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STRONGLY APPROXIMATELY TRANSITIVE GROUP ACTIONS, THE CHOQUET-DENY THEOREM, AND POLYNOMIAL GROWTH

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Let G be a group. A Borel G-space $\mathscr{X}$ with a σ-finite quasi-invariant measure α is called strongly approximately transitive (SAT) if there exists an absolutely continuous probability measure ν such that the closed convex hull $\operatorname{co}(G\nu)$ of the orbit $G\nu$ coincides with the space of absolutely continuous probability measures on $\mathscr{X}$. Call a G-space $(\mathscr{X}, \alpha)$ purely atomic if α is purely atomic. Boundaries of stationary random walks on countable G are always SAT and provide many examples of nonatomic SAT actions. The class of nonatomic SAT G-spaces also includes certain homogeneous spaces of locally compact groups. Every countable nonamenable group and also some amenable groups admit nonatomic SAT actions. However, if G contains a countable nilpotent subgroup of finite index then every SAT G-space is necessarily purely atomic. This implies the Choquet-Deny theorem for such groups. Existence of nonatomic SAT actions is related to growth conditions. A finitely generated solvable group has polynomial growth if and only if it does not admit nonatomic SAT actions.

1. Introduction. Let G be a group and $\mathscr{X}$ a Borel G-space with a σ-finite quasi-invariant measure α. We shall denote by $L^1(\mathscr{X}, \alpha)$ the space of complex measures absolutely continuous with respect to α and by $L^1_1(\mathscr{X}, \alpha) \subseteq L^1(\mathscr{X}, \alpha)$ the subspace of probability measures. For $g \in G$ and $\mu \in L^1_1(\mathscr{X}, \alpha)$ we shall write $g\mu$ for the measure $(g\mu)(A) = \mu(g^{-1}A)$. The action of G on $(\mathscr{X}, \alpha)$ is called *approximately transitive* (AT) if for every pair $\nu_1, \nu_2 \in L^1_1(\mathscr{X}, \alpha)$ and every $\varepsilon > 0$ there exists $\nu \in L^1_1(\mathscr{X}, \alpha)$ such that the total variation norm distances between ν_i, $i = 1, 2$, and the convex hull $\operatorname{co}(G\nu)$ of the orbit $G\nu$ are both less than ε. The concept of approximate transitivity was introduced by Connes and Woods [1] to provide a necessary and sufficient condition for an approximately finite dimensional von Neumann factor to be ITPFI (AT characterizes the flow of weights of ITPFI factors). Connes and Woods also observed [2] that the group action on the Poisson boundary of a (not necessarily stationary) random walk on a locally compact second countable group is approximately transitive. They showed that in the case $G = \mathbb{R}$

or $G = \mathbb{Z}$ every AT standard Borel G-space can be represented as a Poisson boundary. This characterization of AT actions can be extended to countable amenable groups [4] (for countable nonamenable G, amenable approximately transitive actions can be represented as Poisson boundaries [4]). Elementary examples of AT actions are homogeneous spaces of locally compact second countable groups and ergodic rotations on second countable compact abelian groups; certain diffeomorphisms of the circle are also known to be AT [7]. For actions of $\mathbb{Z}$ preserving a finite invariant measure AT implies zero entropy [1]. Every AT action is ergodic.

The present paper is concerned with the following strong version of approximate transitivity. We shall say that a Borel G-space with a σ-finite quasi-invariant measure α is *strongly approximately transitive* (SAT) if there exists a probability measure $\nu \in L^1_1(\mathscr{X}, \alpha)$ such that the convex hull $\mathrm{co}(G\nu)$ is dense in $L^1_1(\mathscr{X}, \alpha)$ with respect to the total variation norm. A measure ν with such property will be called a *SAT measure*. It is obvious that strong approximate transitivity implies approximate transitivity. Ergodic rotations on uncountable compact abelian groups are elementary examples of approximately transitive actions that are not strongly approximately transitive.

A SAT measure ν will be called *trivial* if for every Borel set A, $\nu(A)$ is either 0 or 1. Obviously, every countable transitive G-space admits trivial SAT measures and it is easy to see that it does not admit nontrivial ones. However, some uncountable transitive G-spaces admit nontrivial SAT measures. E.g., for the transitive action of the $ax + b$ group on $\mathbb{R}$ every absolutely continuous probability measure is SAT. The transitive action of $\mathrm{SL}(2, \mathbb{R})$ on the circle $\mathbb{S}$ also admits nontrivial absolutely continuous SAT measures. It turns out that nontrivial SAT measures cannot exist in the presence of a σ-finite *invariant* measure. We show that if G acts with a σ-finite invariant measure α then $(\mathscr{X}, \alpha)$ is SAT if and only if α is purely atomic; i.e., $(\mathscr{X}, \alpha)$ is, essentially, a countable transitive G-space. Henceforth, G-spaces $(\mathscr{X}, \alpha)$ where α is purely atomic will be referred to as *purely atomic*.

As mentioned above, approximate transitivity is a property of boundaries of (not necessarily stationary) random walks on G. Strong approximate transitivity turns out to be a property of boundaries of stationary random walks. We demonstrate the significance of this property by showing that the Choquet-Deny theorem is a direct consequence of the fact that certain groups do not admit nonatomic SAT actions. The boundaries also provide many examples of nonatomic SAT actions.

Let μ be a probability measure on a countable group G. An element $h \in L^\infty(G)$ is called a μ-harmonic function if

$$h(g) = \int_G h(gg')\mu(dg'), \qquad g \in G.$$

It is well known that every μ-harmonic functions can be represented as a bounded Borel function on a certain boundary space [**9**, **10**]. More precisely, there exists a Borel G-space $\mathscr{X}$ with a σ-finite quasi-invariant measure α and an equivariant isometric isomorphism Φ of $L^\infty(\mathscr{X}, \alpha)$ onto the space $\mathscr{H}$ of μ-harmonic functions. It is easy to see that Φ is necessarily given by

$$(1.1) \quad (\Phi f)(g) = \langle g\rho, f\rangle = \int_{\mathscr{X}} f(gx)\rho(dx), \qquad f \in L^\infty(\mathscr{X}, \alpha),$$

where $\rho \in L_1^1(\mathscr{X}, \alpha)$ is a probability measure such that $\mu * \rho = \rho$. The fact that Φ is an isometry implies (via Proposition 2.2) that ρ is a SAT measure. The basic well-known realization of the *μ-boundary* $(\mathscr{X}, \alpha, \rho)$ is obtained as follows. Consider the right random walk of law μ on G. Let $\mathscr{X} = \prod_{n=0}^\infty G$ be the space of paths with the Borel structure given by the invariant (stationary) σ-algebra of the random walk and let α denote the Markov measure on $\mathscr{X}$ defined by μ and a starting probability measure equivalent to the counting measure on G. Equip $\mathscr{X}$ with the G-action $g\{\omega_n\}_{n=0}^\infty = \{g\omega_n\}_{n=0}^\infty$. $\mathscr{X}$ becomes then a Borel G-space and α is a quasi-invariant measure. By the basic theory of Markov chains, $L^\infty(\mathscr{X}, \alpha)$ is isometrically G-isomorphic to the space $\mathscr{H}$ of μ-harmonic functions [**9**, Proposition V.2.4, p. 175], [**10**, Proposition 3.2, p. 82]. The measure ρ of equation (1.1) turns out to be the Markov measure of the random walk started from the identity element $e \in G$. Furthermore, ρ is trivial if and only if every μ-harmonic function is constant on the left cosets of the smallest subgroup $H \subseteq G$ such that $\mu(H) = 1$. In particular, when μ is *adapted*, i.e., $H = G$, then ρ is trivial if and only if every harmonic function is constant.

LEMMA 1.1. *Let μ be a probability measure on a countable group G and $(\mathscr{X}, \alpha, \rho)$ the associated μ-boundary. Then*

(a) *the action of G on $(\mathscr{X}, \alpha)$ is SAT and ρ is a SAT measure,*

(b) *ρ is trivial if and only if every μ-harmonic function is constant on the left cosets of the smallest subgroup $H \subseteq G$ such that $\mu(H) = 1$,*

(c) *when μ is adapted, ρ is trivial if and only if every μ-harmonic function is constant.*

A probability measure μ on G is called a *Choquet-Deny measure* if every μ-harmonic function is constant. A Choquet-Deny measure is necessarily adapted. Consequently, if μ is adapted but not Choquet-Deny then ρ is a nontrivial SAT measure and $(\mathscr{X}, \alpha)$ is a nonatomic SAT G-space. Choquet-Deny measures can exist only on amenable groups **[5, 8]**. However, certain amenable groups admit adapted measures μ that are not Choquet-Deny **[8]**. Thus nonatomic SAT actions exist for every nonamenable G and also for some amenable G.

By the classical Choquet-Deny theorem, for abelian G the boundary measure ρ is trivial for every μ. The most general class of countable groups for which the Choquet-Deny theorem is known to remain true (without any restriction on μ) is the class of almost nilpotent groups (G is *almost nilpotent* if it contains a nilpotent subgroup of finite index). We show that for these groups a result stronger than the Choquet-Deny theorem holds, namely, countable almost nilpotent groups do not admit nonatomic SAT actions.

Finally, we point out a connection between existence of nonatomic SAT actions and the growth of G. Recall that G is said to have polynomial growth if for every finite set $F \subseteq G$ there exists a positive integer r such that the sequence $|F^n|/n^r$, $n = 1, 2, \dots$, is bounded, where $|F^n|$ denotes the cardinality of F^n. By a theorem of Gromov **[6]**, every finitely generated group of polynomial growth is almost nilpotent. Thus groups of polynomial growth do not admit nonatomic SAT actions. This result can be sharpened in the solvable case. Using a result of Rosenblatt **[11]** we show that a finitely generated solvable group has polynomial growth if and only if it does not admit nonatomic SAT actions.

2. Strong approximate transitivity: elementary properties. Throughout the sequel we shall consider actions of a discrete group G. Thus $L^\infty(G) = L^\infty(G, \lambda)$, where λ is the counting measure, is the space of bounded complex valued functions on G equipped with the sup norm, and $L^1(G) = L^1(G, \lambda)$ the space of discrete complex measures. For a general measure space $(\mathscr{X}, \alpha)$, by the weak* topology on $L^\infty(\mathscr{X}) = L^\infty(\mathscr{X}, \alpha)$ we shall mean the $\sigma(L^\infty, L^1)$ topology. When $\mathscr{X}$ and $\mathscr{Y}$ are Borel G-spaces with σ-finite quasi-invariant measures α and β, we shall say that $(\mathscr{X}, \alpha)$ and $(\mathscr{Y}, \beta)$ are *isomorphic* if there exists an equivariant isomorphism of the $*$-algebra $L^\infty(\mathscr{X}) = L^\infty(\mathscr{X}, \alpha)$

onto $L^\infty(\mathscr{Y}) = L^\infty(\mathscr{Y}, \beta)$. We shall say that $(\mathscr{Y}, \beta)$ is a *factor* of $(\mathscr{X}, \alpha)$ if there exists an equivariant isomorphism of $L^\infty(\mathscr{Y})$ onto a unital weakly* closed $*$-subalgebra of $L^\infty(\mathscr{X})$.

REMARK 2.1. It is clear that properties AT and SAT are invariant with respect to isomorphisms and factors.

PROPOSITION 2.2. *Let $\mathscr{X}$ be a Borel G-space with a σ-finite quasi-invariant measure α and let $\rho \in L_1^1(\mathscr{X})$. The following conditions are equivalent:*

(a) *ρ is a SAT measure,*

(b) *for every Borel set A with $\alpha(A) > 0$ and for every $\varepsilon > 0$ there exists $g \in G$ such that $(g\rho)(A) = \rho(g^{-1}A) > 1 - \varepsilon$,*

(c) *the map $\mathscr{R}\colon L^\infty(\mathscr{X}) \to L^\infty(G)$ given by $(\mathscr{R}f)(g) = \langle g\rho, f\rangle = \int_{\mathscr{X}} f\, d(g\rho)$ is an isometry.*

Proof. (a) $\Rightarrow$ (b): Since α is σ-finite we can assume without loss of generality that it is finite. Suppose that ρ is a SAT measure. Define $\beta \in L_1^1(\mathscr{X})$ by $\frac{d\beta}{d\alpha} = \alpha(A)^{-1}\chi_A$. There is a sequence $\{g_i\}_{i=1}^n \subseteq G$ and a sequence $\{p_i\}_{i=1}^n \subseteq [0, 1]$ such that

$$\left\|\beta - \sum_{i=1}^n p_i(g_i\nu)\right\| < \varepsilon \quad \text{and} \quad \sum_{i=1}^n p_i = 1.$$

Hence,

$$1 - \sum_{i=1}^n p_i(g_i\nu)(A) = \left|\beta(A) - \sum_{i=1}^n p_i(g_i\nu)(A)\right| < \varepsilon.$$

This implies that $(g\nu)(A) > 1 - \varepsilon$ for some $g \in G$.

(b) $\Rightarrow$ (c): It is clear that (b) implies that $\|\mathscr{R}f\| = \|f\|$ for every simple function $f \in L^\infty(\mathscr{X})$. Since simple functions are uniformly dense in $L^\infty(\mathscr{X})$ and $\mathscr{R}$ is a contraction, (c) follows.

(c) $\Rightarrow$ (a): Note that $\mathscr{R}$ is a positive weakly* continuous contraction whose dual contraction $\mathscr{R}^*\colon L^1(G) \to L^1(\mathscr{X})$ reads $\mathscr{R}^*\varphi = \varphi * \rho$. Since $\mathscr{R}\mathbb{1} = \mathbb{1}$, we have $\mathscr{R}^*L_1^1(G) \subseteq L_1^1(\mathscr{X})$. It suffices to prove that $\mathscr{R}^*L_1^1(G)$ is dense in $L_1^1(\mathscr{X})$. Let M denote the closure of $\mathscr{R}^*L_1^1(G)$. Suppose there exists $\nu \in L_1^1(\mathscr{X}) - M$. Since M is convex we can use the Hahn-Banach theorem to find a positive element $f \in L^\infty(\mathscr{X})$ and $\varepsilon > 0$ such that $\langle \eta, \mathscr{R}f\rangle = \langle \mathscr{R}^*\eta, f\rangle \le \langle \nu, f\rangle - \varepsilon \le \|f\| - \varepsilon$ for all $\eta \in L_1^1(G)$. But this would imply that $\|\mathscr{R}f\| \le \|f\| - \varepsilon$, in contradiction with (c). □

COROLLARY 2.3. *Strongly approximately transitive actions are ergodic.*

Proof. The isometry $\mathscr{R}: L^\infty(\mathscr{X}) \to L^\infty(G)$ of Proposition 2.2(c) is equivariant and maps constants to constants. Hence, ergodicity follows from ergodicity of the action of G on G. (Since SAT $\Rightarrow$ AT one can also use the fact that AT actions are ergodic **[1]**.)

COROLLARY 2.4. *Let μ be a probability measure on a countable group G. The μ-boundary (as defined in §1) is SAT.*

As explained in §1, the μ-boundaries provide many examples of nonatomic SAT actions. However, these examples are usually complicated to analyse and, with a few exceptions, defy a complete explicit description. Below we show that certain homogeneous spaces of locally compact second countable (lcsc) groups admit nonatomic SAT measures. Let G be a lcsc group and $\mathscr{X}$ its homogeneous space. We shall say that a nonempty open set $U \subseteq \mathscr{X}$ is *contractible* if for every nonempty open $V \subseteq \mathscr{X}$ there exists $g \in G$ with $gU \subseteq V$. If $\mathscr{X}$ admits open contractible sets it will be called *contractive.* It is easy to see that $\mathbb{R}$ is a contractive homogeneous space of the $ax+b$ groups and the circle $\mathbb{S}$ is a contractive homogeneous space of $\mathrm{SL}(2, \mathbb{R})$. In the former case every nonempty open set with compact closure is contractible. Recall that every homogeneous space of a lcsc group admits a unique G-invariant measure class.

COROLLARY 2.5. *A contractive homogeneous space $\mathscr{X}$ of a lcsc group G is SAT with respect to the action of any dense subgroup of G. If every nonempty open set with compact closure is contractible then every absolutely continuous probability measure on $\mathscr{X}$ is a SAT measure for the action of any dense subgroup of G. In particular, the transitive action of the $ax+b$ group on $\mathbb{R}$ and the transitive action of* $\mathrm{SL}(2, \mathbb{R})$ *on $\mathbb{S}$ are SAT.*

Proof. Recall that when $\mathscr{X}$ is a homogeneous space of G, then G acts continuously on $L^1(\mathscr{X})$, i.e., for every $\varphi \in L^1(\mathscr{X})$ the function $G \ni g \to g\varphi \in L^1(\mathscr{X})$ is continuous. Hence, it suffices to prove that the action of G on $\mathscr{X}$ is SAT.

Let $\rho \in L^1_1(\mathscr{X})$. We shall say that ρ is contractible if for every $x \in \mathscr{X}$ there exists a sequence $\{g_n\}_{n=1}^\infty \subseteq G$ such that

$$\lim_{n\to\infty} \int f(g_n x)\rho(dx) = f(x)$$

for every bounded continuous $f: \mathscr{X} \to \mathbb{C}$. It is clear that if ρ is carried on a contractible set, then it is contractible. If every nonempty open set with compact closure is contractible, then every $\rho \in L_1^1(\mathscr{X})$ is contractible. This easily follows from the regularity of ρ: for every $\varepsilon > 0$ there exists a compact $K \subseteq \mathscr{X}$ with $\rho(K) > 1-\varepsilon$. Our corollary will be proven when we show that every contractible $\rho \in L_1^1(\mathscr{X})$ is a SAT measure.

Let $\rho \in L_1^1(\mathscr{X})$ be contractible. Define $\mathscr{R}: L^\infty(\mathscr{X}) \to L^\infty(G)$ by $(\mathscr{R}f)(g) = \langle g\rho, f\rangle$. Contractibility of ρ implies that $\|\mathscr{R}f\| = \|f\|$ for every continuous $f \in L^\infty(\mathscr{X})$. By Proposition 2.2(c) we must prove that $\|\mathscr{R}f\| = \|f\|$ for all $f \in L^\infty(\mathscr{X})$. Let $\{U_n\}_{n=1}^\infty$ be a neighbourhood base at $e \in G$, and let ε_n be an absolutely continuous probability measure on G such that $\varepsilon_n(U_n) = 1$. For every $f \in L^\infty(\mathscr{X})$ the functions

$$f_n(x) = \int_G f(gx)\varepsilon_n(dg), \qquad x \in \mathscr{X},$$

and

$$\hat{f}_n(h) = \int_G (\mathscr{R}f)(gh)\varepsilon_n(dg), \qquad h \in G,$$

are continuous, $\|f_n\| \leq \|f\|$, $\|\hat{f}_n\| \leq \|\mathscr{R}f\|$, and $\hat{f}_n = \mathscr{R}f_n$. Hence, $\|f_n\| = \|\hat{f}_n\| \leq \|\mathscr{R}f\|$. But f_n converges weakly* to f. Since the ball $\{F \in L^\infty(\mathscr{X}); \|F\| \leq \|\mathscr{R}f\|\}$ is weakly* closed, it follows that $\|f\| \leq \|\mathscr{R}f\|$. Since $\mathscr{R}$ is a contraction we have $\|f\| = \|\mathscr{R}f\|$. □

We shall say that an ergodic G-space $\mathscr{X}$ with a σ-finite quasi-invariant measure α is purely atomic if α is purely atomic (i.e., every Borel set of nonzero measure contains an atom of α). When the Borel structure of $\mathscr{X}$ is countably separated, purely atomic actions are, of course, those with countable conull orbits. It is easy to see that $(\mathscr{X}, \alpha)$ is purely atomic if and only if $L_1^1(\mathscr{X})$ contains a trivial (0-1) SAT measure. When $(\mathscr{X}, \alpha)$ is purely atomic it admits only trivial SAT measures (carried on single atoms of α).

PROPOSITION 2.6. *Every SAT action with a σ-finite G-invariant measure α is purely atomic.*

Proof. Let ρ be a SAT measure. By the absolute continuity of ρ there is $\varepsilon > 0$ such that $\rho(E) < 1/2$ whenever $\alpha(E) < \varepsilon$. Suppose that α is not purely atomic. Then there exists a Borel set E with $0 < \alpha(E) < \varepsilon$. Since α is G-invariant we have that $(g\rho)(E) =$

$\rho(g^{-1}E) < 1/2$ for all $g \in G$. This clearly contradicts Proposition 2.2(b). □

COROLLARY 2.7. *There exist approximately transitive actions that are not strongly approximately transitive.*

Proof. An ergodic rotation on an uncountable second countable compact abelian group is an approximately transitive $\mathbb{Z}$-action with a nonatomic invariant measure. □

3. Strong approximate transitivity and polynomial growth.

PROPOSITION 3.1 (0-2 *law*). *Let $\mathscr{X}$ be a Borel G-space with a σ-finite quasi-invariant measure α. If $\rho \in L_1^1(\mathscr{X})$ is a SAT measure then for every $g \in G$ the number*

$$a(g) = \sup_{h \in G} \|h^{-1}gh\rho - \rho\|$$

is either 0 *or* 2.

Proof. Consider the action of G on $L^\infty(\mathscr{X})$. For a given $g \in G$ either $gf = f$ for all $f \in L^\infty(\mathscr{X})$ or there exists $f \in L^\infty(\mathscr{X})$ with $gf \neq f$. In the first case, for every $h \in G$ and $f \in L^\infty(\mathscr{X})$ we have $\langle gh\rho, f\rangle = \langle h\rho, g^{-1}f\rangle = \langle h\rho, f\rangle$, i.e., $gh\rho = h\rho$ and, consequently, $\|h^{-1}gh\rho - \rho\| = \|gh\rho - h\rho\| = 0$ and $a(g) = 0$. Consider the second case. Since simple functions are dense in $L^\infty(\mathscr{X})$ there must exist a Borel set A such that $g\chi_A \neq \chi_A$ (mod α). Consequently, $\alpha(A \triangle gA) \neq 0$ where $\triangle$ denotes the symmetric difference. Thus either $\alpha(gA - A) \neq 0$ or $\alpha(A - gA) \neq 0$. In the first case put $C = gA - A$, and in the second put $C = A - gA$. In either case $\alpha(C) \neq 0$ and $C \cap gC = \varnothing$. Define $\varphi \in L_1^1(\mathscr{X})$ by $\frac{d\varphi}{d\alpha} = \alpha(C)^{-1}\chi_C(x)$ (we can assume that $\alpha(C) < \infty$). Then $\|g\varphi - \varphi\| = 2$. But ρ is a SAT measure. Therefore, by Proposition 2.2(b) for a given $\varepsilon > 0$ we can find $h \in G$ with $(h\rho)(C) = \rho(h^{-1}C) > 1 - \varepsilon/4$. Since C and gC are disjoint, $(h\rho)(gC) < \varepsilon/4$. But C and $g^{-1}C$ are also disjoint; therefore $(gh\rho)(C) = (h\rho)(g^{-1}C) < \varepsilon/4$. Consequently, from the definition of the total variation norm,

$$\begin{aligned}\|h^{-1}gh\rho - \rho\| = \|gh\rho - h\rho\| &\geq |(gh\rho)(gC) - (h\rho)(gC)| \\ &\quad + |(gh\rho)(C) - (h\rho)(C)| \\ &\geq (gh\rho)(gC) + (h\rho)(C) - (h\rho)(gC) - (gh\rho)(C) \\ &= 2(h\rho)(C) - (h\rho)(gC) - (gh\rho)(C) > 2 - \varepsilon.\end{aligned}$$

Since ε is arbitrary, the proof is complete. □

COROLLARY 3.2. *If g is in the center of G, then $\|g\rho - \rho\|$ is either 0 or 2.*

THEOREM 3.3. *Abelian groups do not admit nonatomic SAT actions.*

Proof. Let ρ be a SAT measure on $(\mathscr{X}, \alpha)$. Since G is abelian, by Corollary 3.2 for every $g \in G$ either $g\rho = \rho$ or $\|g\rho - \rho\| = 2$. Let $H = \{g \in G; g\rho = \rho\}$ and let $\{g_i\}_{i\in I}$ be a transversal for G/H. Clearly, the measures $\{g_i\rho\}_{i\in I} \subseteq L_1^1(\mathscr{X})$ are pairwise mutually singular. Since α is σ-finite, I is countable. Consequently, $\beta = \sum_{i\in I} g_i\rho$ is σ-finite and $\beta \ll \alpha$. Note also that if $\beta(A) = 0$ then $\mu(A) = 0$ for every $\mu \in \mathrm{co}(G\rho)$. Since $\mathrm{co}(G\rho)$ is dense in $L_1^1(\mathscr{X})$ it follows that $\alpha(A) = 0$. Therefore $\beta \sim \alpha$. It is clear that β is G-invariant. Thus our claim follows now directly from Proposition 2.6. □

REMARK 3.4. For a general (nonabelian) group the proof shows that a G-space $(\mathscr{X}, \alpha)$ is purely atomic whenever there exists a SAT measure with the 0-2 property: $\|g\rho - \rho\| = 0$ or 2 for every $g \in G$. (Consequently, a SAT measure with the 0-2 property is trivial.)

Our next goal is to extend Theorem 3.3 to countable nilpotent and almost nilpotent groups.

Let G and H be countable groups and $\mathscr{X}$ a Borel G-space. Recall that an H-valued cocycle is a Borel map $\gamma: G \times \mathscr{X} \to H$ such that $\gamma(g_1g_2, x) = \gamma(g_1, g_2x)\gamma(g_2, x)$ for all $x \in \mathscr{X}$, $g_1, g_2 \in G$. When $\mathscr{Y}$ is a Borel H-space, the formula $g(x, y) = (gx, \gamma(g, x)y)$ defines a Borel action of G on $\mathscr{X} \times \mathscr{Y}$. The resulting G-space is denoted by $\mathscr{X} \times_\gamma \mathscr{Y}$ and called the skew product of the G-space $\mathscr{X}$ and the H-space $\mathscr{Y}$ [**12**, p. 75]. If α is a σ-finite quasi-invariant measure on $\mathscr{X}$ and β a σ-finite quasi-invariant measure on $\mathscr{Y}$ then $\alpha \times \beta$ is a quasi-invariant measure on $\mathscr{X} \times_\gamma \mathscr{Y}$.

Let H be a subgroup of G. Choose a cross section $\kappa: G/H \to G$ of the canonical map $\pi: G \to G/H$ and define $\gamma(g, x) = \kappa(gx)^{-1}g\kappa(x)$, $g \in G$, $x \in G/H$. γ is then an H-valued cocycle. When $\mathscr{Y}$ is an H-space, the skew product $G/H \times_\gamma \mathscr{Y}$ is usually called the G-space induced from the H-space $\mathscr{Y}$ [**12**, Definition 4.2.21, p. 75]. With α denoting the counting measure on G/H and β a σ-finite quasi-invariant measure on $\mathscr{Y}$, the measure $\alpha \times \beta$ is a σ-finite quasi-invariant measure on $G/H \times_\gamma \mathscr{Y}$. In the next lemma δ_H denotes the point measure on G/H concentrated in the coset H.

LEMMA 3.5. *For every $\nu \in L_1^1(\mathscr{Y})$ the measure $\rho = \delta_H \times \nu$ is a SAT measure on $G/H \times_\gamma \mathscr{Y}$ if and only if ν is a SAT measure on $\mathscr{Y}$.*

Proof. Suppose ν is a SAT measure, and let A be a Borel set with $(\alpha \times \beta)(A) > 0$. Then

$$\begin{aligned}(g\rho)(A) &= \int_{\mathscr{Y}} \nu(dy)\chi_A(gH, \kappa(gH)^{-1}g\kappa(H)y) \\ &= (\kappa(gH)^{-1}g\kappa(H)\nu)(A^{gH}),\end{aligned}$$

where $A^{gH} = \{y \in \mathscr{Y}; (gH, y) \in A\}$ denotes the section. Note that for a fixed $g \in G$ the map $H \ni h \to \kappa(ghH)^{-1}gh\kappa(H) \in H$ is surjective. Choose g so that $\beta(A^{gH}) \neq 0$. Then for a given $\varepsilon > 0$ choose $h \in H$ with $(\kappa(ghH)^{-1}gh\kappa(H)\nu)(A^{ghH}) = (gh\rho)(A) > 1-\varepsilon$. By Proposition 2.2(b), ρ is SAT. A similar argument shows that if ρ is SAT then so is ν. □

COROLLARY 3.6. *If G is countable and admits only purely atomic SAT actions then every subgroup $H \subseteq G$ also admits only purely atomic SAT actions.*

LEMMA 3.7. *Let $\mathscr{X}$ be a Borel G-space with a σ-finite quasi-invariant measure α (G countable). Let H and K, $K \subseteq H$, be subgroups of the centre of G. Assume that there exists a partition of $\mathscr{X}$ into disjoint Borel sets $\{\mathscr{X}_\xi\}_{\xi \in H/K}$ such that $h\mathscr{X}_\xi = \mathscr{X}_{h\xi}$ for $\xi \in H/K$ and $h \in H$. Moreover, let K act trivially on $\mathscr{X}$ (i.e., $kx = x$ for $k \in K$, $x \in \mathscr{X}$). Then $(\mathscr{X}, \alpha)$ is isomorphic to a skew product $\mathscr{Y} \times_\gamma (H/K)$, where $\mathscr{Y}$ is a G-space with a σ-finite quasi-invariant measure β, H acts trivially on $\mathscr{Y}$, and γ is an (H/K)-valued cocycle.*

Proof. Let $p: \mathscr{X} \to H/K$ be given by $p(x) = \xi$ when $x \in \mathscr{X}_\xi$. Let $\kappa: H/K \to H$ be a cross section of the canonical homomorphism $\pi: H \to H/K$, such that $\kappa(K) = e$. Note that $\kappa(p(x))^{-1}x \in \mathscr{X}_K$ for all $x \in \mathscr{X}$, and $p(hx) = hp(x)$ for $h \in H$ and $x \in \mathscr{X}$. Set $\mathscr{Y} = \mathscr{X}_K$, $\beta =$ the restriction of α to $\mathscr{Y}$. Define $F: \mathscr{X} \to \mathscr{Y}$ by $F(x) = \kappa(p(x))^{-1}x$. As G is countable, it follows that F is a Borel map of $\mathscr{X}$ onto $\mathscr{Y}$. Furthermore, $F(x) = x$ for $x \in \mathscr{X}_K = \mathscr{Y}$, and $F(hx) = x$ for $x \in \mathscr{X}$ and $h \in H$. Moreover, $F\alpha \sim \beta$.

Let $g', g \in G$, $x \in \mathscr{X}$. Since H is central, we have $F(g'gx) = F(g'\kappa(p(gx))F(gx)) = F(\kappa(p(gx))g'F(gx)) = F(g'F(gx))$. It follows that the formula $g \cdot y = F(gy)$ defines a G-action on $\mathscr{Y}$ such that $h \cdot y = y$ for $h \in H$, $y \in \mathscr{Y}$, and $F(gx) = g \cdot F(x)$ for $g \in G$ and $x \in \mathscr{X}$. The measure β is quasi-invariant under this action. Set $\gamma(g, y) = p(gy)$, $g \in G$, $y \in \mathscr{Y}$. γ is then a (H/K)-valued cocycle.

Define $\varphi\colon \mathscr{X} \to \mathscr{Y} \times_\gamma (H/K)$ by $\varphi(x) = (F(x), p(x))$. It easily follows that φ is an equivariant Borel isomorphism (with inverse $\varphi^{-1}(y, \xi) = \kappa(\xi)y$). Moreover, $\varphi\alpha \sim \beta \times \mu$ where μ is the counting measure on H/K. Thus $\mathscr{X}$ and $\mathscr{Y} \times_\gamma (H/K)$ are indeed isomorphic. □

LEMMA 3.8. *Let G be countable and let C be its centre. If $(\mathscr{X}, \alpha)$ is a SAT G-space, then $\mathscr{X}$ is isomorphic to a skew product $\mathscr{Y} \times_\gamma (C/K)$ where $\mathscr{Y}$ is a SAT G-space, C acts trivially on $\mathscr{Y}$, K is a subgroup of G, and γ is a (C/K)-valued cocycle.*

Proof. Let ρ be a SAT measure on $\mathscr{X}$ and let $K = \{g \in C;\, g\rho = \rho\}$. Since $\operatorname{co}(G\rho)$ is dense in $L^1_1(\mathscr{X})$ it follows that K acts trivially on $L^1(\mathscr{X})$ and, consequently, also on $L^\infty(\mathscr{X})$. Let $\mathscr{X}' = \mathscr{X}/K$ be the orbit space and $p\colon \mathscr{X} \to \mathscr{X}'$ the canonical map. Give $\mathscr{X}'$ the quotient Borel structure and the quotient G-action. Set $\alpha' = p\alpha$ and $\rho' = p\rho$. It is clear that $(\mathscr{X}, \alpha)$ and $(\mathscr{X}', \alpha')$ are isomorphic G-spaces, K acts trivially on $\mathscr{X}'$, and ρ' is a SAT measure on $\mathscr{X}'$. Thus without loss of generality we may assume that K acts trivially on $\mathscr{X}$.

We will now show that there exists a family of disjoint Borel sets $\{\mathscr{X}_\xi\}_{\xi\in C/K}$ such that $h\mathscr{X}_\xi = \mathscr{X}_{h\xi}$ for $h \in C$, and $\alpha(\mathscr{X} - \bigcup_{\xi\in C/K} \mathscr{X}_\xi) = 0$. Let $\{c_\xi\}_{\xi\in C/K}$ be a transversal for C/K. We will first prove that for every Borel set $A \subseteq \mathscr{X}$ with $\alpha(A) \neq 0$ there exists a Borel set $\widehat{A} \subseteq \mathscr{X}$ such that $\widehat{A} \subseteq A$, the sets $\{c_\xi \widehat{A}\}_{\xi\in C/K}$ are pairwise disjoint, and $\alpha(\widehat{A}) \neq 0$.

If $\alpha(A) \neq 0$, then by the SAT property, $(g\rho)(A) \neq 0$ for some $g \in G$. By Corollary 3.2 for every $h \in C$ the measures $hg\rho$ and $g\rho$ are either equal or mutually singular. Set $\nu_\xi = c_\xi^{-1} g\rho$, $\xi \in C/K$. Thus from the definition of K the measures ν_ξ and ν_K are mutually singular when $\xi \neq K$. I.e., for every $\xi \neq K$ there exists a Borel set D_ξ with $\nu_K(D_\xi) = 1$ and $\nu_\xi(D_\xi^c) = 1$ (where $D_\xi^c = \mathscr{X} - D_\xi$). Put

$$D = \bigcap_{\substack{\xi\in C/K \\ \xi\neq K}} (D_\xi \cap (c_\xi D_\xi^c)).$$

Then $\nu_K(D) = 1$ and the sets $\{c_\xi D\}_{\xi\in H/K}$ are pairwise disjoint. Set $\widehat{A} = A \cap D$. Then $\nu_K(\widehat{A}) \neq 0$ and $\{c_\xi \widehat{A}\}_{\xi\in H/K}$ are pairwise disjoint.

Consider now the family $\mathscr{E} \subseteq L^\infty(\mathscr{X})$ of nonzero projections of $L^\infty(\mathscr{X})$, i.e., the family of elements of $L^\infty(\mathscr{X})$ given by characteristic functions χ_A of Borel sets with $\alpha(A) \neq 0$. Note that $g\chi_A = \chi_{gA} \pmod{\alpha}$. Let

$$\mathscr{E}_1 = \{\varphi \in \mathscr{E}\,;\ \text{the projections } \{c_\xi \varphi\}_{\xi \in H/K} \text{ are pairwise orthogonal}\}.$$

By the preceding part of the proof ξ_1 is nonempty. Note also that if $\mathscr{F} \subseteq \mathscr{E}_1$ is a linearly ordered subset, then $\bigvee \mathscr{F} \in \mathscr{E}_1$. Hence, by Zorn's lemma, $\mathscr{E}_1$ has a maximal element ψ. We claim that $\bigvee_{\xi \in C/K} c_\xi \psi = \mathbb{1}$. Indeed, if $\mathbb{1} - \bigvee_{\xi \in C/K} c_\xi \psi \neq 0$, then (again by the preceding part of the proof) one can find a projection $\varphi \in \mathscr{E}_1$ with $\varphi \leq \mathbb{1} - \bigvee_{\xi \in C/K} c_\xi \psi$. But this would contradict the maximality. Thus $\bigvee_{\xi \in C/K} c_\xi \psi = \mathbb{1}$. Choosing a representative χ_E of ψ and setting $B = E \cap \bigcap_{\xi \in C/K} c_\xi(E^c)$ and $\mathscr{X}_\xi = c_\xi B$, $\xi \in C/K$, we obtain a family of disjoint Borel sets with $h\mathscr{X}_\xi = \mathscr{X}_{h\xi}$, $h \in C$, and $\alpha(\mathscr{X} - \bigcup_{\xi \in C/K} \mathscr{X}_\xi) = 0$.

Let $\mathscr{X}' = \bigcap_{g \in G} g(\bigcup_{\xi \in C/K} \mathscr{X}_\xi)$, and let $\mathscr{X}'_\xi = \mathscr{X}' \cap \mathscr{X}_\xi$, $\xi \in C/K$. Since G is countable, $\mathscr{X}'$ is a G-invariant conull Borel subset of $\mathscr{X}$, $\mathscr{X}' = \bigcup_{\xi \in C/K} \mathscr{X}'_\xi$, the $\mathscr{X}'_\xi$'s are pairwise disjoint, and $h\mathscr{X}'_\xi = \mathscr{X}'_{h\xi}$, $h \in C$. The G-space $(\mathscr{X}', \alpha)$ is obviously isomorphic to $(\mathscr{X}, \alpha)$. Thus our proof is complete by Lemma 3.7 and Remark 2.1. □

THEOREM 3.9. *Countable nilpotent groups do not admit nonatomic SAT actions.*

Proof. Using Theorem 3.3, Lemma 3.8 and Remark 2.1, the proof is an obvious induction with respect to the class of G. □

LEMMA 3.10. *Let $\nu_1, \nu_2, \dots, \nu_k \in L^1_1(\mathscr{X})$ where $\mathscr{X}$ is a Borel G-space with a σ-finite quasi-invariant measure α. Assume that the convex hull* $\mathrm{co}(G\nu_1, \dots, G\nu_k)$ *is dense in $L^1_1(\mathscr{X})$ and that G is countable. Then there exists $i \in \{1, 2, \dots, k\}$ and a G-invariant Borel set $\mathscr{X}_0 \subseteq \mathscr{X}$ such that $\nu_i(\mathscr{X}_0) = 1$ and ν_i is a SAT measure on $(\mathscr{X}_0, \alpha_0)$ where α_0 is the restriction of α to $\mathscr{X}_0$.*

Proof. Let $\mathscr{F}_i$ denote the collection of Borel sets $A \subseteq \mathscr{X}$ such that $\alpha(A) \neq 0$ and for every $\varepsilon > 0$ and every Borel $B \subseteq A$ with $\alpha(B) \neq 0$ there exists $g \in G$ with $(g\nu_i)(B) > 1 - \varepsilon$. We claim that $\mathscr{F}_i \neq \varnothing$ for at least one $i = 1, 2, \dots, k$. Indeed, it is easy to see that if $\mathscr{F}_i = \varnothing$ for all $i = 1, 2, \dots, k$, then there would exist a sequence $\varepsilon_1, \dots, \varepsilon_n > 0$ and a sequence of Borel sets $B_1 \supseteq B_2 \supseteq \cdots \supseteq B_k$ with $\alpha(B_i) \neq 0$ and

$(g\nu_i)(B_i) \leq 1 - \varepsilon_i$ for all $g \in G$ and $i = 1, 2, \dots, k$. Set $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_k\}$, and define $\varphi \in L^1_1(\mathscr{X})$ by $\frac{d\varphi}{d\alpha} = \alpha(B_k)^{-1}\chi_{B_k}$ (we can assume that α is finite). According to our assumption there exists a convex combination $\sum_{j=1}^N p_j(g_j\nu_{i_j})$ with $\|\varphi - \sum_{j=1}^N p_j(g_j\nu_{i_j})\| < \varepsilon$. Hence, $(g\nu_i)(B_i) \geq (g\nu_i)(B_k) > 1 - \varepsilon \geq 1 - \varepsilon_i$ for some $g \in G$ and some $i = 1, 2, \dots, k$. This would clearly contradict the definition of the sequences $B_1, \dots, B_k$ and $\varepsilon_1, \dots, \varepsilon_k$.

Let $\mathscr{F}_i \neq \varnothing$ and let $\mathscr{E}$ be the corresponding family of projections of $L^\infty(\mathscr{X})$, i.e., $\varphi \in \mathscr{E}$ if and only if $\varphi = \chi_A \pmod{\alpha}$ and $A \in \mathscr{F}_i$. It is clear that $\mathscr{E}$ is a G-invariant family. Therefore $\bigvee \mathscr{E}$ is a G-invariant projection. Since G is countable, it is easy to see that $\psi = \chi_{\mathscr{X}_0} \pmod{\alpha}$ for a G-invariant Borel set $\mathscr{X}_0 \subseteq \mathscr{X}$. It is also easy to see that $\mathscr{X}_0 \in \mathscr{F}_i$. The G-invariance of $\mathscr{X}_0$ then implies that $\nu_i(\mathscr{X}_0) = 1$. By Proposition 2.2(b) and the definition of $\mathscr{F}_i$, ν_i is a SAT measure on $(\mathscr{X}_0, \alpha_0)$. □

THEOREM 3.11. *Let G be a countable group and H a subgroup of finite index. If H admits only purely atomic SAT actions then G also admits only purely atomic SAT actions.*

Proof. Let $\{g_i\}_{i=1}^n$ be a right transversal of H in G. It is clear that if ν is a SAT measure on a G-space $(\mathscr{X}, \alpha)$ then $\mathrm{co}(Hg_1\nu, \dots, Hg_n\nu)$ is dense in $L^1_1(\mathscr{X})$. Considering $\mathscr{X}$ as an H-space our claim follows from Lemma 3.10. □

Recall that by Gromov's theorem **[6]** every finitely generated group of polynomial growth contains a nilpotent subgroup of finite index.

COROLLARY 3.12. *Finitely generated groups of polynomial growth do not admit nonatomic SAT actions.*

THEOREM 3.13. *The following conditions are equivalent for a finitely generated solvable group G:*

(a) *G has polynomial growth,*

(b) *G does not admit nonatomic SAT actions,*

(c) *if μ is a probability measure on G, then every μ-harmonic function is constant on the cosets of the smallest subgroup $H \subseteq G$ such that $\mu(H) = 1$.*

Proof. (a) ⇒ (b) is contained in Corollary 3.12. (b) ⇒ (c) follows from Lemma 1.1(b). To prove that (c) implies (a) we invoke the following result of Rosenblatt **[11]**: A finitely generated solvable group has polynomial growth if and only if it does not contain a free subsemigroup on two generators. Suppose that (c) is true but (a) is false. Then G contains a free subsemigroup S on two generators a, b. Let $\mu = \frac{1}{2}\delta_a + \frac{1}{2}\delta_b$. From (c) every μ-harmonic function h satisfies $h(gs) = h(g)$ for all $g \in G$ and $s \in S$. Let $\mu_n = \frac{1}{n}\sum_{i=1}^{n} \mu^i$, where μ^i is the i-th convolution power. Using **[3**, Théorème 1**]** we have

$$\lim_{n\to\infty} \|g\mu_n - \mu_n\| = \sup\{|h(g) - h(e)|\,;\, h \in \mathscr{H}\,,\ \|h\| \le 1\}$$

for every $g \in G$, where $\|g\mu_n - \mu_n\|$ denotes the total variation norm and $\mathscr{H}$ the space of μ-harmonic functions. Thus

$$\lim_{n\to\infty} \|s\mu_n - \mu_n\| = 0 \quad \text{for every } s \in S.$$

Note that μ_n is a sequence of probability measures carried on S. We can consider these measures as means on S. The set of means is weakly* compact. It follows that every weak* cluster point of the sequence μ_n is a left invariant mean on S. Since S is not amenable we arrive at a contradiction. Thus (c) implies (a). □

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