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PERIODS AND LEFSCHETZ ZETA FUNCTIONS

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The goal of this paper is to obtain information on the set of periods for a transversal self-map of a compact manifold from the associated Lefschetz zeta function in the case when all its zeros and poles are roots of unity.

1. Introduction and statement of the results. One of the most useful theorems for proving the existence of fixed points or, more generally, periodic points of a transversal self-map f of a compact manifold is the Lefschetz fixed point theorem. When studying the periodic points of f, i.e., the set

$$\operatorname{Per}(f) = \{m \in \mathbb{N} \colon f \text{ has a periodic orbit of minimal period } m\},$$

it is convenient to use the Lefschetz zeta function of f, $Z_f(t)$, which is a generating function for the Lefschetz numbers of all iterates of f. The function $Z_f(t)$ is rational in t and can be computed from the homological invariants of f (see §3).

We shall study C^1 self-maps f of a compact manifold which have only transversal periodic points, so called because the graph of f^m is transverse to the diagonal for all $m > 0$. The main contribution of this paper is the study of the periodic orbits of f when its Lefschetz zeta function has a finite factorization into terms of the form $(1 \pm t^n)^{\pm 1}$. A key point is the introduction of the notion of irreducible factor (see §3 for a precise definition). Our main result is the following.

THEOREM A. *Let $f \colon M \to M$ be a transversal map of a compact manifold. Suppose that all the zeros and poles of its Lefschetz zeta function $Z_f(t)$ are roots of unity, and that $Z_f(t)$ has an irreducible factor of the form $(1 \pm t^n)^{\pm 1}$.*

(a) *If n is odd then $n \in \operatorname{Per}(f)$.*

(b) *If n is even then $\{\frac{n}{2}, n\} \cap \operatorname{Per}(f) \neq \varnothing$.*

The proof of this theorem will be given in §3. From Theorem A it follows that each irreducible factor of the form $(1 \pm t^n)^{\pm 1}$ of the

Lefschetz zeta function forces at least one period (n if n is odd, $n/2$ or n if n is even).

The set of periods obtained in this way will be called the *forced set of periods* of f and will be denoted by $\mathrm{FSP}(f)$.

As an application of Theorem A and the algebraic results derived in §2, we obtain an upper bound for the cardinal and for the maximum period of the forced set of periods (see Corollaries 3.3 and 3.4).

Our main basic assumption throughout this work is that all the zeros and poles of the Lefschetz zeta function associated to $f: M \to M$ are roots of unity (for different results under similar assumptions see Franks **[F1]**, **[F3]**, Fried **[Fr]**, Matsuoka **[Mt]** and **[CLN]**). There are three interesting classes of transversal maps which satisfy our basic assumption. First, the set of maps whose set of periods $\mathrm{Per}(f)$ is finite (see Theorem 6 of **[Fr]**). Second, the self-maps of compact connected surfaces with $\mathrm{Per}(f)$ finite or $h(f) = 0$, see Corollaries 4.3 and 4.4. Finally, the self-maps of the n-dimensional torus with $\mathrm{Per}(f)$ finite or $h(f) = 0$, see Corollaries 5.1 and 5.2.

2. Cyclotomic polynomials. As usual, we shall use the notation $c_n(t)$ for the nth *cyclotomic polynomial* given by

$$c_n(t) = \frac{1 - t^n}{\prod_{d|n,\, d<n} c_d(t)}$$

for $n \in \mathbb{N}\backslash\{1\}$ and $c_1(t) = 1 - t$.

Notice that all the zeros of $c_n(t)$ are roots of unity.

A proof of the next proposition may be found in **[L]**.

PROPOSITION 2.1. *Let ξ be a primitive* nth *root of unity and $P(t)$ a polynomial with rational coefficients. If $P(\xi) = 0$ then $c_n(t)|P(t)$.*

Clearly, the degree $\varphi(n)$ of $c_n(t)$ verifies

$$n = \sum_{d|n} \varphi(d)$$

and so $\varphi(n)$ is the Euler function, which may be computed through

$$\varphi(n) = n \prod_{\substack{p|n \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right).$$

Hence, if $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ is the prime decomposition of n, then

$$\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1). \tag{2.1}$$

TABLE 1

$c_1(t) = 1 - t$	$c_2(t) = 1 + t$	$c_3(t) = \dfrac{1-t^3}{1-t}$
$c_4(t) = 1 + t^2$	$c_5(t) = \dfrac{1-t^5}{1-t}$	$c_6(t) = \dfrac{1+t^3}{1+t}$
$c_7(t) = \dfrac{1-t^7}{1-t}$	$c_8(t) = 1 + t^4$	$c_9(t) = \dfrac{1-t^9}{1-t^3}$
$c_{10}(t) = \dfrac{1+t^5}{1+t}$	$c_{11}(t) = \dfrac{1-t^{11}}{1-t}$	$c_{12}(t) = \dfrac{1+t^6}{1+t^2}$
$c_{13}(t) = \dfrac{1-t^{13}}{1-t}$	$c_{14}(t) = \dfrac{1+t^7}{1+t}$	$c_{15}(t) = \dfrac{(1-t^{15})(1-t)}{(1-t^3)(1-t^5)}$
$c_{16}(t) = 1 + t^8$	$c_{17}(t) = \dfrac{1-t^{17}}{1-t}$	$c_{18}(t) = \dfrac{1+t^9}{1+t^3}$
$c_{19}(t) = \dfrac{1-t^{19}}{1-t}$	$c_{20}(t) = \dfrac{1+t^{10}}{1+t^2}$	$c_{21}(t) = \dfrac{(1-t^{21})(1-t)}{(1-t^3)(1-t^7)}$
$c_{22}(t) = \dfrac{1+t^{11}}{1+t}$	$c_{23}(t) = \dfrac{1-t^{23}}{1-t}$	$c_{24}(t) = \dfrac{1+t^{12}}{1+t^4}$
$c_{25}(t) = \dfrac{1-t^{25}}{1-t^5}$	$c_{26}(t) = \dfrac{1+t^{13}}{1+t}$	$c_{27}(t) = \dfrac{1-t^{27}}{1-t^9}$
$c_{28}(t) = \dfrac{1+t^{14}}{1+t^2}$	$c_{29}(t) = \dfrac{1-t^{29}}{1-t}$	$c_{30}(t) = \dfrac{(1+t^{15})(1+t)}{(1+t^3)(1+t^5)}$

In Table 1 we present a list of the first 30 cyclotomic polynomials and their degrees. The following rules follow easily from the definition of cyclotomic polynomials and their properties (see [L]).

$$p \text{ prime} \Rightarrow c_p(t) = \frac{1-t^p}{1-t}, \tag{2.2}$$

$$p = 2^n \Rightarrow c_p(t) = 1 + t^{2^{n-1}}, \tag{2.3}$$

$$p = 2r,\ r \text{ odd} \Rightarrow c_p(t) = c_r(-t), \tag{2.4}$$

$$p = 2^n r,\ r \text{ odd},\ n > 1 \Rightarrow c_p(t) = c_{2r}(t^{2^{n-1}}), \tag{2.5}$$

$$p = p_1 p_2,\ p_1,\ p_2 \text{ prime} \Rightarrow c_p(t) = \frac{c_{p_1}(t^{p_2})}{c_{p_1}(t)} = \frac{c_{p_2}(t^{p_1})}{c_{p_2}(t)}, \tag{2.6}$$

$$p = p_1^\alpha,\ p_1 \text{ prime} \Rightarrow c_p(t) = c_{p_1}(t^{p_1^{\alpha-1}}) = \frac{1-t^{p_1^\alpha}}{1-t^{p_1^{\alpha-1}}}, \tag{2.7}$$

$$c_{p_1^{\alpha_1}\cdots p_k^{\alpha_k}}(t) = c_{p_1\cdots p_k}(t^{p_1^{\alpha_1-1}\cdots p_k^{\alpha_k-1}}), \tag{2.8}$$

$$p \text{ prime}, p \nmid r \Rightarrow c_{pr}(t) = \frac{c_r(t^p)}{c_r(t)}. \tag{2.9}$$

LEMMA 2.2. *Let* $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ *be the prime decomposition of* $n \in \mathbb{N}$. *Then*

(2.10)

$$c_{p_1^{\alpha_1} \cdots p_k^{\alpha_k}}(t) = c_{p_k}(t^{p_k^{\alpha_k - 1}})^{(-1)^{k-1}} \cdot \prod_{j=1}^{k-1} \prod_{1 \le i_1 < \cdots < i_j \le k-1} c_{p_k}((t^{p_{i_1} \cdots p_{i_j}})^{p_1^{\alpha_1 - 1} \cdots p_k^{\alpha_k - 1}})^{(-1)^{k-1-j}}.$$

Proof. Using property (2.8), it is enough to show that

$$c_{p_1 \cdots p_k}(t) = c_{p_k}(t)^{(-1)^{k-1}} \cdot \prod_{j=1}^{k-1} \prod_{1 \le i_1 < \cdots < i_j \le k-1} c_{p_k}(t^{p_{i_1} \cdots p_{i_j}})^{(-1)^{k-1-j}}, \tag{2.11}$$

and we shall prove (2.11) by induction with respect to $k \in \mathbb{N}$. For $k = 1$ it holds trivially. Suppose $k = 2$. By property (2.6) we have

$$c_{p_1 p_2}(t) = c_{p_2}(t)^{-1} c_{p_2}(t^{p_1}),$$

and so (2.11) holds for $k = 2$. Suppose now that (2.11) holds for some $k \in \mathbb{N}$, $k \ge 2$, and consider $c_{p_1 \cdots p_{k+1}}(t)$ with $p_1 < \cdots < p_{k+1}$. Then, applying successively (2.9) and the induction hypothesis,

$$\begin{aligned} c_{p_1 \cdots p_{k+1}}(t) &= \frac{c_{p_2 \cdots p_{k+1}}(t^{p_1})}{c_{p_2 \cdots p_{k+1}}(t)} \\ &= \frac{c_{p_{k+1}}(t^{p_1})^{(-1)^{k-1}}}{c_{p_{k+1}}(t)^{(-1)^{k-1}}} \cdot \prod_{j=1}^{k-1} \prod_{2 \le i_1 < \cdots < i_j \le k} \left[\frac{c_{p_{k+1}}(t^{p_1 p_{i_1} \cdots p_{i_j}})}{c_{p_{k+1}}(t^{p_{i_1} \cdots p_{i_j}})} \right]^{(-1)^{k-1-j}} \\ &= c_{p_{k+1}}(t)^{(-1)^k} c_{p_{k+1}}(t^{p_1})^{(-1)^{k-1}} \cdot \prod_{j=1}^{k-1} \prod_{2 \le i_1 < \cdots < i_j \le k} c_{p_{k+1}}(t^{p_1 p_{i_1} \cdots p_{i_j}})^{(-1)^{k-j-1}} \\ &\qquad \cdot c_{p_{k+1}}(t^{p_{i_1} \cdots p_{i_j}})^{(-1)^{k-j}}. \end{aligned} \tag{2.12}$$

Now it is easy to check that (2.12) is equal to

$$c_{p_{k+1}}(t)^{(-1)^k}\prod_{j=1}^{k-1}\prod_{1\le i_1<\cdots<i_j\le k} c_{p_{k+1}}(t^{p_{i_1}\cdots p_{i_j}})^{(-1)^{k-j}}.$$

Hence, (2.11) holds for $k+1$ and the lemma is proved. □

PROPOSITION 2.3. *Let $c_n(t)$ be of degree $\varphi(n)>2$. Then $c_n(t)$ can be written as*

$$c_n(t)=\prod_{i=1}^{m}(1-\sigma_1(i)t^{q_i})^{\sigma_2(i)}, \tag{2.13}$$

where $m\le\varphi(n)/2$, $q_i\in\mathbb{N}$ and $\sigma_1(i),\sigma_2(i)\in\{-1,1\}$ for $i=1,\dots,m$. Moreover, $q=\max_i q_i$ is smaller than or equal to $n/2$ if n is even or n if n is odd.

Proof. Let $n=p_1^{\alpha_1}\cdots p_k^{\alpha_k}$ be the prime decomposition of n. The case $k=1$ is obvious from (2.7) and (2.3). Suppose first that $k\ge 2$ and that none of the p_i is 2. From Lemma 2.2 and (2.2), $c_n(t)$ factorizes in the form (2.13) with

$$m=2\sum_{j=0}^{k-1}\binom{k-1}{j}=2^k.$$

From (2.1)

$$\varphi(n)=\prod_{i=1}^{k}p_i^{\alpha_i-1}(p_i-1)\ge 2\cdot 4\cdot 2^{k-2}=2^{k+1},$$

and the result follows with $\sigma_1(i)=+1$ for $i=1,\dots,m$.

Suppose now that $k\ge 2$ and that one of the p_i is 2. Applying Lemma 2.2 with $p_k=2$ and (2.3), $c_n(t)$ factorizes in the form (2.13) with $\sigma_1(i)=-1$ for $i=1,\dots,m$ and

$$m=\sum_{j=0}^{k-1}\binom{k-1}{j}=2^{k-1}.$$

But for $k\ge 3$

$$\varphi(n)=\prod_{i=1}^{k}p_i^{\alpha_i-1}(p_i-1)\ge 2\cdot 8\cdot 2^{k-3}=2^{k+1},$$

TABLE 2

d	$n_0(d)$	d	$n_0(d)$	d	$n_0(d)$	d	$n_0(d)$	d	$n_0(d)$
1	2	2	6	3	9	4	12	5	15
6	18	7	21	8	30	9	33	10	37
11	41	12	45	13	48	14	52	15	56
16	60	17	63	18	67	19	71	20	75
21	78	22	82	23	86	24	90	25	93
26	97	27	101	28	105	29	108	30	112
31	116	32	120	33	123	34	127	35	131
36	135	37	138	38	142	39	146	40	150
41	153	42	157	43	161	44	165	45	168
46	172	47	176	48	210	49	214	50	218
51	223	52	227	53	231	54	236	55	240
56	245	57	249	58	253	59	258	60	262
61	266	62	271	63	275	64	280	65	284
66	288	67	293	68	297	69	301	70	306
71	310	72	315	73	319	74	323	75	328
76	332	77	336	78	341	79	345	80	350
81	354	82	358	83	363	84	367	85	371
86	376	87	380	88	385	89	389	90	393
91	398	92	402	93	406	94	411	95	415
96	420	97	424	98	428	99	433	100	437

and the result follows as before. If $k = 2$, $m = 2$. But $\varphi(n)$ is even and hence by hypothesis $\varphi(n) \geq 4$. So, again, $m \leq \varphi(n)/2$.

From the definition of $c_n(t)$ it is clear that $q \leq n$, and from (2.4) and (2.5) it follows that $q \leq n/2$ if n is even. □

Given $d \in \mathbb{N}$, let $n_0(d)$ be defined by

$$n_0(d) = \left[d \prod_{i=1}^{k} \frac{p_i}{p_i - 1} \right] \tag{2.14}$$

where $[\cdot]$ denotes the integer part function and $k \in \mathbb{N}$ is the greatest number of consecutive primes p_i such that $p_1 = 2$ and $(p_1 - 1) \cdots (p_k - 1) \leq d$. See Table 2 for the first one hundred values of $n_0(d)$.

PROPOSITION 2.4. *Given $d \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that for $n > n_0$, $\varphi(n) > d$. Moreover, $n_0(d)$ is the best possible lower bound for n_0.*

Proof. For $d = 8$ we obtain $n_0(d) = 30$ and $\varphi(30) = 8$. So, (2.14) gives the best possible lower bound.

Let $d \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ be given by (2.14). Let n be greater than n_0 so that

$$n > \left[d \prod_{i=1}^{k} \frac{p_i}{p_i - 1} \right] \geq \prod_{i=1}^{k} p_i,$$

and let $n = p_1^{\alpha_1} \cdots p_j^{\alpha_j}$ be the prime decomposition of n. In order to prove that $\varphi(n) > d$ we shall consider two cases. Suppose first that $j > k$. Then, from (2.1)

$$\varphi(n) = \prod_{i=1}^{j} p_{n_i}^{\alpha_i - 1}(p_{n_i} - 1) \geq \prod_{i=1}^{j}(p_{n_i} - 1) \geq \prod_{i=1}^{k+1}(p_i - 1) > d.$$

Suppose now that $j \leq k$. Then

$$\begin{aligned} \varphi(n) = \prod_{i=1}^{j} p_{n_i}^{\alpha_i - 1}(p_{n_i} - 1) &\geq n \prod_{i=1}^{j} \frac{p_{n_i} - 1}{p_{n_i}} \\ &> d \prod_{i=1}^{k} \frac{p_i}{p_i - 1} \prod_{i=1}^{j} \frac{p_{n_i} - 1}{p_{n_i}} \geq d, \end{aligned}$$

because the p_i are consecutive primes and so, for $i = 1, \ldots, j$

$$\frac{p_i}{p_i - 1} \cdot \frac{p_{n_i} - 1}{p_{n_i}} \geq 1. \qquad \square$$

3. Periods forced by the Lefschetz zeta function. Let M be a compact manifold. A map $f\colon M \to M$ is called *transversal* if

(1) f is of class C^1,
(2) $f(M) \subset \operatorname{Int}(M)$,
(3) for every periodic orbit γ of period $p(\gamma)$, $x \in \gamma$ and $m \in \mathbb{N}$

$$\det(Df^{mp(\gamma)}(x) - I) \neq 0.$$

Let $f\colon M \to M$ be a transversal map. Denote by PO the *set of periodic orbits* of f and, given $\gamma \in \mathrm{PO}$, by $p(\gamma)$ the *minimal period* of γ. Following the notation introduced in **[CLN]** we define $u_+(\gamma)$ (resp. $u_-(\gamma)$) as the number of real eigenvalues of $Df^{p(\gamma)}(x)$, $x \in \gamma$, which are strictly greater than one (resp. strictly smaller than one). We also define the following subsets of PO:

$$\begin{aligned}
\mathrm{EE} &= \{\gamma \in \mathrm{PO}\colon u_+(\gamma) \text{ and } u_-(\gamma) \text{ are even}\},\\
\mathrm{EO} &= \{\gamma \in \mathrm{PO}\colon u_+(\gamma) \text{ is even and } u_-(\gamma) \text{ is odd}\},\\
\mathrm{OE} &= \{\gamma \in \mathrm{PO}\colon u_+(\gamma) \text{ is odd and } u_-(\gamma) \text{ is even}\},\\
\mathrm{OO} &= \{\gamma \in \mathrm{PO}\colon u_+(\gamma) \text{ and } u_-(\gamma) \text{ are odd}\}.
\end{aligned}$$

Also, for each odd $r \geq 1$ and $n \geq 0$ let $\mathrm{PO}(2^n r)$ be the set

$$\mathrm{PO}(2^n r) = \{\gamma \in \mathrm{PO}\colon p(\gamma) = 2^n r\},$$

and denote by $\mathrm{EE}_{2^n r}$ (resp. $\mathrm{EO}_{2^n r}$, $\mathrm{OE}_{2^n r}$, $\mathrm{OO}_{2^n r}$) the cardinal of $\mathrm{EE} \cap \mathrm{PO}(2^n r)$ (resp. $\mathrm{EO} \cap \mathrm{PO}(2^n r)$, $\mathrm{OE} \cap \mathrm{PO}(2^n r)$, $\mathrm{OO} \cap \mathrm{PO}(2^n r)$).

Denote by $\mu\colon \mathbb{N} \to \{-1, 0, 1\}$ the *Möbius function* defined by $\mu(1) = 1$ and by the following rule: if $n = p_1^{k_1} \cdots p_j^{k_j}$ is the prime decomposition of n, $\mu(n) = 0$ if $k_i > 1$ for some $i \in \{1, \dots, j\}$, and $\mu(n) = (-1)^j$ otherwise. Denote by $\{\alpha_n\}_{n \in \mathrm{N}}$ the sequence of nonnegative integers defined by

$$\alpha_n = \frac{1}{n} \sum_{k|n} \mu\left(\frac{n}{k}\right), \tag{3.1}$$

and, for each positive odd r and each nonnegative m, let

$$\beta_{2^m r} = \sum_{k=0}^{m} \alpha_{2^k r}. \tag{3.2}$$

Given a continuous self-map of a compact manifold M of dimension n, its *Lefschetz number* is defined as

$$L(f) = \sum_{k=0}^{n} (-1)^k \operatorname{tr}(f_{*k}),$$

where $f_{*k}\colon H_k(M; \mathbb{Q}) \to H_k(M; \mathbb{Q})$ is the endomorphism induced by f on the kth rational homology group of M. The Lefschetz fixed point theorem says that if $L(f) \neq 0$ then f has a fixed point. For the purpose of studying the set $\mathrm{Per}(f)$, it is useful to consider the *Lebschetz zeta function*

$$Z_f(t) = \exp\left(\sum_{m=1}^{\infty} \frac{L(f^m)}{m} t^m\right),$$

which is a generating function for the Lefschetz numbers of all iterates of f and can be computed from the homological endomorphisms f_{*k}

of f as follows:

$$Z_f(t) = \prod_{k=0}^{n} \det(I_{j_k} - t f_{*k})^{(-1)^{k+1}}, \tag{3.3}$$

where $j_k = \dim_{\mathbb{Q}} H_k(M; \mathbb{Q})$, see [F2].

If $f: M \to M$ is transversal, then the Lefschetz numbers of the iterates of f are related in a simple way to the periodic points of f, see [F1] for more details.

With this notation, we have the following theorem, which will be used in subsequent proofs.

THEOREM 3.1 ([CLN]). *Let $f: M \to M$ be a transversal map such that its Lefschetz zeta function is of the form*

$$Z_f(t) = \prod_{p\in P} \prod_{i=1}^{N_p} (1 - \sigma_1(i, p)t^p)^{\sigma_2(i,p)}, \tag{3.4}$$

where P is a finite subset of $\mathbb{N}$, the N_p are natural numbers and $\sigma_1(i, p)$, $\sigma_2(i, p) \in \{-1, 1\}$. Then, for each odd integer $r \geq 1$ and $m = 0, 1, 2, \ldots$ we have

$$\sum_{k=0}^{m} \mathrm{EE}_{2^k r} + \mathrm{EO}_{2^m r} + \gamma(r, m) = \mathrm{OO}_{2^m r} + \sum_{k=0}^{m} \mathrm{OE}_{2^k r}, \tag{3.5}$$

where

(3.6)

$$\gamma(r, m) = \sum_{\substack{q|r \\ 2^m q\in P}} \sum_{i=1}^{N_{2^m q}} \sigma_2(i, 2^m q)\sigma_1(i, 2^m q)\beta_{r/q}$$
$$+ \sum_{j=0}^{m-1} \sum_{\substack{q|r \\ 2^j q\in P}} \sum_{i=1}^{N_{2^j q}} \sigma_2(i, 2^j q) \left(\beta_{2^{m-j} r/q} + \frac{\sigma_2(i, 2^j q) - 1}{2} \beta_{r/q}\right),$$

and we take the second summand equal zero when $m = 0$.

Consider a product of the form

$$\prod_{p\in P} \prod_{i=1}^{N_p} (1 - \sigma_1(i, p)t^p)^{\sigma_2(i,p)}, \tag{3.7}$$

where P is a finite subset of $\mathbb{N}$, N_p is a natural number for each $p \in P$, and $\sigma_1(i, p), \sigma_2(i, p) \in \{-1, 1\}$. A factor in (3.7) of the form $(1-\sigma_1(i, p)t^p)^{\sigma_2(i,p)}$ will be called *irreducible* if it remains after performing the following reductions:

$$(1+t^p)(1+t^p)^{-1} = 1, \tag{R1}$$
$$(1-t^p)(1-t^p)^{-1} = 1, \tag{R2}$$
$$(1+t^p)(1-t^p) = 1-t^{2p}, \tag{R3}$$
$$(1+t^p)^{-1}(1-t^p)^{-1} = (1-t^{2p})^{-1}, \tag{R4}$$
$$(1+t^p)^{-1}(1-t^{2p}) = 1-t^p, \tag{R5}$$
$$(1+t^p)(1-t^{2p})^{-1} = (1-t^p)^{-1}, \tag{R6}$$
$$(1+t^p)(1+t^{2p}) = (1-t^p)^{-1}(1-t^{4p}), \tag{R7}$$
$$(1+t^p)^{-1}(1+t^{2p})^{-1} = (1-t^p)(1-t^{4p})^{-1}. \tag{R8}$$

We remark that the reductions (R1) to (R8) should be performed in correlative order and consequently the whole process of reduction is finite.

THEOREM 3.2. *Let $f: M \to M$ be a transversal map of a compact manifold. Suppose that all the zeros and poles of its Lefschetz zeta function $Z_f(t)$ are roots of unity, and that $Z_f(t)$ has an irreducible factor of the form $(1 \pm t^n)^{\pm 1}$.*

(a) *If n is odd then $n \in \operatorname{Per}(f)$.*

(b) *If n is even then $\{\frac{n}{2}, n\} \cap \operatorname{Per}(f) \neq \varnothing$.*

Proof. From (3.3) and Proposition 2.1, the Lefschetz zeta function of f is of the form

$$Z_f(t) = \frac{\prod_{i=1}^{N} c_{n_i}(t)}{\prod_{i=1}^{Q} c_{q_i}(t)}.$$

Moreover, applying Proposition 2.3 to each $c_{n_i}(t)$, $c_{q_i}(t)$, f is in the hypothesis of Theorem 3.1 and so, given r odd, equation (3.5) holds for $m = 0$. Now, if we have an irreducible factor of the form $(1-\sigma_1 t^r)^{\sigma_2}$, then all the other possible irreducible factors associated to the same power of t are $(1-\sigma_1 t^r)^{\sigma_2}$ or $(1+\sigma_1 t^r)^{-\sigma_2}$ (see reduction rules (R1) to (R4)). Hence,

$$\left|\sum_{i=1}^{N_r} \sigma_2(i, r)\sigma_1(i, r)\right| \neq 0$$

because all the terms in the above sum are equal. So, $r \in \operatorname{Per}(f)$ and statement (a) is proved.

Now we shall prove statement (b). From the definition (3.6) of $\gamma(r, m)$ for $m = 1, 2, \dots$ we have

$$\begin{aligned}\gamma(r, m) - \gamma(r, m-1) = &\sum_{i=1}^{N_{2^m r}} \sigma_2(i, 2^m r)\sigma_1(i, 2^m r) \\ &+ \sum_{i=1}^{N_{2^{m-1} r}} \sigma_2(i, 2^{m-1} r)\left(\frac{1-\sigma_1(i, 2^{m-1} r)}{2}\right),\end{aligned}$$

and subtracting (3.5) for m and $m-1$ we obtain

$$\begin{aligned}\text{(3.8)}\qquad &\mathrm{EE}_{2^m r} + \mathrm{EO}_{2^m r} - \mathrm{EO}_{2^{m-1} r} + \gamma(r, m) - \gamma(r, m-1) \\ &= \mathrm{OE}_{2^m r} + \mathrm{OO}_{2^m r} - \mathrm{OO}_{2^{m-1} r}.\end{aligned}$$

Suppose $n = 2^m r$, $m \geq 1$, r odd. Consider an irreducible factor of the form $1 - t^{2^m r}$. The proof for the other possible irreducible factors $(1 - t^{2^m r})^{-1}$, $1 + t^{2^m r}$ and $(1 + t^{2^m r})^{-1}$ is analogous. For this factor we have $\sigma_1(\cdot, 2^m r)\sigma_2(\cdot, 2^m r) = 1$ and so all the factors of the form

$$(1 - \sigma_1(i, 2^m r)t^{2^m r})^{\sigma_2(i, 2^m r)}$$

that persist after reductions of the type (R1)–(R4) verify $\sigma_1(i, 2^m r) \cdot \sigma_2(i, 2^m r) = 1$. Moreover, the factors of the form

$$(1 - \sigma_1(i, 2^{m-1} r)t^{2^{m-1} r})^{\sigma_2(i, 2^{m-1} r)}$$

that can coexist with $1 - t^{2^m r}$ satisfy

$$\sigma_2(i, 2^{m-1} r)\left(\frac{1-\sigma_1(i, 2^{m-1} r)}{2}\right) \in \{0, 1\}$$

due to the reduction (R5). Then $|\gamma(r, m) - \gamma(r, m-1)| > 0$ and equation (3.8) implies that if $2^{m-1} r \notin \operatorname{Per}(f)$ then $2^m r \in \operatorname{Per}(f)$.

We remark that the proof for the irreducible factors $(1 - t^{2^m r})^{-1}$, $1 + t^{2^m r}$ and $(1 + t^{2^m r})^{-1}$ uses the reductions (R6), (R7) and (R8), respectively, and of course also (R1) to (R4).

Let $P(t)/Q(t)$ be a rational function. We define the *order* of $P(t)/Q(t)$, denoted by $\operatorname{order}(P(t)/Q(t))$, as the number

$$\left[\frac{\operatorname{degree} P(t) + \operatorname{degree} Q(t)}{2}\right] + 1.$$

COROLLARY 3.3. *Let $f: M \to M$ be a transversal map of a compact manifold. Suppose that all the zeros and poles of $Z_f(t)$ are roots of unity. Then*

$$\operatorname{Card}(\operatorname{FSP}(f)) \leq \operatorname{order}(Z_f(t)).$$

Proof. Since the Lefschetz zeta function of f, $Z_f(t)$, is rational, we may write $Z_f(t) = P(t)/Q(t)$ with $P(t)$ and $Q(t)$ polynomials.

By Proposition 2.1, $P(t)$ and $Q(t)$ factorize as the product of cyclotomic polynomials. We shall split $P(t)$ (resp. $Q(t)$) as a product $P(t) = P_1(t)P_2(t)$ (resp. $Q(t) = Q_1(t)Q_2(t)$), where $P_1(t)$ (resp. $Q_1(t)$) factorizes as a product of cyclotomic polynomials of degree strictly greater than 2 and $c_4(t)$, and $P_2(t)$ (resp. $Q_2(t)$) contains the remaining factors, i.e. all the $c_i(t)$ for $i \in \{1, 2, 3, 6\}$. Since the reduction process given by rules (R1)–(R8) does not increase the number of factors of the form $(1 \pm t^n)^{\pm 1}$, from Proposition 2.3 and Theorem 3.2 it follows that the cardinal of the forced set of periods associated to the factors $P_1(t)$ and $Q_1(t)$ is smaller than or equal to (degree $P_1(t)$ + degree $Q_1(t))/2$. Notice that this upper bound is an integer number.

To conclude the proof, it is enough to show that the contribution of the remaining factors ($P_2(t)$ and $Q_2(t)$) to the cardinal of the forced set of periods is smaller than or equal to

$$C = \left[\frac{\operatorname{degree} P_2(t) + \operatorname{degree} Q_2(t)}{2}\right] + 1.$$

Notice that $P_2(t)/Q_2(t)$ is a rational function of the form $c_1(t)^{a_1} \cdot c_2(t)^{a_2} c_3(t)^{a_3} c_6(t)^{a_6}$, where the $a_i \in \mathbb{Z}$. The forced set of periods F associated to a product of this form is contained in $\{1, 3\}$. To prove that $\operatorname{Card}(F) \leq C$ it is enough to consider the case $\operatorname{Card}(F) = 2$ because $C \geq 1$. But if $3 \in F$ then $|a_3| + |a_6| \neq 0$, and hence $C \geq 2$ (see Table 1). □

Let $n \in \mathbb{N}$. We denote by $S(n)$ the set $\{1, 2, \dots, n\}$.

COROLLARY 3.4. *Assume that we are in the hypotheses of Corollary 3.3. Let d be the maximum of the degrees of $P(t)$ and $Q(t)$, where $P(t)/Q(t) = Z_f(t)$. Then*

(a)
$$\operatorname{FSP}(f) \cap \{n \in \mathbb{N} : n \textit{ is odd}\} \subset \begin{cases} S(n_0(d)) & \textit{if } n_0(d) \textit{ is odd}, \\ S(n_0(d) - 1) & \textit{if } n_0(d) \textit{ is even}. \end{cases}$$

(b)
$$\mathrm{FSP}(f) \cap \{n \in \mathbb{N} \colon n \textit{ is even}\} \subset \begin{cases} S(2n_0(d)) & \textit{if } n_0(d) \textit{ is odd}, \\ S(2n_0(d)-2) & \textit{if } n_0(d) \textit{ is even}. \end{cases}$$

Proof. By Proposition 2.1, $P(t)$ and $Q(t)$ factorize as products of cyclotomic polynomials. From Proposition 2.4 it follows that if the cyclotomic polynomial $c_n(t)$ appears in the factorization of $P(t)$ or $Q(t)$, then $n \leq n_0(d)$. Now, the maximum power of t in the decomposition (2.13) of $c_n(t)$ is smaller than or equal to $n/2$ (resp. n) if n is even (resp. odd). Consequently, before applying the reduction procedure given by rules (R1) to (R8), the maximum power of t which appears in the factors $(1 \pm t^j)^{\pm 1}$ of $P(t)/Q(t)$ is $n_0(d)$ (resp. $n_0(d) - 1$) if n is odd (resp. even). Hence, taking into account the reduction rules, the corollary follows. □

4. Transversal surface maps. Throughout this section, M will be a compact connected surface of genus g and $f \colon M \to M$ a transversal map.

Recall that $H_0(M; \mathbb{Q}) \approx \mathbb{Q}$ and that $H_1(M; \mathbb{Q}) \approx \mathbb{Q}^{2g}$, $H_2(M; \mathbb{Q}) \approx \mathbb{Q}$ if M is orientable, and $H_1(M; \mathbb{Q}) \approx \mathbb{Q}^{g-1}$, $H_1(M; \mathbb{Q}) \approx \{0\}$ if M is non-orientable.

PROPOSITION 4.1. *If $h(f) = 0$ then all the eigenvalues of f_{*0}, f_{*1} and f_{*2} are either 0 or roots of unity.*

Proof. Since $f_{*0} = \mathrm{id}$, 1 is the only eigenvalue of f_{*0}.

Let us consider now f_{*2}. If M is non-orientable, then 0 is the only eigenvalue of f_{*2}. If M is orientable, $f_{*2}(1)$ is the degree D of f. From **[MP]** we know that if $|D| > 1$ then $h(f) \geq \log|D|$. Hence, if $h(f) = 0$, $|D| \leq 1$ and so the only possible eigenvalues for f_{*2} are -1, 0 and 1.

Finally, consider f_{*1}. By Theorem 2 of **[Mn]**, if $h(f) = 0$ then all the eigenvalues λ of f_{*1} satisfy $|\lambda| \leq 1$. We claim that every nonzero eigenvalue λ of f_{*1} has modulus 1. Let $\lambda_1, \dots, \lambda_k$ be the nonzero eigenvalues of f_{*1}. Then

$$\det(I - tf_{*1}) = \det(-t(f_{*1} - t^{-1}I)) = (-1)^k \prod_{i=1}^{k} (\lambda_i t - 1). \tag{4.1}$$

Moreover (4.1) must be a polynomial with integer coefficients, because f_{*1} is an integral matrix. Hence, in particular, $\prod_{i=1}^{k} \lambda_i$ must belong to $\mathbb{Z}$. Therefore, $\prod_{i=1}^{k}|\lambda_i| \geq 1$, and the claim follows because $|\lambda_i| \leq 1$.

So (4.1) is a polynomial with integer coefficients, constant term 1 and all its roots have modulus 1. By a standard result in algebra (see Lemma 1.6 of [W]) the proposition follows. □

COROLLARY 4.2. *If* $h(f) = 0$ *or* $\mathrm{Per}(f)$ *is finite, then* $Z_f(t)$ *is of the form*

$$Z_f(t) = \frac{\prod_{i=1}^{N} c_{n_i}(t)}{(1-t)\theta(t)},$$

where $\theta(t)$ *is either* 1, $1-t$ *or* $1+t$ *and* $c_{n_i}(t)$ *is the* n_i*-cyclotomic polynomial.*

Proof. The case $h(f) = 0$ follows directly from the definition of the Lefschetz zeta function, Proposition 4.1 and Proposition 2.1. If $\mathrm{Per}(f)$ is finite, the corollary follows from Theorem 6 of [Fr]. □

The following results improve slightly the statements of Corollaries 3.3 and 3.4 for transversal surface maps.

COROLLARY 4.3. *If* $h(f) = 0$ *or* $\mathrm{Per}(f)$ *is finite, then*

$$\mathrm{Card}(\mathrm{FSP}(f)) \leq \begin{cases} g+1 & \text{if } M \text{ is orientable}, \\ \left[\dfrac{g-1}{2}\right]+1 & \text{if } M \text{ is non-orientable.} \end{cases}$$

Proof. From Corollary 4.2, repeating the arguments of the proof of Corollary 3.3 and taking into account that the contribution of $Q(t) = Q_2(t)$ to $\mathrm{FSP}(f)$ is at most period 1, the result follows. □

The following result is just a restatement of Corollary 3.4.

COROLLARY 4.4. *If* $h(f) = 0$ *or* $\mathrm{Per}(f)$ *is finite, then*

(a)
$$\mathrm{FSP}(f) \cap \{n \in \mathbb{N} : n \text{ is odd}\} \subset \begin{cases} S(n_0(d)) & \text{if } n_0(d) \text{ is odd}, \\ S(n_0(d)-1) & \text{if } n_0(d) \text{ is even}. \end{cases}$$

(b)
$$\mathrm{FSP}(f) \cap \{n \in \mathbb{N} : n \text{ is even}\} \subset \begin{cases} S(2n_0(d)) & \text{if } n_0(d) \text{ is odd}, \\ S(2n_0(d)-2) & \text{if } n_0(d) \text{ is even}, \end{cases}$$

where $d = 2g$ *if* M *is orientable and* $d = g-1$ *if* M *is non-orientable.*

5. Transversal N-torus-maps. In this section we shall derive consequences of the results obtained in Section 3 for transversal self-maps

f on n-dimensional manifolds such that for $k = 2, \dots, n$ all the eigenvalues of f_{*k} can be obtained as products of the eigenvalues of f_{*1}. We remark that these conditions hold for continuous maps of the n-torus T^n and also for continuous maps of many Eilenberg-Mac Lane spaces.

The following corollary is easily obtained from Theorem 3.2 and Corollaries 3.3 and 3.4 repeating the arguments of the proof of Proposition 4.1.

COROLLARY 5.1. *Let $f\colon M \to M$ be a transversal map on a compact manifold M of dimension m with $h(f) = 0$. Suppose that for $k = 2, \dots, m$ all the eigenvalues of f_{*k} can be obtained as products of the eigenvalues of f_{*1}.*

(a) *If $Z_f(t)$ has an irreducible factor of the form $(1 \pm t^n)^{\pm 1}$ with n odd, then $n \in \mathrm{Per}(f)$.*

(b) *If $Z_f(t)$ has an irreducible factor of the form $(1 \pm t^n)^{\pm 1}$ with n even, then $\{n/2, n\} \cap \mathrm{Per}(f) \neq \varnothing$.*

(c) $\mathrm{Card}(\mathrm{FSP}(f)) \leq \mathrm{order}(Z_f(t))$.

(d)

$$\mathrm{FSP}(f) \cap \{n \in \mathbb{N} : n \textit{ is odd}\} \subset \begin{cases} S(n_0(d)) & \textit{if } n_0(d) \textit{ is odd}, \\ S(n_0(d) - 1) & \textit{if } n_0(d) \textit{ is even}. \end{cases}$$

(e)

$$\mathrm{FSP}(f) \cap \{n \in \mathbb{N} : n \textit{ is even}\} \subset \begin{cases} S(2n_0(d)) & \textit{if } n_0(d) \textit{ is odd}, \\ S(2n_0(d) - 2) & \textit{if } n_0(d) \textit{ is even}. \end{cases}$$

Similarly, using Theorem 3.2, Corollaries 3.3 and 3.4 and Theorem 4.1 of [**ABLSS**] we have the following result.

COROLLARY 5.2. *Let $f\colon T^m \to T^m$ be a transversal map and suppose that* Per(f) *is finite. Then,* (a) *to* (e) *of Corollary* 5.1 *hold.*

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