or nicely contained in the manifold (by the polydisc theorem, there is a totally geodesic proper embedding $D^r \to \Omega$, $r = \mathrm{rank}(\Omega)$).

First let us fix some notations. From now on, we shall say that a hermitian manifold $(M, h)$ is *negatively curved*, if it is complete, of bounded torsion, and with bisectional curvature bounded between two negative constants.

Now let $\Omega$ be a bounded domain of holomorphy in $\mathbf{C}^n$. By the results of Cheng-Yau **[C-Y]** and Mok-Yau **[M-Y]**, there exists a unique complete Kähler-Einstein metric on $\Omega$ with Ricci curvature $-1$. Denote it by $g$. Again let $D$ be the unit disc in $\mathbf{C}$.

DEFINITION. $\Omega$ is said to be of *geometric rank* $\geq 2$, if there is a complete Kähler manifold $(M, g_0)$ with Ricci curvature bounded from below, and a holomorphic embedding $f: D \times M \to \Omega$ such that $f_t^*(g) \geq g_0$ for each $t \in D$, where $f_t = f(t, \cdot)$.

In other words, $\Omega$ is of geometric rank $\geq 2$ if it contains a product manifold with bounded second fundamental forms. It is obvious that one can define the actual geometric rank of $\Omega$; however, in this note we shall only be interested in the distinction between the rank one case and the higher rank cases.

For bounded symmetric domains, the polydisc theorem implies that the usual rank dominates the geometric rank.

In §2, we shall prove the following generalization to the above cited result of Mok:

THEOREM A. *Let $\Omega$ be a bounded domain of holomorphy. If it is of geometric rank $\geq 2$, then it cannot be negatively curved.*

We shall also give partial answers in §3 to the question that product manifolds cannot be negatively curved:

THEOREM B. *Let $M = M_1 \times M_2$ be the product of two complex manifolds, with $M_1$ compact. Then there is no (not necessarily complete) hermitian metric on $M$ with bisectional curvature $\leq -1$.*

THEOREM C. *Let $M = M_1 \times M_2$ be the product of two complex manifolds. Suppose that both $M_1$ and $M_2$ admit complete Kähler metrics with Ricci curvature bounded between two negative constants. Then $M$ cannot be negatively curved.*

COROLLARY. *If both $M_1$ and $M_2$ are relatively compact open subsets of some Stein manifolds, then $M = M_1 \times M_2$ cannot be negatively curved.*

*In particular, the product of two bounded domains cannot be negatively curved.*

**2. Bounded domains of holomorphy.** First let us recall the generalized Schwarz lemma of Yau [**Y1**] and Chen-Yang [**C-Y1**].

PROPOSITION 1 ([**Y1**]). *Let $(M, g)$ be a complete Kähler manifold with Ricci curvature $\geq -K_1$, and $(N, h)$ be a hermitian manifold with bounded torsion and with Ricci curvature $\leq -K_2 < 0$. If $\dim(M) = \dim(N)$, and $f\colon M \to N$ is holomorphic, then $f^* dv_h \leq (K_1/K_2)\, dv_g$.*

PROPOSITION 2 ([**C-Y1**]). *Suppose $(M, g)$ is a complete hermitian manifold with bounded torsion, and with second Ricci curvature $\geq -K_1$. Let $(N, h)$ be a hermitian manifold with nonpositive bisectional curvature and with holomorphic sectional curvature $\leq -K_2 < 0$. Then for any holomorphic map $f\colon M \to N$, one has $f^*(h) \leq (K_1/K_2) g$.*

We shall also need the following generalized maximum principle of Yau:

PROPOSITION 3 ([**Y2**]). *If $(M, g)$ is a complete Kähler manifold with Ricci curvature bounded from below, and $\varphi$ is a $C^2$ function on $M$ bounded from above. Then for any $\varepsilon > 0$, there exists $x \in M$ such that: $\varphi(x) > \sup \varphi(M) - \varepsilon$, $|\nabla \varphi(x)| < \varepsilon$, $\Delta\varphi(x) < \varepsilon$.*

Now we are ready to prove Theorem A. The idea comes from Yang's proof in [**Y**] and the basic tool is the Schwarz lemma.

*Proof of Theorem* A. Let $\Omega$ be a bounded domain of holomorphy, with geometric rank $\geq 2$. Let $g$ be the complete Kähler-Einstein metric on it. By definition, there is a complete Kähler manifold $(M, g_0)$ with Ricci curvature bounded from below, and a holomorphic embedding $f\colon D \times M \to \Omega$ such that $f_t^* g \geq g_0$ for each $t \in D$.

Assume that $\Omega$ admits a negatively curved metric $h$. Applying Proposition 2 to the identity map id: $(\Omega, g) \to (\Omega, h)$, one gets $g \geq c' h$, while by Proposition 1 to id: $(\Omega, h) \to (\Omega, g)$ one gets $dv_g \leq c'' dv_h$, with $c$, $c''$ some positive constants. Therefore, $g$ and $h$ dominate each other. Hence $h \geq cg$ with $c > 0$.

Now let $\rho$ be a nonnegative smooth function with compact support in $D$. For $z \in M$, define

$$\varphi(z) = \int_D \rho(t) \cdot f_z^*(\omega_h)$$

where $\omega_h$ is the Kähler form on $h$, and $f_z = f(\cdot\,, z) \colon D \to \Omega$. Then $\varphi$ is a positive smooth function on $M$. It is also bounded from above, since by Schwarz lemma, for any $z \in M$, $f_z^*(\omega_h)$ is dominated by the Poincaré metric on $D$.

Let $z = (z_1, \dots, z_k)$ be a local holomorphic coordinate on $M$, $t = z_0$ a local coordinate on $D$, and $(t, z, z_{k+1}, \dots, z_{n-1})$ a coordinate on $\Omega$. Let $-c' < 0$ be an upper bound for the bisectional curvatures of $h$. Then we have that for each $1 \le i \le k$:

$$h_{t\bar{t}, i\bar{i}} \ge -R_{t\bar{t}i\bar{i}}(h) \ge c' h_{i\bar{i}} h_{t\bar{t}} \ge cc'(g_0)_{i\bar{i}} h_{t\bar{t}}.$$

Therefore,

$$\frac{\partial^2 \varphi}{\partial z_i \partial \overline{z}_i} = \int_D \rho(t) h_{t\bar{t}, i\bar{i}} \frac{\sqrt{-1}}{2}\, dt \wedge d\bar{t} \ge cc'(g_0)_{i\bar{i}} \varphi;$$

hence $\Delta\varphi \ge cc'\varphi$, where the Laplacian is with respect to $g_0$. Let $u = \log\varphi$; then the inequality becomes

$$\Delta u + |\nabla u|^2 \ge cc' > 0.$$

Since $u$ is also bounded from above, by Proposition 3, we get a contradiction. So we conclude that $\Omega$ cannot be negatively curved. □

REMARK. From the proof it is clear that the bounded domain $(\Omega, g)$ can be replaced by any complete hermitian manifold with bounded torsion and with Ricci curvature bounded between two negative constants (or, Ricci $\le -c < 0$ and second Ricci $\ge -c'$), as long as we keep the same condition on the geometric rank. One may also replace the Kählerness of $g_0$ by *hermitian with bounded torsion,* since Proposition 3 (hence Proposition 1) remains valid under such a replacement; here $\Delta$ is the complex Laplacian.

**3. Noncompact product manifolds.** Let $M = M_1 \times M_2$ be the product of two complex manifolds. In this section we shall verify that $M$ cannot be negatively curved under the additional assumptions. First let us quote the following result due to Cheng-Yau **[C-Y]** and Mok-Yau (cf. **[M-Y]**, (3.1)):

PROPOSITION 4 ([C-Y], [M-Y]). *Suppose $X$ is a Stein manifold, $M \subseteq X$ is a relatively compact open subset which is also Stein. If there exists a hermitian metric $h$ on $M$ with holomorphic sectional curvature $\leq -1$ and $h \geq g$ for some Kähler metric $g$ on $X$. Then $M$ admits a complete Kähler-Einstein metric with negative Ricci curvature.*

*Proof of Theorem* B. Assume the contrary, namely assume that there is a hermitian metric on $M$ with bisectional curvature $\leq -1$. Take a small disc $D \subseteq M_2$ and a cut off function $\rho$ in $D$. Then there exists a positive constant $c$ such that $h|_{M_1 \times \{t\}} \geq c \cdot h|_{M_1 \times \{0\}}$ for each $t \in \mathrm{Supp}(\rho)$. Since $M_1$ is compact, the proof of Theorem A gives a contradiction. □

*Proof of Theorem* C. Assume the contrary: there is a negatively curved metric $h$ on $M$. For $i = 1, 2$, let $g_i$ be the complete Kähler metric on $M_i$ with Ricci curvature bounded between two negative constants, and $g = g_1 \times g_2$. Then by applying Propositions 1 and 2 to the identity map on $M$ we get that $h$ and $g$ are dominated by each other. Hence for each $y \in M_2$, $h|_{M_1 \times \{y\}} \geq c \cdot g|_{M_1 \times \{y\}} = c \cdot g_1$, where $c > 0$ is a constant. Take any small disc $D \subseteq M_2$, and the proof of Theorem A goes through. □

REMARK. It is also clear that in Theorem C above one can lose the Kählerness assumption on $g_i$ to the weaker *hermitian with bounded torsion.*

*Proof of Corollary.* Again assume the contrary: there is a negatively curved metric $h$ on $M = M_1 \times M_2$. Then $h|_{M_1 \times \{y\}}$ and $h|_{\{x\} \times M_2}$ give complete hermitian metrics with non-positive holomorphic sectional curvature on $M_1$ and $M_2$, respectively. By [G] or [S], we know that both $M_1$ and $M_2$ are holomorphically convex, hence Stein as they are contained in Stein manifolds. By Proposition 4, they admit complete Kähler-Einstein metrics with negative Ricci curvature. Hence Theorem C applies and one gets a contradiction. □

REMARK. In proving the non-existence of negatively curved metrics on a general product manifold, the main difficulty comes from the fact that on a submanifold with restricted metric, the curvature is not necessarily bounded from below even if the ambient manifold is so. Or equivalently the second fundamental form need not be bounded. While the above line of argument depends on the Schwarz lemma, or eventually the generalized maximum principle, which requires a lower bound on the Ricci curvature, it should be interesting to know whether or not the following holds:

*Question.* On the bidisc $D \times D$, is there any complete Kähler metric with bisectional curvature $\leq -1$?

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