PACIFIC JOURNAL OF MATHEMATICS
Vol. 164, No. 1, 1994

THE INDEX OF TRANSVERSALLY ELLIPTIC OPERATORS FOR LOCALLY FREE ACTIONS

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Let a connected unimodular Lie group G act smoothly and locally freely on a closed manifold X. Assume that the isotropy groups of the action are torsion-free. Let K be the maximal compact subgroup of G. Let T be a G-invariant first order differential operator on X that is elliptic in directions transverse to the G-orbits. Using Kasparov products over C^*G, we prove index formulas equating indices of elliptic operators on $K\backslash X$ with linear combinations of multiplicities of G-representations in $\mathrm{kernel}(T) - \mathrm{kernel}(T^*)$ **.**

Introduction. Let a connected unimodular Lie group G act smoothly on a closed manifold X. Let T be a G-invariant first order differential operator on X that is elliptic in directions transverse to the G-orbits. Kernel(T) and kernel(T^*) need not be finite-dimensional, but they are direct sums of irreducible G-representations, each occurring with finite multiplicity. (We work with assumptions, described in §2, that guarantee that we have Hilbert space structures and unitary G-representations as needed.) The following is then an interesting index problem. For each irreducible G-representation π, calculate the difference:

$$\text{multiplicity of } \pi \text{ in } \ker(T) - \text{multiplicity of } \pi \text{ in } \ker(T^*) .$$

M. Atiyah and I. Singer studied the index theory of invariant operators elliptic in directions transverse to the orbits of a compact Lie group action **[At1]**. They phrased the index problem as the computation of a distribution on G. Let α^+, respectively α^-, be the representation of G on $\ker(T)$, respectively $\ker(T^*)$. The index distribution is then the functional on $C^\infty(G)$ defined for $f \in C^\infty(G)$ by

$$f \to \mathrm{Tr}\left(\int_G f(g)\alpha^+(g)\,dg\right) - \mathrm{Tr}\left(\int_G f(G)\alpha^-(G)\,dg\right).$$

M. Vergne has now given a formula for this distribution in a neighborhood of the identity **[Ve]**. The foundations of this approach to the index problem extend to noncompact G **[Sin]** **[NeZi]**.

In this paper we focus on the direct calculation of the difference of multiplicities when G acts locally freely. For a locally free action the stabilizer of any point is discrete. Because we also assume that the stablilizers are torsion-free, nothing interesting is lost by restricting attention to noncompact G. In §2 we discuss several classes of examples of the situation we study.

Our approach to the multiplicity problem is the following. In §2 we create from the transversally elliptic operator T two Kasparov $(C^*G, \mathbb{C})$-bimodules. One bimodule involves the domain and range Hilbert spaces of T and the operator T. The other involves just the kernels of T and T^*. The final theorem of the section establishes that the bimodules represent the same class in $KK(C^*G, \mathbb{C})$. In this introduction we denote this class by

$$[T] \in KK(C^*G, \mathbb{C}).$$

The next step is motivated by the idea that certain representations of G define classes in $KK(\mathbb{C}, C^*G)$ that can be represented by Kasparov $(\mathbb{C}, C^*G)$-modules constructed from Dirac operators on $K\backslash G$. Here K is a maximal compact subgroup of G. The bimodules constructed from Dirac operators are discussed more fully in §1.c. Different Dirac operators arise by twisting a given Dirac operator by homogeneous vector bundles defined by K-representations V. In this introduction we denote the resulting classes in K theory by

$$[D_V] \in KK(\mathbb{C}, C^*G).$$

For a given irreducible G-representation π, let $m_T(\pi)$ denote the solution to the original index problem

$$m_T(\pi) = \text{mult}.(\pi, \ker(T)) - \text{mult}.(\pi, \ker(T^*)).$$

Suppose $[D_V]$ is related to π as in the preceding paragraph. We are interested in the relationship between the Kasparov product

$$[D_V] \otimes_{C^*G} [T] \in KK(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$$

and $m_T(\pi)$. We discuss this relationship in some detail. It is an equality in some cases. Moreover, our approach provides a way to calculate the above Kasparov product. Our reasoning applies to any Dirac operator D_V on $K\backslash G$.

In §3 we use the $(C^*G, \mathbb{C})$-bimodule defined by the domain and range Hilbert spaces of T and the operator T, to calculate $[D_V] \otimes_{C^*G} [T]$. Theorem 3.18 states that this product equals the class in $KK(\mathbb{C}, \mathbb{C})$ defined by an explicitly described elliptic operator on the compact manifold $K\backslash X$. In this introduction let us denote this elliptic operator by $P(D_V, T)$. Using the natural isomorphism $KK(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$, we see that

$$[D_V] \otimes_{C^*G} [T] = \operatorname{index}(P(D_V, T)).$$

The Atiyah-Singer index theorem calculates the value of this index.

In §4 we calculate the same Kasparov product. This time we use the $(C^*G, \mathbb{C})$-bimodule defined by kernel(T) and kernel(T^*). Now the result is an example of a basically algebraic construction that may be called the index of the graded CCR representation $\ker(T) \oplus \ker(T^*)$ with respect to the elliptic operator D_V. In this introduction we denote this quantity by

$$\operatorname{Index}(\ker(T) \oplus \ker(T^*); D_V).$$

We begin §5 by re-emphasizing in Theorem 5.1 that the Kasparov products of the two preceding sections are equal, i.e.

$$\operatorname{index}(P(D_V, T)) = \operatorname{index}(\ker(T) \oplus \ker(T^*); D_V).$$

We then investigate the implications of this equality. We show that

$$\operatorname{index}(\ker(T) \oplus \ker(T^*); D_V) = \sum_{\beta \in \widehat{G}} \operatorname{Index}(\beta \oplus 0; D_V) \cdot m_T(\beta).$$

For a given D_V and T, there are finitely many nonzero terms in the above summation over the unitary dual of G. However, in order for

$$\operatorname{index}(P(D_V, T)) = \sum_{\beta \in \widehat{G}} \operatorname{Index}(\beta \oplus 0; D_V) \cdot m_T(\beta)$$

to be a useful multiplicity formula, we need explicit information about the coefficients $\operatorname{Index}(\beta \oplus 0; D_V)$.

In §6 we discuss the calculation of these coefficients. If G is amenable, each discrete series representation π defines a class in $KK(\mathbb{C}, C^*G)$, and each such class can also be represented by a Dirac operator, which we denote here by $D_{V(\pi)}$. Then

$$\operatorname{Index}(\beta \oplus 0; D_{V(\pi)}) = \begin{cases} 1 & \text{if } \beta = \pi, \\ 0 & \text{if } \beta \neq \pi, \end{cases}$$

and our index formula becomes

$$\operatorname{index}(P(D_{V(\pi)}, T)) = m_T(\pi).$$

The same phenomenon occurs for many other discrete series representations, including all integrable discrete series representations of linear semisimple G.

At the other extreme, if π is an irreducible unitary principal or complementary series representation of a semisimple G, then $\mathrm{Index}(\pi \oplus 0; D_V) = 0$ regardless of which Dirac operator is used. Our approach never provides information about $m_T(\pi)$ for such π. The vanishing of these coefficients is a consequence of Proposition 6.9, which states roughly that if a representation π can be connected to infinity by a path of CCR representations in the space of all representations, then π defines the zero class in $KK(C^*G, \mathbb{C})$.

In many cases the calculation of $\mathrm{Index}(\beta \oplus 0; D_V)$ can be done using purely representation-theoretic methods. Nonetheless K theory, in particular Proposition 6.9, provides a useful way to organize and extend these methods. In §6 we illustrate this idea by calculating $\mathrm{Index}(\beta \oplus 0; D_V)$ for some irreducible representations β of $\mathrm{SU}(n, 1)$ that occur in the decomposition at the endpoint of a unitary complementary series of representations. For such β our methods provide some information about $m_T(\beta)$. In general our methods are constrained by topological properties of the unitary dual of G.

Finally in §7 we depart from the main thrust of the paper, but not substantially from its spirit or techniques, to prove a multiplicity formula for discrete series representations in quasi-regular representations of amenable, locally compact, second countable, connected topological groups. This result generalizes results appearing in **[MoWo]** and **[R2]**.

REMARK. Our index-theoretic multiplicity formula generalizes the compact case of a formula **[M]** for the index of a Dirac operator on a locally symmetric space. We plan to prove a generalization of the noncompact case in another paper. In fact our paper is written with the point of view that the index-theoretic multiplicities we calculate constitute information that is essentially algebraic in nature. Perhaps the clearest way to restate this vague intuition is that the numbers we compute are invariants of the class in $KK(C^*G, \mathbb{C})$ defined by the transversally elliptic operator. Our index theorem is the statement that Kasparov products over C^*G of Dirac operators with different cycles representing this class lead naturally to different interpretations of these invariants. This point of view puts our work in the same setting as index theorems on (noncompact) locally symmetric spaces. With this in mind, we present in §3 a direct, largely self-contained calculation of the Kasparov product over C^*G of a Dirac operator and a transversally elliptic operator. However, it is worth noting that finer analytic information about the transversally elliptic operator is contained in the class it represents in $KK(C^*(G, C(X)), \mathbb{C})$, [C]. In the compact case that we consider here, one can show that the result of §3 is equivalent to the calculation of a Kasparov product over $C^*(G, C(X))$ that is a special case of results of [HiSk]. (The possible absence of K-orientability can be handled by standard techniques.)

REMARK. Because it is undesirable to assume that the domain and range Hilbert spaces for T, or even kernel(T) and kernel(T^*), are weakly contained in the regular representation of G, we must work with C^*G not C_r^*G. This suggests a need for a better understanding of Dirac induction to the K theory of the full group C^* algebra.

REMARK. To handle situations where $K\backslash G$ fails to have a G-invariant spin structure, one may pass to a double cover G' of G. Then one needs to allow a constant central torsion factor in the isotropy groups of the G'-action on X. The reasoning in this paper extends to this case. In general, we expect that the results of this paper extend to cases where the isotropy groups of G's action on X have nonconstant torsion. However, this extension will require the use of index theory on an orbifold $K\backslash X$.

1. Background.

1.a. *KK theory and Kasparov products.* We recall the definition of KK theory and a method for calculating Kasparov products. G. Kasparov [K4] developed KK theory and its product. A. Connes and G. Skandalis [CSk], [Sk] developed the connection approach to Kasparov products. A detailed exposition of KK theory and its product appears in [B1].

DEFINITION 1.1. For C^* algebras A and B, the set of Kasparov (A, B)-bimodules, $\mathscr{E}(A, B)$, is the set of triples (E, F, ϕ) where:

1. $E = E^0 \oplus E^1$ is a countably generated $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert B-module;
2. ϕ is a homomorphism $\phi: A \to \mathscr{L}(E)$ the algebra of adjointable operators on E (we often omit ϕ from the notation, especially when $A = \mathbb{C}$).
3. F is a degree-one element of $\mathscr{L}(E)$ satisfying for each $a \in A$:
   (a) $\phi(a)(F^2 - I) \in \mathscr{K}(E)$, the algebra of compact operators on E;
   (b) $[\phi(a), F] \in \mathscr{K}(E)$;
   (c) $\phi(a)(F - F^*) \in \mathscr{K}(E)$.

DEFINITION 1.2. $KK(A, B)$ equals $\mathscr{E}(A, B)$, the set of (co)cycles for (A, B), module the equivalence relation generated by homotopy.

Tensor product of modules. Suppose $(E_1, F_1, \phi) \in \mathscr{E}(A, B)$ and $(E_2, F_2, \sigma) \in \mathscr{E}(B, D)$. Let $E = E_1 \otimes_B E_2$ be the graded tensor product. The inner product on E is given by the formula

$$(1.3) \qquad \langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle_E = \langle x_2, \sigma(\langle x_1, y_1 \rangle_{E_1})(y_2) \rangle_{E_2}.$$

For $x \in E_1$ there is an operator $Q_x \in \mathscr{L}(E_2, E)$ defined by $Q_x(y) = x \otimes y$. Its adjoint Q_x^* satisfies $Q_x^*(z \otimes y) = \sigma(\langle x, z \rangle_{E_1})(y)$.

Definition 1.4. Use the notation of the preceding paragraph. An operator $F \in \mathscr{L}(E)$ is called an F_2-connection for E_1 if for every $x \in E_1$, with x of pure degree,

1. $Q_x \circ F_2 - (-1)^{\partial x \partial F_2} F \circ Q_x \in \mathscr{K}(E_2, E)$;
2. $F_2 \circ Q_x^* - (-1)^{\partial x \partial F_2} Q_x^* F \in \mathscr{K}(E, E_2)$.

Definition 1.5. Use the notation of the preceding paragraph. Assume A is separable. Let ϕ' be the map $A \to \mathscr{L}(E)$ arising naturally from ϕ. (E, F, ϕ') is called a Kasparov product of (E_1, F_1, ϕ) and (E_2, F_2, σ) if:

1. F is an F_2-connection for E_1;
2. $(E, F, \phi') \in \mathscr{E}(A, D)$;
3. for each $a \in A$ $\phi'(a)[F_1 \otimes I, F]\phi'(a)^* \geq 0 \bmod \mathscr{K}(E)$.

Notation 1.6. If (E, F, ϕ') satisfies the conditions of Definition 1.5, we write $(E, F, \phi') = (E_1, F_1, \phi) \otimes_B (E_2, F_2, \sigma)$.

Theorem 1.7 **[Sk]**. *$([(E_1, F_1, \phi)], [(E_2, F_2, \sigma)]) \to [(E_1, F_1, \sigma) \otimes_B (E_2, F_2, \sigma)]$ is a well-defined map $KK(A, B) \times KK(B, D) \to KK(A, D)$. We often denote the element of $KK(A, D)$ by $[(E_1, F_1, \phi)] \otimes_B [(E_2, F_2, \sigma)]$.*

1.b. *Completely continuous representations.*

Lemma 1.8. *Let ϕ be a representation of a C^* algebra A in $\mathscr{L}(H)$ for some Hilbert space H. Assume that $\phi(a) \in \mathscr{K}(H)$ for each $a \in A$. Then*

$$\left(H \oplus 0, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \phi \oplus 0\right) \in \mathscr{E}(A, \mathbb{C}).$$

Proof. Check the conditions in the definition of $\mathscr{E}(A, \mathbb{C})$.

Remark 1.9. Let π be a continuous unitary representation of a locally compact group G on a Hilbert space H. For each $f \in L^1(G)$ there is an operator $\sigma_\pi(f) \in \mathscr{L}(H)$ defined by

$$\sigma_\pi(f) = \int_G f(g)\pi(g)\,dg.$$

The map $\sigma_\pi\colon L^1(G) \to \mathscr{L}(H)$ extends to a continuous homomorphism $\sigma_\pi\colon C^*G \to \mathscr{L}(H)$. The image $\sigma_\pi(C^*G)$ is contained in the norm closure of $\sigma_\pi(L^1(G))$.

LEMMA 1.10. *In the setting of the preceding remark, assume* $\sigma_\pi(a) \in \mathscr{K}(H)$ *for every* $a \in C^*G$. *Then*

$$\left(H \oplus 0, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \sigma_\pi \oplus 0\right) \in \mathscr{E}(C^*G, \mathbb{C}).$$

If σ_π *factors through* C_r^*G, *the above data define an element of* $\mathscr{E}(C_r^*G, \mathbb{C})$.

Proof. This lemma is a special case of Lemma 1.8.

1.c. *Dirac induction.* Let G be a connected Lie group with maximal compact subgroup K. Assume that the action of K on $T_e(K\backslash G)$ is spin. (If not satisfied by G, this assumption is satisfied by a double cover of G, which can be used in the arguments of this paper. Alternatively there is a version of Dirac induction that does not require the spin assumption **[K1]**.) Assume that the dimension of $K\backslash G$ is even. (There is also a version of Dirac induction for $\dim(K\backslash G)$ odd.)

Let V be a representation of K, and let D_V be the Dirac operator on $K\backslash G$ twisted by the bundle $V \times_K G$. Let S be the fiber at the identity coset of the spin bundle for $K\backslash G$. Then D_V is defined on smooth compactly supported sections of $(S \otimes V) \times_K G$. Let $E_{S\otimes V}$ be the completion of this set of sections in the norm associated with the C^*G-valued inner product defined on these sections in **[K2]**. One can define an operator $D_V \circ (1 + D_V^2)^{-1/2} \in \mathscr{L}(E_{S\otimes V})$ **[K2]**. This operator has degree one with respect to the grading inherited from the usual grading on S.

LEMMA 1.11 **[K2]**. $(E_{S\otimes V}, D_V \circ (1 + D_V^2)^{-1/2}) \in \mathscr{E}(\mathbb{C}, C^*G)$.

DEFINITION 1.12. The map $R(K) \to KK(\mathbb{C}, C^*G)$ defined by

$$V \to [(E_{S\otimes V}, D_V \circ (1 + D_V^2)^{-1/2})]$$

is called Dirac induction.

REMARK 1.13. Using a C_r^*G-valued inner product and proceeding as above, one can define a map $R(K) \to KK(\mathbb{C}, C_r^*G)$ that is also called Dirac induction. If $p\colon C^*G \to C_r^*G$ is the natural map arising from restriction to the regular representation and if $p_*\colon KK(\mathbb{C}, C^*G)$

$\to KK(\mathbb{C}, C_r^*G)$ is the associated map on K theory, then $p_* \circ (C^*G$ version of Dirac induction) equals the C_r^*G version of Dirac induction.

REMARK 1.14. It is often convenient to view sections of $(S\otimes V)\times_K G \to K\backslash G$ as $S\otimes V$-valued functions on G, $f\colon G\to S\otimes V$, satisfying the K-invariance property $f(kg) = f(g)\cdot k^{-1}$. From this point of view, the C^*G-valued inner product and right C^*G-action on $E_{S\otimes V}$ arise by completion in the C^*G-norm of the following. For f_1 and f_2 smooth compactly support K- invariant $(S\otimes V)$-valued functions on G and for $f\in C_c^\infty(G)$,

$$\langle f_1, f_2\rangle(g) = \int_G \langle f_1(sg^{-1}), f_2(s)\rangle_{S\otimes V}\, ds = \int_G \langle f_1(s), f_2(sg)\rangle_{S\otimes V}\, ds,$$

$$(f_1\cdot f)(g) = \int_G f_1(gs^{-1})f(s)\, ds.$$

The element of $\mathscr{E}(\mathbb{C}, C_r^*G)$ that is implicit in Remark 1.13 can be viewed in an analogous way.

NOTATION 1.15. We denote the set of smooth, compactly supported, K-invariant, $(S\otimes V)$-valued functions on G by

$$C_c^\infty(G, S\otimes V)^K.$$

REMARK 1.16. Give $\mathfrak{g}$, the Lie algebra of G, a metric that is invariant under the adjoint action of K. Let $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$, where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$. Let $\{y_1, \dots, y_n\}$ be an orthonormal basis for $\mathfrak{p}$. Let $\mathrm{cl}(Y_i)$ be the linear map on $S\otimes V$ defined by Clifford multiplication by Y_i on S and the identity on V. For $f\in C_c^\infty(G, S\otimes V)^K$, the identification of Remark 1.14 identifies $(D_V(f))(g)$ with

$$\sum_{i=1}^n \mathrm{cl}(Y_i)\, \frac{d}{dt}\bigg|_{t=0} f(\exp(-tY_i)g).$$

1.d. *Functional calculus.* We recall a technique that can be used in computing Kasparov products in situations where it is not obvious how to apply the standard calculus of pseudodifferential operators. This technique is used to construct Kasparov bimodules in **[BaJ]** and to compute Kasparov products in **[FHRa]**. The observations behind the technique are that the Riemann integral $(1/\pi)\int_0^\infty \lambda^{-1/2}(x+\lambda)^{-1}\, d\lambda$ equals $x^{-1/2}$ and that convergence is uniform in $x\geq 1$. By uniform convergence we mean that for any $\delta > 0$ there exist ε, N and m

such that for any $x \geq 1$ any Riemann sum R of mesh length $\leq m$ for $(1/\pi)\int_{\varepsilon'}^{N'} \lambda^{-1/2}(x+\lambda)^{-1}\,d\lambda$, $0 < \varepsilon' \leq \varepsilon < N \leq N'$, satisfies $|R - x^{-1/2}| < \delta$.

Let H be a Hilbert space. Let P be a self-adjoint operator on H. Then

$$(1.17) \qquad (1+P^2)^{-1/2} = (1/\pi)\int_0^\infty \lambda^{-1/2}(1+P^2+\lambda)^{-1}\,d\lambda$$

in the sense of the norm limit of functional calculus expressions arising from Riemann sums for approximating proper integrals for

$$(1/\pi)\int_0^\infty \lambda^{-1/2}(x+\lambda)^{-1}\,d\lambda.$$

2. The K homology class of a transversally elliptic operator. In this section we discuss transversally elliptic pseudodifferential operators and the classes they represent in Kasparov's operator K-theory. We assume that G is a connected unimodular Lie group acting smoothly on a closed manifold X. We assume that the action of G is locally free (discrete isotropy groups) with torsion-free stabilizers. T is a first-order G-invariant transversally elliptic (Definition 2.3) differential operator mapping sections of a G-vector bundle over X sections of a G-vector bundle over X. We assume that the vector bundles have G-invariant Hermitian structures, that X has a G-invariant measure defined on a σ-algebra containing the Borel sets, and that the sets of smooth sections imbed in the Hilbert spaces of L^2 sections defined using the measures and the Hermitian structures. G acts by unitary transformations on the Hilbert spaces. Let T also denote the L^2 closure of the T defined on smooth sections. The Hilbert space adjoint of T is denoted T^*. The main purpose of this section is to exhibit two elements of $\mathscr{E}(C^*G, \mathbb{C})$ which are determined by T and which represent the same class in $KK(C^*G, \mathbb{C})$.

Definition 2.1. In the situation described above, assume $Y \in \mathfrak{g}$, the Lie algebra of G. Let $\bar{Y}$ be the vector field on X defined by

$$(\bar{Y}f)(x) = \lim_{t\to 0}[f(\exp(-tY)x) - f(x)]/t.$$

Let $\pi\colon T^*X \to X$ be the natural projection from the cotangent bundle. Define T_G^*X to be

$$\{w \in T^*X : w(\bar{Y}|_{\pi(w)}) = 0\ \forall Y \in \mathfrak{g}\}.$$

Remark 2.2. T_G^*X is a closed G-invariant subset of T^*X. $X \subset T_G^*X$.

DEFINITION 2.3. A G-invariant pseudodifferential operator on X is said to be transversally elliptic if its principal symbol is invertible on $T_G^*X - X$.

The preceding definitions are used for arbitrary smooth G actions, but we restrict our attention to actions in which the stabilizers are discrete and torsion-free. When G is compact, this restriction limits us to free actions. When G is not compact, there are many more interesting examples. We discuss a few, complete with G-invariant first-order transversally elliptic operator T mentioned in the first paragraph.

REMARK 2.4. Let Γ be a discrete torsion-free cocompact subgroup of G. Let $X = G/\Gamma$. Use Haar measure do define $L^2(X)$. Let T be the zero operator from $L^2(X)$ to the zero Hilbert space.

EXAMPLE 2.5. For G, Γ, and X as above, let W be a finite-dimensional unitary representation of Γ. Use Haar measure and a Γ-invariant metric on W to define L^2 sections of the bundle $G \times m_\Gamma W \to G/\Gamma$. Let T be the zero operator from L^2 sections of this bundle to the zero Hilbert space.

EXAMPLE 2.6. For G and Γ as above, let M be a closed Riemannian manifold on which Γ acts by isometries. Let $X = G \times_\Gamma M$. Suppose that W_0 and W_1 are Γ-vector bundles over M with Γ-invariant metrics and that T' is a first-order Γ-invariant elliptic differential operator from sections of W_0 to sections of W_1. $G \times_\Gamma W_0$, respectively $G \times_\Gamma W_1$, is a vector bundle over X. We can identify sections σ of $G \times_\Gamma W_i$ with Γ- invariant functions f_σ defined on G and taking values in the set of sections of W_i. (We call a function f_σ Γ-invariant if for all $g \in G$ and $\gamma \in \Gamma$ $f_\sigma(g\gamma) = \gamma^{-1}(f_\sigma(g)$.) Using the identification we can define a transversally elliptic operator T from sections of $G \times_\Gamma W_0$ to sections of $G \times_\Gamma W_1$ by

$$f_{T(\sigma)}(g) = T'(f_\sigma(g)).$$

This identification also allows us to use Haar measure on G and the natural inner product on sections of W_0, respectively W_1, to define the Hilbert space of L^2 sections of $G \times_\Gamma W_0 \to G \times_\Gamma M$, respectively $G \times_\Gamma W_1 \to G \times_\Gamma M$.

REMARK 2.7. Compare the foliation by G-orbits in this and succeeding examples to the Kronecker foliation of the torus.

REMARK 2.8. Later in this section we will show how to use the data G, X, and T to define two elements of $\mathscr{E}(C^*G, \mathbb{C})$. We will show that these two elements represent the same class in $KK(C^*G, \mathbb{C})$ (see Theorem 2.29). In this remark we state these results in the context of Examples 2.5 and 2.6.

The data of Example 2.6 define an element

(2.8i)
$$\left(L^2(G \times_\Gamma W_0) \oplus L^2(G \times_\Gamma W_1), \begin{pmatrix} 0 & T^*(1+TT^*)^{-1/2} \\ T(1+T^*T)^{-1/2} & 0 \end{pmatrix}\right) \in \mathscr{E}(C^*G, \mathbb{C}).$$

Here the action of C^*G on the Hilbert space is the one associated to the unitary representation of G on the Hilbert space by Remark 1.9. The Γ-action in Example 2.6 respects the decomposition of $\{L^2$ section of $W_0 \to M\}$ into eigenspaces of T'^*T' and the decomposition of $\{L^2$ sections of $W_1 \to M\}$ into eigenspaces of $T'T'^*$. Let $\mathscr{U}_0$ be the 0-eigenspace of T'^*T', viewed as a Γ-representation. Let $\mathscr{V}_0$ be the 0-eigenspace of $T'T'^*$, viewed as a Γ-representation. G, Γ, and $\mathscr{U}_0$ or $\mathscr{V}_0$ are data of the type described in Example 2.5, and they define elements

(2.8ii)
$$\left(L^2(G \times_\Gamma \mathscr{U}_0) \oplus 0, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) \quad \text{and} \quad \left(L^2(G \times_\Gamma \mathscr{V}_0) \oplus 0, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)$$

of $\mathscr{E}(C^*G, \mathbb{C})$. The C^*G action arises as above. The content of Theorem 2.29, as applied to these examples, is that the cycle (2.8i) and the difference of the cycles appearing in (2.8ii) represent the same class in $KK(C^*G, \mathbb{C})$.

EXAMPLE 2.9. Consider the product of our Lie group G with another unimodular connected Lie group G_1. Let H_1 be a compact subgroup of G_1. Let T' be a G_1-invariant first-order elliptic differential operator between sections of Hermitian G_1-vector bundles over $H_1\backslash G_1$. For instance if H_1 is a compact Cartan subgroup in G_1, T' could be a Dolbeault operator. If H_1 is a maximal compact subgroup of a noncompact G_1, and if $H_1\backslash G_1$ has a G_1-invariant spin structure, then T' could be a Dirac operator.

Let Γ be a discrete, cocompact, torsion-free subgroup of $G \times G_1$. Let $X = (G \times (H_1\backslash G_1))/\Gamma$, with the natural left G-action. Tensoring T' with the identity operator on functions on G, we define a differential operator on $G \times (H_1\backslash G_1)$ that is invariant under the right

$G \times G_1$-action. The descent of this operator to X is the operator we call T. The Hermitian G_1-bundles on $(H_1 \backslash G_1)$ define Hermitian $G \times G_1$-bundles over $G \times (H_1 \backslash G_1)$, which descend to define Hermitian bundles over X. Haar measure on $G \times G_1$ determines a right $G \times G_1$-invariant measure on $G \times (H_1 \backslash G_1)$, which descends to define a measure on X. The Hermitian structure on the bundles and the measure on X provide what is needed to define the Hilbert spaces of our construction.

The foliation of X by G-orbits is most interesting when Γ is an irreducible lattice in $G \times G_1$ [**Z**, §2.2]. Let $SO(2, 1)$, respectively $SO(3)$, be the subgroup of $SL(3, \mathbb{R})$ leaving invariant the form $x^2 + y^2 - z^2$, respectively $x^2 + y^2 + z^2$. The construction called restriction of scalars provides an irreducible lattice isomorphic to $SO(2, 1)_{\mathbb{Z}[\sqrt{2}]}$ in $SO(2, 1) \times SO(3)$ [**Z**, §2.2, ex. 5.2.12, §6.1]. Because $SO(3)$ is compact, this lattice is cocompact [**BH-C**]. There is a sublattice Γ of finite index that is torsion-free [**B**]. In the notation of our example, we can take $G = SO(2, 1)$ and $G_1 = SO(3)$. An analogous construction, involving restriction of scalars and using the form $x^2 - \sqrt{2}y^2 - \sqrt{3}z^2$, provides an example in which both G and G_1 are noncompact.

We now turn to a general discussion of the G, X, and T described in the first paragraph of this section. We let $\mathbf{F}_0$ and $\mathbf{F}_1$ denote the G-vector bundles over X mentioned in that paragraph.

NOTATION 2.10. We use $\mathbf{F}_0$ and $\mathbf{F}_1$ to define a $\mathbb{Z}/2$-graded complex vector bundle $\mathbf{F} = \mathbf{F}_0 \oplus \mathbf{F}_1$ over X. $L^2(\mathbf{F})$ is the graded Hilbert space of sections of this bundle. $\mathscr{T} = \left(\begin{smallmatrix} 0 & T^* \\ T & 0 \end{smallmatrix}\right)$ is a degree-one unbounded operator on $L^2(\mathbf{F})$.

DEFINITION 2.11. Give $\mathfrak{g}$, the Lie algebra of G, a metric that is invariant with respect to the adjoint action of K. Here K is a maximal compact subgroup of G. Denote by $\mathfrak{k}$ the Lie algebra of K, and by $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$. Let $\{X_1, \dots, X_k\}$ be an orthogonal basis for $\mathfrak{k}$, and let $\{Y_1, \dots, Y_n\}$ be an orthonormal basis for $\mathfrak{p}$. Denote by Δ the element of the universal enveloping algebra defined by $\sum_{i=1}^k X_i^2 + \sum_{j=1}^n Y_j^2$.

LEMMA 2.12. *If π is a unitary representation of G on a Hilbert space H, then the restriction of $\pi(1 - \Delta)$ to the differentiable vectors of π defines an essentially self-adjoint operator on H.*

Proof. This lemma is a theorem of [**NSt**]. One can also consult [**W**, Thm. 4.4.4.3].

REMARK 2.13. We will without further comment use $\pi(1-\Delta)$ to denote the closure of the restriction of $\pi(1-\Delta)$ to the differentiable vectors of π.

LEMMA 2.14. *If π is a unitary representation of G, then $\pi(1-\Delta)$ is a positive operator.*

Proof. See [**W**].

LEMMA 2.15. *Let π be a unitary representation of G on a Hilbert space H.*

(a) *For each $f \in C_c^\infty(G)$ $\exists f'$ and f'' such that $\sigma_\pi(f) = \sigma_\pi(f') \circ (\pi(1-\Delta))^{-1}$ and $\sigma_\pi(f) = (\pi(1-\Delta))^{-1}\sigma_\pi(f'')$.*

(b) *In fact $\sigma_\pi(f) \in \mathscr{K}(H)$ $\forall f \in C^*G$ (i.e. π is a completely continuous representation) if and only if $\pi(1-\Delta)$ has a completely continuous inverse.*

Proof. See [**W**, vol. 1, pp. 255, 304]. (We take adjoints to get (a) in the first form.) The full strength of (b) is due to [**NSt**].

NOTATION 2.16. Denote by ρ the representation of G on $L^2(\mathbf{F})$, $(\rho(g)\xi)(x) = g \cdot (\xi(g^{-1}x))$.

LEMMA 2.17. *$\rho(1-\Delta) + \mathscr{T}^2$ defines a second-order elliptic operator on X. (With respect to the grading on $C^\infty(\mathbf{F})$, this operator has degree zero.)*

Proof. Recall that T is transversally elliptic and that $\rho(1-\Delta)$ is elliptic along the G-orbits. The full ellipticity of $\rho(1-\Delta) + \mathscr{T}^2$ is a consequence of calculations with principal symbols and of the non-negativity of $\rho(1-\Delta)$ and of $\mathscr{T}^2$.

LEMMA 2.18. *View $\rho(1-\Delta) + \mathscr{T}^2$ as an unbounded operator on $L^2(\mathbf{F})$ with domain $C^\infty(\mathbf{F})$, the smooth sections of $\mathbf{F}$. Then $\rho(1-\Delta) + \mathscr{T}^2$ is symmetric.*

Proof. Recall that $\pi(X)^*$ is an extension of $\pi(-X)$ for $X \in \mathfrak{g}$ [**W**]. Calculate using Stokes' theorem on the closed manifold X.

LEMMA 2.19. (a) *$\rho(1-\Delta) + \mathscr{T}^2$, with domain $C^\infty(\mathbf{F})$, is essentially self-adjoint on $L^2(\mathbf{F})$ (we use the same notation $\rho(1-\Delta) + \mathscr{T}^2$ to denote its closure).*

(b) *$\rho(1-\Delta) + \mathscr{T}^2$ has compact inverse;*

(c) $L^2(\mathbf{F}) = \oplus_\lambda H_\lambda$ *(Hilbert space direct sum) where:*

(i) H_λ *is the eigenspace for* $\rho(1-\Delta)+\mathscr{T}^2$ *associated to eigenvalue* λ;

(ii) *each* H_λ *is a finite-dimensional subspace of* $C^\infty(\mathbf{F})$;

(iii) *each* $\lambda \in [1, \infty]$, *and* $\forall N$ $\{\lambda : \lambda \leq N\}$ *is finite.*

Proof. By Lemmas 2.17 and 2.18, (a) follows from **[T**, p. 54]. The rest of the lemma is a consequence of Rellich's lemma, elliptic regularity, and the nature of compact self-adjoint operators, as well as the observation that $\rho(-\Delta)$ and $\mathscr{T}^2$ are non-negative

LEMMA 2.20. *There exists an orthonormal basis of* $L^2(\mathbf{F})$ *consisting of eigenvalues of* $\mathscr{T}^2$ *that are contained in* $C^\infty(\mathbf{F})$.

Proof. On $C^\infty(\mathbf{F})$ $\mathscr{T}^2$ commutes with $\rho(1-\Delta)+\mathscr{T}^2$. Thus $\mathscr{T}^2$ maps each H_λ to itself. $\mathscr{T}^2$ is symmetric on $C^\infty(\mathbf{F})$. Thus each H_λ has an orthonormal basis of eigenvalues for $\mathscr{T}^2$.

LEMMA 2.21. $\mathscr{T}^2$ *is essentially self-adjoint on* $C^\infty(\mathbf{F})$.

Proof. Each eigenvector described in Lemma 2.20 is an analytic vector for $\mathscr{T}^2$. Apply Nelson's analytic vector theorem **[ReSi]**.

LEMMA 2.22. *Each* H_λ *has an orthonormal basis of vectors that are eigenvectors for both* $\rho(1-\Delta)$ *and* $\mathscr{T}^2$. *Consequently each* H_λ *decomposes as* $\bigoplus_{\mu,\nu} H_{\mu,\nu}$ *where for each* μ *and* ν

$$H_{\mu,\nu} = \{\vec{h} \in L^2(\mathbf{F}) : \rho(1-\Delta)\vec{h} = \mu\vec{h} \text{ and } \mathscr{T}^2\vec{h} = \nu\vec{h}\}.$$

Proof. On $C^\infty(\mathbf{F})$ $\rho(1-\Delta)$ is symmetric and $\rho(1-\Delta)$ commutes with $\rho(1-\Delta)+\mathscr{T}^2$ and with $\mathscr{T}^2$.

COROLLARY 2.23. *The spectra of* $\rho(1-\Delta)$ *and of* $\mathscr{T}^2$ *are pure point.*

Proof. This follows from Lemmas 2.19 and 2.22.

PROPOSITION 2.24. *Let* $\mathscr{H}_{\nu_0}$ *be the eigenspace for* $\mathscr{T}^2$ *associated to an arbitrary eigenvalue* ν_0 *of* $\mathscr{T}^2$. *Then the restriction of* $\rho(1-\Delta)$ *to* H_{ν_0} *has compact inverse.*

Proof. Recall that $\rho(-\Delta)$ and $\mathscr{T}^2$ are non-negative. Fix ν_0. $\mathscr{H}_{\nu_0} = \bigoplus_\mu H_{\mu,\nu_0}$ where the H_{μ,ν_0} are those described in Lemma 2.22. Fix μ_0.

$$\text{Dimension}\left(\bigoplus_{\mu\le\mu_0} H_{\mu,\nu_0}\right) \le \text{dimension}\left(\bigoplus_{\lambda\le\mu_0+\nu_0} H_\lambda\right).$$

The latter dimension is finite by Lemma 2.19, where the H_λ are defined.

We record at this point a lemma on symmetric transversally elliptic first order differential operators. We will not pursue this subject further in this paper.

LEMMA 2.25. *Let G, X, and T be as in the first paragraph of this section except that we assume T maps sections of an ungraded vector bundle $\mathbf{F}$ to sections of $\mathbf{F}$ and that T is symmetric on $C^\infty(\mathbf{F})$. Then T is essentially self-adjoint on $C^\infty(\mathbf{F})$ and the spectrum of T is pure point.*

Proof. We did not use the grading on the Hilbert space in our analysis of $\rho(1-\Delta)+T^2$. Thus the same analysis applies to the operator $\rho(1-\Delta)+T^2$ arising from the symmetric operator T of this lemma. Because T commutes with $\rho(1-\Delta)+T^2$ on $C^\infty(\mathbf{F})$ and because T is symmetric on $C^\infty(\mathbf{F})$, T restricts to define a symmetric operator on each of the finite-dimensional H_λ. Thus $L^2(\mathbf{F})$ has an orthonormal basis of smooth eigenvectors for T. Nelson's analytic vector theorem implies that T is essentially self-adjoint.

We now return to the non-trivially graded case where $T\colon L^2(\mathbf{F}_0)\to L^2(\mathbf{F}_1)$. We use the notation of 2.10, 2.16, and 1.9.

THEOREM 2.26. $(L^2(\mathbf{F}), \mathscr{T}\circ(1+\mathscr{T}^2)^{-1/2}, \sigma_\rho)\in\mathscr{E}(C^*G, \mathbb{C})$.

Proof. We verify explicitly Conditions 3.a, 3.b, and 3.c of Definition 1.1.

3.a. $(\mathscr{T}\circ(1+\mathscr{T}^2)^{-1/2})^2 - I = -(1+\mathscr{T}^2)^{-1}$. Because $C_c^\infty(G)$ is norm dense in C^*G, it suffices to prove that $\sigma_\rho(a)\circ(1+\mathscr{T}^2)^{-1}$ is compact for $a\in C_c^\infty(G)$. Choose an arbitrary $a\in C_c^\infty(G)$. By Lemma 2.19. (b) $\sigma_\rho(a)\circ(\rho(1-\Delta)+1+\mathscr{T}^2)^{-1}$ is compact. We finish the proof by showing that

$$\sigma_\rho(a)\circ(1+\mathscr{T}^2)^{-1} - \sigma_\rho(a)\circ(\rho(1-\Delta)+1+\mathscr{T}^2)^{-1}$$

is compact.

$$\begin{aligned}
&\sigma_\rho(a)\circ(1+\mathscr{T}^2)^{-1}-\sigma_\rho(a)\circ(\rho(1-\Delta)+1+\mathscr{T}^2)^{-1}\\
&\quad=\sigma_\rho(a)[(1+\mathscr{T}^2)^{-1}-(\rho(1-\Delta)+1+\mathscr{T}^2)^{-1}]\\
&\quad=\sigma_\rho(a)\circ(1+\mathscr{T}^2)^{-1}(\rho(1-\Delta)+1+\mathscr{T}^2-(1+\mathscr{T}^2))\\
&\qquad\circ(\rho(1-\Delta)+1+\mathscr{T}^2)^{-1}\\
&\quad=\sigma_\rho(a)\circ(1+\mathscr{T}^2)^{-1}\circ\rho(1-\Delta)\circ(\rho(1-\Delta)+1+\mathscr{T}^2)^{-1}.
\end{aligned}$$

Lemma 2.15.(a) shows that there is an $a'\in C_c^\infty(G)$ for which the above equals $\sigma_\rho(a')\circ(1+\mathscr{T}^2)^{-1}\circ(\rho(1-\Delta)+1+\mathscr{T}^2)^{-1}$, which is compact by Lemma 2.19.(b).

3.b. Because $\mathscr{T}$ commutes with the action of G, this commutator is always zero.

3.c. $\mathscr{T}\circ(1+\mathscr{T}^2)^{-1/2}$ is self-adjoint.

REMARK 2.27. A more general version of Theorem 2.26 appears in [C]. However, there is some value in recording the calculations that are special to our setting.

THEOREM 2.28. *Let $\mathscr{H}_0$ be the 0-eigenspace for $\mathscr{T}^2$. $\mathscr{H}_0$ inherits a grading from $L^2(\mathbf{F})$ and an action σ_ρ of C^*G from $L^2(\mathbf{F})$. Then*

$$\left(\mathscr{H}_0,\begin{pmatrix}0&0\\0&0\end{pmatrix},\sigma_\rho\right)\in\mathscr{E}(C^*G,\mathbb{C}).$$

Proof. Because the operator is the zero operator, we focus on property 3.a of Definition 1.1. Proposition 2.24 and Lemma 2.15.(b) show that the restriction of σ_ρ to $\mathscr{H}_0$ is completely continuous.

THEOREM 2.29. *The cycles in the two preceding theorems represent the same class in $KK(C^*G,\mathbb{C})$. I.e.*

$$[(L^2(\mathbf{F}),\mathscr{T}\circ(1+\mathscr{T}^2)^{-1/2},\sigma_\rho)]=\left[\left(H_0,\begin{pmatrix}0&0\\0&0\end{pmatrix},\sigma_\rho\right)\right].$$

Proof. $L^2(\mathbf{F})=\mathscr{H}_0\oplus\bigoplus_{\nu\neq0}\mathscr{H}_\nu$ where ν runs over the nonzero eigenvalues of $\mathscr{T}^2$. As in the proofs of the preceding theorems, we can show that $(\bigoplus_{\nu\neq0}\mathscr{H}_\nu,\mathscr{T}\circ(1+\mathscr{T}^2)^{-1/2},\sigma_\rho)\in\mathscr{E}(C^*G,\mathbb{C})$.

$$\begin{aligned}
&[(L^2(\mathbf{F}),\mathscr{T}\circ(1+\mathscr{T}^2)^{-1/2},\sigma_\rho)]\\
&\quad=\left[\left(\mathscr{H}_0,\begin{pmatrix}0&0\\0&0\end{pmatrix},\sigma_\rho\right)\right]+\left[\left(\bigoplus_{\nu\neq0}\mathscr{H}_\nu,\mathscr{T}\circ(1+\mathscr{T}^2)^{-1/2},\sigma_\rho\right)\right].
\end{aligned}$$

$\mathscr{T} \circ (1+\mathscr{T}^2)^{-1/2}$ is block diagonal with respect to the decomposition of $\bigoplus_{\nu \neq 0} \mathscr{H}_\nu$ into eigenspaces of $\mathscr{T}^2$. On each $\mathscr{H}_\nu$, $\mathscr{T} \circ (1+\mathscr{T}^2)^{-1/2}$ is of the form

$$\begin{pmatrix} 0 & u_\nu^* \\ u_\nu & 0 \end{pmatrix} \nu^{1/2}(1+\nu)^{-1/2},$$

where u_ν is the unitary operator $\mathscr{T} \circ \nu^{-1/2}$. If for each $\nu \neq 0$, we replace $\mathscr{T} \circ (1+\mathscr{T}^2)^{-1/2}$ by

$$\mathscr{F}_\nu(t) = \begin{pmatrix} 0 & u_\nu^* \\ u_\nu & 0 \end{pmatrix} ((1-t)\nu(1+\nu)^{-1/2} + t),$$

then over $0 \leq t \leq 1$,

$$\left(\bigoplus_{\nu \neq 0} \mathscr{H}_\nu, \bigoplus_{\nu \neq 0} \mathscr{F}_\nu(t), \sigma_\rho \right)$$

defines a homotopy between $(\oplus_{\nu \neq 0} \mathscr{H}_\nu, \mathscr{T} \circ (1+\mathscr{T}^2)^{-1/2}, \sigma_\rho)$ and a degenerate element of $\mathscr{E}(C^*G, \mathbb{C})$.

3. Kasparov product of a Dirac operator and a transversally elliptic operator. Let $[(E_{S\otimes V}, D_V \circ (1+D_V^2)^{-1/2})] \in KK(\mathbb{C}, C^*G)$ be a K theory class defined by Dirac induction as described in §1. Let $[(L^2(\mathbf{F}), \mathscr{T} \circ (1+\mathscr{T}^2)^{-1/2}, \sigma_\rho)] \in KK(C^*G, \mathbb{C})$ be the K homology class defined by a transversally elliptic operator as described in Theorem 2.26. In this section we exhibit an elliptic operator on $K\backslash X$ that represents the class in $KK(\mathbb{C}, \mathbb{C})$ defined by the Kasparov product $[(E_{S\otimes V}, D_V \circ (1+D_V^2)^{-1/2})] \otimes_{C^*G} [(L^2(\mathbf{F}), \mathscr{T} \circ (1+\mathscr{T}^2)^{-1/2}, \sigma_\rho)]$. We use the notation appearing in the discussion of Dirac induction in §1. X, G, T, $\mathbf{F}$, $\mathscr{T}$, and ρ are as in §2. For simplicity we frequently write W for $S \otimes V$.

DEFINITION 3.1. Let $\mathbf{W}$ denote the product bundle $W \times X \to X$. The action of K defined by $k \cdot (w, x) = (w \cdot k^{-1}, kx)$ gives $\mathbf{W}$ the structure of a Hermitian K-vector bundle over X. The grading on W defines a grading on $\mathbf{W}$.

DEFINITION 3.2. Let $\mathbf{W} \otimes \mathbf{F}$ denote the bundle over X whose fiber at x, $(\mathbf{W} \otimes \mathbf{F})_x$, equals $\mathbf{W} \otimes \mathbf{F}_x$. $\mathbf{W} \otimes \mathbf{F}$ receives a grading as the graded tensor product of $\mathbf{W}$ and $\mathbf{F}$. The tensor product action gives $\mathbf{W} \otimes \mathbf{F}$ the structure of a Hermitian K-vector bundle over X.

Recall that $C^\infty(\mathbf{F})$ refers to the smooth sections of $\mathbf{F}$. Recall also that the set of smooth compactly supported sections of $W \times_K G \to$

$K\backslash G$ can be identified with the set of smooth compactly supported K-invariant W-valued functions on G, $C_c^\infty(G, W)^K$.

DEFINITION 3.3. Define a map

$$Q\colon C_c^\infty(G, W)^K \otimes C^\infty(\mathbf{F}) \to C^\infty(\mathbf{W} \otimes \mathbf{F})$$

by

$$Q(f \otimes \xi)(x) = \int_G f(g) \otimes (\rho(g)\xi)(x)\, dg.$$

LEMMA 3.4. *For each $k \in K$ and for any f, ξ, and x as above*

$$[k \cdot (Q(f \otimes \xi))](x) = [Q(f \otimes \xi)](x);$$

i.e. the image of Q lies in the set of K-invariant elements of $C^\infty(\mathbf{W} \otimes \mathbf{F})$, $C^\infty(\mathbf{W} \otimes \mathbf{F})^K$.

Proof. $k \cdot [Q(f \otimes \xi)(k^{-1}x)] = k \cdot [\int_G f(g) \otimes g \cdot \xi(g^{-1}k^{-1}x) dg]$. Setting $u = kg$, we see that the above equals

$$k \cdot \left[\int_G f(k^{-1}u) \otimes k^{-1}u \cdot \xi(u^{-1}x)\, du\right].$$

We now use the K-invariance property of f and then bring the action of k inside the integral to set the above equal to

$$k \cdot \left[\int_G f(u) \cdot k \otimes k^{-1}u \cdot \xi(u^{-1}x)\, du\right] = \int_G f(u) \otimes u \cdot \xi(u^{-1}x)\, du.$$

PROPOSITION 3.5. $E_W \otimes_{C^*G} L^2(\mathbf{F}) \cong L^2(\mathbf{W} \otimes \mathbf{F})^K$.

Proof. The isomorphism arises from the extension to completions of Q. Using approximations to delta functions, one can check that the range of Q contains a dense subset of $L^2(\mathbf{W} \otimes \mathbf{F})^K$. Note that $L^2(\mathbf{W}\otimes\mathbf{F})^K$ is a closed subset of $L^2(\mathbf{W}\otimes\mathbf{F})$. To complete the proof it suffices to check that for $f_1, f_2 \in C_c^\infty(G, W)^K$ and $\xi_1, \xi_2 \in C^\infty(\mathbf{F})$, $\langle Q(f_1 \otimes \xi_1), Q(f_2 \otimes \xi_2)\rangle_{L^2(\mathbf{W}\otimes\mathbf{F})}$ equals $\langle f_1 \otimes \xi_1, f_2 \otimes \xi_2\rangle_{E_W \otimes_{C^*G} L^2(\mathbf{F})}$. The former inner product equals

$$\int_X \left\langle \int_G f_1(g)\otimes(\rho(g)\xi_1)(x)\, dg, \int_G f_2(h)\otimes(\rho(h)\xi_2)(x)\, dh \right\rangle_{(\mathbf{W}\otimes\mathbf{F})_x} dx.$$

The latter inner product equals

$$\begin{aligned}\langle \xi_1, \sigma_\rho(\langle f_1, f_2\rangle_{E_w})(\xi_2)\rangle_{L^2(\mathbf{F})} &= \int_X \langle \xi_1(y), [\sigma_\rho(\langle f_1, f_2\rangle_{E_w})(\xi_2)](y)\rangle_{\mathbf{F}_y}\, dy \\ &= \int_X \langle \xi_1(y), \int_G \langle f_1, f_2\rangle_{E_w}(s)(\rho(s)\cdot\xi_2)(y)\rangle_{\mathbf{F}_y}\, ds\, dy \\ &= \int_X \langle \xi_1(y), \int_G\int_G \langle f_1(ts^{-1}), f_2(t)\rangle_W (\rho(s)\cdot\xi_2)(y)\rangle\, dt\, ds\, dy.\end{aligned}$$

Rewriting the inner product on $(\mathbf{W}\otimes\mathbf{F})_x$ as the product of inner products on factors, replacing $\langle \rho(g)\xi_1(x), \rho(h)\xi_2(x)\rangle_{\mathbf{F}_x}$ by $\langle \xi_1(x), \rho(g^{-1})\rho(h)\xi_2(x)\rangle_{\mathbf{F}_x}$, and changing variables, we see that these inner products are equal.

REMARK 3.6. The isotropy groups for G's action on X are torsion-free. Therefore, K acts freely on X, and $K\backslash X$ is a closed smooth manifold. Let

$$q\colon X \to K\backslash X$$

be the quotient map. Using the given measure on X, we assign a measure to $K\backslash X$ so that $\|f\|_{L^2(K\backslash X)} = \|f\circ q\|_{L^2(X)}$. $K\backslash(\mathbf{W}\otimes\mathbf{F}) \to K\backslash X$ is a vector bundle with Hermitian structure inherited from that on $\mathbf{W}\otimes\mathbf{F}$. It follows that if we extend q to a quotient map $\mathbf{W}\otimes\mathbf{F} \to K\backslash(\mathbf{W}\otimes\mathbf{F})$, then we can identify $L^2(\mathbf{W}\otimes\mathbf{F})^K$ with $L^2(K\backslash(\mathbf{W}\otimes\mathbf{F}))$ by

$$\begin{gathered}R\colon L^2(\mathbf{W}\otimes\mathbf{F})^K \underset{\rightarrow}{\cong} L^2(K\backslash(\mathbf{W}\otimes\mathbf{F})), \\ R(\sigma)(q(x)) = q(\sigma(x)).\end{gathered}$$

DEFINITION 3.7. Let $Y \in \mathfrak{g}$. Recall that there is a linear map $\mathrm{cl}(Y)\colon S\otimes V \to S\otimes V$ given by the tensor product of Clifford multiplication by Y on S and the identity on V. Recall also that $(\mathbf{W}\otimes\mathbf{F})_x = W\otimes\mathbf{F}_x$. Define a vector bundle map

$$c(Y)\colon \mathbf{W}\otimes\mathbf{F} \to \mathbf{W}\otimes\mathbf{F}$$

by

$$c(Y)_x = \mathrm{cl}(Y)\otimes \mathrm{id}_{\mathbf{F}_x}.$$

DEFINITION 3.8. Let $Y \in \mathfrak{g}$. Define a differential operator $d(Y)$ on $C^\infty(\mathbf{W}\otimes\mathbf{F})$ by

$$[d(Y)\eta](x) = c(Y)\left.\frac{d}{dt}\right|_{t=0} \exp(tY)\eta(\exp(tY)^{-1}x).$$

DEFINITION 3.9. Let $\{Y_1, \dots, Y_n\}$ be an orthogonal basis for $\mathfrak{p}$, as in Remark 1.16. Define a differential operator $\underline{\mathscr{D}}_V$ on $C^\infty(\mathbf{W}\otimes\mathbf{F})$ by

$$\underline{\mathscr{D}}_V = \sum_{i=1}^{n} d(Y_i).$$

Use the same notation $\underline{\mathscr{D}}_V$ to denote the closure, as an operator on $L^2(\mathbf{W}\otimes\mathbf{F})$, of the original $\underline{\mathscr{D}}_V$.

REMARK 3.10. $\underline{\mathscr{D}}_V$ commutes with the action of K. Let $\underline{\mathscr{D}}_V^K$ denote the restriction of $\underline{\mathscr{D}}_V$ to $L^2(\mathbf{W}\otimes\mathbf{F})^K$. Define an operator $\mathscr{D}_V$ on $L^2(K\backslash(\mathbf{W}\otimes\mathbf{F}))$ by

$$\mathscr{D}_V(R(\sigma)) = \underline{\mathscr{D}}_V^K(\sigma).$$

PROPOSITION 3.11. *If $f\in C_c^\infty(G, W)^K$ and $\xi\in C^\infty(\mathbf{F})$, then*

$$\underline{\mathscr{D}}_V(Q(f\otimes\xi)) = Q(D_V(f)\otimes\xi).$$

Proof. The change of variable $u=\exp(tY_i)^{-1}g$ shows that

$$\begin{aligned}\int_G & f(\exp(tY_i)^{-1}g)\otimes g\cdot\xi(g^{-1}x)\,dg\\ &= \int_G f(u)\otimes\exp(tY_i)u\cdot\xi(u^{-1}\exp(tY_i)^{-1}x)\,du.\end{aligned}$$

It follows that

$$\begin{aligned}\int_G & \sum_{i=1}^{n}\mathrm{cl}(Y_i)\left.\frac{d}{dt}\right|_{t=0} f(\exp(tY_i)^{-1}g)\otimes g\otimes g\cdot\xi(g^{-1}x)\,dg\\ &= \sum_{i=1}^{n} d(Y_i)\left(\int_G f(u)\otimes u\cdot\xi(u^{-1}x)\,du\right).\end{aligned}$$

REMARK 3.12. Because $L^2(\mathbf{W}\otimes\mathbf{F})\cong W\otimes L^2(\mathbf{F})$, the transversally elliptic operator $\mathscr{T}$ on $L^2(\mathbf{F})$ can be used to define a transversally elliptic operator $\underline{\mathscr{T}}_W$ on $L^2(\mathbf{W}\otimes\mathbf{F})$ by $\underline{\mathscr{T}}_W(w\otimes\xi) = (-1)^{\partial w}w\otimes\mathscr{T}(\xi)$, for w an element of W having pure degree. Because $\mathscr{T}$ is K-invariant, $\underline{\mathscr{T}}_W$ defines an operator $\underline{\mathscr{T}}_W^K$ on $L^2(\mathbf{W}\otimes\mathbf{F})^K$ and an operator $\mathscr{T}_W$ on $L^2(K\backslash(\mathbf{W}\otimes\mathbf{F}))$ as in Remark 3.10.

PROPOSITION 3.13. *If $f\in C_c^\infty(G, W)^K$ and $\xi\in C^\infty(\mathbf{F})$, then*

$$\underline{\mathscr{T}}_W(Q(F\otimes\xi)) = Q(f\otimes\mathscr{T}(\xi)).$$

Proof. Compare Remark 1.9 and Definition 3.3. Recall that $\mathscr{T}$ is assumed to commute with the action of G on $C^\infty(\mathbf{F})$.

DEFINITION 3.14. Define an operator $\mathscr{P}_W$ on $L^2(K\backslash(\mathbf{W}\otimes\mathbf{F}))$ by

$$\mathscr{P}_W = \mathscr{D}_V + \mathscr{T}_W .$$

PROPOSITION 3.15. *For* $f \in C_c^\infty(G, W)^K$ *and* $\xi \in C^\infty(\mathbf{F})$

$$\mathscr{P}_W \circ R \circ Q(f\otimes\xi) = R\circ Q\circ(D_V\otimes 1 + 1\otimes\mathscr{T})(f\otimes\xi).$$

(*Recall that notation* $\otimes$ *refers to graded tensor products.*)

Proof. See Propositions 3.11 and 3.13.

PROPOSITION 3.16. $\mathscr{P}_W$ *is a degree-one first-order elliptic differential operator on* $K\backslash X$. $\mathscr{P}_W = \mathscr{P}_W^*$. (*Recall that we use the same notation for an operator and its closure.*)

Proof. Because D_V and $\mathscr{T}$ are degree-one first-order differential operators, $\mathscr{P}_W$ is also. To prove ellipticity observe that the cotangent space at a point y in $K\backslash X$ is identified by q^* with $T_K^*(X)_x$ for any $x \in q^{-1}(y)$. $T_K^*(X)_x$ can be written $\mathfrak{p}^* \oplus T_G^*(X)_x$. The pullback to $T^*(K\backslash X)_y$ of $K\backslash(\mathbf{W}\otimes\mathbf{F})_y$ can be identified with $\mathbf{W}\otimes\mathbf{F}_x$.

The principal symbol of the lower left corner of D_V, at any point $\eta_1 \in \mathfrak{p}^*$, defines a linear map α_{η_1} from the even part of W to the odd part of W. The principal symbol of the lower left corner of $\mathscr{T}$, at any point $\eta \in T_K^*(X)_x$, defines a linear map β_η from the even part of $\mathbf{F}_x$ to the odd part of $\mathbf{F}_x$.

Suppose $\eta = (\eta_1, \eta_2) \in \mathfrak{p}^* \oplus T_G^*(X)_x \cong T_K^*(X)_x$. Using the identification given in the first paragraph of this proof, one can calculate that the principal symbol of the lower left corner of $\mathscr{P}_W$, at η, is the linear map from the even part of $\mathbf{W}\otimes\mathbf{F}_x$ to the odd part of $\mathbf{W}\otimes\mathbf{F}_x$ given by the sharp product of α_{η_1} with β_η. The principal symbol of the upper right corner of $\mathscr{P}_W$ is the adjoint of the principal symbol of the lower left corner. Because D_V and $\mathscr{T}$ are G-invariant, we have described the symbol of $\mathscr{P}_W$ explicitly, in spite of our use of various identifications.

The map α_{η_1} is invertible for $\eta_1 \in \mathfrak{p}^* - \{\vec{0}\}$. Also, β_η is invertible for $\eta \in T_G^*(X)_x - \{\vec{0}\}$. It follows from the properties of the sharp product that the principal symbol of $\mathscr{P}_W$ is invertible off the zero section of $T^*(K\backslash X)$, i.e. that $\mathscr{P}_W$ is elliptic.

The self-adjointness of $\mathscr{P}_W$ follows from the formal self-adjointness of $\mathscr{D}_V$ and $\mathscr{T}_W$ on the compact manifold $K\backslash X$.

COROLLARY 3.17. $(L^2(K\backslash(\mathbf{W}\otimes\mathbf{F})), \mathscr{P}_W\circ(1+\mathscr{P}_W^2)^{-1/2}) \in \mathscr{E}(\mathbb{C}, \mathbb{C})$.

Proof. Because $K\backslash X$ is compact, this result is a consequence of Proposition 3.16.

THEOREM 3.18. $[(E_{S\otimes V}, D_V \circ (1 + D_V^2)^{-1/2})] \otimes_{C^*G} [(L^2(\mathbf{F}), \mathscr{T} \circ (1 + \mathscr{T}^2)^{-1/2}, \sigma_\rho)] = [(L^2(K\backslash(\mathbf{W} \otimes \mathbf{F})), \mathscr{P}_W \circ (1 + \mathscr{P}_W^2)^{-1/2})] \in KK(\mathbb{C}, \mathbb{C})$.

Proof. Corollary 3.17 establishes property 2 of Definition 1.5. The positivity and connection properties required of a Kasparov product are established in Propositions 3.27 and 3.28, the proofs of which require several lemmas.

LEMMA 3.19. *Suppose $f \in C^\infty(G, W)^K$ is such that $(1 + D_V^2)f \in C_c^\infty(G, W)^K$. Then $f \in E_W$. Assume $\xi \in C^\infty(\mathbf{F})$. Then*

$$R \circ Q((1 + D_V^2)f \otimes \xi) = (1 + \mathscr{D}_V^2)(R \circ Q(f \otimes \xi)).$$

Proof. Because $1 + \mathscr{D}_V^2$ is a closed operator, this lemma is a consequence of Remark 3.10, Proposition 3.11 and the proof of Theorem 2 of **[K2]**.

LEMMA 3.20. *The identification of Hilbert spaces $E_W \otimes_{C^*G} L^2(\mathbf{F})$ and $L^2(K\backslash(\mathbf{W}\otimes\mathbf{F}))$ given by Proposition* 3.5 *and Remark* 3.6 *identifies $(1 + D_V^2)^{-1/2} \otimes 1$ with $(1 + \mathscr{D}_V^2)^{-1/2}$.*

Proof. Because the operators in question are bounded, it suffices to show that for $f \otimes \xi \in C_c^\infty(G, W)^K \otimes C^\infty(\mathbf{F})$,

$$R \circ Q((1 + D_V^2)^{-1/2} f \otimes \xi) = (1 + \mathscr{D}_V^2)^{-1/2}(R \circ Q(f \otimes \xi)).$$

Formula (1.17) permits us to reduce the problem to a comparison of $R \circ Q((1 + D_V^2)^{-1} f \otimes \xi)$ with $(1 + \mathscr{D}_V^2)^{-1}(R \circ Q(f \otimes \xi))$. By Lemma 3.19 these are equal.

LEMMA 3.21. *Under the identification of modules mentioned in Lemma* 3.20,

$$D_V(1 + D_V^2)^{-1/2} \otimes 1 = \mathscr{D}_V(1 + \mathscr{D}_V^2)^{-1/2}.$$

Proof. By Lemma 3.20 the bounded operators $(1+D_V^2)^{-1/2}D_V \otimes 1$ and $(1+\mathscr{D}_V^2)^{-1/2}\mathscr{D}_V$ agree on $C_c^\infty(G, W)^K \otimes C^\infty(\mathbf{F})$.

LEMMA 3.22. *There is an orthonormal basis for* $L^2(K\backslash(\mathbf{W}\otimes\mathbf{F}))$ *that consists of smooth eigenvectors for* $\mathscr{P}_W^2$. *The eigenspace associated to each eigenvalue of* $\mathscr{P}_W^2$ *is a finite-dimensional subspace of* $C^\infty(K\backslash(\mathbf{W}\otimes\mathbf{F}))$.

Proof. $\mathscr{P}_W^2$ is a self-adjoint second-order elliptic differential operator on the closed manifold $K\backslash X$.

NOTATION 3.23. Let U_α denote the eigenspace for $\mathscr{P}_W^2$ associated with eigenvalue α. Let U denote the algebraic direct sum

$$U = \bigoplus_\alpha U_\alpha .$$

Note that U is dense in $L^2(K\backslash(\mathbf{W}\otimes\mathbf{F}))$.

LEMMA 3.24. *On* $C^\infty(K\backslash(\mathbf{W}\otimes\mathbf{F}))$, $\mathscr{P}_W^2$ *commutes with* $\mathscr{D}_V$ *and* $\mathscr{T}_W$. *Also* $\mathscr{D}_V\mathscr{T}_w = -\mathscr{T}_W\mathscr{D}_V$.

Proof. The proof is a computation using the definition of $\underline{\mathscr{D}}_V$, Definition 3.9, and the G-invariance of $\mathscr{T}$.

LEMMA 3.25. *For each* α, $\mathscr{D}_V$ *and* $\mathscr{T}_W$ *are symmetric operators mapping* U_α *to* U_α. *As maps from* U *to* U, $\mathscr{D}_V$ *and* $\mathscr{T}_W$ *are symmetric operators.*

Proof. Because $\mathscr{D}_V$ and $\mathscr{T}_W$ are symmetric on $C^\infty(K\backslash(\mathbf{W}\otimes\mathbf{F}))$, this lemma follows from Lemma 3.24.

LEMMA 3.26. *The operators* $\mathscr{D}_V$, $\mathscr{T}_W$, $(1+\mathscr{D}_V^2)^{-1/2}$ *and* $(1+\mathscr{T}_W^2)^{-1/2}$ *commute with* $\mathscr{P}_W^2$ *and* $(1+\mathscr{P}_W^2)^{-1/2}$ *as maps from* U_α *to* U_α *and from* U *to* U. *Also* $\mathscr{T}_W$ *commutes with* $\mathscr{D}_V^2$ *and* $(1+\mathscr{D}_V^2)^{-1/2}$ *as maps from* U_α *to* U_α *and from* U *to* U.

Proof. By the finite-dimensional spectral theorem, this lemma is a consequence of Lemma 3.25.

PROPOSITION 3.27. $[D_V(1+D_V^2)^{-1/2}\otimes 1, \mathscr{P}_W(1+\mathscr{P}_W^2)^{-1/2}] \geq 0$ *modulo compact operators. This establishes property* 3 *of Definition* 1.5 *for the Kasparov product of Theorem* 3.18.

Proof. Recall that square brackets denote graded commutators, which are the same as ordinary commutators unless both operators have degree one.

It suffices to establish non-negativity of this bounded operator on the dense subset U. The following formal manipulations of operators are justified by the preceding lemmas when the operators are applied to elements of U.

$$\begin{aligned}
&[D_V(1+D_V^2)^{-1/2}\otimes 1\,,\,\mathscr{P}_W(1+\mathscr{P}_W^2)^{-1/2}] \\
&\quad= [D_V(1+D_V^2)^{-1/2}\otimes 1\,,\,\mathscr{P}_W](1+\mathscr{P}_W^2)^{-1/2} \\
&\qquad - \mathscr{P}_W[D_V(1+D_V^2)^{-1/2}\otimes 1\,,\,(1+\mathscr{P}_W^2)^{-1/2}] \\
&\quad= [\mathscr{D}_V(1+\mathscr{D}_V^2)^{-1/2}\,,\,\mathscr{P}_W](1+\mathscr{P}_W^2)^{-1/2} \\
&\qquad - \mathscr{P}_W[\mathscr{D}_V(1+\mathscr{D}_V^2)^{-1/2}\,,\,(1+\mathscr{P}_W^2)^{-1/2}] \\
&\quad= [\mathscr{D}_V(1+\mathscr{D}_V^2)^{-1/2}\,,\,\mathscr{D}_V](1+\mathscr{P}_W^2)^{-1/2} \\
&\qquad + [\mathscr{D}_V(1+\mathscr{D}_V^2)^{-1/2}\,,\,\mathscr{T}_W](1+\mathscr{P}_W^2)^{-1/2} \\
&\qquad - \mathscr{P}_W[\mathscr{D}_V(1+\mathscr{D}_V^2)^{-1/2}\,,\,(1+\mathscr{P}_W^2)^{-1/2}].
\end{aligned}$$

We analyze the terms after the last equals sign.

By Lemma 3.26 $\mathscr{P}_W[\mathscr{D}_V(1+\mathscr{D}_V^2)^{-1/2}\,,\,(1+\mathscr{P}_W^2)^{-1/2}]=0$.

$$\begin{aligned}
&[\mathscr{D}_V(1+\mathscr{D}_V^2)^{-1/2}\,,\,\mathscr{T}_W](1+\mathscr{P}_W^2)^{-1/2} \\
&\quad= [(1+\mathscr{D}_V^2)^{-1/2}\mathscr{D}_V\,,\,\mathscr{T}_W](1+\mathscr{P}_W^2)^{-1/2} \\
&\quad= (1+\mathscr{D}_V^2)^{-1/2}[\mathscr{D}_V\,,\,\mathscr{T}_W](1+\mathscr{P}_W^2)^{-1/2} \\
&\qquad + [\mathscr{T}_W\,,\,(1+\mathscr{D}_V^2)^{-1/2}]\mathscr{D}_V(1+\mathscr{P}_W^2)^{-1/2}.
\end{aligned}$$

By Lemmas 3.24 and 3.26, each term after the last equals sign is zero. Similarly

$$\begin{aligned}
&[\mathscr{D}_V(1+\mathscr{D}_V^2)^{-1/2}\,,\,\mathscr{D}_V](1+\mathscr{P}_W^2)^{-1/2} \\
&\quad= (1+\mathscr{D}_V^2)^{-1/2}[\mathscr{D}_V\,,\,\mathscr{D}_V](1+\mathscr{P}_W^2)^{-1/2} \\
&\qquad + [\mathscr{D}_V\,,\,(1+\mathscr{D}_V^2)^{-1/2}]\mathscr{D}_V(1+\mathscr{P}_W^2)^{-1/2} \\
&\quad= (1+\mathscr{D}_V^2)^{-1/2}2\mathscr{D}_V^2(1+\mathscr{P}_W^2)^{-1/2}.
\end{aligned}$$

That $\langle(1+\mathscr{D}_V^2)^{-1/2}2\mathscr{D}_V^2(1+\mathscr{P}_W^2)^{-1/2}u\,,\,u\rangle\geq 0$ for each $u\in U$ follows from Lemma 3.26 and the finite dimensional spectral theorem.

PROPOSITION 3.28. *For $f\in E_W$, Let Q_f denote the map $L^2(\mathbf{F})\to E_W\otimes_{C^*G}L^2(\mathbf{F})\cong L^2(K\backslash(\mathbf{W}\otimes\mathbf{F}))$ defined by $Q_f(\xi)=f\otimes\xi$. Then for $f\in C_c^\infty(G\,,\,W)^K$ of pure degree,*

$$Q_f\circ\mathscr{T}(1+\mathscr{T}^2)^{-1/2}-(-1)^{\partial f}\mathscr{P}_W(1+\mathscr{P}_W^2)^{-1/2}\circ Q_f$$

is compact. Because the ideal of compact operators is norm-closed and because the adjoint of a compact operator is compact, this result establishes that the connection condition for the Kasparov product of Theorem 3.18 *is satisfied.*

Proof. Because $\mathscr{T}$ commutes with the action of G,

$$\begin{aligned}Q_f\mathscr{T}(1+\mathscr{T}^2)^{-1/2} &- (-1)^{\partial f}\mathscr{P}_W(1+\mathscr{P}_W^2)^{-1/2}Q_f\\ &= (-1)^{\partial f}(\mathscr{T}_W(1+\mathscr{T}_W^2)^{-1/2} - \mathscr{P}_W(1+\mathscr{P}_W^2)^{-1/2})Q_f.\end{aligned}$$

$$\begin{aligned}(\mathscr{T}_W(1+\mathscr{T}_W^2)^{-1/2} &- \mathscr{P}_W(1+\mathscr{P}_W^2)^{-1/2})Q_f\\ &= \mathscr{T}_W((1+\mathscr{T}_W^2)^{-1/2} - (1+\mathscr{P}_W^2)^{-1/2})Q_f - \mathscr{D}_V(1+\mathscr{P}_W^2)^{-1/2}Q_f.\end{aligned}$$

$$\mathscr{D}_V(1+\mathscr{P}_W^2)^{-1/2}Q_f = [\mathscr{D}_V\,,\,(1+\mathscr{P}_W^2)^{-1/2}]Q_f + (1+\mathscr{P}_W^2)^{-1/2}\mathscr{D}_VQ_f.$$

$[\mathscr{D}_V\,,\,(1+\mathscr{P}_W^2)^{-1/2}]$ is compact because it is pseudodifferential of negative order on a compact manifold. The same is true of $(1+\mathscr{P}_W^2)^{-1/2}$, and $\mathscr{D}_VQ_f$ is bounded by Lemma 2.15(a).

Using (1.17) and the observations that $\mathscr{T}$ is closed and all compositions are bounded, we write

$$\begin{aligned}&\mathscr{T}_W((1+\mathscr{T}_W^2)^{-1/2} - (1+\mathscr{P}_W^2)^{-1/2})Q_f\\ &\quad= \left(\frac{1}{\pi}\right)\int_0^\infty \lambda^{-1/2}\mathscr{T}_W((1+\mathscr{T}_W^2+\lambda)^{-1} - (1+\mathscr{P}_W^2+\lambda)^{-1})Q_f\,d\lambda\\ &\quad= \left(\frac{1}{\pi}\right)\int_0^\infty \lambda^{-1/2}\mathscr{T}_W(1+\mathscr{T}_W^2+\lambda)^{-1}\\ &\qquad\qquad\times((1+\mathscr{P}_W^2+\lambda) - (1+\mathscr{T}_W^2+\lambda))(1+\mathscr{P}_W^2+\lambda)^{-1}Q_f\,d\lambda\\ &\quad= \left(\frac{1}{\pi}\right)\int_0^\infty \lambda^{-1/2}\mathscr{T}_W(1+\mathscr{T}_W^2+\lambda)^{-1}\mathscr{D}_V^2(1+\mathscr{P}_W^2+\lambda)^{-1}Q_f\,d\lambda\\ &\quad= \left(\frac{1}{\pi}\right)\int_0^\infty \lambda^{-1/2}\mathscr{T}_W(1+\mathscr{T}_W^2+\lambda)^{-1}(1+\mathscr{P}_W^2+\lambda)^{-1}\mathscr{D}_V^2Q_f\,d\lambda\\ &\qquad+\left(\frac{1}{\pi}\right)\int_0^\infty \lambda^{-1/2}\mathscr{T}_W(1+\mathscr{T}_W^2+\lambda)^{-1}[\mathscr{D}_V^2,\,(1+\mathscr{P}_W^2+\lambda)^{-1}]Q_f\,d\lambda.\end{aligned}$$

After the last equals sign, the first term is compact because $(1+\mathscr{P}_W^2+\lambda)^{-1}$ is compact and, by Lemma 2.15(a), $\mathscr{D}_V^2Q_f$ is bounded; the second term is compact because the commutator is pseudodifferential of negative order.

4. The index of a CCR representation with respect to an elliptic operator. Let $[(E_{S\otimes V}\,,\,D_V(1+D_V^2)^{-1/2})] \in KK(\mathbb{C}\,,\,C^*G)$ be a K theory class defined by Dirac induction as described in §1. Let

$[(\mathscr{H}_0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \sigma_\rho)] \in KK(C^*G, \mathbb{C})$ be the K homology class defined by the kernel of a transversally elliptic operator and of its adjoint as in Theorem 2.28. In this section we relate the Kasparov product

$$[(E_{S\otimes V}, D_V(1+D_V^2)^{-1/2})] \otimes_{C^*G} \left[\left(\mathscr{H}_0, \begin{pmatrix}0&0\\0&0\end{pmatrix}, \sigma_\rho\right)\right] \in KK(\mathbb{C}, \mathbb{C})$$

to an algebraic construction that may be called the index of H_0 with respect to D_V. Among the papers in which such a construction has played a role are **[M]**, **[P]**, **[S]**, **[VZu]**, and **[Wil]**. We follow the exposition of **[M]**, except for some left/right conventions. The only property of (H_0, σ_ρ) required in this section is that (H_0, σ_ρ) be a CCR representation. We state our theorem in that generality.

NOTATION 4.1. Let $\mathscr{U}$ denote the universal enveloping algebra of the complexification of $\mathfrak{g}$. We view the elements of $\mathscr{U}$ as right- invariant differential operators on G.

NOTATION 4.2. Let W^+ and W^- be finite-dimensional right unitary K-representations. $\operatorname{Hom}(W^+, W^-)$ becomes a left K-representation by

$$(k \cdot A)(w) = A(wk) \cdot k^{-1}.$$

The restriction of Ad_G to K defines a left K-representation on $\mathscr{U}$. The tensor product of these two representations defines a representation of K on $\mathscr{U} \otimes \operatorname{Hom}(W^+, W^-)$. We denote by

$$(\mathscr{U} \otimes \operatorname{Hom}(W^+, W^-))^K$$

the set of K-invariant elements of $\mathscr{U} \otimes \operatorname{Hom}(W^+, W^-)$.

REMARK 4.3 **[M]**. The sets of smooth sections of the homogeneous bundles $W^\pm \times_K G$ can be identified with $(C^\infty(G) \otimes W^\pm)^K \cong C^\infty(G, W^\pm)^K$. (Recall Notation 1.15.) The set of right-invariant differential operators from $C^\infty(G, W^+)^K$ to $C^\infty(G, W^-)^K$ corresponds to $(\mathscr{U} \otimes \operatorname{Hom}(W^+, W^-))^K$ via the convention of Notation 4.1.

DEFINITION 4.4. An element of $(\mathscr{U} \otimes \operatorname{Hom}(W^+, W^-))^K$ is called elliptic if the corresponding right-invariant differential operator is elliptic.

REMARK 4.5 **[M]**. The map on $\mathfrak{g}^{\mathbb{C}}$ given by $X \to -\bar{X} = X^*$ extends to a conjugate linear antiautomorphism of $\mathscr{U}$, which we denote $u \to$

u^*. For $D = \sum u_i \otimes A_i \in (\mathscr{U} \otimes \mathrm{Hom}(W^+, W^-))^K$ the formal adjoint D^* is given by

$$\left(\sum u_i \otimes A_i\right)^* = \sum u_i^* \otimes A_i^*.$$

DEFINITION 4.6. Let $D = \sum u_i \otimes A_i \in (\mathscr{U} \otimes \mathrm{Hom}(W^+, W^-))^K$.

Let π be a unitary representation of G on the Hilbert space $H(\pi)$. Let $H^\infty(\pi)$ denote the set of smooth vectors for this representation. (See [W].)

Define

$$\pi(D): H^\infty(\pi) \otimes W^+ \to H^\infty(\pi) \otimes W^-$$

by

$$\pi(D) = \sum \pi(u_i) \otimes A_i.$$

Define

$$D_\pi: (H^\infty(\pi) \otimes W^+)^K \to (H^\infty(\pi) \otimes W^-)^K$$

to be the restriction of $\pi(D)$ to the sets of K-invariants.

NOTATION 4.7. When the notation for the original operator already contains a subscript, e.g., D_V, the notations for the operator of Definition 4.6 and its restriction to K-invariants will be $\pi(D_V)$ and $D_{V,\pi}$ respectively.

PROPOSITION 4.8 [M]. *Assume that D is elliptic. Then the kernel of D_π is the orthogonal complement in $(H(\pi) \otimes W^+)^K$ of* image$((D^*)_\pi)$. *Consider $D_\pi: (H(\pi) \otimes W^+)^K \to (H(\pi) \otimes W^-)^K$ as an unbounded operator between Hilbert spaces.*

COROLLARY 4.9 [M]. *If D is elliptic, then the closure of $(D^*)_\pi$ coincides with the Hilbert space adjoint of $\mathscr{D}_\pi$. Consequently, we use the notation D_π^* without ambiguity.*

REMARK 4.10. We describe a graded version of the preceding construction. Let $W = W^+ \oplus W^-$ grade W. Assign a grading to $(\mathscr{U} \otimes \mathrm{Hom}(W, W))^K$ by using the natural grading on $\mathrm{Hom}(W, W)$. Let D be an odd degree element of $(\mathscr{U} \otimes \mathrm{Hom}(W, W))^K$. Thus D consists of

$$D^+ \in (\mathscr{U} \otimes \mathrm{Hom}(W^+, W^-))^K \quad \text{and} \quad D^- \in (\mathscr{U} \otimes \mathrm{Hom}(W^-, W^+))^K.$$

Assume H is a graded Hilbert space, $H = H_{\mathrm{even}} \oplus H_{\mathrm{odd}}$, on which G acts by a unitary representation π that respects the grading.

$$D_\pi: \begin{matrix} (H^\infty_{\mathrm{even}}(\pi) \otimes W^+)^K \\ \oplus \\ (H^\infty_{\mathrm{odd}}(\pi) \otimes W^-)^K \end{matrix} \to \begin{matrix} (H^\infty_{\mathrm{odd}}(\pi) \otimes W^+)^K \\ \oplus \\ (H^\infty_{\mathrm{even}}(\pi) \otimes W^-)^K \end{matrix}$$

is given by the matrix

$$D_\pi = \begin{pmatrix} 0 & D^-_{\pi_{\text{odd}}} \\ D^+_{\pi_{\text{even}}} & 0 \end{pmatrix}.$$

REMARK 4.11. Using the domain space and range space of D_π as the even and odd parts of $(H^\infty(\pi) \otimes W)^K$, respectively, we can define a degree one operator

$$\mathscr{D}_\pi \colon (H^\infty(\pi) \otimes W)^K \to (H^\infty(\pi) \otimes W)^K$$

by

$$\mathscr{D}_\pi = \begin{pmatrix} 0 & D^*_\pi \\ D_\pi & 0 \end{pmatrix}.$$

By Corollary 4.9 $\mathscr{D}_\pi$ is essentially self-adjoint on $(H(\pi) \otimes W)^K$.

REMARK 4.12. The Dirac operator D_V on $C^\infty(G, S \otimes V)^K$ corresponds to $\sum Y_i \otimes A_i \in (\mathscr{U} \otimes \mathrm{Hom}(S \otimes V, S \otimes V))^K$. Here $\{Y_i\}$ is a basis for $\mathfrak{p}$, orthonormal with respect to a K-invariant inner product, and A_i is the tensor product of Clifford multiplication by Y_i on S with the identity operator on V. $D^-_V \colon C^\infty_c(G, S^- \otimes V)^K \to C^\infty_c(G, S^+ \otimes V)^K$ is the formula adjoint of $D^+_V \colon C^\infty_c(G, S^+ \otimes V)^K \to C^\infty_c(G, S^- \otimes V)^K$. We will use the notation D_V for both the differential operator and for its realization in $(\mathscr{U} \otimes \mathrm{Hom}(S \otimes V, S \otimes V))^K$, and we will treat the notation D^+_V and D^-_V in the same way.

PROPOSITION 4.13. *Let $\mathscr{H}_0$ be the kernel of the square $\mathscr{T}^2$ of the transversally elliptic operator of* §2. *Let ρ denote the representation of G on $\mathscr{H}_0$. Give $\mathscr{H}_0$ the grading inherited from $L^2(\mathbf{F})$. More generally let $(\mathscr{H}_0, \rho)$ be a $\mathbb{Z}/2$-graded CCR representation of G on a Hilbert space. Let D_V be the realization of a Dirac operator in $(\mathscr{U} \otimes \mathrm{Hom}(W, W))^K$. (Recall the notation $W = S \otimes V$.) Then following Remark* 4.11 *to define an operator $\mathscr{D}_{V,\rho}$, we get a Kasparov bimodule*

$$((\mathscr{H}_0(\rho) \otimes W)^K, \mathscr{D}_{V,\rho} \circ (1 + \mathscr{D}^2_{V,\rho})^{-1/2}) \in \mathscr{E}(\mathbb{C}, \mathbb{C}).$$

Proof. By Remark 4.11, we need only show that

$$(1 + \mathscr{D}^2_{V,\rho})^{-1} \in \mathscr{K}((\mathscr{H}_0(\rho) \otimes W)^K).$$

Let $\Delta \in \mathscr{U}^K$ be as in Definition 2.11. Because W is finite-dimensional and ρ is CCR, Lemma 2.15 implies that it suffices to show that range $((1 + \mathscr{D}^2_{V,\rho})^{-1})$ is contained in domain $(\rho(1 - \Delta) \otimes 1)$. Thus it suffices to show that $(\rho(1-\Delta)\otimes 1)(1+\mathscr{D}_{V,\rho}^2)^{-1}$ is bounded. We proceed by showing that $(1+\mathscr{D}_{V,\rho}^2)(\rho(1-\Delta)\otimes 1)^{-1}$ is bounded away from zero. We accomplish this with the help of a definition and two lemmas. Definition 4.14 and Lemma 4.15 are from [S], although our context is not exactly the same as that of [S].

DEFINITION 4.14. Let π be a unitary representation of G. For $\sum Z_i \otimes A_i + 1 \otimes B \in \mathscr{U} \otimes \mathrm{Hom}(W, W)$, with $Z_i \in \mathfrak{g}$, the associated $\sum \pi(Z_i) \otimes A_i + 1 \otimes B$ is called a first-order operator.

LEMMA 4.15. *Let π be a unitary representation of G and let Δ be as in Definition 2.11. Let P be a first-order operator. Then $P \circ (1-\pi(\Delta))^{-1/2}$ is bounded.*

Proof. It suffices to show that there is a constant c such that for $v \otimes w \in H^\infty(\pi) \otimes W$ $\|P(1-\pi(\Delta))^{-1/2}(v\otimes w)\|^2 \le \|v \otimes w\|^2$. Replacing v by $(1-\pi(\Delta))^{1/2}\xi$, we can change this inequality to $\|P(\xi \otimes w)\|^2 \le c\|(1-\pi(\Delta))^{1/2}\xi \otimes w\|^2 = c\langle(1-\pi(\Delta))\xi, \xi\rangle \|w\|^2$. Because W is finite-dimensional, we may assume P is of the form $\sum_i \pi(Z_i) \otimes A_i$, where there is an upper bound on $\{\|A_i\|\}$ and where $\{Z_i\}$ is a basis for $\mathfrak{g}$ of the kind discussed in Definition 2.11.

$$\begin{aligned}\|P(\xi \otimes w)\|^2 &= \sum_{i,j} \langle \pi(Z_i)\xi, \pi(Z_j)\xi\rangle \langle A_i w, A_j w\rangle \\ &\le k\|w\|^2 \sum_{i,j} \langle \pi(Z_i)\xi, \pi(Z_j)\xi\rangle \\ &\le k\|w\|^2 \sum_{i,j} \|\pi(Z_i)\xi\|\, \|\pi(Z_j)\xi\| \\ &\le c\|w\|^2 \sum_i \|\pi(Z_i)\xi\|^2 = c\|w\|^2 \langle -\pi(\Delta)\xi, \xi\rangle \\ &\le c\|w\|^2 \langle (1-\pi(\Delta))\xi, \xi\rangle .\end{aligned}$$

LEMMA 4.16. *In the setting of Proposition 4.13, $\mathscr{D}_{V,\rho}^2 = -\rho(\Delta) \otimes 1 + A$, where A is a first-order operator.*

Proof. This is a consequence of the usual computation of the square of the Dirac operator and the observation that the restriction to K-invariants of $\rho(\sum X_i^2)$, where $\{X_i\}$ is a basis for $\mathfrak{k}$, equals $1 \otimes B$ for some $B \in \mathrm{Hom}(W, W)$.

We now finish the proof of Proposition 4.13.

$$\begin{aligned}(1+\mathscr{D}_{V,\rho}^2)\circ(\rho(1-\Delta)\otimes 1)^{-1} &= (\rho(1-\Delta)\otimes 1 + A)(\rho(1-\Delta)\otimes 1)^{-1}\\ &= (\rho(1-\Delta)\otimes 1)(\rho(1-\Delta)\otimes 1)^{-1}\\ &\quad + A(\rho(1-\Delta)\otimes 1)^{-1/2}(\rho(1-\Delta)\otimes 1)^{-1/2}.\end{aligned}$$

The first term after the last equals is the identity, and Lemma 4.15 and the compactness of $(\rho(1-\Delta)\otimes 1)^{-1/2}$ imply that the second term has norm strictly less than one on the complement of some finite-dimensional subspace.

PROPOSITION 4.17. *Let π be a unitary representation of G on the Hilbert space H, and let σ_π be the associated representation of C^*G. Let W be a finite-dimensional unitary right K-representation, and let E_W be the Hilbert C^*G-module that is the completion of the set of smooth compactly supported sections of $W\times_K G\to K\backslash G$. ($E_W$ is as described in the discussion of Dirac induction in* §1.) *Then*

$$E_W\otimes_{C^*G}H\cong (W\otimes H)^K.$$

Proof. Define a map $Q\colon W\otimes C_c^\infty(G)\otimes H\to w\otimes H$ by

$$Q(w\otimes f\otimes\xi)=w\otimes\sigma_\pi(f)(\xi)=w\otimes\int_G f(g)\pi(g)\xi\,dg.$$

Using an approximate identity for C^*G, one can show that Q has dense range. An explicit calculation shows that the restriction of Q to $(W\otimes C_c^\infty(G))^K\otimes H$ has image in $(W\otimes H)^K$. To show that this restriction has image dense in $(W\otimes H)^K$, one uses the preceding density result, the observation that any $\alpha\in(W\otimes H)^K$ equals its own K-average, and a calculation that the K-average of $Q(\beta\otimes\gamma)$ equals $Q((K\text{-average of }\beta)\otimes\gamma)$. Here $\beta\otimes\gamma\in(W\otimes C_c^\infty(G))\otimes H$, and the K-average of an element refers to the integral over all $k\in K$ of the images of the element under the action of k.

We finish the proof of this proposition by showing that the restriction of Q intertwines the inner products. Suppose $\sum w_i^1\otimes f_i^1\in(W\otimes C_c^\infty(G))^K$ and $\xi^1\in H$ and similarly for $\sum w_j^2\otimes f_j^2$ and ξ^2.

$$\begin{aligned}&\left\langle Q\left(\sum w_i^1\otimes f_i^1\otimes\xi^1\right), Q\left(\sum w_j^2\otimes f_j^2\otimes\xi^2\right)\right\rangle_{(W\otimes H)^K}\\ &\quad=\sum_{i,j}\langle w_i^1, w_j^2\rangle_W\langle\sigma_\pi(f_i^1)\xi^1, \sigma_\pi(f_j^2)\xi^2\rangle_H\\ &\quad=\sum_{i,j}\langle w_i^1, w_j^2\rangle_W\langle\xi^1, \sigma_\pi(f_i^{1*}f_j^2)\xi^2\rangle_H.\end{aligned}$$

$$\left\langle \sum w_i^1 \otimes f_i^1 \otimes \xi^1 , \sum w_j^2 \otimes f_j^2 \otimes \xi^2 \right\rangle_{E_W \otimes_{C^*G} H}$$
$$= \left\langle \xi^1 , \sigma_\pi \left(\left\langle \sum w_i^1 \otimes f_i^1 , \sum w_j^2 \otimes f_j^2 \right\rangle_{E_W} \right) \xi^2 \right\rangle_h$$
$$= \sum_{i,j} \langle \xi^1 , \sigma_\pi (f_i^{1*} f_j^2) \xi^2 \rangle_H \langle w_i^1 , w_j^2 \rangle_W .$$

THEOREM 4.18. *Let* $[(E_W , D_V \circ (1 + D_V^2)^{-1/2})] \in KK(\mathbb{C}, C^*G)$ *be the* K *theory class defined by Dirac induction as in* §1. *Let*

$$\left[\left(\mathscr{H}_0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \sigma_\rho \right) \right] \in KK(C^*G, \mathbb{C})$$

be the K*-homology class of Theorem* 2.28. *More generally let* $[(\mathscr{H}_0, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \sigma_\rho)]$ *be the class in* $KK(C^*G, \mathbb{C})$ *defined by a graded CCR representation. Let*

$$[((\mathscr{H}_0(\rho) \otimes W)^K , \mathscr{D}_{V,\rho} \circ (1 + \mathscr{D}_{V,\rho}^2)^{-1/2})] \in KK(\mathbb{C}, \mathbb{C})$$

be the class described in Proposition 4.13. *Then*

$$[(E_W , D_V \circ (1 + D_V^2)^{-1/2})] \otimes_{C^*G} \left[\left(\mathscr{H}_0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \sigma_\rho \right) \right]$$
$$= [((\mathscr{H}_0(\rho) \otimes W)^K , \mathscr{D}_{V,\rho} \circ (1 + \mathscr{D}_{V,\rho}^2)^{-1/2})] \in KK(\mathbb{C}, \mathbb{C}) .$$

Proof. Propositions 4.13 and 4.17 imply that we need check only properties 1 and 3 of Definition 1.5. Property 1 follows from the observations that W is finite-dimensional and ρ is a CCR representation. Property 3 follows from the observation that $D_V \circ (1 + D_V^2)^{-1/2} \otimes 1 = \mathscr{D}_{V,\rho} \circ (1 + \mathscr{D}_{V,\rho}^2)^{-1/2}$. If we replace $C^\infty(\mathbf{F})$ in the proof of Lemma 3.21 by $\mathscr{H}_0^\infty(\rho)$, the smooth vectors in $H_0(\rho)$, then the reasoning used to prove Lemma 3.21 provides a proof of the above observation.

COROLLARY 4.19. *Let* f *denote the natural map* $f \colon C^*G \to C_r^*G$ *defined by restriction to the regular representation. Let* f_* *denote the map induced by* f, $f_* \colon KK(\mathbb{C}, C^*G) \to KK(\mathbb{C}, C_r^*G)$. *Suppose there is a map* $\sigma_\rho^r \colon C_r^*G \to \mathscr{L}(\mathscr{H}_0)$ *such that* $\sigma_\rho = \sigma_\rho^r \circ f$. *Then*

$$f_*([(E_W , D_V \circ (1 + D_V^2)^{-1/2})]) \otimes_{C_r^*G} \left[\left(\mathscr{H}_0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \sigma_\rho^r \right) \right]$$
$$= [((\mathscr{H}_0(\rho) \otimes W)^K , \mathscr{D}_{V,\rho} \circ (1 + \mathscr{D}_{V,\rho}^2)^{-1/2})] \in KK(\mathbb{C}, \mathbb{C}) .$$

Proof. Define $[f] \in KK(C^*G, C_r^*G)$ by

$$[f] = \left[\left(C_r^*G \otimes 0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, M_f \right) \right] .$$

Here $M_f(\phi) =$ multiplication by $f(\phi)$. Kasparov product by $[f]$ on the right realizes f_*. Under our assumption on σ_ρ, $[(\mathscr{H}_0, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \sigma_\rho)] = [f] \otimes_{C_r^*G} [(H_0, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \sigma_\rho^r)]$, and the result follows from Theorem 4.18 by associativity of the Kasparov product.

5. Index formula for multiplicities. Theorems 2.29, 3.18, and 4.18 imply an equality in $KK(\mathbb{C}, \mathbb{C})$ that we shall call an index theorem. We state some corollaries that describe implications of this equality. In particular we use $KK(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$ to interpret the index theorem as a numerical equation.

THEOREM 5.1. *Use the notation of Theorems* 3.18 *and* 4.18. *In particular, we choose* V *and* $W = S \otimes V$. *Then in* $KK(\mathbb{C}, \mathbb{C})$

$$\begin{aligned} &[(L^2(K\backslash(\mathbf{W} \otimes \mathbf{F})), \mathscr{P}_W \circ (1 + \mathscr{P}_W^2)^{-1/2})] \\ &\qquad = [((\mathscr{H}_0(\rho) \otimes W)^K, \mathscr{D}_{V,\rho} \circ (1 + \mathscr{D}_{V,\rho}^2)^{-1/2})]. \end{aligned}$$

Proof. By Theorems 3.18 and 4.18 both of these elements are Kasparov products of a fixed element in $KK(\mathbb{C}, C^*G)$ with elements in $KK(C^*G, \mathbb{C})$. By Theorem 2.29 the elements in $KK(C^*G, \mathbb{C})$ are equal.

COROLLARY 5.2. *For a degree one operator such as* $\mathscr{P}_W$ *or* $\mathscr{D}_{V,\rho}$, *use the superscript* $+$ *to denote that part of the operator mapping even elements to odd elements.* (*In the notation of Remark* 4.10, $\mathscr{D}_{V,\rho}^+ = D_{V,\rho}$.) *Then in* $\mathbb{Z}$

$$\operatorname{index}(\mathscr{P}_W^+) = \operatorname{index}(\mathscr{D}_{V,\rho}^+).$$

Proof. Use the standard isomorphism $KK(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$.

NOTATION 5.3. Let $\{H(\beta) \colon \beta \in B\}$ denote the set of distinct irreducible G-representations that occur with nonzero multiplicity in $\mathscr{H}_0(\rho)$. For each $\beta \in B$ let $H(\underline{\beta})$ denote the direct sum of all copies of $H(\beta)$ that occur in $\mathscr{H}_0(\rho)$. (Because $\mathscr{H}_0(\rho)$ is CCR, each $H(\beta)$ occurs finitely many times.) Give each $H(\underline{\beta})$ the grading inherited from $\mathscr{H}_0(\rho)$.

COROLLARY 5.4. *With notation as above,*

$$\operatorname{Index}(\mathscr{P}_W^+) = \sum_{\beta \in B} \operatorname{index}(D_{V,\underline{\beta}}^+).$$

(By Proposition 4.13 the right-hand side includes only finitely many nonzero terms.)

Proof. All constructions decompose with respect to the decomposition of $\mathscr{H}_0(\rho)$ into subrepresentations.

NOTATION 5.5. For $\beta \in B$ let $m^{\text{even}}(\beta)$, respectively $m^{\text{odd}}(\beta)$, denote the multiplicity of $H(\beta)$ in the even part, respectively the odd part, of $\mathscr{H}_0(\rho)$.

COROLLARY 5.6. *With notation as above,*

$$\operatorname{Index}(\mathscr{P}_W^+) = \sum_{\beta \in B} (m^{\text{even}}(\beta) \operatorname{index}(D_{V,\beta}^+) + m^{\text{odd}}(\beta) \operatorname{index}(D_{V,\beta}^-)).$$

Proof. Recall the construction of $D_{V,\underline{\beta}}^+$ in Remarks 4.10 and 4.11.

NOTATION 5.7. For $\beta \in B$ let $m(\beta) = m^{\text{even}}(\beta) - m^{\text{odd}}(\beta)$.

COROLLARY 5.8. *With notation as above,*

$$\operatorname{Index}(\mathscr{P}_W^+) = \sum_{\beta \in B} m(\beta) \operatorname{index}(D_{V,\beta}^+).$$

Proof. Because D_V^- is the adjoint of D_V^+, Corollary 4.9 implies that $\operatorname{index}(D_{V,\beta}^-) = -\operatorname{index}(D_{V,\beta}^+)$.

REMARK 5.9. The analysis of the right-hand side of the formula in Theorem 5.1 remains true for any graded CCR representation $\mathscr{H}_0(\rho)$.

REMARK 5.10. By Corollary 4.19, if the CCR representation $\mathscr{H}_0(\rho)$ is weakly contained in the regular representation, then the right-hand side of the formula in Theorem 5.1 can arise as a Kasparov product over C_r^*G.

6. Calculation of coefficients. Because the left side of the formula in Corollary 5.8 is the index of an elliptic operator on a compact manifold, we like to think that Corollary 5.8 provides a way to calculate multiplicities of representations in kernel(T)−kernel(T^*). (Here T is the invariant transversally elliptic operator of §2.) Information about the coefficients, $\operatorname{index}(D_{V,\beta}^+)$, appearing in Corollary 5.8 is needed to justify this point of view. In this section we discuss the calculation of these coefficients. Parts of the discussion lie purely in representation theory, but other parts involve K theory. At the least K theory provides a point of view that vastly increases the accessibility of these coefficients; at best it may provide truly new insights.

The following summarizes the results of this section. For β in a broad class of discrete series representations, the methods of this paper determine $m(\beta)$. For β among the other discrete series representations and "nearby" representations, Corollary 5.8 equates the index of an elliptic operator with a linear combination of $m(such\ \beta)$, at least for special G. Thus the methods of this paper may give some information about the presence of such β in kernel(T) and kernel(T^*). Finally the methods of this paper will never direct the presence of representations from the complementary or irreducible principal series. The dependence of K groups of C^*G on the topology of G's unitary dual provides the fundamental limitation on our techniques.

REMARK 6.1. The relationship between the tensor product and the functor Hom identifies $(H(\pi) \otimes W)^K$ with a space of K-intertwining linear maps.

REMARK 6.2. (See **[W]**.) When K is a large compact subgroup of G, the domain and range spaces $(H(\beta) \otimes W^{\pm})^K$ of every $D^+_{V,\beta}$ are finite-dimensional. Thus the coefficient index$(D^+_{V,\beta})$ equals the difference of these dimensions. In principle these coefficients can be calculated from a thorough understanding of the K-types of irreducible representations $H(\beta)$. If a connected semisimple G has finite center, then its maximal compact subgroup K is large.

REMARK 6.3. Let G be connected and semisimple with finite center. Assume that rank G equals rank K so that G has discrete series representations.

There is a correspondence between the set of discrete series representations of G and a subset of the set of irreducible representations of K **[AtS]**. Under our conventions this correspondence arises from the right action of G on the kernel of D^+_V. We refer to this correspondence in the following may: D^+_V realizes the discrete series π_V.

REMARK 6.4. In **[Wi1]** and **[Wi2]** F. Williams discusses a class of discrete series satisfying a certain positivity condition. All integrable discrete series lie in this class. It seems plausible that this class is exactly the set of discrete series that are isolated in the unitary dual of G.

The following theorem of F. Williams arises form the point of view discussed in Remark 6.2. Our side and sign conventions differ slightly from those of Williams.

THEOREM 6.5 [**Wi1**]. *Suppose G is a linear connected semisimple Lie group. Suppose* $\operatorname{rank}(K) = \operatorname{rank} G$. *Suppose π_V is a discrete series representation for G that lies in the class of Remark* 6.4. *Then if β is an irreducible unitary G-representation,*

$$\operatorname{index}(D^+_{V,\beta}) = \begin{cases} 1 & \text{if } \beta = \pi_V, \\ 0 & \text{if } \beta \neq \pi_V. \end{cases}$$

COROLLARY 6.6. *The methods of our paper calculate $m(\pi_V)$ for discrete series π_V in the class of Remark* 6.4. *Under the assumptions of the preceding theorem, with $W = V \otimes S$, Corollary* 5.8 *reads*

$$\operatorname{Index}(\mathscr{P}^+_W) = m(\pi_V).$$

Proof. Apply Theorem 6.5 to calculate coefficients in Corollary 5.8.

The following "lemma" summarizes our application of K theory to the calculation of coefficients other than those discussed above. This "lemma" is offered not as something requiring a difficult proof but rather as a description of the framework in which we work.

LEMMA 6.7. *Let G be a connected unimodular Lie group. Let $H(\pi)$ be a CCR representation of G. Let $H(\pi) = \bigoplus_j H(\pi_j)$ be the decomposition of $H(\pi)$ into a direct sum of irreducible representations. Assume* $[(H(\pi)\oplus 0, \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \sigma_\pi)] = 0 \in KK(C^*G, \mathbb{C})$, *Then for any Dirac operator D_V on $K\backslash G$, $\sum_j \operatorname{index}(D^+_{V,\pi_j}) = 0$.*

Proof. This lemma is a consequence of Theorem 4.18 and Remark 5.9 and of the linearity of the Kasparov product.

Proposition 6.9 and its applications show that there exist interesting CCR representations that define the zero class in $KK(C^*G, \mathbb{C})$. We have been informed that the idea behind this proposition appears as Corollary 3.1.14 of A. Valette's thesis [**Va**], where attention is focused on paths in the reduced dual of a semisimple Lie group G.

REMARK 6.8 [**D2 3.9.8**]. Let G be a second countable, locally compact group. Let H be an infinite-dimensional separable Hilbert space. Let Y be the quotient of $\operatorname{Rep}(C^*G, H)$ by the equivalence relation generated by intertwining partial isometries (which identify essential subspaces). The unitary dual $\widehat{G}$ of G can be identified with a subspace of Y.

PROPOSITION 6.9. *Use the notation of the preceding remark. Let* $\gamma: [0, \infty) \to Y$ *be a continuous map such that:*

1. *for each* $t \in [0, \infty)$ *the nondegenerate representation associated to* $\gamma(t)$ *is CCR;*
2. *there exists a compact subset* C *of* $[0, \infty)$ *for which* $t \notin C$ *implies* $\gamma(t) \in \widehat{G}$;
3. *as* $t \to \infty$, $\gamma(t) \to \infty$ *in* $\widehat{G}$.

Then $[(H \oplus 0, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \gamma(0))]$ *equals zero in* $KK(C^*G, \mathbb{C})$.

Proof. $((H \oplus 0) \otimes C([0, \infty]), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \gamma(t)) \in \mathscr{E}(C^*G, C([0, \infty]))$. (Compare Lemma 1.8.) Here, by assumptions 2 and 3 and by **[Fe]**, we may extend γ so that $\gamma(\infty)$ is the zero representation. Then zero representation defines the zero class in $KK(C^*G, \mathbb{C})$.

REMARK 6.10. The preceding proposition implies that any irreducible unitary representation of a connected, nilpotent Lie group G represents the zero class in $KK(C^*G, \mathbb{C})$. The same is true for an irreducible principal series or irreducible unitary complementary series representation of a noncompact, connected, semisimple Lie group G with finite center. In what follows we discuss more involved applications of this proposition. These are offered as examples, not as a complete discussion of the applications of this proposition.

REMARK 6.11. Let $G = \mathrm{SU}(n, 1)$. Let P be a minimal parabolic subgroup of G. $P = MAN$ with $M = U(n-1)$ and $A = \{\exp(tH) : t \in \mathbb{R}\}$. If $\sigma \in \widehat{M}$ is an irreducible representation of M on V_σ and $\nu \in \mathbb{C}$, let $\sigma \otimes \nu$ be the representation of MAN on V_σ defined by

$$(\sigma \otimes \nu)(M \cdot \exp(tH) \cdot n) = \sigma(m)e^{\nu t}.$$

The principal series representation $\pi(\sigma, \nu)$ of G is defined by

$$\pi(\sigma, \nu) = \mathrm{ind}_P^G(\sigma \otimes \nu).$$

If ν is imaginary, then $\pi(\sigma, \nu)$ is a unitary principal series representation of G. If ν is imaginary and nonzero, $\pi(\sigma, \nu)$ is irreducible. Whether $\pi(\sigma, 0)$ is irreducible depends on σ.

When $\pi(\sigma, 0)$ is irreducible, complementary series representations exist. In particular there is a maximal interval $[0, \lambda_\sigma] \subset \mathbb{R}$, $\lambda_\sigma > 0$, such that $\nu \in [0, \lambda_\sigma]$ implies that $\pi(\sigma, \nu)$ is infinitesimally equivalent to a unitary representation which is also irreducible for $\nu \in [0, \lambda_\sigma)$. The unitary representation associated to $\nu = \lambda_\sigma$ is a finite direct sum of irreducible unitary representations. In our case,

$G = \mathrm{SU}(n, 1)$, the decomposition at $\nu = \lambda_\sigma$ is described in **[Kr]** or **[Ze]**. In general, there is a Vogan calculus **[V]** for such decompositions.

LEMMA 6.12. *Use the notation of Remark* 6.11. *In particular choose a* $\sigma \in \widehat{M}$ *for which* $\pi(\sigma, 0)$ *is irreducible. Let* Y *be as in Remark* 6.8. *Let* $\gamma\colon [0, \infty) \to Y$ *be defined in a natural affine manner subject to the following conditions*:

1. $\gamma(0) =$ *the unitary representation associated to* $\nu = \lambda_\sigma$;
2. *As* t *moves from* 0 *to* 1, $\gamma(t)$ *moves from the unitary representation associated to* $\nu = \lambda_\sigma$ *to the unitary representation associated to* $\nu = 0$;
3. *As* t *moves from* 1 *to* ∞, $\gamma(t)$ *equals* $\pi(\sigma, i(t-1))$.

Then γ *satisfies the conditions of Proposition* 6.9, *and the unitary representation associated with* $\nu = \lambda_\sigma$ *defines the zero class in* $KK(C^*G, \mathbb{C})$.

Proof. The continuity of γ follows from direct calculations and a result of **[Fe]**, which is also discussed in **[D2]**. That $i\nu \to i\infty$ corresponds to going to infinity in $\widehat{G}$ is a result of **[L]**.

REMARK 6.13. The kind of calculations needed above are done explicitly for the deSitter group in **[BoMa]**.

REMARK 6.14. Again let $G = \mathrm{SU}(n, 1)$. The following table of representations with trivial infinitesimal character is taken from **[Ze]**.

$$\begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \end{pmatrix} \begin{pmatrix} \alpha_{1,2} & \alpha_{1,3} \\ \alpha_{2,2} & \alpha_{2,3} \end{pmatrix} \cdots \begin{pmatrix} \alpha_{1,n-1} & \alpha_{1,n} \\ \alpha_{2,n-1} & \alpha_{2,n} \end{pmatrix} \begin{pmatrix} \alpha_{1,n} & \alpha_{1,n+1} \\ \alpha_{2,n} & \end{pmatrix}$$

$$\begin{pmatrix} \alpha_{2,1} & \alpha_{2,2} \\ \alpha_{3,1} & \alpha_{3,2} \end{pmatrix} \begin{pmatrix} \alpha_{2,2} & \alpha_{2,3} \\ \alpha_{3,2} & \alpha_{3,3} \end{pmatrix} \cdots \begin{pmatrix} \alpha_{2,n-1} & \alpha_{1,1} \\ \alpha_{3,n-1} & \end{pmatrix}$$

$$\vdots \qquad\qquad \vdots$$

$$\begin{pmatrix} \alpha_{n-1,1} & \alpha_{n-1,2} \\ \alpha_{n,1} & \alpha_{n,2} \end{pmatrix} \begin{pmatrix} \alpha_{n-1,2} & \alpha_{n-1,3} \\ \alpha_{n,2} & \end{pmatrix}$$

$$\begin{pmatrix} \alpha_{n,1} & \alpha_{n,2} \\ \alpha_{n+1,1} & \end{pmatrix}.$$

Each $\alpha_{i,j}$ is an irreducible unitary G-representation. If $i+j = n+2$, $\alpha_{i,j}$ belongs to the discrete series. Each set of parentheses corresponds to some $\sigma \in \widehat{M}$. The direct sum of representations occurring within a given set of parentheses is the decomposition associated with $(\sigma, \nu = \lambda_\sigma)$ in Remark 6.11.

LEMMA 6.15. *Let $\bigoplus \alpha_{i,j}$ correspond to one of the sets of parentheses in the preceding table. Let D_V^+ be a Dirac operator on $K\backslash G$. Then*

$$\sum_{i,j} \operatorname{index}(D^+_{V,\alpha_{i,j}}) = 0.$$

Proof. By Remark 6.14, Lemma 6.12 applies.

LEMMA 6.16. *Let G be a noncompact, connected semisimple Lie group with finite center. Let D_V^+ realize a discrete series representation π_r of G. Let π_s be a discrete series representation of G. Then*

$$\operatorname{index}(D^+_{V,\pi_s}) = \begin{cases} 1 & \text{if } \pi_r = \pi_s, \\ 0 & \text{if } \pi_r \neq \pi_s. \end{cases}$$

Proof. The reduction to C_r^*G used in Corollary 4.19 applies here. Much information about the class in $KK(\mathbb{C}, C_r^*G)$ determined by D_V can be deduced from the behavior of D_V on L^2 sections of the bundle $W \times_K G \to K\backslash G$. The Hilbert C_r^*G-module defining the class in K theory arises as the C_r^*G-norm completion of the set of smooth compactly supported sections of this bundle. G-invariant operators can be regarded as fields of operators of G's reduced dual, from which we observe that an operator on the Hilbert C_r^*G-module defines an operator on the Hilbert space of L^2 sections. Each discrete series representation π contributes a summand of compact operators $\mathscr{K}_\pi$ to C_r^*G. Combining the relationship between operators on the Hilbert module and operators on the Hilbert space with the realization theorem of **[AtS]**, we see that the image of D_V's class in $K_0(\mathscr{K}_\pi)$ is zero if D_V does not realize π and equals the class of a rank one projection if D_V realizes π. (Here image is with respect to the map induced on K theory by the natural projection of C_r^*G onto a direct summand $\mathscr{K}_\pi$.) An explicit description of these KK cycles appears in **[FH3]**.

PROPOSITION 6.17. *Let $G = \mathrm{SU}(n, 1)$. Let D_V^+ realize a discrete series representation of G. Let $\alpha_{i,j}$ be a representation appearing in the table of Remark* 6.14. *Then the preceding information determines* $\operatorname{index}(D^+_{V,\alpha_{i,j}})$.

Proof. Lemma 6.15 determines a system of linear equations whose terms are either $\operatorname{index}(D^+_{V,\alpha_{i,j}})$, $i+j < n+2$, or constants determined by Lemma 6.16. Isolating the constants, we get a system of equations with invertible coefficient matrix.

REMARK 6.18. In the long run the ideas of this section should be roughly as effective in calculating the values of $\operatorname{index}(D^+_{V,\beta})$ when the group G is an arbitrary connected unimodular Lie group having discrete series representations as they are when G is semisimple. At this time, however, the general case is less well understood than the semisimple case. In what follows we indicate how the ideas in this section should generalize.

REMARK 6.19. We outline some aspects of the theory of connected unimodular Lie groups having discrete series representations. These results are due to N. Anh, in whose papers **[A1]**, **[A2]** can be found a careful version of the following discussion. Anh defines something called an H-group, which is roughly a unimodular solvable Lie group that has square integrable representations and whose representation theory behaves like that of nilpotent Lie groups with square integrable representations.

Suppose G is a unimodular connected Lie group that has square integrable (i.e. discrete series) representations. Then G is the semidirect product of an H-group H, with compact center Z that is central in G, and a connected reductive Lie group S having compact center.

$$\{e\} \to H \to G \to S \to \{e\}.$$

The center of G is of the form $Z \cdot C$, where C is central in S.

Let π be an irreducible discrete series representation of a group G, where G is as in the preceding paragraph. Then there exists a discrete series representation $(\tau, H(\tau))$ of H, an extension $\bar{\tau}$ to a representation $(\bar{\tau}, H(\tau))$ of G, and an irreducible discrete series representation $(\sigma, H(\sigma))$ of S such that π can be realized on the Hilbert space $H(\sigma) \otimes H(\tau)$ by

$$\pi(s, h) = \sigma(s) \otimes \bar{\tau}(sh).$$

REMARK 6.20. For the groups of Remark 6.19, the problem of realizing discrete series representations by using elliptic operators appears not to have been solved in general. Under certain assumptions, whose details we omit, J. Rosenberg **[R1]** has used harmonic induction (Dolbeault operators) to solve this realization problem. By §6 of **[FH1]** the class in $KK(\mathbb{C}, C^*G)$ represented by such a Dolbeault operator can also be represented by a Dirac operator.

PROPOSITION 6.21. *Suppose that* G *is of the type discussed in Remark* 6.19:

$$\{e\} \to H \to G \to S \to \{e\}.$$

Suppose further that the reductive factor S *is either compact or locally isomorphic to the product of a compact group and a finite number of groups locally isomorphic to* $\mathrm{SO}(2n, 1)$ *or* $\mathrm{SU}(n, 1)$. *Let* π_V *be a discrete series representation of* G *that is isolated in the full unitary dual of* G. (*If* G *is amenable all discrete series representations are isolated* **[Gr]**.) *Then*

(a) *Associated to* π_V *is a direct summand of compact operators in* C^*G. *The class in* $KK(\mathbb{C}, C^*G)$ *defined by a rank one projection in this summand can also be represented by the class of a Dirac operator* D_V.

(b) *For any irreducible* CCR *representation* β

$$\mathrm{index}(D^+_{V,\beta}) = \begin{cases} 1 & \text{if } \beta = \pi_V, \\ 0 & \text{if } \beta \neq \pi_V. \end{cases}$$

Proof. (a) A discrete series representation is CCR **[D2**, 14.4.3**]**. By **[K1]**, **[K3]**, and **[JK]** Dirac induction is an isomorphism.

(b) Compute the Kasparov product by using the rank one projection to represent the class in $KK(\mathbb{C}, C^*G)$.

REMARK 6.22. We extend the argument involving paths of $\mathrm{SU}(n, 1)$-representations to this setting. Assume G fits in the following sequence:

$$\{e\} \to N \to G \to S \to \{e\}.$$

Here N is a Heisenberg group with compact center, and we assume that the action of S factors through the symplectic group. The realization theorem discussed in Remark 6.20 holds in this case **[R1]**.

REMARK 6.23. Let τ be a discrete series representation of N. Then there is an extension of τ to a representation $\bar{\tau}$ of G (or perhaps of G', where G' is a double cover of G formed by using a double cover S' of S). (See, e.g., **[A1]**, **[A2]**.) For each representation σ of S (or S'), the representation $\pi(s, n) = \sigma(s) \otimes \bar{\tau}(sn)$ is irreducible and CCR. Write $\pi = \phi_\tau(\sigma)$.

THEOREM 6.24 **[D1]**. *The map* $\phi_\tau \colon \widehat{S} \to \mathrm{Prim}(G)$, *or* $\widehat{S'} \to \mathrm{Prim}(G')$, *is a homeomorphism from* $\widehat{S}$, *or* $\widehat{S'}$, *to an open and closed subset of* $\mathrm{Prim}(G)$, *or* $\mathrm{Prim}(G')$.

REMARK 6.25. The table of Remark 6.14 is the same for a double cover of $\mathrm{SU}(n, 1)$ as it is for $\mathrm{SU}(n, 1)$.

PROPOSITION 6.26. *Suppose that for G as in Remark 6.22, $S = \mathrm{SU}(n, 1)$. Let τ be a discrete series representation of G, let σ be a discrete series representation of S, and let D_V be the Dirac operator realizing $\phi_\tau(\sigma)$. (As always we may have to pass to G' and S'.) Then for the representation $\alpha_{i,j}$ of Remark 6.14, we can compute* $\operatorname{index}(D^+_{V,\sigma_\tau(\alpha_{i,j})})$.

Proof. Theorem 6.24 provides us with a system of equations just like that used for $\mathrm{SU}(n, 1)$. The realization theorem implies that the values at discrete series $(i + j = n + 2)$ are just as before.

7. Multiplicity formula for certain quasi-regular representations. The Kasparov product of §3, applied in the single orbit case, is an important step in the proof of a rather general formula for the multiplicity of a discrete series representation in a quasi-regular representation. The formula does not involve explicitly the index of an elliptic operator; but because its short proof places it so clearly in the realm of index-theoretic multiplicity formulas, we include it here. The formula has been established previously in the following special cases: G a simply connected nilpotent Lie group **[MoWo]** and G a simply connected exponential solvable Lie group **[R2]**.

THEOREM 7.1. *Let G be an amenable, locally compact, second countable, connected group with discrete series representations. Let Γ be a discrete, torsion-free, cocompact subgroup of G. Let π be a discrete series representation of G with formal degree $d(\pi)$. Let $m(\pi)$ be the multiplicity of π in $L^2(G/\Gamma)$. Then*

$$m(\pi) = \operatorname{vol}(G/\Gamma) \cdot d(\pi).$$

Here $\operatorname{vol}(G/\Gamma)$ is defined using Haar measure, and $d(\pi)$ is defined as formal dimension in **[D2,** 14.3.4].

Proof. Because G/Γ is compact, the quasi-regular representation λ is *CCR*. Thus

$$\left(L^2(G/\Gamma) \oplus 0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \sigma_\lambda\right) \in \mathscr{E}(C^*G, \mathbb{C}). \tag{7.2}$$

Because G is connected, standard structure theory says that there is a compact normal subgroup K_0 such that $K_0\backslash G$ is a Lie group.

Thus, for K the maximal compact subgroup, it makes sense to talk about Dirac induction involving a Dirac operator on $K\backslash G$. (As usual one may need to use a double cover of $K_0\backslash G$.)

Because G is amenable its reduced dual equals its full dual. The discrete series representation π, which is CCR **[D2**, 14.4.3], defines a primitive ideal that is both open **[Gr]** and closed in the primitive ideal space and that has a unique preimage in the dual of G **[D2**, 4.1.10]. Thus π contributes a direct summand of compact operators to C^*G. Let e_π denote a rank one projection in this summand and $[e_\pi]$ the associated class in $KK(\mathbb{C}, C^*G)$.

Representing $[e_\pi]$ by an element of $\mathscr{E}(\mathbb{C}, C^*G)$ whose Hilbert C^*G-module is the compact operators on the Hilbert space associated to π (or, when necessary, a countable direct sum of such), one can compute that

$$[e_\pi] \otimes_{C^*G} \left[\left(L^2(G/\Gamma), \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \sigma_\lambda\right)\right] = m(\pi). \tag{7.3}$$

(Here we identify $KK(\mathbb{C}, \mathbb{C})$ with $\mathbb{Z}$ in the standard way.) Such a calculation is done explicitly in **[FH3]**.

Because G is amenable, there is a twisted Dirac operator $\widetilde{D}$ on $K\backslash G$ whose class $[\widetilde{D}] \in KK(\mathbb{C}, C^*G)$ arising from Dirac induction satisfies

$$[\widetilde{D}] = [e_\pi] \cdot [K1]. \tag{7.4}$$

By the calculation of §3

$$[\widetilde{D}] \otimes_{C^*G} \left[\left(L^2(G/\Gamma), \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \sigma_\lambda\right)\right] = \operatorname{index}(D), \tag{7.5}$$

where D is the descent of $\widetilde{D}$ to $K\backslash G/\Gamma$. By **[At2]**

$$\operatorname{index}(D) = \operatorname{trace}_\Gamma(e_\pi). \tag{7.6}$$

It follows from (7.3), (7.4), and (7.6) that

$$m(\pi) = \operatorname{trace}_\Gamma(e_\pi). \tag{7.7}$$

As mentioned in **[AtS]**

$$\operatorname{trace}_\Gamma(e_\pi) = \operatorname{vol}(G/\Gamma) \cdot \operatorname{trace}_G(e_\pi). \tag{7.8}$$

Here trace_G refers to the extension to K theory of the natural trace on C_r^*G. This trace satisfies, for $f \in L^1(G) \cap L^2(G)$, $\operatorname{trace}_G(f^*f) = \|f\|^2_{L^2(G)}$. By **[D2**, 14.4.2] and the observation that $e_\pi^* e_\pi = e_\pi$,

$$\operatorname{trace}_G(e_\pi) = d(\pi). \tag{7.9}$$

Thus (7.7), (7.8), and (7.9) imply

$$m(\pi) = \mathrm{vol}(G/\Gamma) \cdot d(\pi). \tag{7.10}$$

PROPOSITION 7.11. *Let G be a connected K-amenable Lie group with discrete cocompact torsion-free subgroup Γ. Let π be a discrete series representation of G that is isolated in the full unitary dual of G. Then with notation as in Theorem* 7.1,

$$m(\pi) = \mathrm{vol}(G/\Gamma) \cdot d(\pi).$$

Proof. By Anh's characterization of connected unimodular Lie groups with a discrete series, discussed in Remark 6.19, G is the semidirect product of an amenable Lie group with a semisimple Lie group. By **[JVa]** the semisimple Lie group is K-amenable. It follows from **[JVa]**, **[JK]**, and **[K3]** that Dirac induction is an isomorphism. Thus there are representatives $[\widetilde{D}]$ and $[e_\pi]$ of the same class in $KK(\mathbb{C}, C^*G)$. We then proceed as in the proof of Theorem 7.1.

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Received May 11, 1992. The material by the first author is based upon work supported by the National Science Foundation under Grant No. DMS-8903472. The material by the second author is based upon work supported by the National Science Foundation under Grant No. DMS-8901436.

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