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ON A CONSTRUCTION OF PSEUDO-ANOSOV DIFFEOMORPHISMS BY SEQUENCES OF TRAIN TRACKS

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Pseudo-Anosov diffeomorphisms are representable by sequences of train tracks with some property. We introduce the necessary and sufficient condition for a sequence of train tracks under which the sequence represents a pseudo-Anosov diffeomorphism.

1. Introduction. Thurston compactified the Teichmüller space of a surface M by adding the measured foliation space $\mathcal{MF}$ as its boundary. Diffeomorphisms of M are classified by their natural actions on the compactified Teichmüller space. A diffeomorphism is called pseudo-Anosov when it acts so that an arational element of $\mathcal{MF}$, which has no connections of singularities, is invariant up to multiplication by a scalar not equal to 1 ([**1**]). Such a folation is called a pseudo-Anosov foliation.

A train track is a 1-dimensional CW complex drawn on M with a certain property (for example, see Fig. 8.4c). A train track is regarded as a coordinate system of $\mathcal{MF}$ by giving thickness to its edges (see Fig. 1.9). The coordinate transformations are piecewise linear. The set of all train track gives thus a PL structure on $\mathcal{MF}$ ([**3**]).

The operations split, shift and collapse give rise to a new train track from a train track (Fig. 1.6a). The coordinate neighborhood defined by one train track contains that of the other ([**6**], [**3**]). A sequence of so related train tracks, or, a sequence of such operations, is called a word. A pseudo-Anosov foliation is characterized as the intersection of the coordinate neighborhoods of the train tracks of iterations of a certain word ([**4**]). On the other hand, an arbitrary pseudo-Anosov diffeomorphism is representable by a train track and a word. In fact, a filling (see Definition 3.1) train track τ and a word from τ to a train track σ represents a mapping class of M when σ is the image of τ under that mapping class (see Proposition 3.3 and its preceding paragraphs). In particular, an arbitrary pseudo-Anosov diffeomorphism has such a representation ([**4**], see Proposition 3.5) and the iteration of the word determines the pseudo-Anosov foliation. Therefore in order to construct all the pseudo-Anosov diffeomorphisms, one has only to study the condition for a word under which the iteration of the word determines a pseudo-Anosov foliation. There are only finitely many combinatorial types of train tracks up to a homeomorphism of the surface, and the words satisfying the condition correspond to the conjugacy classes of pseudo-Anosov diffeomorphisms (see Section 4 and Proposition 3.3). The aim of this paper is to describe this condition in a combinatorial way. We will obtain a finite procedure to determine whether a given word represents a pseudo-Anosov diffeomorphism.

The main results of this paper are Theorems 5.2, 6.10 and 7.17. Roughly speaking, they are like as follows.

First, because only splits play an important role in determining whether a word represents a pseudo-Anosov, we introduce the category of "shiftless type" (Section 3, see Fig. 2.3) to avoid a needless complication.

A word is examined by the "carries" of train tracks which it induces. When a word is given, we look at the first train track τ and the last one σ. Then σ is carried by τ, that is, σ can be drawn inside a regular neighborhood of τ (Fig. 1.11a). There is some ambiguity about the way that σ is carried (Fig. 2.7). Theorem 5.2 pays attention to it. For a certain trainpath of σ (Fig. 1.3), among all the carries induced by the word there is a carry which stretches that trainpath maximally (Fig. 2.8). Theorem 5.2 says that if and only if there is a trainpath of σ which coincides to the corresponding part of τ when we take a carry which stretches the trainpath maximally, the word does not represent a pseudo-Anosov (of some type, see Definition 4.2).

Theorem 6.10 examines the matrix (Fig. 1.12) determined by a carry (any carry which comes from the given word will do). Its eigenvector determines values given to the capillaries (branches in the shiftless category, see Fig. 2.2) and the prongs (Fig. 1.1) of the train track (see Definition 6.1 and 6.2). Theorem 6.10 says that if and only if these values make some figure like as Fig. 6.6, the word does not represent a pseudo-Anosov (of some type).

In Theorem 7.17, we consider the cone of polyhedron (W_τ) determined by train track τ. W_τ is the quotient space of the positive part of the real vector space spanned by the prongs and the capillaries of τ, taken quotient by the subspace spanned by the vectors coming from the vertices (Fig. 2.2) of τ (see Definition 7.3). A carry (any carry will do) determines an action $\overline{A}$ (Definition 7.2) to W_τ. Theorem 7.17 says that if and only if $\overline{A}$ maps certain generating lines (which corresponds to the capillaries) of W_τ to the interior of W_τ, the word represents a pseudo-Anosov (of some type).

In Theorems 5.2 and 6.10, we have to examine, in addition, the positivity condition (PC) (stated before Definition 4.1) concerning the eigenvector of the matrix determined by the word, but in Theorem 7.17, we don't have to. The proofs for Theorem 5.2 and 6.10 are achieved independently, but they are intrinsically the same. The proof for Theorem 7.17 uses Theorem 5.2 and Proposition 6.7 (6.7 is used in the proof of Theorem 6.10). These three theorems are intrinsically the same, but Theorem 7.17 is a "sophisticated" version.

2. Preliminaries. Let M be a closed orientable surface of genus ≥ 2. We refer the readers to [**1**] for the details concerning measured foliations on M and also to [**6**] and [**3**] for those concerning train tracks.

Here are short explanations of terminologies used in the theory of train tracks, with which the new terminologies used in this paper are introduced (they are marked with †in Section 2).

A *train track* τ on M (abbreviated *"track"*) is a one-dimensional finite CW-complex embedded in M which is C^1 in the sense of Fig. 1.1 and no components of $M - \tau$ are 0,1,2-gons or annuli (in the sense of C^1, see Fig. 1.2). A 1-cell is called a *branch*, a 0-cell, a *switch*. The angle between two branch ends is called a *prong*† (Fig. 1.1).

A *trainpath* (abbreviated *"path"*) of a track τ is a C^1-immersion from a closed interval to τ whose initial and terminal points both lie on switches. Two trainpaths are identified by an orientation-preserving transformation of the intervals. When both of its endpoints lie on the same switch and the connection is C^1, a trainpath is called *closed.* A trainpath c *starts from*† a prong p when the initial point of c lies on the switch associated with p and c lies on the opposite side of p near the initial point (each switch has two sides naturally). Similarly, c *ends into*† a prong q when the inverse c^{-1} of c starts from q (Fig. 1.3).

A track τ is *recurrent* if, for each branch b of τ, there exists a closed trainpath which passes through b.

An *orientation* of τ is a set of orientations on the switches and branches of τ such that the orientations of the branches coincide with those of the switches at their endpoints. τ is orientable if and only if there is no trainpath starting from a prong and ending into the same prong.

A track is *generic* if each switch has only three branch ends. (Such a switch is called *trivalent.*) See Fig. 1.4.

CONVENTION 1. In the following, we consider only tracks each of whose switches does not have two or more branch ends on *both* sides of it (Fig. 1.5). □

The operations on a generic track called a *"split"* and a *"shift"* are indicated in Fig. 1.6a. In a split, two prongs "pass each other". There are two cases, a *"left split"* and a *"right split"*. In a shift, a prong "overtakes" another one.

In this paper we would like to use the terminologies "split" and "shift" in a slightly generalized sense. A *"comb"* is an operation deforming an r-valent switch to an $(r-1)$-valent switch and a trivalent switch as in Fig. 1.6b.

A *"shift"* on a non-generic track is a composition of a shift, combs and their inverses. It is not important in the following discussion whether the composition is counted as one shift or each of the operations is counted as one shift one by one. (Then a shift is as in Fig. 1.6c.)

A *"split"* on a non-generic track is a split composed with combs and their inverses. A split is performed between two prongs. They are pushed ahead by combs, split between each other, then regulated by inverse combs as indicated in Fig 1.6d. (The last regulation might be omitted, but this is not important as before.) Reference to "one" split make sense.

A sequence of (generalized) splits and shifts is called a *word*,or, referring to the initial track τ, a *word from* τ. Each operation is thought to be a *"letter"*. We say *"tracks appear from a word"*† when they are obtained by applying the letters in order they appear in the word. When the word is finite, the last track τ' is *"obtained by applying the word to the initial track* τ*"*. Then the word is denoted *"word:* $\tau \longrightarrow \tau'$*"*.

CONVENTION 2. In the following, we consider only recurrent tracks and words which result in recurrent tracks. □

A *fibered neighborhood* $N(\tau) \subset M$ of a track τ ia a neighborhood of τ with a retraction $N(\tau) \downarrow \tau$. The fibers of this retraction form a foliation $\mathcal{T}$ on $N(\tau)$. Each leaf of $\mathcal{T}$ is called a *tie*, and the ties which pass through the cusps of $N(\tau)$ are called the *singular ties* (Fig. 1.7).

$\mathcal{MF}$ denotes the space of the equivalence classes of the measured foliations on M. $\mathcal{F} \in \mathcal{MF}$ is *carried by* τ when there exists a partial foliation F representing $\mathcal{F}$ whose support is contained in $N(\tau)$ and is transverse to the tie foliation $\mathcal{T}$. This is denoted $\mathcal{F} \prec \tau$ (Fig. 1.8). The set of elements of $\mathcal{MF}$ carried by τ is denoted V_τ.

We consider non negative wights on the branches of τ. $E(\tau)$ denotes the set of such weights that satisfy the following condition: at each switch, the sum of the weights of the branches on one side equals to that of the other side (the switch condition). $E(\tau)$ is naturally a subset of R^I, where I is the number of branches of τ. $E(\tau)$ is a convex cone whose vertex is O in R^I ([**3**]). There is a natural correspondence from $E(\tau)$ to V_τ as follows. For a point r of $E(\tau)$, we foliate each *branch* of $N(\tau)$† (a subset of $N(\tau)$ retracting to a branch of τ) by a parallel leaves transverse to ties and give a transverse measure equal to the entry of r corresponding to the branch. Because of the switch condition, we can paste the branches of $N(\tau)$ so that their parallel foliations give rise to a partial measured foliation F on M (Fig. 1.9). r corresponds to the measured foliation $\mathcal{F}$ represented by F. This correspondence is, in fact, bijective ([**6**]). In the following, we often consider a foliation $\mathcal{F} \in V_\tau$ to be represented as a partial foliation F as above.

A *positive*† weight is a weight which is positive on each branch. A positive weight corresponds to a foliation represented by a partial foliation whose support is all of $N(\tau)$. Such a foliation is called a *positive foliation of* τ†.

τ is *suited to* $\mathcal{F}$ if $\mathcal{F}$ is a positive foliation of τ and, when represented by a partial foliation whose support is $N(\tau)$, it has no leaf connecting two cusps of $N(\tau)$ (Fig. 1.10).

Let τ and σ be two tracks. σ is *carried by* τ if σ is isotopic on M to $\sigma' \subset N(\tau)$ such that σ' is tie-transverse. This is denoted by $\tau \succ \sigma$ (Fig. 1.11a). In this case, $N(\sigma)$ can be drawn in $N(\tau)$ with fibers induced from those of $N(\tau)$ (Fig. 1.11b). Hence $V_\tau \supset V_\sigma$, which induces the inclusion map $E(\tau) \hookleftarrow E(\sigma)$.

Let σ be carried by τ. We will use the terminology "carry" also as a noun. A *carry of* σ *onto* τ† is a tie transverse image of σ in $N(\tau)$. Two carries σ_1 and σ_2 are *carry isotopic*† (or simply *isotopic*†) if there exists an isotopy of $N(\tau)$ which maps σ_1 to σ_2 among carries of σ onto τ (Fig. 1.12).

Choose a tie in each branch of $N(\tau)$ arbitrarily and call it the *central tie* of the branch. For a carry of σ, perturbing it by a carry isotopy, one can arrange so that no switches of σ intersect the central ties. Thus we have the *intersection matrix* A, whose (i,j) entry is the (geometric) intersection number of the central tie of the i-th branch of τ and the j-th branch of σ. This depends on the carry (Fig. 1.12). The matrix A determines a linear map from R^J to R^I, where I and J denote the numbers of branches of τ and σ respectively, which is an extension of the inclusion map $E(\sigma) \hookrightarrow E(\tau)$ introduced above.

An *unzipping of* $N(\tau)$ *along a path* is the operation as follows. Take a tie-transverse simple path $c \subset N(\tau)$ starting from a cusp of $N(\tau)$ so that near the initial point of c, c traverses ties which lie on the opposite side of the cusp. Furthermore we require that c intersects $\partial N(\tau)$ only at the starting point s. A new track σ is obtained by retracting each component of the ties of $N(\tau) - (\operatorname{int} c \cup s)$. $N(\tau) - (\operatorname{int} c \cup s)$ is regarded as a fibered neighborhood $N(\sigma)$ of σ when completed by adding a double of $(\operatorname{int} c \cup s)$ as its boundary segments. (That is, $N(\sigma)$ can be homotoped to $N(\tau)$ doubled on $(\operatorname{int} c \cup s)$.) Thus we "unzipped" $N(\tau)$ along c getting $N(\sigma)$. Conversely, we can "zip" $N(\sigma)$ (along a pair of paths with the common starting point at a cusp) getting $N(\tau)$ (Fig. 1.13). Clearly $\tau \succ \sigma$. If c meets no singular ties except at the starting point, then $\tau = \sigma$ (mapped each other by an isotopy on M).

A composition of several consecutive unzippings may also be referred to as a single unzipping. It is made of unzippings along paths as above which are disjoint each other. Similarly, a composition of consecutive zippings may be referred to as a single zipping. Then, in general, an unzipping corresponds to a word: $\tau \to \sigma$ (Fig. 1.13).

REMARK. In unzipping a fibered neighborhood of a track, one has to take care in taking c not to obtain a non-recurrent track or a track having a switch each of both sides of which has two or more branch ends (see the Convention 1). □

PROPOSITION 1.1. *The followings are equivalent:*

(i) $\tau \succ \sigma$ *and they have the same number of prongs.*

(ii) $N(\tau)$ *unzips to* $N(\sigma)$.

(iii) *There exists a word:* $\tau \to \sigma$.

Proof. We only show that (i) implies (ii). The others are clear. We represent $N(\sigma)$ in $N(\tau)$ with the ties induced from those of $N(\tau)$ and take two paths used in the zipping as follows. Assume two paths a and b on $\partial N(\sigma)$ make a cusp c of $N(\sigma)$. The initial points of a and b is the cusp c. Near c, each point of a is on the same tie as a point of b. Thus there is a correspondence between points of a near c and points of b near c. This correspondence breaks down when a and b meet a singular tie t associated with a cusp C of $N(\tau)$ (Fig. 1.14). The segments of a and b from c to t are the required paths.

Thus each cusp c of $N(\sigma)$ corresponds to a cups C of $N(\tau)$ (because $M - \sigma$ has no bigons. See Fig. 1.14). Clearly this is injective, hence bijective as there are the same number of cusps (which corespond to the prongs bijectively). Taking the converse of this operation, one has the unzipping from $N(\tau)$ to $N(\sigma)$. □

REMARK. This proof is taken from [**4**] 2.2 (Lemma 2.1 and Proposition 2.2). □

From this proof, the prongs of σ correspond to prongs of τ if $\sigma \prec \tau$. This correspondence is called the *prong correspondence by the carry*†. It is determined uniquely by the isotopy class of the carry.

Given a word: $\tau \to \sigma$, the (isotopy class of the) carry of σ onto τ is determined and hence its prong correspondence is determined. In this situation, this is bijective. In particular, this correspondence is called the *prong correspondence by the word*†.

PROPOSITION 1.2. *If $\tau \succ \sigma \succ \mathcal{F}$ and $\mathcal{F}$ is a positive foliation of σ, and if $N(\sigma)$ is obtained by an unzipping of $N(\tau)$, then this unzipping is realized as an unzipping along a path on a leaf of F which represents $\mathcal{F}$.*

Proof. F is a partial foliation representing $\mathcal{F}$ on $N(\sigma)$ whose support is all of $N(\sigma)$. We zip it to obtain $N(\tau)$ keeping the partial foliations on the fibered neighborhoods on the way. Then the zipper in $N(\tau)$ is the path in question (Fig. 1.15). □

PROPOSITION 1.3. *If $\tau \succ \sigma \succ \mathcal{F}$ and τ is suited to $\mathcal{F}$, there exists a word: $\tau \dashrightarrow \sigma$.*([**4**] Proposition 2.2.)

3. Shiftless caregory. Consider a track τ. For each prong of τ, consider a trainpath which starts from the prong and ends into some prong. Such paths exist because τ is assumed to be recurrent.

DEFINITION 2.1. For a prong of τ,the shortest path which starts from the prong and ends into some prong is called the *canonical path* from the prong ([**6**]). The prongs which the canonical path ends into are called the *opposing prongs* of the prong. The canonical path is unique, so only the prongs at the switch at the terminal point of it are the opposing prongs (Fig. 2.1). □

DEFINITION 2.2. A track is *shiftless type* (or, shortly, *shiftless*) if all the canonical paths are length one (i.e. each canonical path passes through only one branch).

In this case, the branches of τ are classified into two types: the branches which connect two (or more) opposing prongs, and the other branches. The former ones are called the *vertices* and considered to or not to contain the switches on their ends depending on the situation. The others are called the *capillaries*. The prongs and capillaries at the switches at a vertex are called the ones belonging to (or, simply, ones at) the vertex (Fig. 2.2). □

Clearly the next lemma holds.

LEMMA 2.3. *From an arbitrary (recurrent) track τ, we get a shiftless track by a word consisting of only shifts. Furthermore this is uniquely determined.*

It is called the shiftless type of τ (Fig. 2.3). In the following, in figures, the vertices of shiftless tracks are often drawn very short (as if they were switches).

As a matter of convenience, we use the terms "left" and "right" of a vertex of a track to distinguish the two sides of the vertex each of which consists of the capillaries with zero angles and the prongs they make. Once fixed the left-right orientation (which is a local orientation), then the terms "upper" or "lower" make sense (Fig. 2.4a).

DEFINITION 2.4. Instead of the operations splits and shifts introduced in the Section 2, we adopt the next ones.

(i) A (shiftless) *split:* a left (respectively right) split between the left undermost (resp. uppermost) prong and one of the right prongs of a vertex or a left (resp. right) split between the right uppermost (resp. undermost) prong and one of the left prongs of a vertex, each followed by the regulation to the shiftless type (Fig. 2.4b).

(ii) A *slide:* a left split between one of the left non-undermost prong and one of the right non-uppermost prong of a vertex or a right split between one of the left non-uppermost prong and one of the right non-undermost prong of a vertex, each followed by the regulation to the shiftless type (Fig. 2.4c). This seems to be a slide of the "upper half" and the "lower half" of the vertex along a fault in an earthquake.

Words made of these operations are called *shiftless words* if necessary to distinguish them from ordinary words. □

Clearly the next lemma holds.

LEMMA 2.5. *For each ordinary word: $\tau \to \sigma$, there exists the corresponding shiftless word:* (the shiftless type of τ) $\to$ (the shiftless type of σ).

We further have:

PROPOSITION 2.6. *Every shiftless word is represented as a composition of the operations of the following types:*

(i) *The left or the right split between the left prong and the right prong of a vertex with only two prongs (one on each side).*

(ii) *The left (resp. right) split between the left (resp. right) uppermost and non-undermost prong and the right (resp. left) uppermost prong of a vertex or the right (resp. left) split between the left (resp. right) undermost and non-uppermost prong and the right (resp. left) undermost prong of a vertex.*

(iii) *the slide which comes from the left (resp. right) split between the left (resp. right) uppermost (and hence non-undermost from the definition of slides) prong and the right (resp. left) undermost (and non-uppermost) prong of a vertex.*

See Fig. 2.5.

Proof. An arbitrary slide is decomposable as follows. Assume the slide is left between the left m-th prong from the top and the right n-th prong from the bottom of a vertex. First we perform the left slide of the type (iii). Then perform the left split of the type (ii) between the right bottom and the left second highest (which is now the uppermost at the new vertex). We repeat such splits $(m-1)$ times (between the right bottom and the left i-th from the top where $1 < i \leq m$), followed by the symmetrical $(n-1)$ left splits (between the left m-th from the top and the right j-th from the bottom where $1 < j \leq n$). Thus we have decomposed the given slide. It is similar when the given slide is right.

For a given split, we decompose it as follows. Assume the split is left between the right uppermost prong and the left m-th prong p from the top of a vertex. We make the left split between the left uppermost and the right uppermost (denoted q), which is of the type (ii). Then successively we make the left split between q and the left second highest (which is now the uppermost). In this way the successive m splits of the type (ii) make the desired split. This argument is not applicable to only two cases. The one is that the right uppermost capillary has its other end on the left side of the same vertex and it is higher than p, in which case we might repeat the splits above infinitely and cannot reach p. But when this occurs, the left split between p and q results in a non-recurrent track, which is not allowed by Convention 2 (Fig. 2.6a).The other case is that p is the left undermost, in which case we might violate the stipulate "non-undermost" in (ii). In this case, we make successive $(m-1)$

splits, then turning our eye to the right undermost, make successive $(r-1)$ splits where r denotes the number of the right prongs. Now the vertex is of the type in (i) and we can make the left split of the type (i) between p and q. Again this is not applicable to the case that the left side of the vertex has many prongs after the last $(r-1)$ splits. But, by the repeating application of the argument above, we can reach the decomposition of the given split except when the right uppermost capillary comes back to the right second highest and the left undermost capillary comes back to the left second lowest, in which case again we might repeat infinitely many splits and cannot reach p, q or the type (i). But in this case, the given split again results in a non-recurrent track (Fig. 2.6b).

Thus we always represent a split or a slide as a composition of the operations of the types (i), (ii) or (iii). □

Suppose we are in shiftless category.

DEFINITION 2.7. *Capillaries and vertices of* $N(\tau)$ are portions corresponding to ones of τ. □

When $\tau \succ \sigma$, we consider only carries of σ each vertex of σ of which is in one vertex or capillary of $N(\tau)$. (All the carries can be isotoped to this type.)

DEFINITION 2.8. Two carries are *equivalent* if they can be isotoped to each other preserving their vertices inside the vertices or capillaries of $N(\tau)$ they initially locate. The equivalence class of a carry is again called a *carry.* A carry isotopy induces an isotopy of these classes, again called a (*carry*) isotopy. □

DEFINITION 2.9. A *left* (*or right*) *carry isotopy of a vertex* of σ is an isotopy in which the vertex is moved along a trainpath which starts at the vertex and runs left (or right). (It is not unique.) (Fig. 2.7.) □

This means that we move the prongs on the left (or right) side of the vertex along the paths in $N(\tau)$ which make correspondence of the prongs and the cusps of $N(\tau)$. When one of the prongs moves all over the path, the cusp "sticks" in the prong and the left (or right) isotopy is over. If the same occurs on some vertex on a trainpath starting from the opposing prongs, the isotopy is obstructed by that prong-and-cusp sticking similarly (Fig. 2.7).

We will consider an end of a trainpath as a switch or a vertex depending on the situation.

DEFINITION 2.10. A *maximally isotoped carry* of a trainpath c of σ is a carry of σ such that the left end of c (which is on a vertex of σ) is left isotoped maximally and the right end of σ is right isotoped maximally. □

A maximally isotoped carry is not unique, but in it c is unique as a point set (Fig. 2.8). If both the ends cannot be isotoped at the same time (for example, when c is not simple), there is no maximally isotoped carry.

DEFINITION 2.11. For each carry of σ onto τ which lets each vertex of σ inside a vertex of $N(\tau)$, the *intersection matrix* A is defined as follows.

The (i,j)-entry of A is the number of the components of the intersection of the j-th capillary of σ and the i-th capillary of $N(\tau)$ in the carry. □

This definition of A is a special case of the definition given in Section 2 and the next follows.

LEMMA 2.12. *Suppose $\tau \succ \sigma \succ \xi$. Denote the intersection matrix of $\tau \succ \sigma$ (respectively $\sigma \succ \xi$ and $\tau \succ \xi$) A (respectively B and C). Then $C = AB$. (A repeated carry is represented by the product of the matrices.) (Clear.)*

4. Representing diffeomorphisms by words.

We do not restrict ourselves in the shiftless category in this section.

DEFINITION 3.1. If $M - \tau$ is simply connected, that is, each component of $M - \tau$ is n-gon ($n > 2$), we say τ is *filling.* □

DEFINITION 3.2. (i) Two tracks τ_1, τ_2 are *equal* if they are isotopic on M. We denote it by $\tau_1 = \tau_2$.

(ii)We say that two tracks τ_1, τ_2 are *"isomorphic"* if there exists a diffeomorphism $f : M \to M$ such that $f(\tau_1) = \tau_2$. It is denoted $\tau_1 \approx \tau_2$. If we want to specify the diffeomorphism f, we denote $\tau_1 \approx_f \tau_2$. Then there is a correspondence of the branches, switches and prongs of τ_1 and τ_2. We call this the *correspondence by f (or by $\tau_1 \approx_f \tau_2$).* f is called the *representing diffeomorphism.*

An equivalence class of isomorphic tracks is called a *combinatorial track.* The class of τ is also called the *combinatorial form* of τ. τ *realizes* the combinatorial track. □

Whether τ is filling is determined by $M-\tau$, so it is a combinatorial property. If $\tau_1 \approx_f \tau_2$ is filling, a diffeomorphism $g : M \to M$ such that $g(\tau_1) = \tau_2$ and g has the same prong correspondence as the representing diffeomorphism f is determined uniquely up to isotopy of M, because if the prongs are determined, such diffeomorphism is determined on the edges of the n-gons of the complementary region, hence is determined on the inside of the n-gons.

Given a word from τ, the obtained track τ' is uniquely determined. This also holds in the combinatorial sense, i.e. the combinatorial form of τ' is uniquely determined by the combinatorial form of τ.

Hence the next holds.

PROPOSITION 3.3. *Let τ be a filling combinatorial track. A word from τ such that the obtained combinatorial track τ' is the same combinatorial track as τ determines, when their prong correspondence is specified, a unique conjugacy class of diffeomorphisms: $M \to M$.*

Proof. Taking one of the realization of τ and applying the word to it, we obtain a new realization and a representing diffeomorphism. This diffeomorphism is unique up to conjugacy. If we change the initial realization, then the representing diffeomorphism change by conjugacy. □

PROPOSITION 3.4. *If $\tau_1 \approx_f \tau_2$ and $\tau_1 \succ \tau_2$, there exists $\mathcal{F} \in \mathcal{MF}$ such that $\tau_1 \succ \tau_2 \succ \mathcal{F}$ and $f(\mathcal{F}) = \lambda\mathcal{F}$ $(\lambda > 0)$.*

Such $\mathcal{F}$ is called an invariant foliation of f (and $\tau_1(\approx \tau_2)$). ($\mathcal{F}$ is not neccessarily unique.)

Proof. We identify $E(\tau_1)$ with $E(\tau_2)$ by the branch correspondence by f. From this arises the identification of $\mathcal{F} \in V_{\tau_1}$ and $f(\mathcal{F}) \in V_{\tau_2}$ in $\mathcal{MF}$. Composing $f : E(\tau_1) \to E(\tau_2)$ with the *inclusion*: $E(\tau_2) \hookrightarrow E(\tau_1)$, we get a linear map from $E(\tau_1)$ to $E(\tau_1)$. Considering in the projective space, it induces a continuous map from a compact convex set to itself, and there exists a fixed point by Brouwer's fixed point theorem. This fixed point represents (the projective class of) an invariant foliation $\mathcal{F}$. □

REMARK. This proof is taken from [**4**].

PROPOSITION 3.5. *If a diffeomorphism $f : M \to M$ is pseudo-Anosov, then there exist a filling generic track τ and a word: $\tau \to f(\tau)$ such that the word contains at least one split letter (letters are splits or shifts because we consider generic tracks) and τ is suited to the unstable foliation $\mathcal{F}$ of f.* ([**4**] Theorem 4.1.)

Conversely, the next holds.

PROPOSITION 3.6. *Assume that filling generic tracks $\tau_1 \approx_f \tau_2$ and a word: $\tau_1 \to \tau_2$ containing at least one split are given. If $\tau_1 \approx \tau_2$ is suited to one of the invariant foliations of ($\tau_1 \approx \tau_2$ and) f, denoted by $\mathcal{F}$, then f is pseudo-Anosov with the unstable foliation $\mathcal{F}$.* (From [**4**].)

From above, in order to construct all the conjugacy classes of pseudo-Anosov diffeomorphisms, we only have to gather following triples: a filling generic combinatorial track, a word from it returning to itself and a prong correspondence by the representing diffeomorphism between the initial track and the obtained track (in comparison with the correspondence by the word).

They must also satisfy following two conditions:

1. The word contains a split.
2. (Let τ be one of the realizations of the filling generic track and τ' be obtained by applying the word to τ. Because τ' is isomorphic to τ and they are filling, the prong correspondence determines a diffeomorphism f.) τ is suited to an invariant foliation $\mathcal{F}$ of f.

We want to express that condition 2 is easier.

REMARK. The last remark of [**4**] says that a word represents a pseudo-Anosov diffeomorphism only if: the linear map from $E(\tau)$ to itself induced by the word is primitive irreducible. □

However, this is only a necessary condition. In fact, the next holds.

PROPOSITION 3.7. *Even though the linear map from $E(\tau)$ to itself* *(defined by the word: $\tau \to f(\tau)$) is primitive irreducible, τ is not neccessarily suited to the invariant foliation.*

Proof. Take a pseudo-Anosov diffeomorphism f of an orientable surface M' which has genus $g > 0$ and two boundaries. From the unstable foliation of f we obtain a filling generic track τ and a word: $\tau \to f(\tau)$ containing a split such that the unstable foliation is a positive foliation of τ and it has no leaf connecting cusps of $N(\tau)$. This is done similarly to the proof of Proposition 3.5 (see [**4**]). Here "filling" means that the two boundaries are contained in different components of $M - \tau$ and each of them is an m-gon minus a smooth disk ($m > 0$) and the other components are n-gons ($n > 2$). Now we sew the boundaries to make a orientable genus $g + 1$ surface M. In sewing, we keep $f(\tau)$ carried by τ obtaining $\hat{\tau}$ and $\widehat{f(\tau)}$ (Fig. 3.1b). We further deform τ and $f(\tau)$ as in Fig. 3.1a before sewing. Let $\hat{f}$ be the diffeomorphism mapping $\hat{\tau}$ to $\widehat{f(\tau)}$. It exists since the two are isomorphic even after the deformation above. Thus we have a word: $\hat{\tau} \to \hat{f}(\hat{\tau})$ on M.

The invariant foliation $\hat{\mathcal{F}}$ of $\hat{f}$ is obtained by sewing the invariant foliation $\mathcal{F}$ of f along the boundaries, so it has a closed leaf cycle. On the other hand, $\mathcal{F}$ is uniquely ergodic since it is the unstable foliation, so is $\hat{\mathcal{F}}$.

By [**4**] Theorem 3.1 $\bigcap_{n=0}^{\infty} V_{\hat{f}^n(\hat{\tau})} = \Delta_{\hat{\mathcal{F}}}$. ($\Delta_{\hat{\mathcal{F}}}$ is the subspace of $\mathcal{MF}$ consisting of all the measured foliations whose underlying (topological) foliations are the same as $\hat{\mathcal{F}}$.) But since $\hat{\mathcal{F}}$ is uniquely ergodic, $\Delta_{\hat{\mathcal{F}}} = \{\hat{\mathcal{F}}\}$. Furthermore, $\hat{\mathcal{F}}$ is clearly positive from the construction. Hence an arbitrary element of $V_{\hat{\tau}}$ is mapped to a positive foliation of $\hat{\tau}$ by repeating the linear map from $E(\hat{\tau})$ to itself sufficiently many times. Hence the linear map is primitive irreducible. But $\hat{\tau}$ is not suited to the invariant foliation. (Hence $\hat{f}$ is not pseudo-Anosov.) An example of this track and word is given in Section 8. $\square$

REMARK. Naturally the word: $\hat{\tau} \to \hat{f}(\hat{\tau})$ does not satisfy the condition in theorems in later sections. For example, the first condition of (ii) of Theorem 5.2 does not hold (i.e. ($\star$) holds). The segment of $\hat{\tau}$ which was superposed on the boundaries of M' before sewing is mapped onto itself by $\hat{f}$. $\square$

5. The 1-vertex case. A pair of a generic track and a (generic) word can be translated to a pair of a shiftless track and a shiftless word. Their representing diffeomorphisms coincide (up to isotopy) because deforming the generic track only in a (small) simply connected neighborhood as in Fig. 4.1 causes no difference between the representing diffeomorphisms f and f' defined by $\tau_1 \approx_f \tau_2$ and $\tau_1' \approx_{f'} \tau_2'$ respectively, for f and f' differ only in the neighborhood.

So the argument in Section 4 above can be modified for the shiftless category. Condition 1 in Section 4 (right after Proposition 3.6) is changed to:

1$'$. The word is not vacant (has at least one letter).

CONVENTION 3. In the following sections throughout this paper, we consider in the shiftless category unless otherwise mentioned. □

If there is a *word*: $\tau \to f(\tau)$, then $\tau \succ f(\tau)$, so (one of) the intersection matrix A is defined as in Definition 2.11. Composing with the change of basis which maps the j-th capillary of $f(\tau)$ to the j-th capillary of τ, (again writing it A,) we set the (i,j)-entry of A to be the number of the components of the intersection of the i-th capillary of τ and the image by f of the j-th capillary of τ.

A represents a linear map: $R^I \to R^I$, where I denotes the number of the capillaries of τ. If x is a column vector of R^I, its image by f is Ax. Hence the eigenvectors of A (of a positive eigenvalue) satisfying the switch condition (i.e. belonging to $E(\tau)$) correspond bijectively to the invariant foliations of f and τ.

The next condition is equivalent to the existence of the positive foliations of f and τ.

Positivity condition (PC): There exists an eigenvector of a positive eigenvalue of A which satisfies the switch condition and all its entries are positive.

DEFINITION 4.1. Consider a word: $\tau \to \sigma$. We apply the letters in the word one by one. For each letter being applied, we take the carry which left the prongs not concerning to the letter fixed (the carry is determined uniquely). Composing these carries, we have a carry of σ onto τ. This is called the *natural(-ly determined) carry* (*of σ onto τ*). See Fig. 4.2.

If $\sigma = f(\tau)$, multiplying the intersection matrix by the basis changing matrix (depending on f) as before, we get the matrix A as before. We call it the *(intersection) matrix naturally determined by the word and* f. $\square$

DEFINITION 4.2. The *singularity type* of a pseudo-Anosov diffeomorphism f is the sequence of numbers which represents the numbers of the singularities classified by the numbers of the separatrices from them in the canonical model (which has no connections of singularities) of the unstable foliation of f.

The *polygonal type* of a filling track τ is the sequence of numbers which represents the numbers of the polygons of $M - \tau$ classified by the numbers of the edges of them.

We say f *is the type of* τ if the singularity type of f is the same as the polygonal type of τ (where the numbers of the separatrices correspond to the numbers of the edges). $\square$

Now in the following in this section, we consider the case where τ has only one vertex. (A special case is argued in [**2**]. See [**2**] Theorem 11.) We allow the tracks appearing from the word to have many vertices if it is not the last track.

THEOREM 4.3. *Let* τ *be a filling 1-vertex track and consider a non-vacant word:* $\tau \to f(\tau)$. *Then the followins are equivalent.*

(i) f *is pseudo-Anosov and is the type of* τ.

(ii) *(PC) holds for the matrix naturally determined (by the word and* f*) and the maximal transparent set (determined by the word and* f*) has at most one element.*

The new terminologies are defined below.

DEFINITION 4.4. (i) For each capillary of the tracks appearing from a word, we define inductively whether it is *positive* or *zero*.

Let τ_2 be obtained from τ_1 by a letter. Take the natural carry of $\tau_1 \succ \tau_2$. If a capillary of τ_2 passes through at least one positive capillary of τ_1, then it is positive. Otherwise zero. Set all the capillaries of the initial track of the word positive.

(The definition above is equivalent to the following. For each track appearing from the word, take the natural carry of it onto the initial track. (Let them be $\tau \succ \sigma$. τ is the initial track of the word.) If a capillary of σ passes through at least one capillary of $N(\tau)$, it is positive, otherwise zero.)

For the word: $\tau \to f(\tau)$, since $f(\tau)$ is 1-vertex, all its capillaries are positive.

(ii) For each prong of the tracks appearing from a word, we define whether it is *positive* or *zero.*

Let τ_2 be obtained from τ_1 by a letter. Then their prongs correspond bijectively. If the letter is a slide, a prong of τ_2 is positive if and only if it corresponds to a positive prong of τ_1. If the letter is a split, a prong of τ_2 is positive if and only if it corresponds to a positive prong of τ_1 or it has just "travelled" along a positive capillary of τ_1 at the split (Fig. 4.3). Set all the prongs of the initial track of the word zero.

(The definition above is equivalent to the following. As in (i), carry σ onto τ naturally. Then $N(\sigma)$ is represented by an unzipping of $N(\tau)$. For each prong of σ, take the corresponding unzipping path. If the path passes through at least one capillary of $N(\tau)$, the prong is positive, otherwise zero.)

(iii) Consider a word: $\tau \to f(\tau)$. A *transparent set determined by the word and* f is a subset T of $\{$*the prongs of* $\tau\}$ satisfying the following two conditions:

(1) T is invariant under the composition of the correspondence by the word: $\{$*the prongs of* $\tau\} \to \{$*the prongs of* $f(\tau)\}$ and the correspondence by f^{-1}: $\{$*the prongs of* $f(\tau)\} \to \{$*the prongs of* $\tau\}$.

(2) On each track appearing from the word, every prong corresponding (by the word) to T is zero.

The union of all the transparent sets is also transparent. This is the *maximal transparent set.* □

REMARK. In Theorem 4.3, it is not necessary to take the natural carry if we adjust the definition of transparency appropriately. □

PROOF OF THEOREM. If condition (i) holds, there are only two foliation classes in $\mathcal{MF}$ which are invariant under f; the stable and the unstable foliation. Hence the invariant foliation $\mathcal{F}$ of τ and f is the type of τ. This does not occur if $\mathcal{F}$ is not a positive foliation of τ, so (PC) holds. On the other hand, condition (ii) implies (PC). Hence in both cases we can represent the word: $\tau \to f(\tau)$ by an unzipping along a leaf of (F representing) $\mathcal{F}$ which is the positive invariant foliation of τ and f. Since the word is not vacant, the unique vertex is divided by the unzipping (Fig. 4.4). Hence the vertex of $N(f(\tau))$ is confined to (strictly) thinner part than that of $N(\tau)$. $N(\tau)$ maps to $N(f(\tau))$ by f because the vertex is unique, thus $f(\mathcal{F})$ has the same measure on thinner part than $\mathcal{F}$, hence it has the measure multiplied by $\lambda(> 1)$.

Suppose (i) holds. $\mathcal{F}$ is the unstable foliation of f (not the stable since $\lambda > 1$) and, its type being the same as τ, it has no cusp connection when represented as a partial foliation F whose support is $N(\tau)$, i.e. τ is suited to $\mathcal{F}$. Suppose the maximal transparent set T has two or more elements. By repeating the word: $\tau \rightarrow f(\tau)$ (n times), we can set the prong correspondences by f^n and $word^n$ to coincide. Then the prongs in T never "travel" along capillaries of $N(\tau)$. Take two prongs in T and the separatrices from them. When they meet the same tie for the first time, the measure between them is positive because τ is suited to $\mathcal{F}$. Naturally the corresponding part of $N(f^n(\tau))$ has the same measure (measured by the measure of $f^n(\mathcal{F})$), but the two separatrices won't be unzipped, so the corresponding part of $N(f^n(\tau))$ is the identical as the original part. If measured by the measure of $f^n(\mathcal{F})$, it has the measure multiplied by λ^n. Thus we are led to contradiction. Hence T has at most one element and (ii) holds. (Fig. 4.5.)

Conversely suppose (ii) holds. Suppose $\mathcal{F}$ has a cusp connecting leaf. The number of the capillaries of $N(\tau)$ passed by the leaf (with multiplicity) is finite. Since T has at most one element, one of the two ends of the leaf unzips. If we repeat f, we unzip infinite capillaries of $N(\tau)$ along the leaf (since an unzipping is done at least once per f^n), which leads to contradiction. Hence τ is suited to $\mathcal{F}$, so $\mathcal{F}$ is invariant arational, and since $\lambda > 1$, $\mathcal{F}$ is the unstable foliation of f and it is the type of τ. Thus (i) holds. □

REMARK. An arbitrary pseudo-Anosov diffeomorphism is, if taken the power which leaves each separatrix invariant, described by a word from a 1-vertex track (like as in [**2**]). □

6. General case. In the following sections we consider the general case not restricted to the 1-vertex case.

LEMMA 5.1. *By repeating a word* $w : \tau \to f(\tau)$ *(n times), we can arrange so that the prong correspondences by* w^n *and by* f^n *coincide.*

Above n is bounded by some number determined by the number of the prongs.

(Clear.)

THEOREM 5.2. *For a word* $w(\neq \emptyset) : \tau \to f(\tau)$ *where* τ *is filling, the following are equivalent.*

(i) f *is pseudo-Anosov of the type of* τ.

(ii) The positivity condition (PC) holds, and the next condition does not hold:

$(\star)$ *When the word is repeated (n times) so that the prong correspondences by* w^n *and by* f^n *coincide, there is a trainpath c passing through each vertex of* τ *at most once such that the image* $f^n(c)$ *on a maximal isotoped carry of* $f^n(c)$ $\big($*on the carry* $\tau \succ f^n(\tau)\big)$ *is projected* onto (surjectively) c *when* $N(\tau)$ *is retracted to* τ *(Fig. 5.1).*

LEMMA 5.3. *For an invariant foliation* $\mathcal{F}$ *of a word* $(\neq \emptyset) : \tau \to f(\tau)$*, the dilatation* λ $(f(\mathcal{F}) = \lambda\mathcal{F})$ *satisfies* $\lambda \geq 1$*. If* $\mathcal{F}$ *is a positive foliation of* τ, $\lambda > 1$.

Proof. When $\mathcal{F}$ is represented as a partial foliation F whose support coincides with $N(\tau)$, the vertex of $N(\tau)$ with the maximal measure has positive measure μ since F is not zero. The measure of its image by f is μ/λ, but it is the maximal one, hence $\mu \geq \mu/\lambda$ and $\lambda \geq 1$.

Suppose F is positive and $\lambda = 1$.

In the special case where there is only one vertex v which has the maximal measure, f must map v to itself and $f(v)$ occupies the whole height of v. Therefore other vertices cannot be mapped to v by f, hence unzippings cannot occur at v (since F is positive) (Fig. 5.2).

In the general case where there are several vertices of maximal measure, they are permuted among each other, and also unzippings cannot occur at them.

Similarly any vertex of the next maximal measure is occupied by the image of one vertex. Inductively, each vertex is occupied by the image of one vertex, hence no vertex is unzipped, contracting the fact that word is not vacant. □

LEMMA 5.4. *Let $\mathcal{F}$ and $\mathcal{F}'$ be invariant foliations of a word ($\neq \emptyset$) : $\tau \to f(\tau)$, where τ is filling. τ is suited to $\mathcal{F}$ if and only if τ is suited to $\mathcal{F}'$.*

Proof. The dilatation λ is larger than one by Lemma 5.3. Hence f is pseudo-Anosov of the type of τ and a foliation invariant under f is either the stable foliation of f or the unstable foliation. Since "invariant foliations of f and τ" are of course invariant under f, they must be either the stable foliation or the unstable foliation, so must be the type of τ. A foliation to which τ is not suited is not the type of τ (no matter whether it is not positive or it has cusp connection), hence is not "invariant of f and τ". □

LEMMA 5.5. *The existence of an invariant foliation of a word ($\neq \emptyset$) : $\tau \to f(\tau)$ to which τ is suited is equivalent to the existence of an invariant foliation of $w^m : \tau \to f^m(\tau)$ (the word m-times repeated by the prong and capillary correspondence by f) to which τ is suited. (m is arbitrary.)*

Proof. Clearly the former implies the latter. Conversely if there is an invariant foliation of f and τ to which τ is not suited, it is invariant of f^m and τ. This lemma follows from Lemma 5.4. □

PROOF OF THEOREM. From Lemma 5.5, we may assume that the prong correspondences by w and by f coincide.

As in Theorem 4.3, there is a positive invariant foliation $\mathcal{F}$. $f(\mathcal{F}) = \lambda\mathcal{F}$ and $\lambda > 1$ from Lemma 5.3. $\mathcal{F}$ is represented by F whose support is $N(\tau)$. Suppose there is a cusp connecting leaf. If it passes through some vertex twice or more times, there are tie segments whose endpoints are in the leaf. Since the leaf is finite, there is a segment with the minimal measure of F. But since $\lambda > 1$, the measure (of F) of the image (by f^k) of the segment converges to zero as k tends to infinity. The leaf is invariant under f since the prong correspondences coincide, which is a contradiction. Hence cusp connecting leaves, if any, pass through each vertex at most once.

Suppose (i) holds. Then (PC) holds. Suppose ($\star$) holds. We consider the trainpath c in ($\star$). c is embedded in τ. Let $N(c)$ denote the portion of $N(\tau)$ retracting to c. Since $f(c) = c$ when maximally isotoped, one of the prongs from which c starts is stuck into by the corresponding cusp of $N(\tau)$ (and similar on the terminal point of c). We may assume that $f(N(\tau))$ realizes the maximally isotoped carry as above and $f(N(\tau))$, $f^2(N(\tau)), \ldots$ are realized so as to match F. The meaning is that all their boundaries $\partial f(N(\tau)), \partial f^2(N(\tau)), \ldots$ are on the leaves of F at the cost that they might be doubled. Since $\lambda > 1$, any capillary or vertex has measure (of F) converging to zero. Then $f^k(N(c))$ converges, as k tends to infinity, to a cusp connecting leaf between the two cusps which sticks into the prongs on the ends of $f(c)$ (Fig. 5.3). This contradicts to (i), hence ($\star$) fails, thus (ii) holds.

Conversely suppose (ii) holds. If F has a leaf l connecting two cusps s_1 and s_2, it corresponds to a path c of τ which passes through each vertex at most once. Let p_i denote the prong of τ which corresponds to s_i. The image of l by f is contained in l itself. l may be unzipped, so the image does not always occupy the whole of itself. Each of the two complementary segments l_i of l make the prong correspondence (by the word) between p_i and $f(p_i)$. Then we can left isotope $f(c)$ so as to contract l_1. If the isotopy must stop before reaching p_1, there is a cuspprong sticking at one of the vertices v' on the way of c. Since $f(N(v'))$ contains l in its interior (because v' is on c), each tie of $N(v')$ contains two points of l, the one from l_1 and the other from int $f(N(v'))$. (Here $N(v')$ denotes the vertex of $N(\tau)$ corresponding to v'.) This implies that l passes through v' twice, contracting to the hypothesis. Thus $f(c)$ can be left isotoped until p_1 sticks into $f(p_1)$. Since c is imbedded in τ, we can right isotope $f(c)$ similarly fixing $f(p_1)$. Thus we have one of the maximally isotoped carries of $f(c)$, on which $f(c)$ is projected to all over c by the retraction, contradicting to (ii) (Fig. 5.4). Hence f has an arational invariant foliation and a dilatation larger than one, hence is pseudo-Anosov. $\square$

REMARK. Since the number of capillaries of τ is bounded by some number determined by the genus, the procedure above is bounded by some number determined by the genus.

7. Length and travel length (values given to capillaries and prongs). Consider a track τ.

DEFINITION 6.1. We give a nonnegative real number to each capillary of τ with no relation and call it a *length* of the capillary. □

A vector of capillary lengths is represented by a point of R^I, where I denotes the number of the capillaries of τ. Possible vectors form the first quadrant of R^I and its boundary.

When $\tau \succ \sigma$, by choosing a carry, we get the intersection matrix A (Definition 2.11). For each capillary of σ, its length is induced from the carry. It is the sum, with multiplicity, of the lengths of the capillaries of τ which the capillary of σ passes through. When we write the length vector of τ by the row vector $x \in R^I$ and that of σ by $y \in R^J$, where J is the number of the capillaries of σ, we have $y = xA$.

When $\tau \succ \sigma$, we give a nonnegative real number to each prong of σ as follows: each prong of σ corresponds to a cusp of $N(\tau)$ by a path as in Proposition 1.1, and we give it the sum, with multiplicity, of the lengths of the capillaries of $N(\tau)$ which the path passes through.

Or, more generally, we consider that nonnegative real numbers are also given to the prongs of τ. Then we give to each prong of σ the sum of the value above and the number given to the prong of τ corresponding to the prong (Fig. 6.1).

DEFINITION 6.2. These numbers are called *travel lengths* of the prong. □

An $(I+P)$-tuple ($I(P)$ denotes the number of capillaries (prongs)) of the values of lengths and travel lengths given to a track is also called a value.

If $\tau \succ \sigma \succ \xi$, if σ is given the value induced from a value of τ, the value of ξ induced from the value of σ coincide to that induced from τ.

Given a value of τ, we transform it as follows:

At a vertex of τ, lengths and travel lengths of all the prongs and all the capillaries on one side of it is added the same value $v(t)$, and on the other side, added $-v(t)$. Then resulting lengths and travel lengths form another value of τ if no prongs or capillaries have negative lengths or travel lengths. $v(t)$ varies from zero to a nonnegative real number when t varies from 0 to 1. We make such transformations on each vertex at the same time. If the values are nonnegative for all t, the transformation is made in the set of all the values of τ (Fig. 6.2).

DEFINITION 6.3. A transformation like above is called a *value isotopy* between the values of τ at $t = 0$ and at $t = 1$. □

When $\tau \succ \sigma$, a carry isotopy causes a value isotopy of the values induced from the carries of σ onto τ. Conversely the next holds.

LEMMA 6.4. *Assume all the prongs of τ are given travel lengths zeroes. Given a value of σ induced from a carry $\tau \succ \sigma$, each value isotopy of that value is realized by some carry isotopy (where we follow the convention that if a vertex v of σ is in a capillary of $N(\tau)$, the value of σ is an intermediate value between the values induced from carries in which v is in vertices of $N(\tau)$ (Fig. 6.3)).*

Proof. Given a carry $\tau \succ \sigma$, two trainpaths of σ are "parallel" if there is a homeomorphism from one to the other which maps each point to a point lying on the same tie of $N(\tau)$. (The orientations of the trainpaths are ignored.)

Thus, when a prong p of σ has a positive induced travel length, the two edges e_1 and e_2 making p (where an "edge" means a smooth edge of a component of $M - \tau$, made of some branches) have parallel subsegments s_1 and s_2. One endpoint of s_i is at p, and the other endpoint lies on the singular tie passing through the cusp of $N(\tau)$ corresponding to p (see Fig. 1.14, where σ is drawn as $N(\sigma)$ and the homeomorphism between $a = s_1$ and $b = s_2$ is indicated by vertical arrows). s_i travels through branches of $N(\tau)$ the sum of whose lengths is as large as the positive value of p. Such situation is referred to that e_1 and e_2 are parallel as long as they travel the positive value of p.

Since to be parallel is an equivalence relation, a series of adjacent prongs ("adjacent" two prongs have a common capillary making them) has a set of parallel subsegments of the edges making them, one endpoint of each of which is at the common switch, and which have the same lengths as the smallest value of the prongs. Hence if all left (resp. right) prongs of a vertex have positive values $\geq \alpha$, all left (resp. right) edges are parallel to each other as long as they travel α.

Now suppose that we are given a value of σ induced from a carry $\tau \succ \sigma$ and a value isotopy of that value. Because a value isotopy operates on vertices, we only have to carry-isotope each vertex to realize the value isotopy. At each vertex, say v, value isotopy contributes negative value $-v(t)$ on one side of v, say left side. Then all left prongs have positive values $\geq v(t)$, hence all left edges, which make the left prongs, are parallel as long as they travel $v(t)$. We may left carry-isotope v so as to shorten the parallel subsegments of the edges simultaneously (and at the same time extend the right capillaries). We cannot left carry-isotope only on the following two cases. (i) One of the left prongs, say p, is stuck in by the corresponding cusp. In this case, the travel length of p is zero and the given value isotopy is over. (ii) One of the left capillaries, say b, becomes very short and v bumps against the next left vertex v'. In this case, the length of b is zero and if the given value isotopy goes on, it must contributes negative value also on the left side of v'. Hence all left prongs (and maybe capillaries) of v' have positive values. Then v' may be left carry-isotoped. By repeating this discussion, we obtain the whole realization. The only case which might be an obstruction is that we repeat case (ii) and the chain of "the next left vertex" is circular (the next left of v is v', the next left of v' is v'', ..., the next left of $v^{(n)}$ is v). In this situation we have a circle of 0-capillaries and any left prong of any vertex on that circle has a positive value. For example, see Fig. 7.5b, where we set $a = 0$ and $p > 0$. But this cannot occur because then the edges making p can only travel around the circle of 0-capillaries and never attain the positive value of p. (A detailed proof is given in the last three paragraphs of the proof of Proposition 7.13, where positivity is assumed for all right prongs, not left.) $\square$

We consider a word $(\neq \emptyset) : \tau \to f(\tau)$ and (one of) the intersection matrix A obtained from it.

LEMMA 6.5. *Among the vectors of the capillary lengths of τ, there exists an f-invariant one. That is, if vectors are described as row vectors in R^I, there exists $y \in (R_+ \cup \{0\})^I$ such that $yA = \lambda' y$ and*

$\lambda' > 0$. And then $\lambda' \geq 1$. Moreover if the positivity condition (PC) holds, $\lambda' > 1$.

Proof. Since A maps the convex cone $(R_+ \cup \{0\})^I$ into itself, from Brouwer's fixed point theorem, there is a fixed point projectively, which is y above.

The sum Y of all the entries of y, which is not zero, does not decrease when y is mapped by A because each capillary of $N(\tau)$ is passed through by at least one capillary of $f(\tau)$. Hence $\lambda' \geq 1$.

Let (PC) hold. Take a positive invariant foliation $\mathcal{F}$ of f and τ. When we describe $f(\tau)$ as an unzipped $N(\tau)$ along a leaf of F which represents $\mathcal{F}$ and whose support is $N(\tau)$, since $\lambda > 1$ from Lemma 5.3, its capillary has measure (of F) converging to zero when repeatedly mapped by f. Hence, by repeating f infinitely, each capillary of $N(\tau)$ is passed through by infinite capillaries. Since y is not zero, there is a capillary of τ with a positive length. It is passed through infinitely, so $Y^{(k)}$ (which is the sum of entries of yA^k) tends to infinity as k tends to infinity. Since $Y^{(k)} = \lambda'^k Y$, it follows $\lambda' > 1$. □

The vectors of the lengths of $f(\tau), f^2(\tau), \ldots, f^k(\tau), \ldots$ induced by this vector y are $\lambda' y,\ \lambda'^2 y, \ldots, \lambda'^k y, \ldots$.

If we take the n-th power of f and the word as in Lemma 5.1 so that the prong correspondences by f and the word coincide, and if the prongs of τ are given travel lengths zeros, then the vector of the travel lengths of $f^{nk}(\tau)$ is $(1 + \lambda'^n + \cdots + \lambda'^{n(k-1)})p$ where p denotes the vector of the travel lengths of $f^n(\tau)$.

DEFINITION 6.6. Assume that $\tau \succ \sigma$. For a prong p of σ and a capillary (or a vertex) B of τ, we say that p is *able to pass B (m times)* if we can arrange the carry by deforming it by a carry isotopy so that it satisfies the following condition: the path which makes the correspondence between p and a cusp of $N(\tau)$ passes through B (m *times*).

Similarly, we say that a capillary b of σ is *able to pass B (m times)* if we can arrange that b passes through B (m times) by a carry isotopy. □

PROPOSITION 6.7. *Consider a word $(\neq \emptyset) : \tau \to f(\tau)$ where τ is filling. Assume that a natural number m is given arbitrarily. If f*

is pseudo-Anosov of the type of τ, then for each prong P of τ, by taking a sufficiently large number k, the corresponding (by the word) prong p of $f^k(\tau)$ is able to pass every capillary or vertex of τ m times. (Then, for an arbitrary number k' larger than k, this holds.)

Proof. Represent the unstable foliation $\mathcal{F}$ of f as a partial foliation F whose support is $N(\tau)$ and regard the word as an unzipping along leaves of F. p is represented as the cusp c of $N(f^k(\tau))$ which the separatrix from P meets for the first time. We denote the segment s_k on the separatrix from the initial point (i.e. the cusp of $N(\tau)$ representing P) to the furthest point which can be reached by c in carry isotoping. "Be able to pass (m times)" means that s_k passes through the capillary (or vertex) in question m times.

$\{s_k\}$ is a monotone increasing family. Suppose that there is a number m such that some prong of τ is not able to pass some capillary (or vertex) m times even though k tends to infinity. Then $s := \cup s_k$ is bounded. (For, since F is arational, every separatrix passes through all the capillaries and vertices arbitrarily many times. See [**1**].) That is, there is a bound z on the separatrix which is the furthest point reached by s_k's. z is on a vertex of $N(\tau)$ and a "sticking" occurs at the vertex for any k such that $s_k = s$, and the sticking prevents s_k from overpassing z. z' denotes the first intersection after z of the separatrix and the singular tie where the cusp(s) which causes the sticking lies. (The tie is the left (or right) end of the vertex.) The measure between z' and the nearest (measured by F) cusp on the tie is positive since there is no cusp connection. On the other hand, since f is pseudo-Anosov, the measures (measured by F) of the vertices of $f^k(\tau)$ converge uniformly to zero. Hence the sum of those becomes smaller than the measure (measured by F) between z' and the nearest cusp above when k is sufficiently large. In this situation, stickings which prevent s_k from reaching further point than z' cannot occur, which is a contradiction (Fig. 6.4). □

When $\tau \succ \sigma$ and all the prongs of τ are given travel lengths zeroes (and the capillaries of τ are given fixed capillary lengths), and when we isotope σ by a carry isotopy so that some prong p of σ has its largest possible induced travel length, the induced value of σ is as follows: there exists a path from p travelling only capillaries with length zero (0-capillaries) and ends into a prong with travel length zero (0-prong) (Fig. 6.5).

Next we consider a word: $\tau \to f(\tau)$ satisfying (PC). Denote one of the positive invariant foliations $\mathcal{F}$. Taking some carry, applying Lemma 6.5, we have one of the eigenvectors y. We fix capillary lengths to be y, all travel lengths zero. We further assume that the n-th power of f has the prong correspondences coincide as before. Then the next follows.

LEMMA 6.8. *If τ is suited to $\mathcal{F}$, the largest travel length of p on $f^n(\tau)$ mentioned above is positive (where p is arbitrary).*

Proof. From the proof of Proposition 6.7, if (PC) holds and k is sufficiently large, whether a prong $p^{(k)}$ of $f^k(\tau)$ is able to pass a capillary (or a vertex) of τ depends on whether the separatrix of F (which represents $\mathcal{F}$ and whose support is $N(\tau)$) passing $p^{(k)}$ passes through the capillary (or vertex). Now τ is suited, so F is arational and every leaf is dense in $N(\tau)$. So if k is sufficiently large, $p^{(k)}$ is able to pass every capillary of τ, especially positive one. Hence, on the induced value of $f^k(\tau)$, either $p^{(k)}$ already has a positive value (this means that $p^{(k)}$ passes through a positive capillary in fact) or no path starting from $p^{(k)}$ and travelling only 0-capillaries ends into a 0-prong (this means that $p^{(k)}$ is only "able" to pass positive capillaries). As mentioned just before Definition 6.6, whether a prong (or a capillary) of $f^{nk}(\tau)$ is positive or zero is determined by whether that of $f^n(\tau)$ is positive or zero. So the situation above is the same on $f^n(\tau)$, hence either p has a positive value or no path starting from p and travelling only 0-capillaries ends into a 0-prong. In the first case p passes through a positive capillary in the carry. In the second case one can value-isotope the value so that p has a positive value, which means $f^n(\tau)$ can be carry-isotoped so that p has a positive value (Lemma 6.4). In any case, the largest travel length of p is positive. □

LEMMA 6.9. *Again fix a value of τ as above. If the separatrix from a prong p of $f^n(\tau)$ is cusp connecting, the largest induced value of p is zero.*

Proof. A cusp connecting separatrix passes only finite capillaries, so it travels only finite length, even if mapped by f^k where k tends to infinity. And since the prongs on its ends can travel only along it, their travel lengths are both bounded. So is the length of the path corresponding to the separatrix. But as mentioned just before Definition 6.6, since $\lambda' > 1$, if bounded then zero. Hence we have a path from p travelling only 0-capillaries and ending into a 0-prong. p has zero travel length and its largest value is zero. □

With these result, we have:

THEOREM 6.10. *For a word* $(\neq \emptyset) : \tau \to f(\tau)$ *where* τ *is filling, the followings are equivalent:*

(*i*) f *is pseudo-Anosov of the type of* τ.

(*ii*) *The positivity condition* (*PC*) *holds and if* (*fixing some carry of* $f(\tau)$ *onto* τ,) *we set a value of* τ *as one of the eigenvectors* y (*and travel lengths zero*) *as above, the induced value of* $f^n(\tau)$ *satisfies the following: there is no path which starts from a* 0*-prong, travels only along* 0*-capillaries and ends into a* 0*-prong* (*Fig. 6.6*).

8. The quotient space by value isotopy. We take some carry of σ onto τ which maps the vertices of σ into the vertices of τ. In Definition 2.11 or right after Definition 6.1, we have a matrix representing the map between the vectors of the capillary lengths of τ and σ. In this section, we add new rows and columns to A to represent the map between the values (i.e. the vectors of the capillary lengths and the travel lengths) of τ and σ.

DEFINITION 7.1. We can think that all the values of τ make the first quadrant and its boundary of R^{I+P}, where I and P denote the numbers of the capillaries and the prongs of τ respectively. Denote it $F(\tau)$.

An axis of R^{I+P} which is of capillary (i.e. one of the first I axes) is called a *capillary axis*, that of a prong a *prong axis*. □

DEFINITION 7.2. Take a carry $\tau \succ \sigma$ mapping the vertices to the vertices. We define a new $(I+P)\times(J+Q)$ matrix $\overline{A}$ as follows.

$(\overline{A}_{ij}) = (A_{ij})$ if $i \in I,\ j \in J$;
$(\overline{A}_{pj}) = 0$ if $p \in P,\ j \in J$;
$(\overline{A}_{iq}) =$ the number of the components of
$$(c_q \cap (\text{the } i\text{-th capillary of } \tau))$$
if $i \in I,\ q \in Q$, where c_q is the path connecting the q-th prong of σ to the corresponding cusp of τ;
$(\overline{A}_{pq}) = 1$ if the p-th prong of τ corresponds to the q-th prong of σ,
$= 0$ otherwise,
if $p \in P, \quad q \in Q$.

□

If a row vector $x \in R^{I+P}$ denotes a value of τ and $y \in R^{J+Q}$ denotes the value of σ induced from x by the carry, then $x\overline{A} = y$. To ignore the variance in taking a carry, we make the next definition.

DEFINITION 7.3. For each vertex v of τ, the *vertex vector* of v is the vector in R^{I+P} which has entry 1 on each of the capillaries and the prongs at one side of v, -1 on those at the other side of v, 0 on the others, where, if a capillary has its both ends at v, we add those effects. H_τ denotes the subspace of R^{I+P} spanned by all the vertex vectors.

(Because each prong belongs to only one vertex, the vertex vectors are all linearly independent and dim $H_\tau =$ (the number of the vertices).)

The quotient space of R^{I+P} by H_τ is denoted G_τ. Its dimension equals to the number of the branches of τ when it is deformed to be generic by combs (which is, $3P/2$).

The image of $F(\tau)$ in G_τ is a convex cone from O. We denote it W_τ. □

LEMMA 7.4. *The (set of points in the) equivalence class of a point of $F(\tau)$ intersects $F(\tau)$ at a compact convex set, called the "possible set" of the point.*

Proof. The equivalence class is a subspace, so it is closed convex. So is $F(\tau)$. Then so is the intersection.

If it is not compact, there exist a point $p \in F(\tau)$ and a sequence $h_k \in H_\tau$ such that $p + h_k \in F(\tau)$ and some entry of h_k tends to infinity. If the entry is of a prong (i.e. of the latter P entries of R^{I+P}), then the entry of any opposing prong tends to minus infinity, contradicting that $p + h_k \in F(\tau)$. If the entry is of capillary b, take one of the ends of b and the vertex v it belongs to, and take the entry of one of the prongs at the opposite side of v. Take the one on the other end of b. The sum of them tends to minus infinity as k tends to infinity, again leading to contradiction. □

LEMMA 7.5. *The possible set of a point in the intersection of $F(\tau)$ and the space spanned by all the capillary axes is singleton.*

Proof. The possible set of a point consists of all the points which can be reached from the point by a value isotopy (since it is convex), but the point has all the prong entries zero, from which no value isotopy can be done. □

LEMMA 7.6. *The intersection of $F(\tau)$ and the space spanned by all the prong axes is called the* prong space. *The possible set of a point in the prong space is contained in the prong space, and is singleton (i.e. itself only) if τ is non-orientable, and a segment if τ is orientable.*

Proof. Let $s \in$ *the prong space.* We calculate $h \in H_\tau$ such as $s + h \in F(\tau)$. h is represented as $h = a_1 v_1 + \cdots + a_K v_K$ where $a_i > 0$ for $1 \leq i \leq k$ and $a_i = 0$ for $k+1 \leq i \leq K$ (K denotes the number of the vertices of τ). If $h \neq 0$, take a capillary b on which v_1 has a minus entry. Since the contribution of $a_1 v_1$ to h on b is $-a_1 < 0$ (or $-2a_1$, in which case b has both ends on one side of v_1 and τ is not orientable), b must be added a' ($a' \geq a_1$) on the other end, say v_{i_2}. Then $a' = a_{i_2} \geq a_1 > 0$.

If τ is orientable, thus we have a closed trainpath which passes through $v_1, v_{i_2}, \ldots, v_{i_m}$ and comes back to v_1. Then $a_1 \leq a_{i_2} \leq \ldots \leq a_{i_m} \leq a_1$, hence $a_1 = a_{i_2} = \cdots = a_{i_m}$. Since we can reach every vertex of τ by a closed trainpath, we have $k = K$ and $a_1 = a_i$ for all i. Hence $h \in \{a \sum v; a \in R\}$, where the sum is taken over all the vertices (with appropriate orientation). This is a one dimensional subspace of H_τ with all the capillary entries zero. Hence the possible set of s is contained in the prong space. Since $s + h$ belongs to the prong space, all of such h's make a compact convex subset of the one dimensional subspace, that is, a segment.

Similarly, if τ is non orientable, we have a trainpath coming back to v_1 with the opposite direction (this is not called a closed trainpath). The contributions on v_1 is not compatible, hence $a_1 = 0$ and $h = 0$.

Above we took a capillary b with minus entry. If such a capillary does not exist, then all the capillaries on the (minus) side have their other ends on the opposite (plus) side. Since τ is recurrent, there is no other capillaries. Then τ is orientable and the above argument applies. □

LEMMA 7.7. *The interior points of $F(\tau)$ project to the interior points of W_τ. Conversely the interior points of W_τ is the images of the interior points of $F(\tau)$.*

Proof. Projections are submersions. Hence that is clear since $F(\tau)$ is a convex $(I + P)$ dimensional manifold. □

COROLLARY 7.8. *For a point p in $F(\tau)$, that the image of p is in the interior of W_τ is equivalent to that some interior point of $F(\tau)$ is in the possible set of p. (Clear from convexity.)*

PROPOSITION 7.9. *A value of τ is projected to a boundary point of W_τ if and only if it has one of the followings:*

(i) *a circle of* 0*-capillaries* (*Fig. 7.1a*),

(ii) *a path from a* 0*-prong into a* 0*-prong made of only* 0*-capillaries* (*Fig. 7.1b*) (*including "opposing* 0*-prongs"*).

Proof. If there is one of the above types, such a value clearly cannot be isotoped to a positive value. Conversely, if there is no such types, then we can reduce the number of the 0-capillaries as follows.

Take some 0-capillary. Starting from it, we travel 0-capillaries arbitrarily as long as possible. Such a travel must stop, that is, there is a capillary end whose opposite capillary ends (capillary ends which are on the opposite side of the vertex the capillary end is located) are all positive because there are only finitely many capillaries and there is no circle of 0-capillaries. If one of the opposite prongs of the capillary end above is a 0-prong, then we travel inversely and again reach such an end as above. But since there are no paths like (ii), this end has no 0-prongs as an opposite prong. Now we can isotope the value by adding small scalar multiple of vertex vector of the vertex such that the last 0-capillary is isotoped to be positive and none of the positive capillaries and prongs are isotoped to be non positive (like in Fig. 7.2a).

Thus we reduce to the case that only prongs are zero. But there is no opposing 0-prongs, we can isotope the value by adding small scalar multiple of vertex vectors of each vertex such that the 0-prongs are all isotoped to be positive and no other capillaries or prongs are isotoped to be non positive (Fig. 7.2b). □

Clearly the set of 0-capillaries and 0-prongs of the type (i) or (ii) above is invariant by an isotopy. Then we have a cellular decomposition of W_τ. The cell of the largest dimension consists of values which have no such patterns of the types above. It is also the interior of W_τ. Cells of the second largest dimension have only one pattern, and so on. If a cell is defined by a set of such patterns which is a subset of another set, it is on the boundary of the cell defined by this set. See Fig. 7.3.

PROPOSITION 7.10. *Two cells are in general position if one is not the boundary of the other.*

Proof. C_1 and C_2 denotes the two cells. We show that if a point of C_2 is represented as a linear combination of points of C_1, then C_2 is a boundary of C_1. Suppose that a point c_2 of C_2 is represented as $\sum x_i c_1^i$ where c_1^i is a point of C_1 and x_i is a real coefficient. Let v_1^i, v_2 be the preimages of c_1^i, c_2. Then $v_2 = \sum x_i v_1^i + h$ where $h \in H_\tau$. The segment connecting $\sum x_i v_1^i$ and $\sum |x_i| v_1^i$ is in a side face L of $F(\tau)$ and its possible set (union of possible sets of points in it) is also in L because 0-capillaries -and -prongs pattern is invariant by an isotopy near $\sum |x_i| v_1^i$. Hence v_2 is in L. Thus c_2 is in a boundary of C_1. □

Thus W_τ is a convex cone from 0, which has each image of axis as a generating line if τ is non orientable, and one of the edges (a prong axis) is dependent (i.e. is not a generating line essentially) if τ is orientable. Fig. 7.4 gives examples.

LEMMA 7.11. *Consider some carry $\tau \succ \sigma$. The matrix $\overline{A}$: $R^{I+P} \to R^{J+Q}$ induces a linear map: $G_\tau \to G_\sigma$.*

Proof. We have to show that two H_τ-equivalent values of τ are mapped by $\overline{A}$ to H_σ-equivalent values of σ. Take $h \in H_\tau$ which makes two values of τ equivalent. Let $h' \in H_\sigma$ be as: its coefficient of each vertex vector v' of σ is the same as the coefficient of the vertex vector of τ in which vertex v' of σ is contained on the carry. □

DEFINITION 7.12. Assume $\tau \succ \sigma$ and fix some value of τ. A capillary (or a prong) is *"able to be positive"* if there is a carry isotopy such that for the induced value of σ, the entry of the capillary (or the prong) is positive. □

PROPOSITION 7.13. *Assume $\tau \succ \sigma$ and fix capillary lengths of τ (travel lengths all zero). If all the prongs of σ are able to be positive, then all the capillaries of σ are also able to be positive.*

Proof. Take an arbitrary capillary a of σ and denote its left end e_1 and the right end e_2. At e_1, a makes a prong p on its upper or lower side with another capillary b (which might be a itself).

As mentioned just before Lemma 6.8, we isotope σ left at e_1 maximally so that p takes the largest value. Then there exists a 0-capillary path ending into a 0-prong. Furthermore we isotope σ right at e_2 fixing e_1. Then there are two cases.

The first is that we can reach the maximally isotoped carry of a. t denotes the value of a at this time. Then there exists a 0-capillary path ending into a 0-prong q. We can isotope σ so that q has the largest value, but this value is at most t. Since q is able to be positive, $t > 0$ and a is able to be positive (Fig. 7.5a).

The second case is that we are "hooked" at e_1 which is fixed before e_2 is maximally right isotoped. Then there exists a 0-capillary path between e_1 and e_2 and together with a it forms an embedded closed path c (Fig. 7.5b). Suppose that a has a value zero at this time. Then any right prong of any vertex of c has its largest value, that is, a positive value by assumption. But this cannot occur when c is a circle of 0-capillaries, as stated below. Thus a must be positive.

As mentioned in the proof of Lemma 6.4, the edges making a positive prong p have parallel subsegments s_1 and s_2, which have a common initial point at p and have a positive value.

Now in Fig. 7.5b, a and b makes a positive prong p. s_1 (resp. s_2) denotes the subsegment of the edge which consists of a (resp. b) and consecutive capillaries (and vertices). s_1 and s_2 are parallel and have the same positive length. s_i might be a subsegment of a or b, but since the length of a is zero, a is a proper subset of s_1. Hence there is no capillary as indicated by the broken line in Fig. 7.5b and b' is the second subsegment of s_1 (again s_1 might be a subsegment of the path $a \cdot b'$). Then s_2 travels parallel to $a \cdot b'$ (or a subsegment of it). Since all the next prongs (on the right side of the next vertex) are again positive and a' is zero, a' and b' (or its subsegment) are parallel. Hence s_2 travels parallel to $a \cdot a'$.

Since s_i travelled only 0-capillaries in the above discussion, there still remains a subsegment of s_i to travel. But, by repeating the above discussion, s_i can only travel around c, which is a circle of 0-capillaries, and can never escape from the circle. Thus s_i cannot reach its terminal point without travelling infinitely many capillaries of $N(\tau)$, which contradicts that σ is obtained by a (finite) word from τ. □

COROLLARY 7.14. *Consider a word:* $\tau \to f(\tau)$. *If*

(α) *For each capillary of* τ, *there exists a natural number* k *such that each prong of* $f^k(\tau)$ *is able to pass the capillary, then each capillary of* $f^k(\tau)$ *is also able to pass each capillary of* τ.

Proof. Give length 1 to the capillary of τ, 0 to the other capillaries and prongs. By (α), the hypothesis of Proposition 7.13 holds, then each capillary of $f^k(\tau)$ is able to be positive and the conclusion holds. □

PROPOSITION 7.15. *Consider a word: $\tau \to f(\tau)$ where τ is filling. If (α) holds, then f is pseudo-Anosov of the type of τ.*

Proof. We show the positivity condition (PC). Take one of the eigenvectors U in (PC) (a measure eigenvector, representing measures of capillaries). Take some capillary b with a positive measure (there exists one since U is not zero). For a capillary b' of τ, take a number k in (α). Then $f^k(b)$ is able to pass b' and b' has a positive measure (a measure greater than or equal to that of b multiplied by the positive eigenvalue). As b' is arbitrary, (PC) holds. Applying Theorem 5.2, since ($\star$) denies (α), the proposition is proved. $\square$

With Proposition 6.7, we have:

PROPOSITION 7.16. *Consider a word: $\tau \to f(\tau)$ where τ is filling. Then f is pseudo-Anosov of the type of τ if and only if (α) holds.*

THEOREM 7.17. *Consider a word: $\tau \to f(\tau)$ where τ is filling. Then f is pseudo-Anosov of the type of τ if and only if (β) holds.*

(β) By repeating the linear map: $G_\tau \to G_\tau$ induced from $\overline{A}$ (and the prong and capillary correspondence by f), every (positive part of) capillary axis of R^{I+P} is mapped into the interior of W_τ.

Proof. (α) is equivalent to: for an arbitrary vector ($\neq 0$) of capillary lengths of τ, there exists a natural number k such that every prong of $f^k(\tau)$ is able to be positive. This is equivalent to (γ) from Proposition 7.13 and Corollary 7.14.

(γ) For an arbitrary vector ($\neq 0$) of capillary lengths of τ, there exists a natural number k such that every prong or capillary of $f^k(\tau)$ is able to be positive.

(γ) is equivalent to:

(δ) For an arbitrary vector ($\neq 0$) of capillary lengths of τ, there exists a natural number k such that the induced value of $f^k(\tau)$ has an interior point of $F(f^k(\tau))$ in its possible set.

(δ) implies (γ) because the interior point can be realized as the value induced by some carry (Lemma 6.4). Conversely, if (γ) holds, the induced value of $f^k(\tau)$ has, in its possible set, points each of which has positive entry on each prong or capillary. Since the possible set is convex, it also contains the barycenter of these points, which is in the interior of $F(f^k(\tau))$. Thus (δ) holds.

Clearly (δ) implies (β). (β) implies (δ) because an arbitrary vector $(\neq 0)$ of capillary lengths of τ is in the space spanned by all capillary axses and W_τ is convex. □

A word: $\tau \to f(\tau)$ is, writing each letter, $\tau = \tau_0 \to \tau_1 \to \tau_2 \to \cdots \to \tau_{m-1} \to \tau_m = f(\tau)$, where an arrow $(\to)$ represents aletter. Consider $G_l = G_{\tau_l}$, $W_l = W_{\tau_l}$. By mapping by the linear map, we consider that all W_l are in G_m (they map into W_m). Now let us trace back the word from W_m to W_0. We call the images of (the positive parts of) the capillary or prong axes of τ_l as the *capillary or prong generatrices* of W_l. For each letter, we make some sum of the generatrices of W_l to get the generatrices of W_{l-1}. Examples are given in Fig. 7.6.

In (a), the generatrices a', b', p' of W_{l-1} are obtained from those a, b, p of W_l as $a' = a$, $b' = a + b + p$, $p' = p$ and the others are the same as their corresponding ones (or, "invariant"). In (b), W_{l-1} has one more generatrix. In (c), one generatrix is lost. Thus the image of a generatrix of W_l in W_m is represented as a linear combination of the generatrices of W_m. The generatrices of W_0 correspond to those of W_m by f^{-1}, and we can repeat the same operations. f is pseudo-Anosov if and only if all the capillary generatrices are thus mapped into the interior of W_m when repeatedly mapped.

Since each cell of W_0 is mapped into one (closed) cell of W_m (from Proposition 7.10), considering the inclusion relation of the cells of W_0 into those of W_m, we can know whether a generatrix is kept on boundaries even if the word is infinitely repeated. Thus we can avoid an infinite procedure.

As to the linear combination above, we only have to note whether the coefficients are zero or not, ignoring the exact values since cells are known by their interior points.

COROLLARY 7.18. *Consider a word $w : \tau_0 \to \tau_m$. If all the capillary generatrices of W_0 is mapped into the interior of W_m by the operation like above, then the word $w'' : \tau_0 \to \tau_{m'}$ which is composed by adding a word $w' : \tau_m \to \tau_{m'}$ $(\approx \tau_0)$ to w, represents a* *pseudo-Anosov diffeomorphism of the type of* τ_0.

Similarly, if add a word: $(\tau_0 \approx)\tau_{0'} \to \tau_0$, *holds the same.*

Proof. In tracing a word, W becomes greater and greater. Hence the fact that the generatrices are in the interior does not change. □

9. Examples. Here we describe the procedure of constructing (shiftless) filling tracks and give an example of an (isotopy class of a) pseudo-Anosov diffeomorphism.

Let τ be a filling track. On each complementary polygon, we deform its edges slightly inward keeping the vertices (i.e. the endpoints of the edges) fixed (Fig. 8.1a). The union of all the complementary crescents (shaded on Fig. 8.1a) of all the polygons forms a union of bigons and annuli on M (Fig. 8.1b). The spires of the bigons correspond to the capillary ends of neither uppermost nor undermost of the vertices of τ. The pairs of the edges of the bigons and the pairs of the boundaries of the annuli correspond to the (maybe closed) trainpaths of τ which go from the uppermost (resp. undermost) to the uppermost (resp. undermost) at each vertex of τ they pass.

We fix an orientation of M so that the edges of the polygons are oriented, for example, counterclockwise. The edges of the bigons and the boundaries of the annuli then give oriented sequences of the edges of the polygons realized as intervals or circles, called *"lines"*. The lines appear in pairs and there is an orientation reversing homeomorphism between the two lines in each pair. Furthermore the homeomorphism does not map the vertices (i.e. the endpoints of the edges in the line) to the vertices except at the ends of the line (when the pair is of a bigon). We call such homeomorphisms *"pasting homeomorphisms"* of the pairs. Pasting homeomorphisms represent the way of retracting the bigons and the annuli to obtain τ.

Now conversely assume that we have a set of pairs of lines and pasting homeomorphisms. (Hence an interval and a circle cannot make a pair.) We consider a disk neighborhood of the terminal point v of an edge e_0 in a line as follows.

Assume that each edge of each polygon appears once and only once in the lines. e_0 has, in the polygon it belongs to, the next edge e_1, which makes an angle with e_0. If the initial point of e_1 is also the initial point of the line it belongs to, the neighborhood of the initial point in e_1 is pasted to the neighborhood of the terminal point of the other line in the pair. Denote the last edge e_2, which is pasted with e_1 near the terminal point. If the initial point of e_1 is in the interior of the line, e_1 and its predecessor e_4 on the line are pasted near their connection point with an edge e_3 on the other line in the pair. Thus we successively determine the edges gathering around v. We take sectorial neighborhoods near the angles like between e_0 and e_1 and half disk neighborhoods near the edges like e_3. We further take care so that the boundaries of the neighborhoods coincide when they are pasted. Since there are only finitely many edges, this procedure terminates and the union of the neighborhoods above is a disk neighborhood of v (Fig. 8.2a).

A set of pairs of lines and pasting homeomorphisms are obtained from a (maybe non recurrent) track if and only if:

(i) all the edges appear once and only once in the set and

(ii) for each vertex of each line, its disk neighborhood has exactly two half disk neighborhoods as its constituents (Fig. 8.2b).

We may identify two pasting homeomorphisms if they map the endpoints of one line to the other line in the same order in comparison with the endpoints of the other line.

REMARK. For the sake of constructing all the combinatorial tracks, we have to identify two sets of pairs of lines and pasting homeomorphisms if they map each other by a permutation of the polygons (of the same type) and rotations inside the polygons. However, this corresponds topologically to a permutation of separatrices of a measured foliation and we have to note these transformations when we consider words and their effects on tracks. □

Words change lines, pairings and pasting homeomorphisms.

Now we describe an example of a pseudo-Anosov diffeomorphism. It is a diffeomorphism of Σ_2 (genus 2) and has the unstable foliation with one singularity with six separatrices.

Consider a hexagon with the counterclockwise orientation and name its edges $1, 2, \ldots, 6$ (Fig. 8.3). The angle between the edges i and $i+1$ is named p_i (which will be a prong when pasted). We take a combinatorial track $\tau_0 = (15-2, 64-3)$. This is a set of pairs $(15, 2)$ and $(64, 3)$, where xy is a line made of edges x and y connecting in this order, i.e. x terminates at the initial point of y. The pasting homeomorphisms are deterrmined since one of the lines in each pair has no interior connections (i.e. endpoints of edges). See Fig. 8.4a. This set satisfies the conditions (i), (ii) above. See Fig. 8.4b. It is realized, for example, by the track indicated in Fig. 8.4c. It is recurrent.

Now we apply a word. We use the following notation. Recall the types of the letters used in Proposition 2.6. We denote the left (resp. right) split of the type (i) between prongs p and q L_p (resp. R_p) or L_q (resp. R_q). The left (resp. right) split of the type (ii) between the left (resp. right) uppermost prong p and the right (resp. left) uppermost prong and the left (resp. right) split of the type (ii) between the left (resp. right)undermost prong and the right (resp. left) undermost prong p are denoted L_p (resp. R_p). The left (resp. right) slide of the type (iii) described as between the left (resp. right) uppermost prong p and the right (resp. left) undermost prong q is denoted G_p (resp. D_p) or G_q (resp. D_q). We apply a word $w = G_{p_3}L_{p_5}R_{p_6}L_{p_6}L_{p_4}R_{p_1}$ to τ_0 where the ordering of prongs is induced from the prong correspondence by the word on each track appearing from the word. The obtained track is $\tau_6 = (53-6, 42-1)$ and is isomorphic to τ_0 by the rotation of the hexagon which takes the edge 1 of τ_6 to the edge 3 of τ_0.

τ_0 has four capillaries, which we order as: the capillary corresponding to the edge 5 (and a half of 2) is denoted b_1. Similarly denote b_2 for 4 (and 3), b_3 for 1 (and 2) and b_4 for 6 (and 3) (Fig. 8.4d). Let f be the (isotopy class of the) diffeomorphism on Σ_2 determined by $w : \tau_0 \to \tau_6 = f(\tau_0)$ (which is unique in this case). Then the intersection matrix A determined naturally by w and f is

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

where the i-th row corresponds to b_i and the j-th column to $f(b_j)$.

First let us check the positivity condition (PC). The eigenvalues of A are the solutions of the equation

$$\lambda^4 - 2\lambda^3 - 2\lambda + 1 = 0$$

and the eigenvector for λ is

$$(\lambda^2, \lambda^3 - \lambda^2 - 1, \lambda^2 + 1, \lambda).$$

There are two solutions $0 < \lambda_1 < 1$ and $2 < \lambda_2 < 3$ and the eigenvector for λ_2 is positive, hence (PC) holds. (The switch condition is automatically satisfied in this case.) Since τ_0 has only one vertex, Theorem 4.3 applies as follows. Recall the definition of the maximal transparent set T. The prongs p_2, p_3 and p_5 of τ_6 are zero, but there is no invariant set including them. Hence T is vacant and, applying Theorem 4.3, we know that f is a pseudo-Anosov diffeomorphism fixing a foliation with one singularity with six separatrices. Also we can check the conditions in Theorem 5.2 easily.

Next we consider the track $\tau_1 = (15 - 2, c46 - c3A)$ (which appears on the way of applying w above), where c denotes that the line is circular and A is a suffix to represent the passing homeomorphism. It pastes the endpoints of $c3$ to the interior of 4 (Fig. 8.5a). Of course τ_1 is recurrent combinatorial track. The word $w' = L_{p_5}R_{p_6}L_{p_6}L_{p_4}R_{p_1}G_{p_1}$ takes τ_1 to $f'(\tau_1) = \tau_7 = (53 - 6, c24 - c1A)$ where A represents that the endpoints of $c1$ is pasted to the interior of 2. (f', in fact, equals to f.) They are isomorphic by the rotation taking 1 to 3 as before.

The capillaries of τ_1 is ordered as follows. The capillary made of the edge 5 (and a half of 2) is b_1, the one made of the first quarter of 3 and the first half of 4 is b_2, the one made of 1 (and a half of 2) is b_3, the one made of 6 and the middle half of 3 is b_4 and the one made of the last quartet of 3 and the last half of 4 is b_5 (Fig. 8.5b). The intersection matrix A' determined naturally is

$$A' = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

($f(b_5)$ is contained in a vertex, hence the last column is zero.) (Fig. 8.5c.)

Another (and this is the only other) intersection matrix A'' is

$$A'' = \begin{pmatrix} 0\,1\,1\,0\,0 \\ 1\,1\,0\,1\,0 \\ 0\,1\,0\,0\,1 \\ 1\,0\,0\,0\,0 \\ 1\,1\,1\,1\,0 \end{pmatrix}.$$

See Fig. 8.5d.

Also the extended intersection matrix $\overline{A}'$ is

$$\overline{A}' = \begin{pmatrix} 0\,1\,1\,0\,0\,0\,1\,0\,0\,0\,0 \\ 1\,1\,0\,1\,0\,0\,1\,1\,0\,0\,0 \\ 0\,1\,1\,1\,0\,0\,1\,0\,0\,0\,1 \\ 1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 1\,1\,1\,1\,0\,1\,1\,1\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1 \\ 0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0 \end{pmatrix}.$$

The eigenvalues of A' is the same as A above. The (column) eigenvector for λ_2 is

$$(\lambda_2^2, \lambda_2^3 - \lambda_2^2 - 1, \lambda_2^2 + 1, \lambda_2, \lambda_2^2 + \lambda_2 + 1),$$

which is positive and satisfies the switch conditions $m(b_1)+m(b_5) = m(b_1) + m(b_3) + m(b_4)$ and $m(b_2) + m(b_3) + m(b_4) = m(b_2) + m(b_5)$ where $m(b_i)$ denotes the transverse measure of b_i induced from the eigenvector, and (PC) holds. The length eigenvector (i.e. the row eigenvector) for λ_2 is

$$\left(\lambda_2 - 1, \frac{\lambda_2^3 - \lambda_2^2 - 1}{\lambda_2 + 1}, \lambda_2, \frac{\lambda_2^2(\lambda_2 - 2)}{\lambda_2 - 1}, 0\right)$$

and the vector of travel lengths of $f'(\tau_1)$ is then

$$(0, \text{positive}, \text{positive}, 0, 0, \text{positive})$$

where the k-th entry is for $f'(p_k)$. If we take the third power f'^3 of f' so that the prong correspondences coincide, the vector P of travel lengths of $f'^3(\tau_1)$ is positive, which is easily checked using $\overline{A}'$.

Now we apply Theorem 6.10. $f'^3(\tau_1)$ has only one 0-capillary, namely $f'^3(b_5)$, and no 0-prongs. Hence $f'^3(\tau_1)$ has no such patterns of 0-capillaries and 0-prongs described in Theorem 6.10 and we see that f' is pseudo-Anosov.

Next we apply Theorem 7.17 using the cellular decomposition of W_{τ_1}. Let us represent an (open) cell by the set of the capillaries and the prongs which can be made positive by a value isotopy applied to a value in the cell. The generatrices of W_{τ_1} is mapped by w' as follows. Assume that the generatrices of W_{τ_7} are ordered by the correspondence by f'. (The generatrix form) b_1 of W_{τ_1} is mapped to the sum $b_2 + b_3 + p_2$ of the generatrices of W_{τ_7} (where the sum is taken in G_{τ_7}), b_2 to $b_1 + b_2 + b_4 + p_2 + p_3$, b_3 to $b_2 + \cdots + b_4 + p_2 + p_6$, b_4 to b_1, b_5 to $b_1 + \cdots + b_4 + p_1 + \cdots + p_3$, p_1 to p_3, p_2 to $p_4, \ldots, p_6$ to p_2. Hence, for example, b_1 is mapped to the cell $\{b_2, b_3, p_2\}$ of W_{τ_7}, b_3 to $C = \{b_2, \ldots, b_5, p_2, p_3, p_6\}$ because the values with positive entries on b_2, b_3, b_4, p_2 and p_6 can be isotoped to the values with positive entries on $b_2, \ldots, b_5$, p_2, p_3 and p_6 (Fig. 8.5e).

The cell C is mapped by w' to the cell $\{b_1, \ldots, b_5, p_1, \ldots, p_6\}$ which is the largest cell consisting of the interior points of W_{τ_7}. Hence b_3 is mapped into the interior by w'^2. A similar calculation shows us that all the capillaries are mapped into the interior (by w'^3). Hence f' is pseudo-Anosov. (We do not have to check (PC).) Above applications of Theorem 6.10 and Theorem 7.17 are similar if we take A'' instead of A'.

An example of an application of the theorems to a diffeomorphism which is not pseudo-Anosov is given in the following. $\sigma_0 = (1-4, 2-5, c3-c6)$ is a recurrent combinatorial track. The word

$$w_1 = L_{p_5} R_{p_5} L_{p_2} D_{p_2} R_{p_6} D_{p_2} R_{p_1} R_{p_4} R_{p_2} R_{p_3} R_{p_6} L_{p_3} L_{p_6} : \sigma_0 \to \sigma_{13}$$

defines the (isotopy class of the) diffeomorphism f_1 satisfying $f_1(\sigma_{13}) = \sigma_0$ with the prong correspondence coinciding with that of w_1. The ordering of the capillaries of σ_0 is: b_1 corresponds to the first half of 3 and that of 6, b_2 to 2 (and 5), b_3 to 1 (and 4) and b_4 to the last half of 3 and that of 6. See Fig. 8.6, where the word is represented as an unzipping.

The intersection matrix A_1 is

$$A_1 = \begin{pmatrix} 2&1&0&0 \\ 2&2&1&0 \\ 0&0&2&1 \\ 3&2&0&0 \end{pmatrix}.$$

A_1 is Perron-Frobenius (primitive irreducible). In fact,

$$A_1^3 = \begin{pmatrix} 20&14&6&1 \\ 31&22&14&4 \\ 16&11&10&4 \\ 34&24&11&2 \end{pmatrix}.$$

Therefore there is a positive eigenvector with a positive eigenvalue and (PC) holds. (The switch condition is satisfied automatically in this case.) The maximal transparent set T_1 is $\{p_1, p_4\}$ and hence f_1 is not pseudo-Anosov. (This is also an example for the Remark at the end of Section 4.)

Using the track $\sigma_4 = (1-4, c2-c5, c3-c6)$ appearing on the way of w_1, we make another example. Assume

$$w_1' = R_{p_6} D_{p_2} R_{p_1} R_{p_4} R_{p_2} R_{p_3} R_{p_6} L_{p_3} L_{p_6} L_{p_5} R_{p_5} L_{p_2} D_{p_2} :$$

$$\sigma_4 \to \sigma_{17} = f_1'(\sigma_4)$$

has the prong correspondence coinciding with that of f_1'. (f_1' is, in fact, equal to f_1.) The capillaries of σ_4 are denoted as: b_1 corresponds to the last half of 2 (and that of 5), b_2 to the first half of 2 (and that of 5), b_3 to 1 (and 4), b_4 to the last half of 3 (and that of 6) and b_5 to the first half of 3 (and that of 6). Then the intersection matrix A_1' is

$$A_1' = \begin{pmatrix} 0&1&0&0&0 \\ 2&2&1&4&0 \\ 2&0&2&3&0 \\ 1&0&0&2&0 \\ 5&1&3&9&0 \end{pmatrix}.$$

Again (PC) holds. (That goes without saing because w_1' is intrinsically the same as w_1.) The length eigenvector is

$$(3, \sqrt{3}, 1, 4+\sqrt{3}, 0)$$

and the eigenvalue is $2+\sqrt{3}$. The vector of travel lengths is

$$(0, 6+\sqrt{3}, 1+\sqrt{3}, 0, 6+4\sqrt{3}, 1+\sqrt{3}).$$

The 0-capillary b_5 starts from one of the 0-prongs p_1 and ends into the other p_4. Hence we know again that f_1' is not pseudo-Anosov.

Using the cellular decomposition as before, this is shown as follows. The generatrix b_1 of W_{σ_4} is mapped to b_2 of $W_{\sigma_{17}}$. b_2 is mapped to $2b_1+2b_2+b_3+4b_4+3p_2+p_3+3p_5+p_6$ which is in the cell $C_1 = \{b_1, \ldots, b_4, p_2, p_3, p_5, p_6\}$ of $W_{\sigma_{17}}$. b_3 is mapped to $2b_1+2b_3+3b_4+2p_2+p_3+2p_5+p_6$, which is in the cell $C_2 = \{b_1, b_3, b_4, p_2, p_3, p_5, p_6\}$. b_4 is mapped to $b_1+2b_4+p_2+p_5$, which is in the cell $C_3 = \{b_1, b_4, p_2, p_5\}$. b_5 is mapped to $5b_1+b_2+3b_3+9b_4+6p_2+2p_3+p_4+5p_5+3p_6$, which is in the cell $\{b_1, \ldots, b_5, p_1, \ldots, p_6\}$, i.e. the interior of $W_{\sigma_{17}}$. C_1 is mapped into itself, thus the generatrix b_2 cannot go into the interior even if w_1' and f_1' is infinitely many times repeated. (C_2 is mapped into C_1, C_3 is mapped into $C_4 = \{b_1, b_2, b_4, p_2, p_5\}$ and C_4 is mapped into C_1.) Hence f_1' is not pseudo-Anosov.

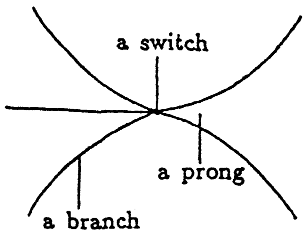


FIGURE 1.1

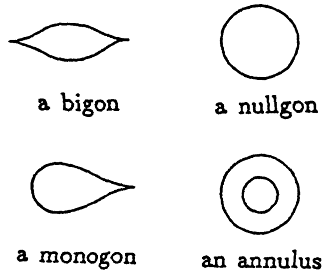
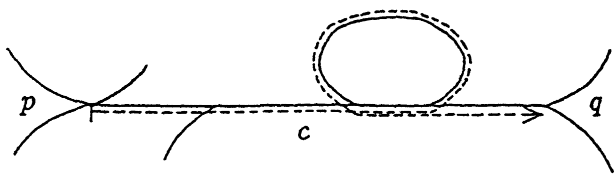


FIGURE 1.2



c starts from p and ends into q.

FIGURE 1.3

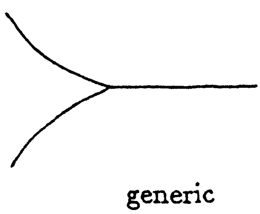


FIGURE 1.4

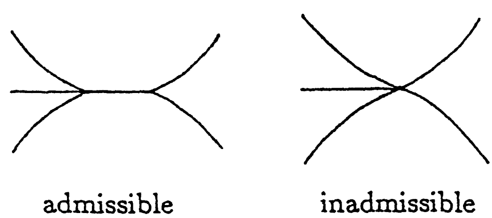


FIGURE 1.5

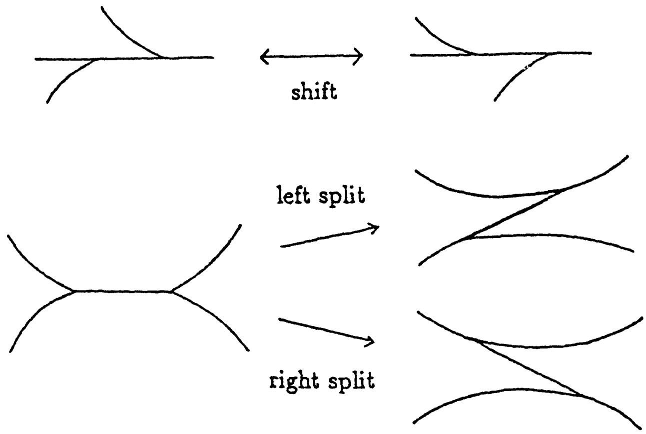


FIGURE 1.6 a

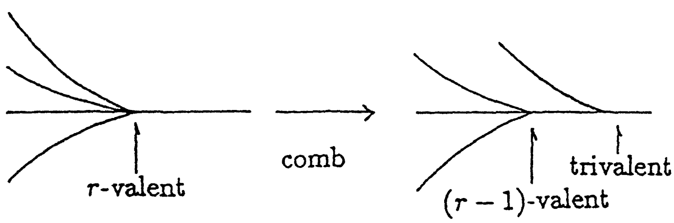


FIGURE 1.6 b

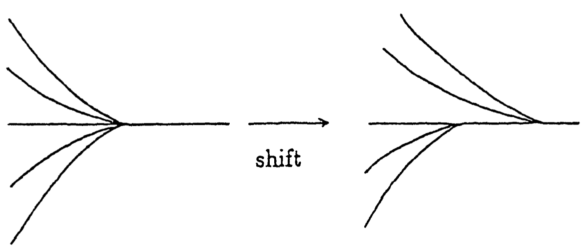


FIGURE 1.6 c

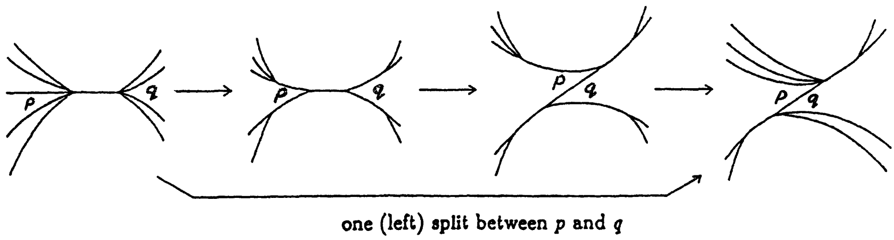


FIGURE 1.6 d

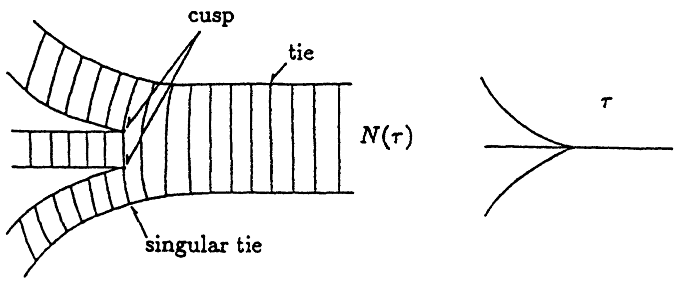


FIGURE 1.7

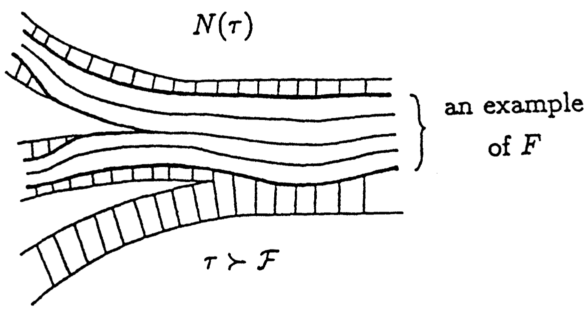


FIGURE 1.8

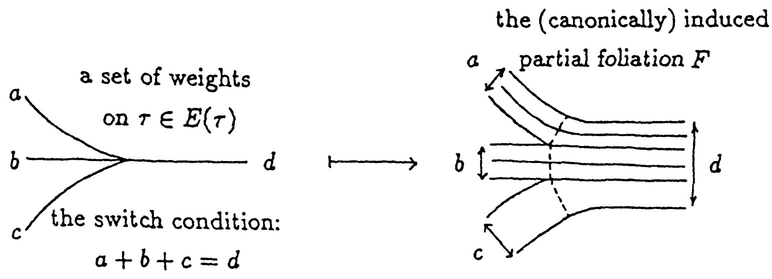
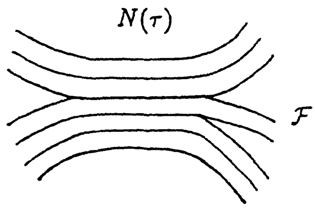


FIGURE 1.9



an example of $\mathcal{F}$ (represented as a partial foliation) and τ not suited to $\mathcal{F}$

FIGURE 1.10

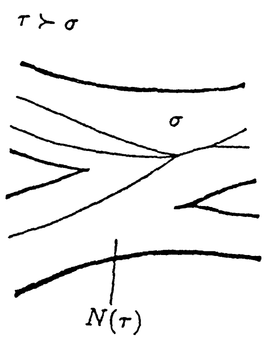


FIGURE 1.11 a

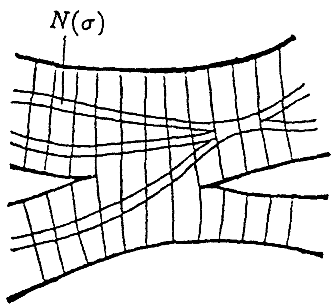
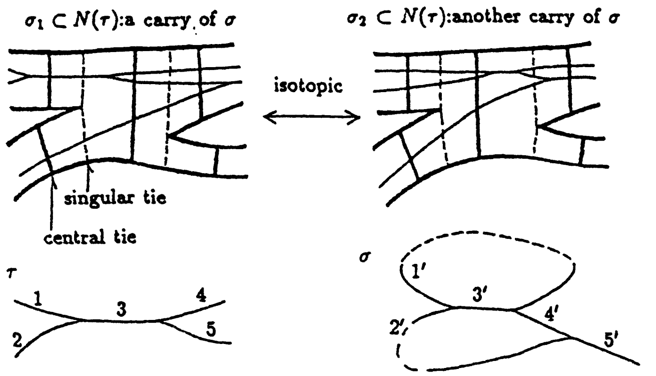


FIGURE 1.11 b



the intersection matrix by σ_1

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1' & 2' & 3' & 4' & 5' \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

the intersection matrix by σ_2

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1' & 2' & 3' & 4' & 5' \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

FIGURE 1.12

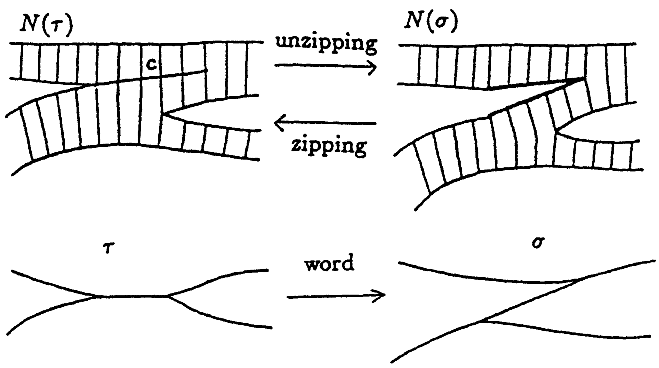


FIGURE 1.13

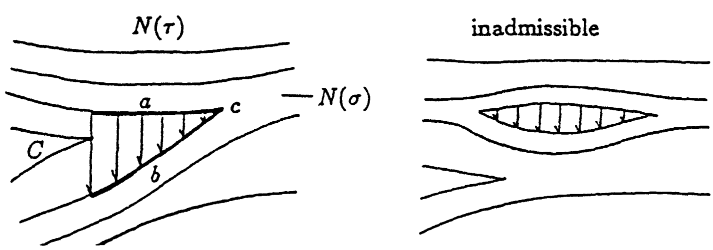


FIGURE 1.14

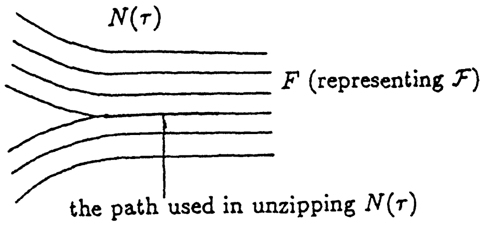


FIGURE 1.15

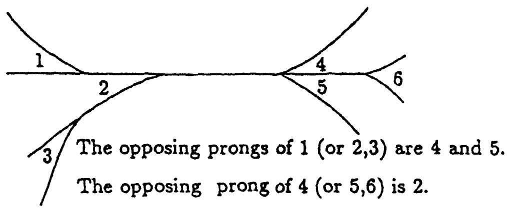


FIGURE 2.1

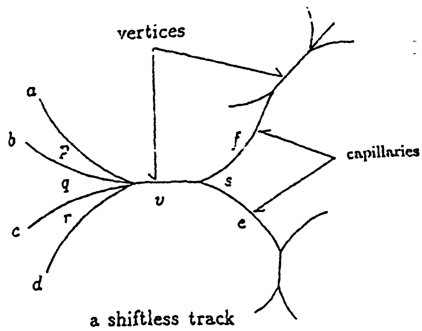


FIGURE 2.2

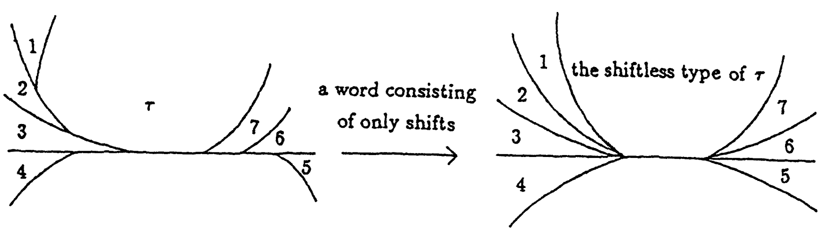


FIGURE 2.3

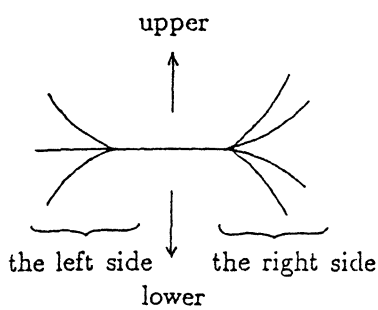


FIGURE 2.4 a

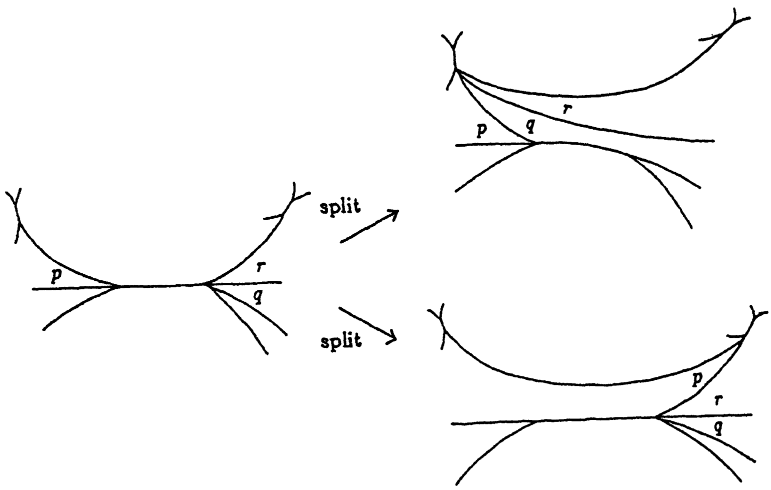


FIGURE 2.4 b

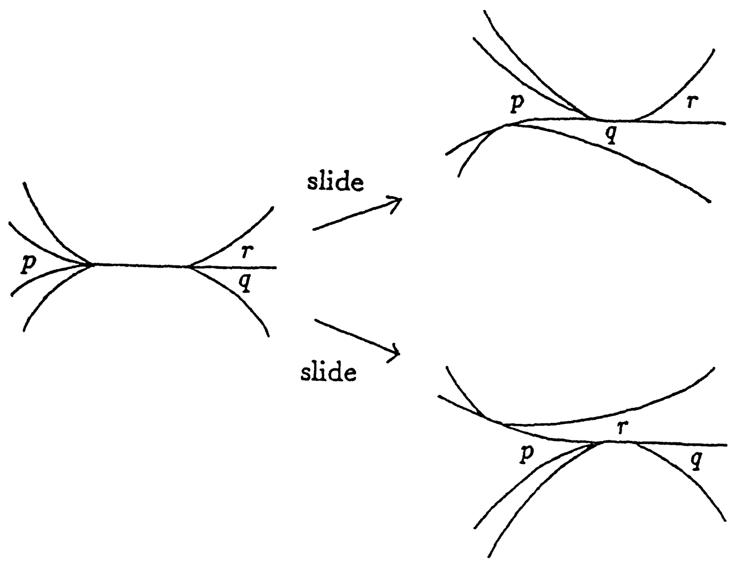


FIGURE 2.4 c

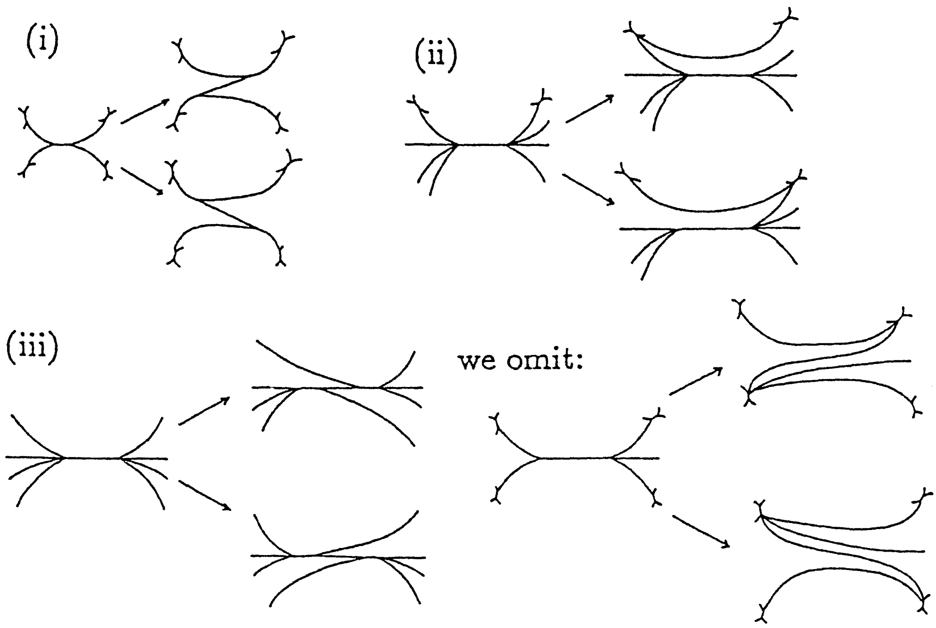


FIGURE 2.5

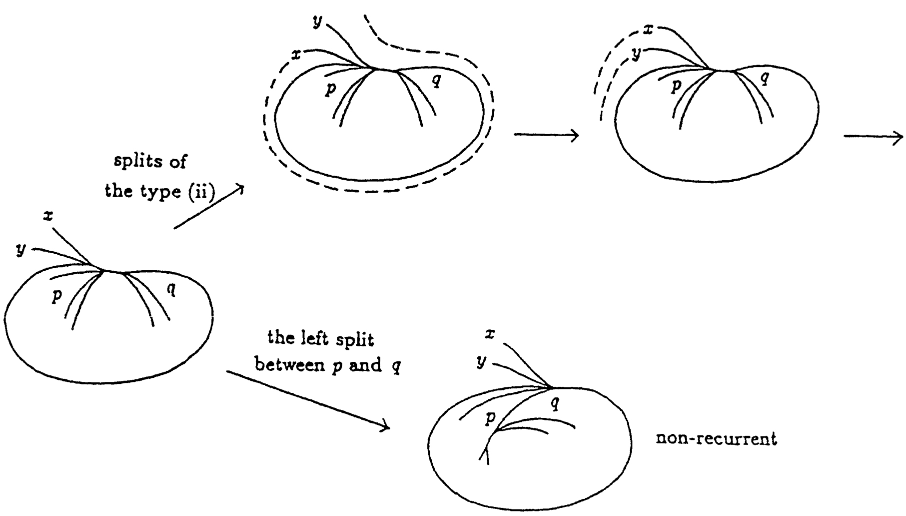


FIGURE 2.6 a

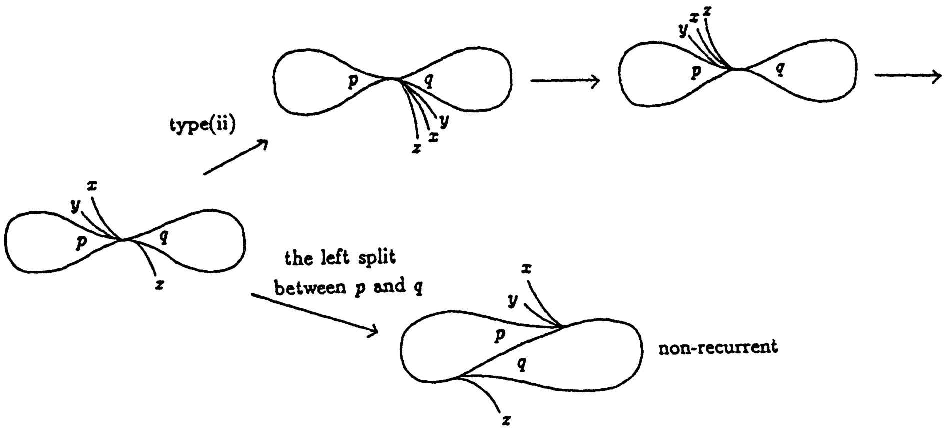
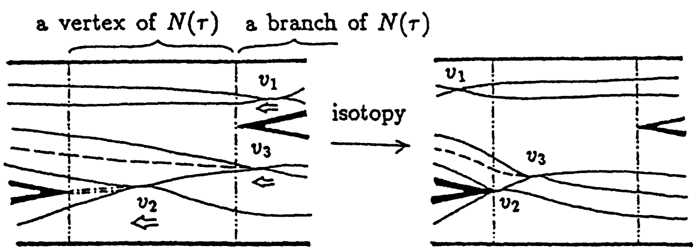
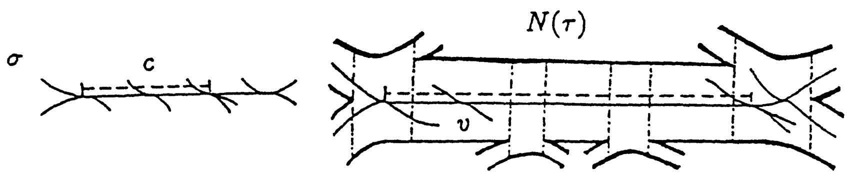


FIGURE 2.6 b



v_1 is left isotoped.
v_2 has a sticking on its left prong and cannot be left isotoped. (The path making the correspondence between the prong and a cusp is drawn by =:==:==:== .)
v_3 has no stickkings on its prong, but since v_2 sticks, v_3 is left isotoped maximally in the right figure.

FIGURE 2.7



an example of a maximally isotoped carry of c
There are many possible location of v.

FIGURE 2.8

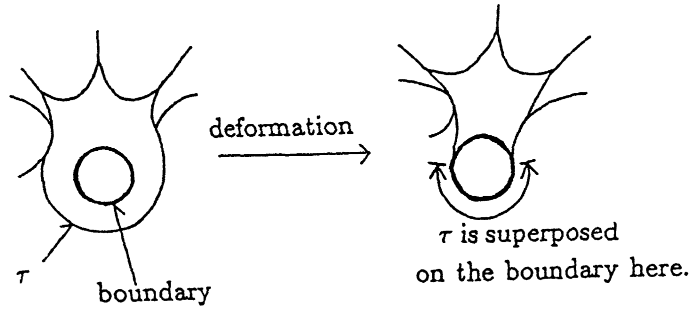


FIGURE 3.1 a

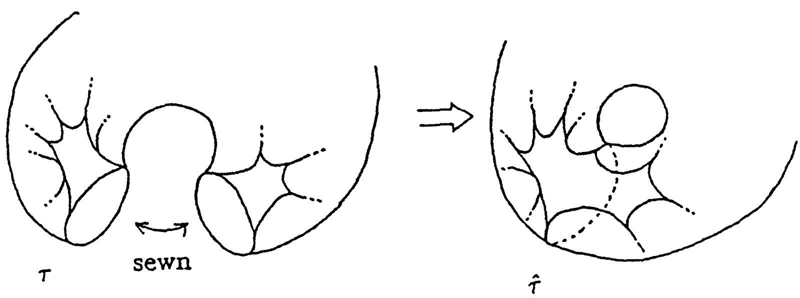


FIGURE 3.1 b

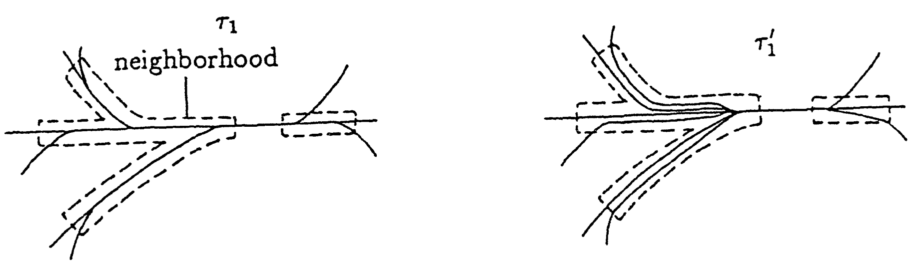


FIGURE 4.1

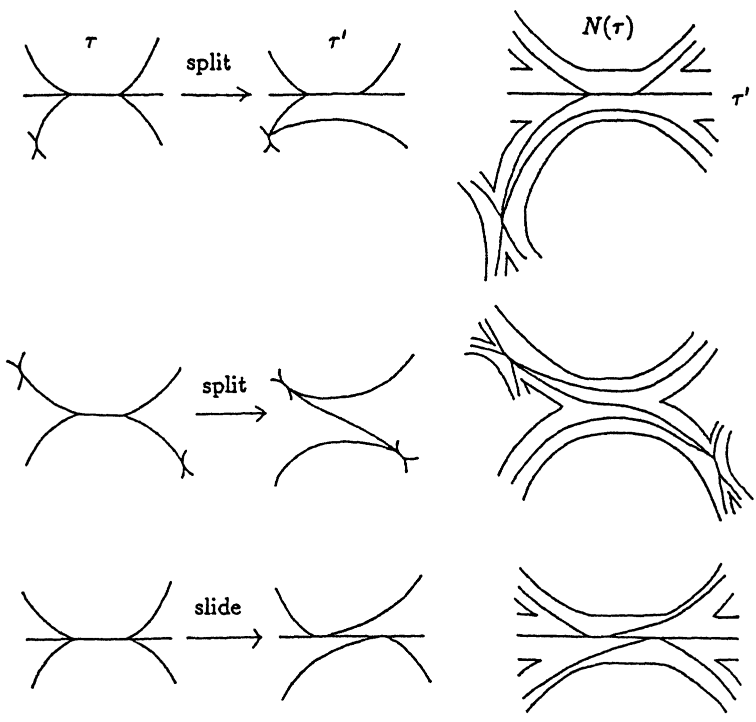
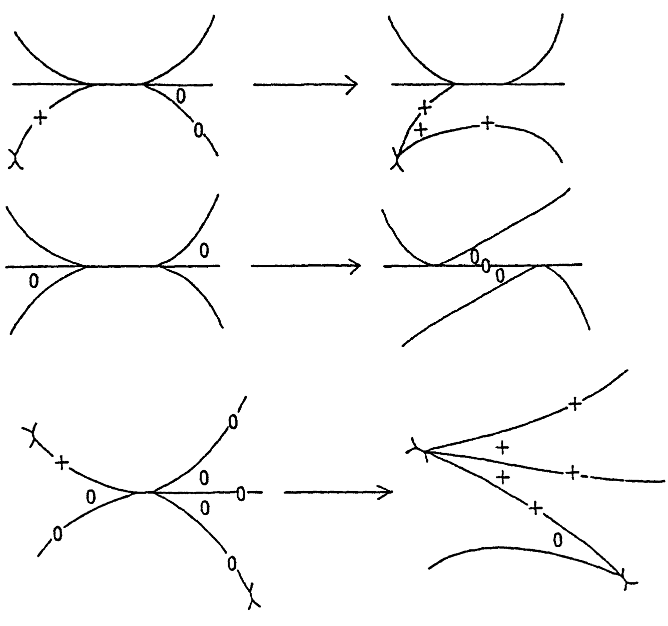


FIGURE 4.2



Positive capillaries and prongs are marked with +, zero ones with 0.

FIGURE 4.3

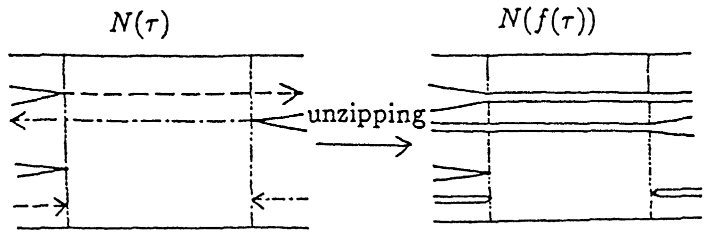


FIGURE 4.4

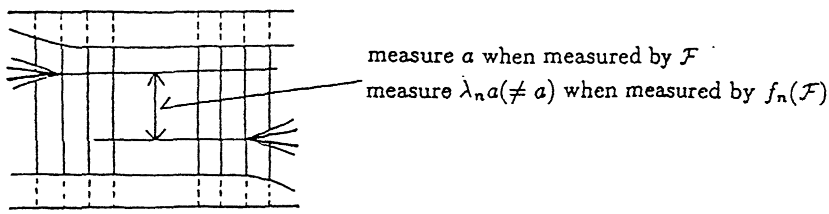


FIGURE 4.5

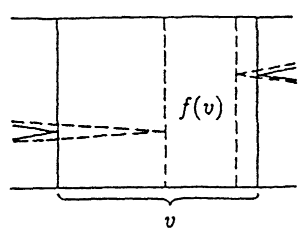


FIGURE 5.2

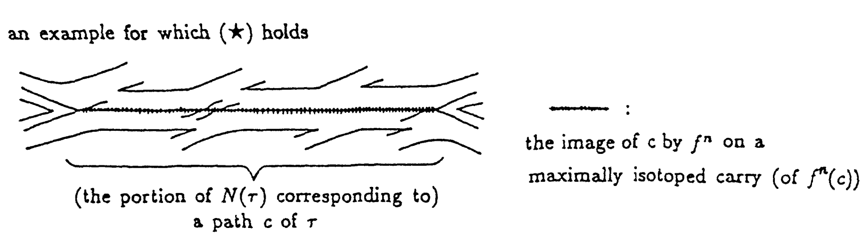


FIGURE 5.1

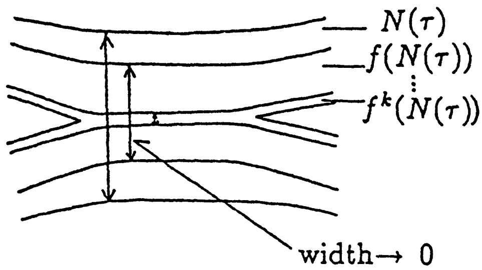


FIGURE 5.3

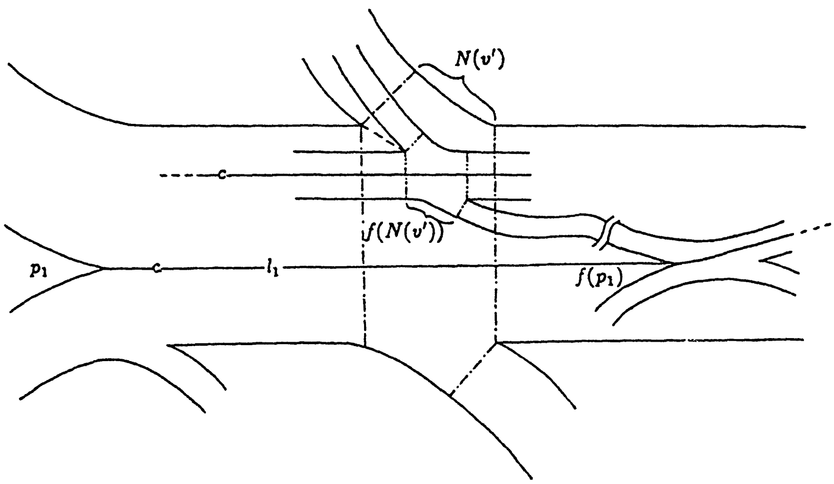


FIGURE 5.4

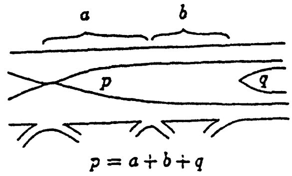
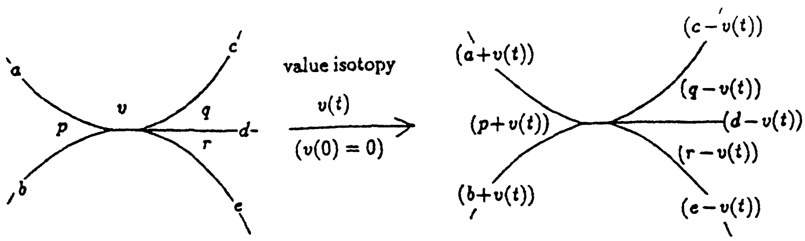
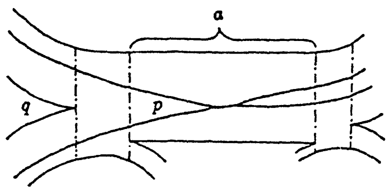


FIGURE 6.1



$(a+v(t)),\ldots,(r-v(t))$ are all non-negative for $t \in [0,1]$.

FIGURE 6.2



$q < p < a+q$

FIGURE 6.3

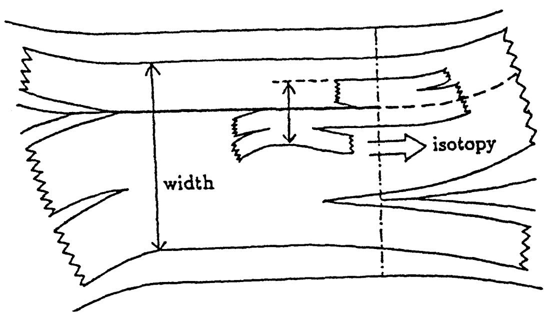


FIGURE 6.4

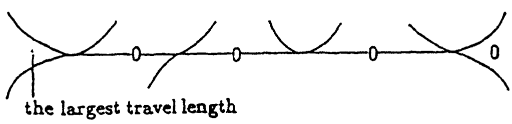


FIGURE 6.5

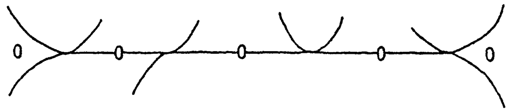
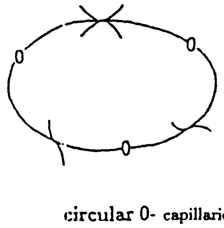


FIGURE 6.6



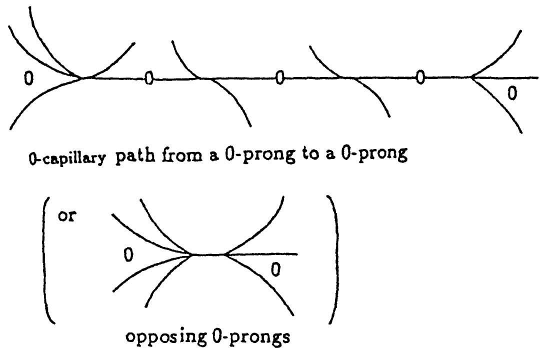
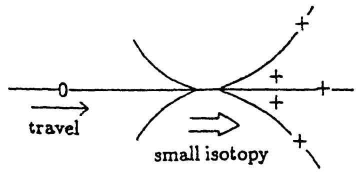


FIGURE 7.1 a

FIGURE 7.1 b



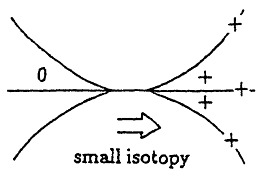


FIGURE 7.2 a

FIGURE 7.2 b

FIGURE 7.4 a

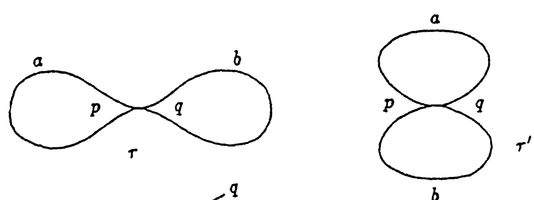
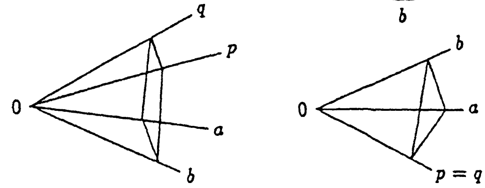


FIGURE 7.4 b



the image W_τ of $F(\tau)$ in $G_\tau \cong R^3$ the image $W_{\tau'}$ of $F(\tau')$ in $G_{\tau'} \cong R^3$

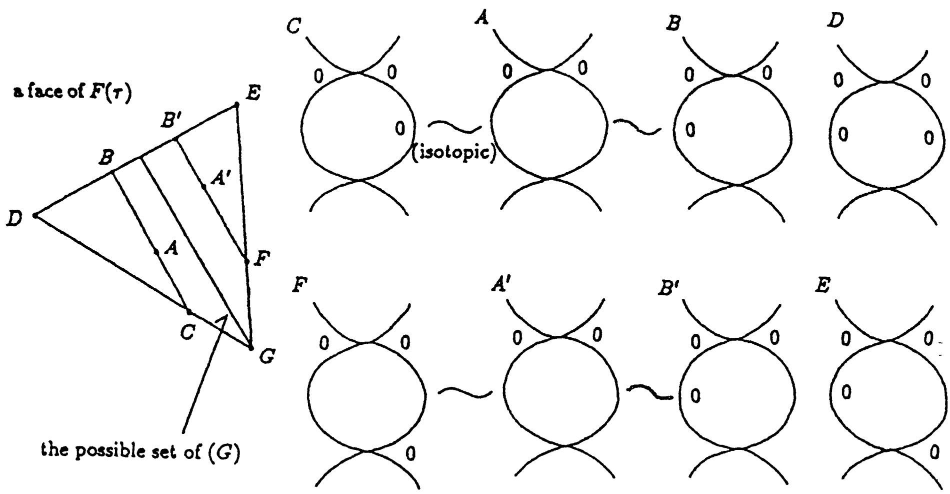


FIGURE 7.3

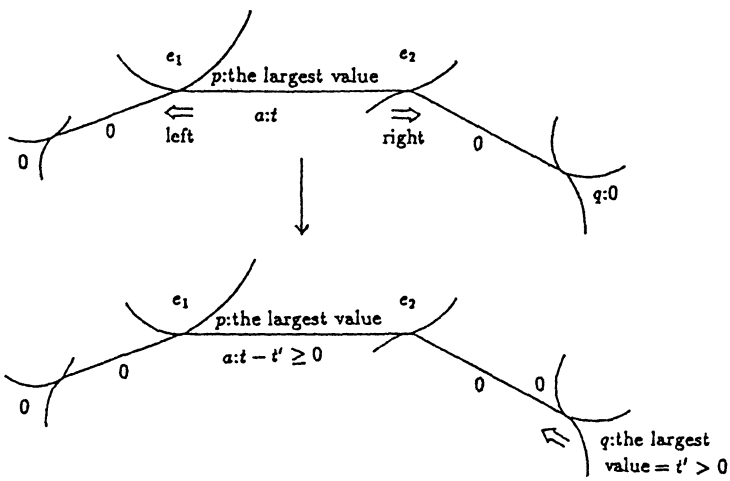


FIGURE 7.5 a

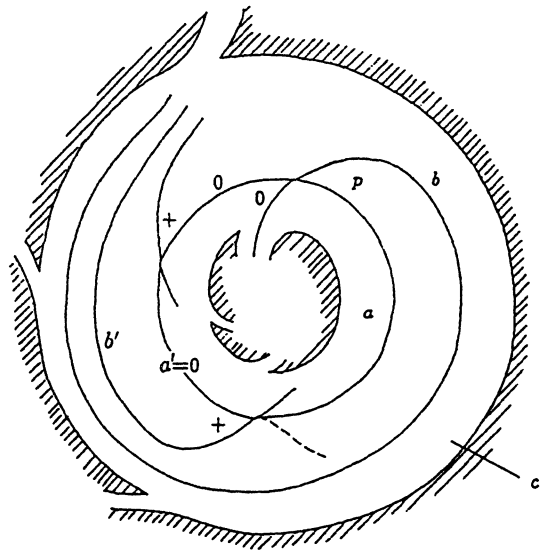


FIGURE 7.5 b

FIGURE 7.6 a

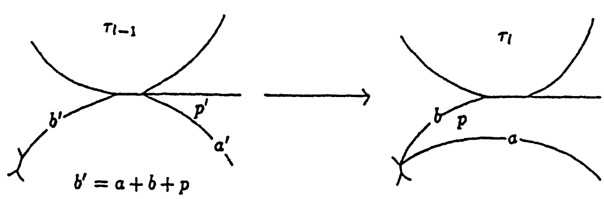


FIGURE 7.6 b

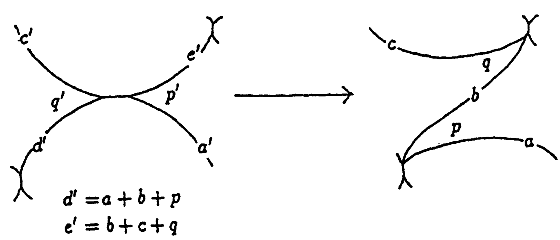
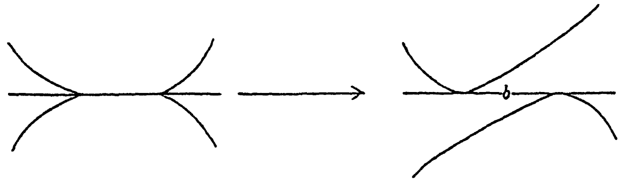


FIGURE 7.6 c



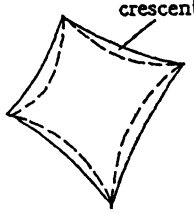


FIGURE 8.1 a

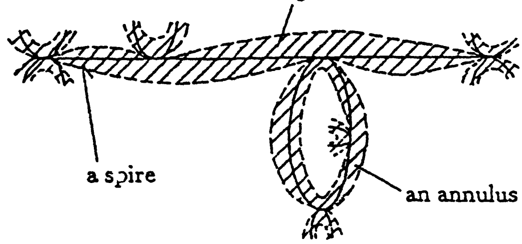


FIGURE 8.1 b

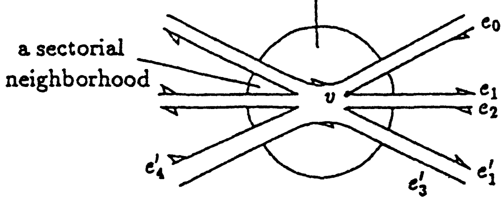


FIGURE 8.2 a

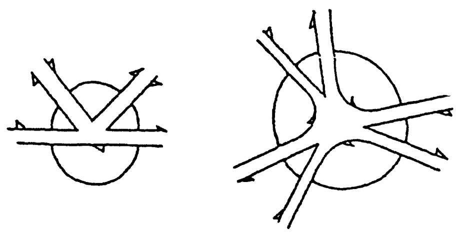


FIGURE 8.2 b

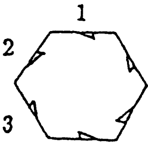


FIGURE 8.3

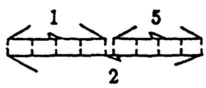


FIGURE 8.4 a

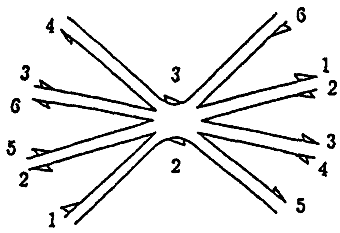


FIGURE 8.4 b

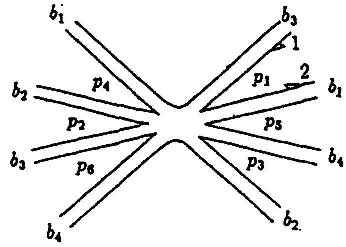


FIGURE 8.4 d

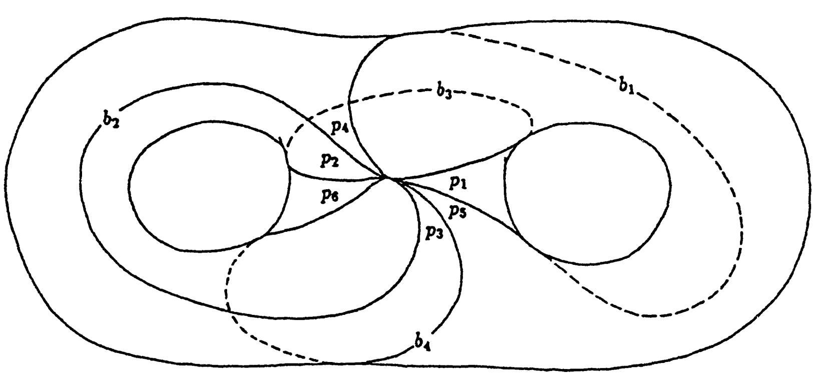


FIGURE 8.4 c

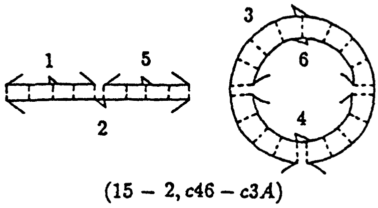


FIGURE 8.5 a

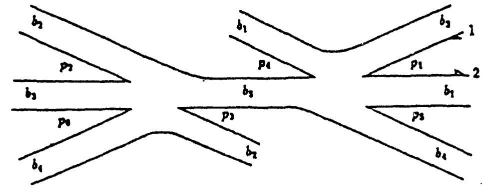


FIGURE 8.5 b

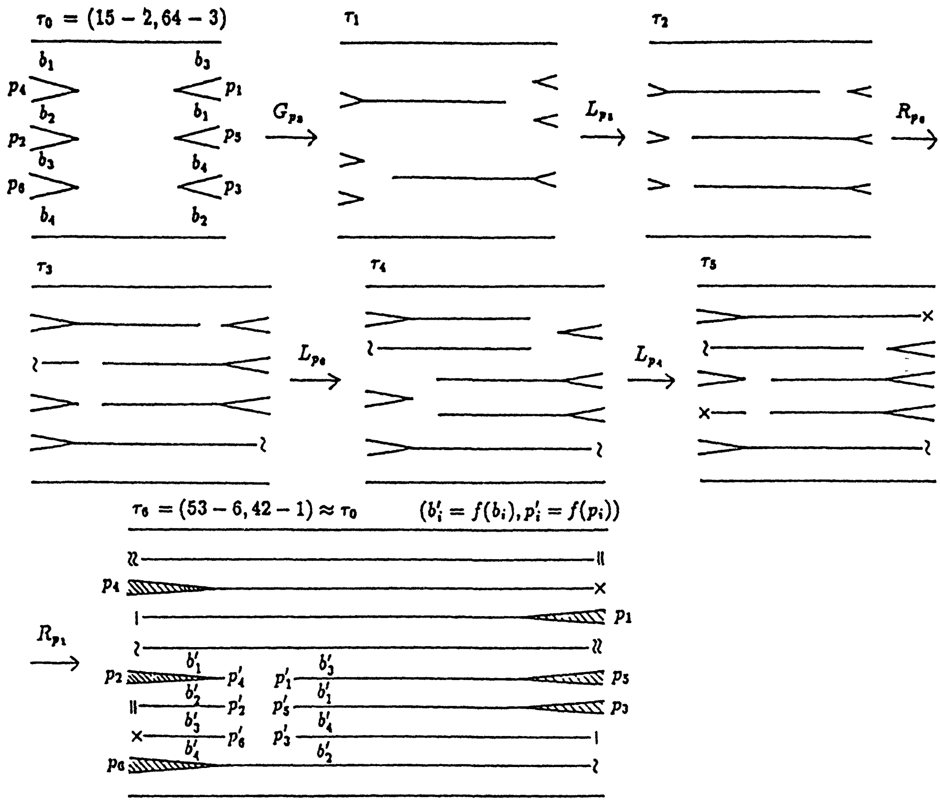

$\tau_0 = (15-2, 64-3)$
τ_1
τ_2
τ_3
τ_4
τ_5
$\tau_6 = (53-6, 42-1) \approx \tau_0$
$(b_i' = f(b_i), p_i' = f(p_i))$
G_{p_5}
L_{p_1}
R_{p_6}
L_{p_6}
L_{p_4}
R_{p_1}

FIGURE 8.4 e

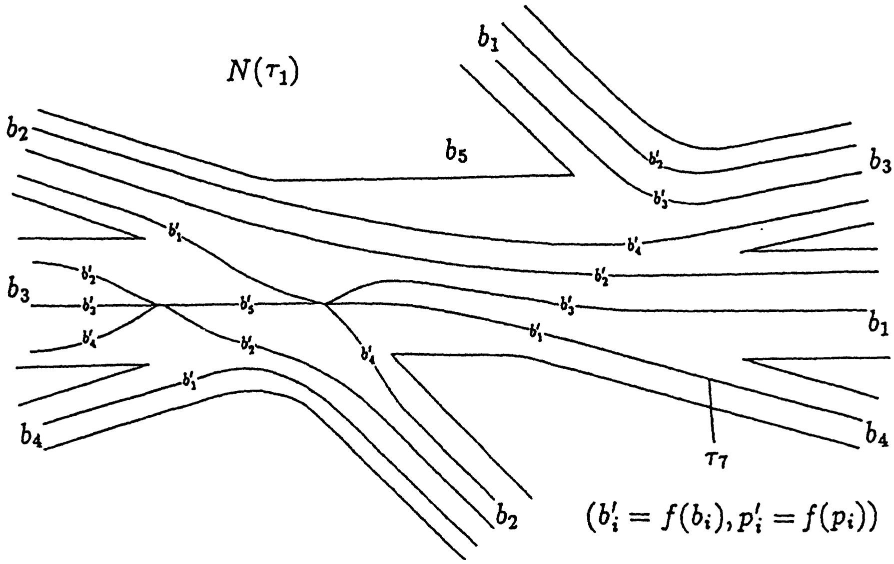


FIGURE 8.5 c

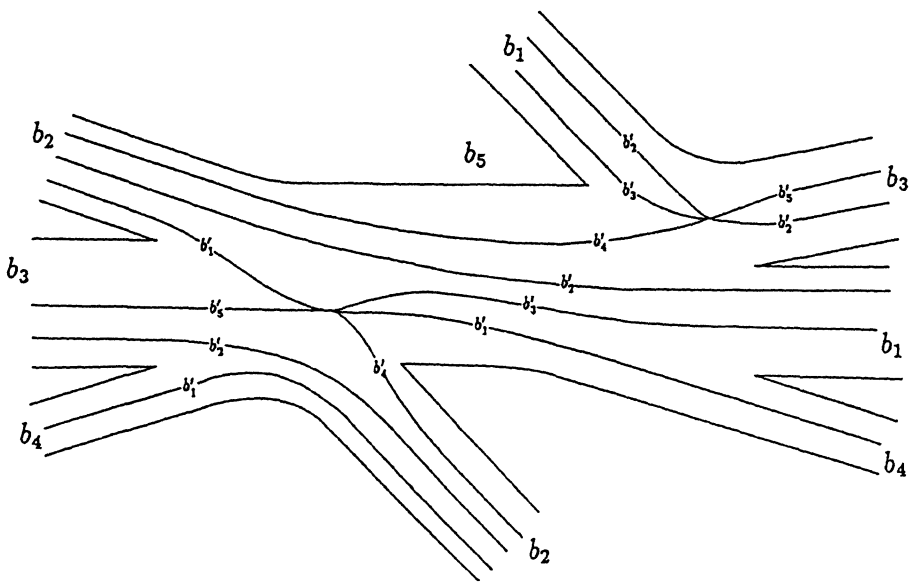


FIGURE 8.5 d

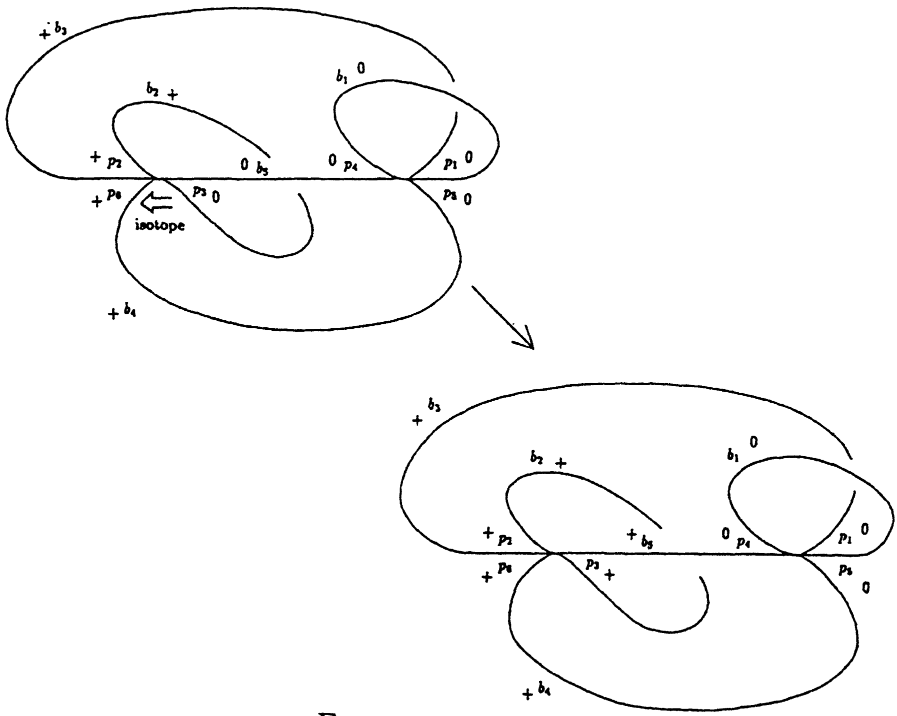


FIGURE 8.5 e

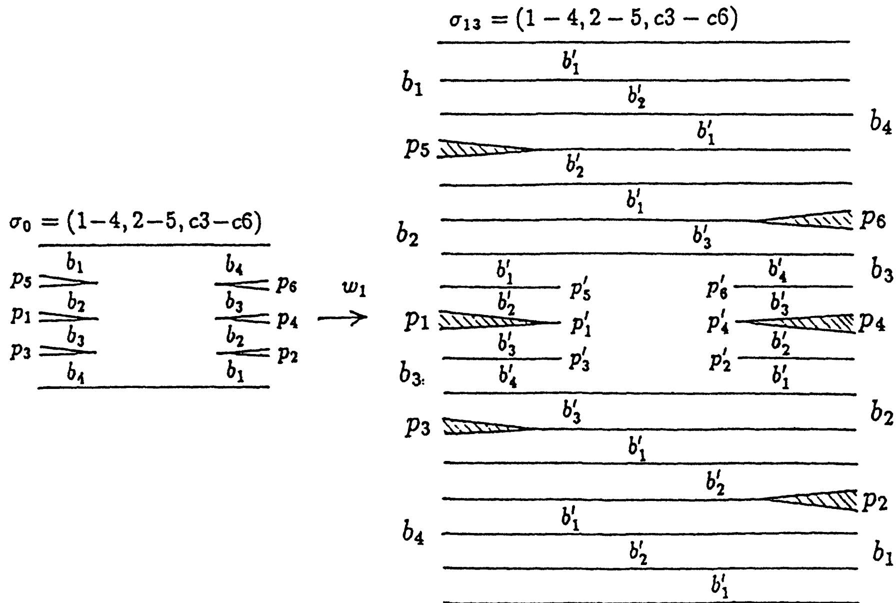
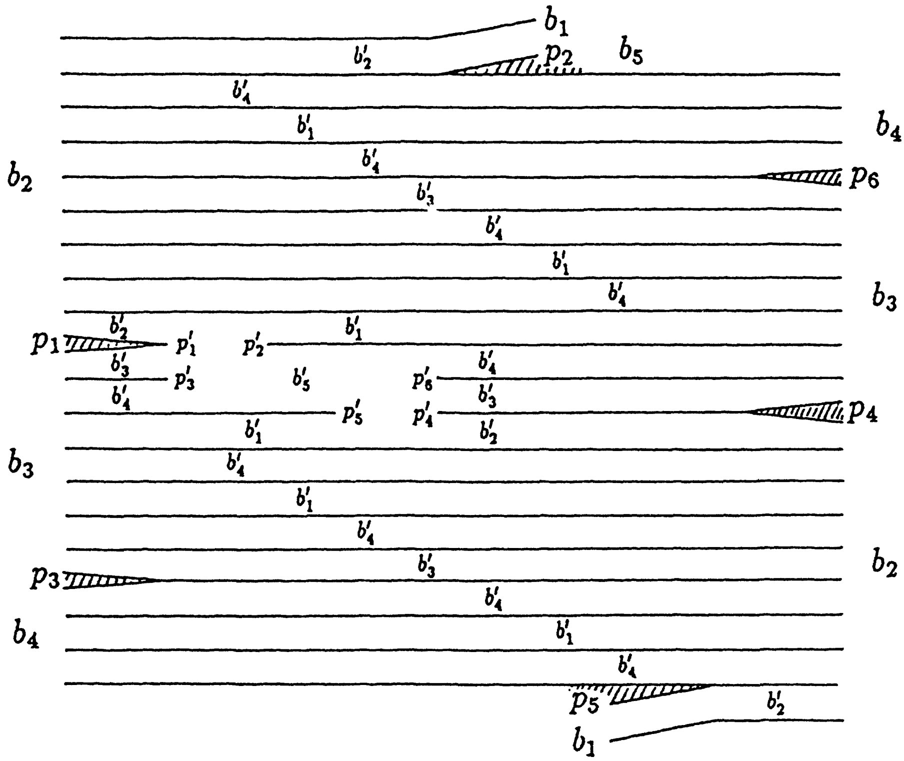


FIGURE 8.6



b_1
b'_2
p_2
b_5
b'_4
b'_1
b_4
b'_4
b_2
p_6
b'_3
b'_4
b'_1
b'_4
b_3
b'_2
p_1
p'_1
p'_2
b'_1
b'_3
p'_3
b'_5
p'_6
b'_4
b'_4
p'_5
p'_4
b'_3
p_4
b'_1
b'_2
b_3
b'_4
b'_1
b'_4
p_3
b'_3
b_2
b'_4
b_4
b'_1
b'_4
p_5
b'_2
b_1

FIGURE 8.7

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