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PAIRED CALIBRATIONS APPLIED TO SOAP FILMS, IMMISCIBLE FLUIDS, AND SURFACES OR NETWORKS MINIMIZING OTHER NORMS

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In this paper we introduce a new method for proving area-minimization which we call "paired calibrations." We begin with the simplest application, the cone over the tetrahedron, which appears in soap films. We then discuss immiscible fluid interfaces, crystal surfaces, and one-dimensional networks minimizing other norms.

1. Introduction In her classification of soap-film singularities [**T1**], Jean Taylor proved only by the process of elimination that the cone over the edges of the regular tetrahedron minimizes area among surfaces separating the four faces. We give a direct proof which applies to regular simplices in all dimensions. See Figure 1.0.1.

Configurations of several immiscible fluids try to minimize an energy proportional to interfacial surface area, but the constant of proportionality varies for each pair of fluids. Chapter 2 proves that certain cones minimize such weighted areas.

The surface energy of a crystal depends on direction, as given by a norm Φ on unit normals. Chapter 3 proves certain cones Φ-minimizing, such as a cone over a triangular prism. The hypotheses involve basic geometric questions, such as the number of possible cardinalities of equilateral sets (i.e., sets of pairwise equidistant points) for a norm on R^n.

We also consider 1-dimensional Φ-minimizing networks for differentiable norms Φ. It is well-known that length-minimizing networks meet in threes at 120° angles. Chapter 4 classifies the singularities in Φ-minimizing networks in R^n and establishes $n+1$ as the sharp bound on the number of segments that can meet at a point.

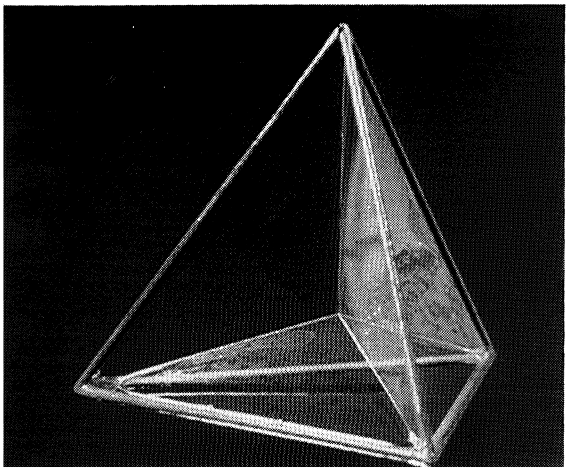

FIGURE 1.0.1. The cone over the tetrahedron provides the least-area soap film which separates the four regions. Photo by F. Goro.

1.1 The regular simplex cone is area-minimizing. As an illustration of paired calibrations, we now sketch a proof that the truncated cone C over the $(n-2)$-skeleton of the regular simplex centered at the origin in R^n is area-minimizing among hypersurfaces separating the $(n-1)$-dimensional faces F_i of the simplex.

Let p_i be the vertices of the dual regular simplex with unit length edges. Each p_i lies on the ray from the origin through the center of the face F_i, and all p_i are at the same distance from the origin. Note that $p_j - p_i$ is the unit normal to a piece of the cone C.

Consider a competing surface M, dividing the simplex into regions R_i containing F_i. (If any region is a "bubble" containing no F_i, just call it part of R_1.) Let M_{ij} be the surface separating R_i from R_j, oriented with normal pointing into R_j. Since p_i is a constant vectorfield, its flux through the boundary of R_i is zero by the divergence theorem. Thus,

$$\sum_i (\text{Flux of } p_i \text{ through } F_i) = -\sum_{i \neq j} (\text{Flux of } p_i \text{ through } M_{ij})$$

$$= \sum_{i<j} (\text{Flux of } p_j - p_i \text{ through } M_{ij}) \leq \sum_{i<j} \mathit{area}\ M_{ij}.$$

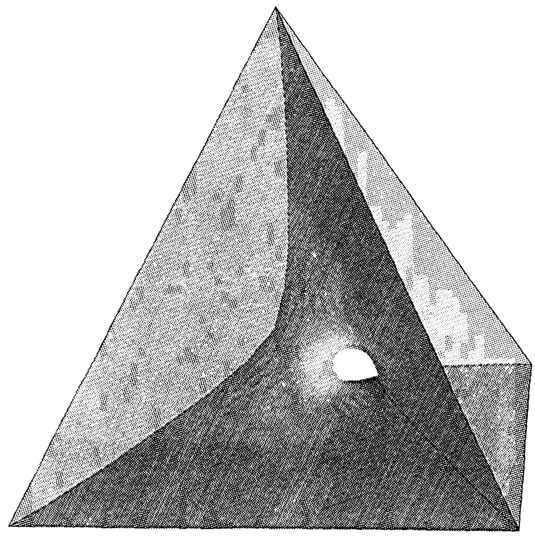

FIGURE 1.1.1. It is an open question whether the tetrahedron bounds a smaller soap film which does not separate the four regions. This figure was done by Jean Taylor of Rutgers University and The Geometry Center, following an idea due to Bob Hardt.

The first term is independent of M, and we get equality if $M = C$, so that

$$\textit{area } C \leq \textit{area } M.$$

We call the p_i paired calibrations because on each piece of surface we are considering the combined effect of two fluxes. (In place of flux we could have used differential forms, as in the standard theory of calibrations.)

For this simplex cone there is an interesting variation on the proof, using projections onto the faces of the simplex. For each i, project $M \cap \partial(R_i)$ orthogonally onto F_i. Each regular point of M gets projected onto two faces, say F_i and F_j. The sum of the two stretch factors (signed Jacobians) is maximized when M_{ij} is perpendicular to $p_j - p_i$, which is true everywhere if $M = C$. Since C is stretched the most, it must have the least area.

It is an open question whether the tetrahedral frame bounds a smaller stable or unstable soap film which does not separate the four regions. See Figure 1.1.1.

A related open question asks whether the standard triple bubble is the least-area way to enclose three given volumes. See Figure 1.1.2.

We remark that a hypersurface minimizes area among separators if and only if it is size-minimizing (for some orientation with multiplicities; cf. [**M8** 2.8]).

Among separating hypersurfaces, area-minimizing of course implies $(M, 0, \infty)$-minimal (the "area-minimizing" conditions of [**T1**, I. (8)]). The converse holds in R^n for $n \geq 4$, as follows by the methods of B. White [**W1**]; it fails for $n = 3$, although it does hold for $n = 2$.

Ken Brakke discovered our fundamental idea independently and has developed it further (see [**B1**], [**B2**], [**B3**]). For a partial extension to curvey minimal surfaces and constant-mean-curvature surfaces see [**M11**].

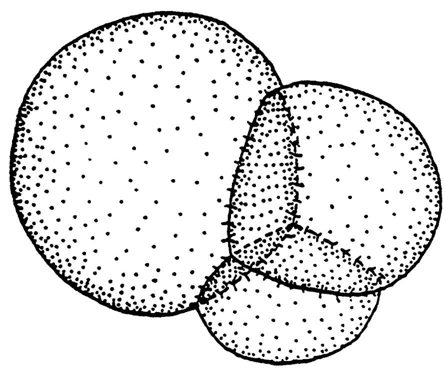

FIGURE 1.1.2. It is an open question whether the standard triple bubble is the least-area way to enclose three given volumes. (Jim Bredt) [**M10**].

1.2 Immiscible fluids (Chapter 2). A configuration of immiscible fluids $F_1, \ldots, F_m$, such as air, benzene, mercury, and water, tends (in the absence of gravity) to minimize an interface energy. This energy is proportional to area, with a different constant of proportionality a_{ij} for each pair of fluids.

Theorem 2.5 gives a sufficient condition for energy minimization for a hypersurface H consisting of planar pieces H_{ij} with unit normals n_{ij} separating F_i from F_j. H is energy minimizing if whenever

k hyperplane pieces $H_{i_1 i_2}, H_{i_2 i_3}, \ldots, H_{i_k i_1}$ meet along a codimension-2 plane, we have the balancing condition

$$(1) \qquad a_{i_1 i_2} n_{i_1 i_2} + \cdots + a_{i_k i_1} n_{i_k i_1} = 0,$$

and for any distinct integers $1 \leq j_1, \ldots, j_k \leq m$,

$$(2) \qquad |a_{j_1 j_2} n_{j_1 j_2} + \cdots + a_{j_{k-1} j_k} n_{j_{k-1} j_k}| \leq a_{j_1 j_k}.$$

The proof parallels the flux proof sketched in 1.1, with the points p_i chosen so that $p_j - p_i = a_{ij} n_{ij}$. Thus the essential step involves finding an "equilateral set" of points p_i at prescribed distances from each other (cf. 2.1, 2.3).

Examples include the cone over the 1-skeleton of the cube in R^3, which minimizes surface energy if an interface between opposite regions would be $\sqrt{2}$ times as costly as between adjacent regions (cf. 2.2, 2.6).

1.3 General norms (Chapter 3). The energy $\Phi(S)$ of a crystal surface S is given by an integral $\int_S \Phi(n)$ in which the weighting of area depends on the unit normal n at each point. (The same symbol Φ is used both for the norm $\Phi(n)$ and for the associated total surface energy $\Phi(S)$.) Chapter 3 generalizes our earlier results to general norms Φ.

Theorem 3.9 gives a sufficient condition for a hypersurface H consisting of planar pieces H_{ij} separating regions R_i, R_j to minimize $\sum \Phi_{ij}(H_{ij})$. Let Φ^*_{ij} denote the norm dual to Φ_{ij} (see 3.1 for definitions). Let n_{ij} denote the unit normal to H_{ij}, and let $n^*{}_{ij}$ denote a Φ^*_{ij}-unit dual to n_{ij}. Then H minimizes $\sum \Phi_{ij}(H_{ij})$ if whenever k hyperplane pieces

$$H_{i_1 i_2}, \ldots, H_{i_k i_1}$$

meet along a codimension-2 plane,

$$(1) \qquad n^*_{i_1 i_2} + \cdots + n^*_{i_k i_1} = 0,$$

and for any distinct integers $1 \leq i_1, \ldots, i_k \leq m$,

$$(2) \qquad \Phi^*_{i_1 i_k}(n^*_{i_1 i_2} + \cdots + n^*_{i_{k-1} i_k}) \leq 1.$$

Again the proof parallels the flux proof sketched in 1.1, with the points p_i chosen so that $p_j - p_i = n^*{}_{ij}$. Thus the essential step involves for example finding points p_i at unit distance from each other in the Φ^* norm (cf. 3.2).

C.M. Petty [**P**, Theorem 4] proved that for any norm Φ^*, there are 4 equidistant points ("an equilateral tetrahedron") in R^3. Consequently the cone over the 1-skeleton of a certain dual tetrahedron is minimizing. It is an open question whether there are always $n+1$ equilateral points in R^n (cf. 3.3).

For some smooth, strictly convex norms Φ^* on R^3 there are 5 equidistant points (3.4). Consequently certain cones over the 1-skeleton of triangular prisms are Φ-minimizing (3.5). In these new singular cones, nine surfaces and six curves meet at a point.

1.4 Existence and regularity. The existence of minimizers for area or any single norm, as boundaries of top-dimensional currents, is an easy application of geometric measure theory. When the various interfaces are assigned different weightings (of a single norm), as with immiscible fluids, existence theory requires the methods of F. Almgren [**A**], with certain stringent additional hypotheses to avoid "frothing." (See [**A**, VI.1 (7)]. We remark that for existence, these additional hypotheses may be relaxed to a triangle inequality $\sigma_{ik} \leq \sigma_{ij} + \sigma_{jk}$.)

For all of these problems, almost everywhere regularity follows by the methods of Almgren [**A**], with improvements in certain cases by J. Taylor [**T1, T2**] and B. White [**W2**].

No one has worked out extensions of the existence and regularity theory to the case of different norms for different interfaces.

1.5 Minimizing networks (Chapter 4). Generalizations of the classical Steiner or Fermat problem (cf. [**CR**, pp. 356-361]) ask for the shortest network connecting a finite set of "boundary" points in R^n. It is well known that the solution consists of finitely many straight line segments, generally meeting at auxiliary nodes in threes at 120° angles.

Replace length by a differentiable norm Φ and minimize $\int \Phi(T)$, where T is the unit tangent vector. Again there is a Φ-minimizing network consisting of finitely many straight line segments, generally meeting at auxiliary nodes.

The main structural question asks how segments can meet at a node. In partial analogy with our results on hypersurfaces, we give necessary and sufficient conditions for a collection C of rays a_j emanating from the origin to be Φ-minimizing. Let a_j^* denote the Φ^*-unit duals to the a_j. Then C is Φ-minimizing if and only if

$$\sum a_j^* = 0 \tag{1}$$

and any subcollection of the a_j^* satisfies

$$\Phi^*(\sum_{j\in J} a_j^*) \leq 1. \tag{2}$$

To show these conditions necessary, one considers variations (1) displacing the origin of all the vectors, or (2) displacing the origin of some vectors and connecting the new origin to the old.

To show the conditions sufficient, in analogy to the hypersurface case, one uses the $p_j = a_j^*$ as calibrations.

Using these results, Theorem 4.5 gives a complete characterization of nodes in Φ-minimizing networks in R^n, including the fact that at most $n+1$ segments meet at a point. For example, the network connecting the vertices of a regular tetrahedron to the center of mass is Φ-minimizing for certain Φ.

1.6 References. Expositions of our results appear as [**M6**], [**M9**], [**LM**], and [**M5**]. For an introduction to rectifiable sets and geometric measure theory see [**M4**]. For a survey on calibrations, see [**M1**] or [**M2**].

2. Immiscible fluids. This chapter provides examples of cones which minimize total interface energy, as for immiscible fluids. These cones serve as models for general singular structure. Examples include cones over simplices and cubes.

Immiscible fluids $F_1, \ldots, F_m$ tend to occupy (disjoint) regions $R_1, \ldots, R_m$ in such a way as to minimize the total interface energy. This energy is proportional to area, but the constant of proportionality or *interface energy* a_{ij} depends on which two fluids F_i, F_j are separated by the interface.

The first theorem starts with any configuration of m points in R^n and produces an associated energy-minimizing partition of the unit ball into m regions.

2.1 Immiscible Fluids Theorem I.
Given real numbers ("interface energies") $a_{ij} = a_{ji} > 0$ for $1 \leq i \neq j \leq m$, suppose there are points $p_1, \ldots, p_m \in R^n$ such that

$$|p_j - p_i| = a_{ij}.$$

Let $C \subset B(0,1)$ be a hypersurface which divides $B(0,1)$ into regions $R_1, \ldots, R_m$ separated by pieces of hyperplanes H_{ij} normal to $p_j - p_i$.

Then for any other hypersurface $T = \cup T_{ij}$ (a closed set which is a C^1 manifold almost everywhere) which also separates the $R_i \cap S(0,1)$ from each other in $B(0,1)$ (with R_i facing R_j across T_{ij}),

$$\sum a_{ij} Area\, H_{ij} \leq \sum a_{ij} Area\, T_{ij}.$$

Proof. Let $S_i = R_i \cap S(0,1)$. Then

$$\begin{aligned}\sum_{i<j} a_{ij} Area\,(H_{ij}) &= \sum_{i<j} (\text{Flux of } p_j - p_i \text{ through } H_{ij}) \\ &= \sum_i (\text{Flux of } p_i \text{ through } S_i) \\ &= \sum_{i<j} (\text{Flux of } p_j - p_i \text{ through } T_{ij}) \\ &\leq \sum_{i<j} a_{ij} Area\,(T_{ij}).\end{aligned}$$

□

REMARK. We can allow more general competitors T; select the regions R_i in such a way that their topological boundaries have finite area, and let T be the union of reduced boundaries. Almost everywhere, T will separate exactly two regions and will have a well-defined approximate tangent plane.

2.2 Examples for immiscible fluids. In Theorem 2.1, if P is the polytope with vertices p_i, C could be the cone over the $(n-2)$-skeleton of the dual polytope. For example, if P is a unit regular octahedron in R^3, the distance a_{ij} between adjacent points is 1, while the distance a_{ij} between opposite points is $\sqrt{2}$. Consequently, for these interface energies, the dual cone C over the 1-skeleton of the cube is minimizing. See Figure 2.2.1.

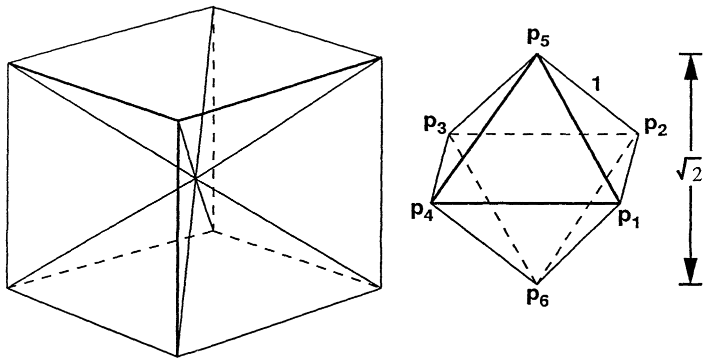

FIGURE 2.2.1. The cone over the cube is energy-minimizing if interfaces between opposite regions are $\sqrt{2}$ times as costly as between adjacent regions. The proof uses the dual polyhedron, the unit regular octahedron, where opposite points are a distance $\sqrt{2}$ apart.

The same result holds for the hypercone over the $(n-2)$-skeleton of the cube in R^n. Ken Brakke [**B1**] proves stronger results by generalizing our constant vectorfields p_i to variable divergence-free vectorfields. He proves that for $n \geq 4$, the cones are actually area-minimizing. More specifically, let $a(n)$ denote the least value of the interface energy between opposite regions for which the hypercone over the cube in R^n is minimizing. Then $a(3) = \sqrt{2}$, $0.545 < a(4) < 0.94$, and $a(7) = 0$. Thus the cone over the cube in R^7 is area-minimizing even if we do not require opposite regions of space to be separated.

Given a set of interface energies a_{ij} between four immiscible fluids in R^3, Theorem 2.1 applies if there are points p_i with $|p_j - p_i| = a_{ij}$, i.e., if there is a tetrahedron with edge lengths a_{ij}. The following generalization of the triangle inequality, due to Schoenberg [**S**], tells whether or not there is an n-simplex with prescribed edge lengths. Schoenberg's theorem also gives a criterion for embedding more than $n+1$ points in R^n with prescribed distances between them. An interesting discussion of these results appears in Blumenthal [**B**], Section 4.3. Blumenthal includes another criterion in R^3 in terms of the three angles at one of the vertices of the hypothesized tetrahedron.

2.3 Proposition ([**S**], Theorem 1). *Given positive numbers* $a_{ij} = a_{ji}$, $0 \leq i, j \leq n$, *with* $a_{ii} = 0$, *there are points* $p_0, \dots, p_n$ *in* R^n

such that $dist(p_i, p_j) = a_{ij}$ *if and only if the matrix* Q *with entries*

$$q_{ij} = \frac{1}{2}(a_{0i}^2 + a_{0j}^2 - a_{ij}^2)$$

is positive semidefinite.

If Q *has rank* s, *then the points can be located in* R^s *but not in* R^{s-1}. *In particular, the points will be in general position in* R^n *if and only if* Q *is positive definite.*

Proof. Suppose we have the $n+1$ points in R^n. Let $w_i = p_i - p_0$ for $1 \leq i \leq n$. Form an n by n matrix W whose columns are w_i. Then $W^T W = Q$, which is therefore positive semidefinite. More generally, if the $n+1$ points are in general position in R^s, then W will be an s by n matrix of rank s, so that Q will have rank s.

Conversely, if Q is positive semidefinite of rank s, we can find an s by n matrix W such that $W^T W = Q$; then let p_i be the i^{th} column of W, with $p_0 = 0$. □

2.4 Remark. In Theorem 2.1, of course if $H_{ij} = \emptyset$, C remains minimizing for $a'_{ij} > a_{ij}$, so that the hypothesis may be weakened to $|p_j - p_i| \leq a_{ij}$ for any such pair i, j. It follows for example that for a nearly flat tetrahedron or other pyramid the minimizer is the set of faces not including the base. The theorem also admits the possibility that some $R_i = \emptyset$, i.e., that we are allowing in competition fluids which need not occur in the minimizer.

The following reformulation of Theorem 2.1 gives easily checked sufficient conditions for a configuration of immiscible fluids to minimize interface energy. The conditions are not necessary in general (see Example 2.2).

2.5 Immiscible Fluids Theorem II. Given real numbers ("interface energies") $a_{ij} = a_{ji} > 0$ *for* $1 \leq i \neq j \leq m$, *let* $C \subset B(0,1) \subset R^n$ *be a hypersurface which divides* $B(0,1)$ *into nonempty regions* $R_1, \ldots, R_m$ *separated by pieces of hyperplanes* H_{ij}, *oriented with unit normals* $n_{ij} = -n_{ji}$ *pointing from* R_i *into* R_j.

Suppose that whenever k *hyperplane pieces* $H_{i_1 i_2}, H_{i_2 i_3}, \ldots, H_{i_k i_1}$ *meet along a co-dimension-2 plane,*

$$a_{i_1 i_2} n_{i_1 i_2} + \cdots + a_{i_k i_1} n_{i_k i_1} = 0. \tag{1}$$

Further suppose that for any distinct integers $1 \leq i_1, \ldots, i_s \leq m$,

$$\left| a_{i_1 i_2} n_{i_1 i_2} + \cdots + a_{i_{s-1} i_s} n_{i_{s-1} i_s} \right| \leq a_{i_1 i_s}, \tag{2}$$

whenever the $n_{i_j i_{j+1}}$ *are all defined because* $H_{i_j i_{j+1}}$ *occurs.*

Then for any other hypersurface $M = \cup M_{ij}$ *(a closed set which is a* C^1 *manifold almost everywhere) which also separates the* $R_i \cap S(0,1)$ *from each other in* $B(0,1)$ *(with* R_i *facing* R_j *across* M_{ij}*),*

$$\sum a_{ij} Area\, H_{ij} \leq \sum a_{ij} Area\, M_{ij}.$$

Proof. We will apply Theorem 2.1 with Remark 2.4. Put $p_1 = 0$. To define p_j for $1 < j \leq m$, consider a generic path γ_0 from R_1 to R_j passing through distinct regions $R_{i_1} = R_1, R_{i_2}, \ldots, R_{i_s} = R_j$. Let

$$p_j = a_{i_1 i_2} n_{i_1 i_2} + \cdots + a_{i_{s-1} i_s} n_{i_{s-1} i_s}. \tag{3}$$

From (1), the definition of p_j is independent of the choice of path γ_0. Moreover, if γ is a generic path from R_i to R_j passing through distinct regions $R_{k_1}, \ldots, R_{k_s}$, then

$$p_j - p_i = a_{k_1 k_2} n_{k_1 k_2} + \cdots + a_{k_{s-1} k_s} n_{k_{s-1} k_s}.$$

By (2), $|p_j - p_i| \leq a_{ij}$. If H_{ij} occurs, then there is a direct path from R_i to R_j and

$$p_j - p_i = a_{ij} n_{ij}.$$

The result follows by 2.1 with 2.4. □

3. General norms. This chapter provides examples of cones which minimize hypersurface energies given by general norms Φ_{ij} on the space of normal vectors, as in the surface energy of crystals. Such cones serve as models for general singular structure. The case when all of the norms Φ_{ij} are equal is of primary interest. Examples include a cone over a triangular prism (Proposition 3.5).

3.1 Definitions. A *norm* Φ in R^n is a homogeneous convex function on R^n, positive except at 0. That is,

$$\Phi(ax) = |a|\Phi(x),$$
$$\Phi(x+y) \leq \Phi(x) + \Phi(y),$$
$$\Phi(x) > 0 \quad if \quad x \neq 0.$$

The associated energy $\Phi(S)$ of a hypersurface S is given by the integral $\int_S \Phi(n)$ of the norm of the unit normal n.

The *dual norm* to Φ, denoted Φ^*, is given by

$$\Phi^*(w) = \sup\{w \cdot v : \Phi(v) = 1\}.$$

Then

$$|v \cdot w| \leq \Phi(v)\Phi^*(w).$$

If equality holds, we say that w is *dual* to v.

Geometrically, a vector w is dual to a given vector v (say $\Phi(v) = 1$) if w is an outward-pointing normal (of any length) to the unit Φ-ball at v. See Figure 3.1.1. If the unit Φ-ball is not differentiable at v, then the direction of w is not uniquely determined; w only needs to be normal to any supporting hyperplane.

The following facts are true about dual norms (cf. [**M3**, Prop. 3.3]).

(1) $\Phi^{**} = \Phi$

(2) Φ^* is differentiable if and only if Φ is strictly convex

(3) Φ^* is $C^{1,1}$ if and only if Φ is uniformly convex

(4) Φ^* is smooth (C^∞) and uniformly convex if and only if Φ is smooth and uniformly convex.

One often imposes conditions (2) or (3); see Remarks 3.7, 3.8.

Note that the relation "w is dual to v" is symmetric only if understood properly: If w is dual to v with respect to the norm Φ, then v is dual to w with respect to the norm Φ^*.

The following theorem starts with m Φ_{ij}^*-equidistant points in R^n and produces an associated energy-minimizing partition of the unit ball into m regions.

3.2 General Norms Theorem I. Let $\Phi_{ij} = \Phi_{ji}$ *be norms on* R^n *for*

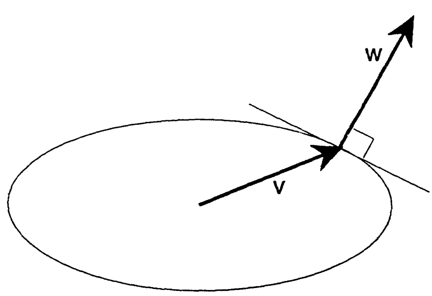

FIGURE 3.1.1. The vector w is dual to v.

$1 \leq i \neq j \leq m$. *Suppose there are points* $p_1, \ldots, p_m \in R^n$ *such that*

$$\Phi_{ij}^*(p_j - p_i) = 1.$$

Let $C = \cup H_{ij} \subset B(0,1)$ *be a hypersurface which divides* $B(0,1)$ *into regions* $R_1, \ldots, R_m$ *separated by pieces of hyperplanes* H_{ij} *with unit normals* n_{ij} *dual to* $p_j - p_i$ *(i.e.,* $n_{ij} \cdot (p_j - p_i) = \Phi_{ij}(n_{ij})$*).*

Then for any other hypersurface $M = \cup M_{ij}$ *(see remark following Theorem 2.1) which also separates the* $R_i \cap S(0,1)$ *from each other in* $B(0,1)$ *(with* R_i' *facing* R_j' *across* M_{ij}*),*

$$\sum \Phi_{ij}(H_{ij}) \leq \sum \Phi_{ij}(M_{ij}).$$

Further, if it happens that two regions R_i *and* R_j *do not face each other across a surface* H_{ij} *of positive area, i.e.,* $H_{ij} = \emptyset$*, then we can allow* $\Phi_{ij}^*(p_j - p_i) \leq 1$ *for any such* i *and* j.

Proof. Let $S_i = R_i \cap S(0,1)$. Then

$$\sum_{i<j} \Phi_{ij}(M_{ij}) = \sum_{i<j} \int_{M_{ij}} \Phi_{ij}(n) = \sum_{i<j} \int_{M_{ij}} \Phi_{ij}^*(p_j - p_i)\Phi_{ij}(n)$$

$$\geq \sum_{i<j} \int_{M_{ij}} (p_j - p_i) \cdot n = \sum_{i<j} (\text{Flux of } p_j - p_i \text{ through} M_{ij})$$

$$= \sum_i (\text{Flux of } p_i \text{ through } S_i)$$

with equality if $M_{ij} = H_{ij}$. □

3.3 Remarks. Of course for $\Psi = \Psi^*$ the standard Euclidean norm on R^n, there is an "equilateral" set of $n+1$ points (at the vertices of a regular simplex) satisfying the hypothesis of Theorem 3.2. It is an open question whether there are $n+1$ such equidistant points for any norm Ψ on R^n, even for Ψ smooth and uniformly convex and $n = 4$. It is true in R^3. Indeed, C.M. Petty [**P**, Theorem 4] proves that for any norm Ψ on R^n ($n \geq 3$), any *maximal* equilateral set S satisfies

$$4 \leq card\, S \leq 2^n.$$

By maximal we mean a set to which we cannot add another equidistant point; the same norm may have larger equilateral sets.

Both bounds are sharp. The second equality holds for the non-smooth, non-uniformly-convex ℓ^∞ norm with cubical unit ball and the 2^n equidistant points at the vertices of the cube (the only example, up to linear equivalence).

Section 3.4 will give a smooth, uniformly convex norm Ψ on R^3 with 5 equidistant points. Petty [**P**, p. 373] observes that it follows from work of Grünbaum [**G**] that 5 is the upper bound for norms on R^3 not satisfying a certain "Property P," in particular, for uniformly convex norms. We conjecture that 5 is the upper bound for differentiable norms in R^3 too.

Thus the possible cardinality and combinatorial structure of Ψ-equilateral sets for norms Ψ on R^n remains open, even for differentiable norms on R^3, the case of greatest physical interest. Also see Kusner [**K**].

3.4 Examples. Here we describe some norms Ψ on R^3, including one with *five* points all a unit Ψ-distance from each other.

Let T be the standard regular tetrahedron in R^3 with three vertices p_1, p_2, p_3 in the xy-plane and the fourth p_4 on the positive z-axis, and let $p_5 = -p_4$; see Figure 3.4.1.

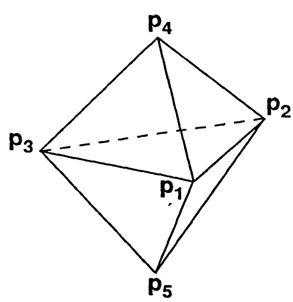

FIGURE 3.4.1. The regular tetrahedron T and its reflection.

Let Ψ_0 be the norm such that the top half of the unit Ψ_0-sphere is the truncated cone over the unit circle in the plane with vertex $(0,0,z)$, with $z > 0$ chosen so that $p_4 - p_1$, $p_4 - p_2$, and $p_4 - p_3$ are on the cone, and therefore the vertices of T are Ψ_0-equidistant. See Figure 3.4.2.

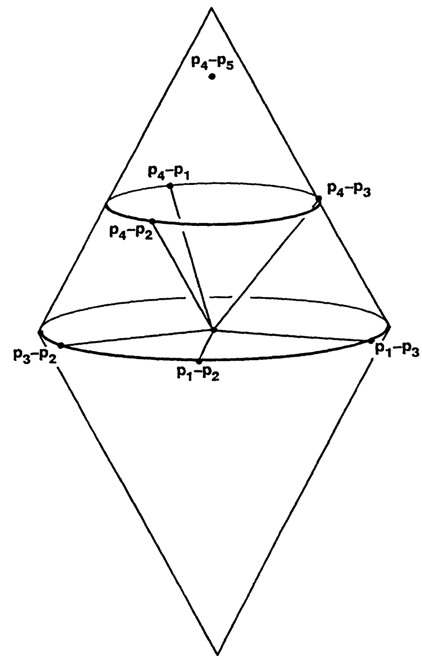

FIGURE 3.4.2. The unit Ψ_0-ball. All six sides of the regular tetrahedron have Ψ_0 length 1 as well.

Simple trigonometry shows that

$$\Psi_0(p_4) = 1 - \frac{1}{\sqrt{3}} < \frac{1}{2}.$$

The norm Ψ_0 can be smoothed to uniformly convex norms Ψ_1 and Ψ_2 with

$$\Psi_1(p_4) < \Psi_2(p_4) = \frac{1}{2},$$

maintaining symmetry under rotations about the z-axis and keeping the two circles of Figure 3.4.2 in the unit sphere. See Figure 3.4.3. Then the five points $p_1, \dots p_5$ all satisfy

$$\Psi_2(p_i - p_j) = 1 \qquad \text{for } i \neq j.$$

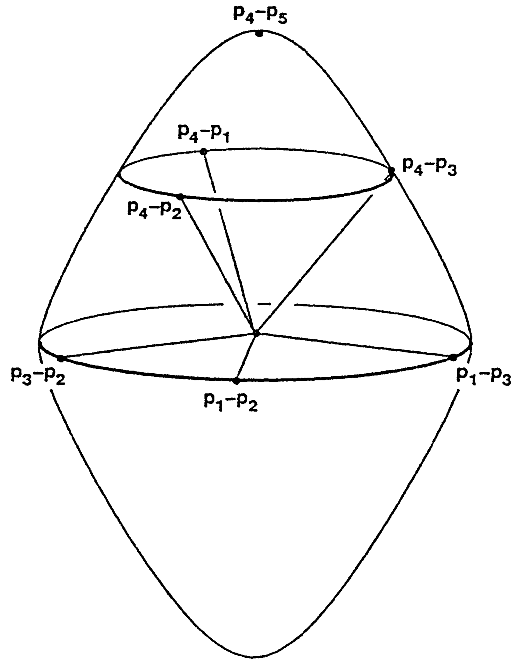

FIGURE 3.4.3. The unit Ψ_2-ball. All five vertices of the regular tetrahedron and its reflection are unit Ψ_2-distance apart.

The following proposition gives a new minimizing cone in R^3. See Figure 3.5.1.

3.5 Proposition. The cone over the 1-skeleton of any triangular prism is Φ-minimizing for some smooth, uniformly convex norm Φ. At the origin, nine surfaces and six curves meet at a point.

Proof. Let Φ be the dual of the norm Ψ_2 of Example 3.4. Let p_i be the vertices of the regular tetrahedron and its reflection as in Figure 3.4.1, so that $\Phi^*(p_j - p_i) = 1$. Let n_{ij} be the unit vectors dual to the $p_j - p_i$. These vectors n_{ij} include the unit normals to the cone C over the edges of a certain vertical right triangular prism P, which by 3.2 is Φ-minimizing. Any other triangular prism is affinely equivalent to P, and hence its cone is minimizing for some norm.

Note that since the closures of the two regions of space corresponding to the points p_4 and p_5 only intersect at the vertex of C, we can allow $\Phi^*(p_4 - p_5) \leq 1$, so that C is also Φ-minimizing if Φ

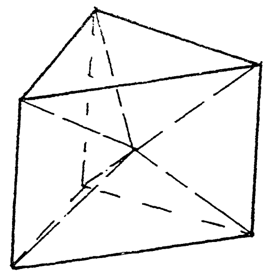

FIGURE 3.5.1. The cone over the triangular prism is Φ-minimizing.

is the dual of the norm Ψ_1 of Example 3.4. □

The methods of this paper do not require the piecewise planar surface to be a cone. For the integrand Ψ_2 and its dual Φ just considered, there is a one-parameter family of surfaces which we can prove are Φ-minimizing. The boundaries are the edges of taller and shorter triangular prisms. They are described as follows. For concreteness, scale the above Φ-minimizing cone C so that its height is 1, and let its vertex be the origin.

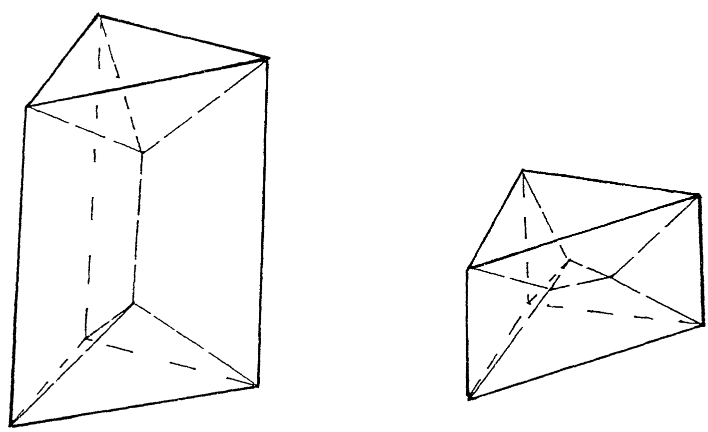

FIGURE 3.5.2. Other Φ-minimizing surfaces.

To obtain a Φ-minimizing surface of height $L > 1$, cut off the half of C where $z > 0$ and lift it up a distance $L - 1$. Connect the two pieces with the Cartesian product of a horizontal Y with a vertical line segment of length $L-1$. See Figure 3.5.2. This surface is composed of three vertical trapezoids and six triangles.

To obtain a Φ-minimizing surface of height $L < 1$ (say $L = 1 - 2d$), cut out the portion of C where $-d < z < d$, shift the upper remaining piece down a distance $2d$, and add a horizontal triangle to fill the hole in the middle. See Figure 3.5.2. This surface has six trapezoids and four triangles.

3.6 Proposition.Each of the one-parameter family of surfaces described above is Φ-minimizing for the smooth, uniformly convex norm $\Phi = \Psi_2^*$.

Proof. The normals n_i are the same as for the cone C, with one additional normal $n_{10} = n_{10}^*$ for the shorter surfaces, which have the horizontal triangle. □

3.7 Remarks. More singular examples are provided by norms Φ which fail to be strictly convex (which means that the dual norm Φ^* fails to be differentiable). For example, let Φ^* be the ℓ^∞ norm on R^n, which has a cubical unit ball, whose 2^n vertices are equidistant. By Theorem 3.2, the union of all the axis hyperplanes (the cone over the generalized octahedron) is Φ-minimizing.

As a second example, let Φ^* be the ℓ^1 norm, whose unit ball is the regular octahedron or "cross-polytope." The $2n$ vertices of the octahedron are equidistant in this norm. Again by Theorem 3.2, the cone over the cube is Φ-minimizing. (In R^3, the cone over the cube is also Φ-minimizing if the unit Φ^*-ball is a hexagonal prism, whose height is adjusted so that six of its twelve vertices are the corners of a regular octahedron.)

If the unit Φ-ball in R^3 has a large flat face on top, so that the unit Φ^* ball comes to a sharp point on top, then the graph of any function on the disc with Lipschitz constant at most 1 is Φ-minimizing. Thus there are infinitely many Φ-minimizing ways to split the cylinder.

Remark. As in Remark 2.4, for any pair i, j for which $H_{ij} = \emptyset$, the hypothesis of Theorem 3.2 may be weakened to $\Phi_{ij}^*(p_j - p_1) \leq 1$. The theorem also admits the possibility that some $R_i = \emptyset$.

The following reformulation of Theorem 3.2 gives easily checked sufficient conditions for a configuration to minimize the energies given by norms. Again the case where all the norms Φ_{ij} are equal is of primary interest.

3.9 General Norms Theorem II. Let $\Phi_{ij} = \Phi_{ji}$ be norms on R^n for $1 \leq i \neq j \leq m$. Let $C \subset B(0,1) \subset R^n$ be a piecewise planar hypersurface which divides $B(0,1)$ into nonempty regions $R_1, \ldots, R_m$ separated by pieces of hyperplanes H_{ij} oriented with unit normal $n_{ij} = -n_{ji}$ pointing from R_i into R_j. Let ${n^*}_{ij} = -{n^*}_{ji}$ be a Φ^*_{ij}-unit dual to n_{ij}, i.e.,

$$ {n^*}_{ij} \cdot v \leq \Phi_{ij}(v), $$

with equality for $v = n_{ij}$.

Suppose that whenever k hyperplane pieces $H_{i_1 i_2}, H_{i_2 i_3}, \ldots, H_{i_k i_1}$ meet along a codimension-2 plane,

$$ (1) \qquad {n^*}_{i_1 i_2} + \cdots + {n^*}_{i_k i_1} = 0. $$

Further suppose that for any distinct integers $1 \leq i_1, \ldots, i_k \leq m$,

$$ (2) \qquad \Phi^*_{i_1 i_k}\left({n^*}_{i_1 i_2} + \cdots + {n^*}_{i_{k-1} i_k}\right) \leq 1, $$

whenever the $n_{i_j i_{j+1}}$ are all defined because $H_{i_j i_{j+1}}$ occurs.

Then for any other hypersurface $M = \cup M_{ij}$ (see remark following Theorem 2.1) which also separates the $R_i \cap S(0,1)$ from each other in $B(0,1)$ (with R_i facing R_j across M_{ij}),

$$ \sum \Phi_{ij}(H_{ij}) \leq \sum \Phi_{ij}(M_{ij}). $$

Proof. Replace $a_{ij} n_{ij}$ by ${n^*}_{ij}$ in the proof of Theorem 2.5 to obtain p_i such that $p_j - p_i = {n^*}_{ij}$ (when H_{ij} occurs) and more generally $\Phi^*_{ij}(p_j - p_i) \leq 1$ for all i and j. The result follows by Theorem 3.2 and Remark 3.8. □

Remarks. If the Φ_{ij} are differentiable at n_{ij}, then condition (1) is necessary for the first variation to vanish (cf. Lemma 4.1). If Φ_{ij} is not differentiable at n_{ij}, then ${n^*}_{ij}$ is not uniquely determined (Φ^*_{ij} is not *strictly* convex; cf 3.1(2)).

In the statements and proofs of Theorems 3.2 and 3.5, the assumption implicit in the definition of a norm that $\Phi(-v) = \Phi(v)$ is unnecessary; the hypothesis $\Phi_{ij} = \Phi_{ji}$ must merely be replaced by $\Phi_{ij}(v) = \Phi_{ji}(-v)$.

4. Minimizing networks. This chapter charecterizes singularities in energy-minimizing networks. In particular, Theorem 4.5 shows that in R^n at most $n+1$ segments meet at a point. A good reference is [**M7**, Chapter 10].

For any norm Φ on R^n, for any piecewise differentiable curve C with unit tangent vector T, define the energy

$$\Phi(C) = \int_C \Phi(T).$$

Any finite set of "boundary" points can be connected by a Φ-minimizing network, consisting of finitely many straight line segments, possibly meeting at auxiliary nodes (cf. [**A2**]).

The following lemma gives a useful formula for the first variation of Φ-energy.

4.1 Lemma. Let Φ be a differentiable norm on R^n, and let $a \in R^n$. The first variation in $\Phi(a)$ satisfies

$$\delta(\Phi(a)) = \frac{a^*}{\Phi^*(a^*)} \cdot \delta a.$$

Here a^* is a vector dual to a, so that equality holds in the general inequality $v \cdot w \leq \Phi(v)\Phi^*(w)$ when $v = a$ and $w = a^*$.

Proof. Differentiating

$$\Phi(a) = \frac{a^*}{\Phi^*(a^*)} \cdot a$$

yields

$$\delta(\Phi(a)) = \frac{a^*}{\Phi^*(a^*)} \cdot \delta a + \delta\left(\frac{a^*}{\Phi^*(a^*)}\right) \cdot a,$$

and the second term vanishes because

$$\frac{b^*}{\Phi^*(b^*)} \cdot a \leq \Phi(a),$$

with equality when $b = a$. □

The following theorem gives a characterization of Φ-minimizing network cones.

4.2 Theorem. *Let Φ be a differentiable norm on R^n, and let Φ^* denote the dual norm. Let $a_1, \dots, a_k \in R^n$, normalized so that $\Phi(a_j) = 1$, and let $a_1^*, \dots a_k^*$ denote the duals such that $\Phi^*(a_j^*) = 1$ and $a_j \cdot a_j^* = 1$. Then the network C consisting of rays from the origin to $a_1, \dots, a_k$ is Φ-minimizing if and only if*

$$a_1^* + \cdots + a_k^* = 0 \tag{1}$$

and any subcollection of the a_j^* has a sum of Φ^*-norm at most 1:

$$\Phi^*(\sum_{j \in J} a_j^*) \leq 1. \tag{2}$$

Remarks. Condition (1) is the equilibrium condition for k segments meeting at the origin. By convexity, such an equilibrium is a minimum (for fixed topological type). Cf. [**CG**, (10) and Theorem 3]. Condition (2) deals with other topological types.

If Φ is not differentiable, the dual vectors a_j^* are not uniquely determined. If (1) and (2) hold for some choice of the a_j^*, then C is Φ-minimizing.

The converse fails for nondifferentiable Φ. Alfaro et. al. [**A1, A2**] show that the cone consisting of four vectors along the axes in R^2 is minimizing for a certain piecewise C^∞, uniformly convex norm Φ, but conditions (1) and (2) hold for no choice of the a_j^*. Conger [**Con**] proves an analogous result for six vectors along the axes in R^3.

For the nondifferentiable, non-uniformly-convex ℓ^∞ norm on R^n, with cubical unit ball, the 2^n vectors from the center to the vertices of the unit cube form a minimizing network. Cf. [**H**].

Proof of Theorem 4.2. First suppose C is Φ-minimizing. For variations in C displacing the center an amount δa, since Φ is differentiable, Lemma 4.1 gives a first variation of

$$(\sum a_j^*) \cdot (-\delta a),$$

so that (1) holds. Similarly for variations displacing the origin of the segments $\{\overline{0a_j} : j \in J\}$ an amount δa and adding a line segment joining the old origin to the new, the first variation is

$$\left(\sum_{j\in J} a_j^*\right) \cdot (-\delta a) + \Phi(\delta a),$$

so that (2) holds.

Second, suppose (1) and (2) hold. Let N be any network connecting the points a_j. For $j \geq 2$, let P_j be paths in N from a_1 to a_j, such that no P_j overlaps itself. For each segment S_i of N, let $\{P_j : j \in J_i\}$ be paths containing S_i. Then

$$\begin{aligned}
\Phi(C) = k = \sum_{j=1}^{k} a_j^* \cdot a_j &= \sum_{j=2}^{k} a_j^* \cdot (a_j - a_1) && \text{by (1)} \\
&= \sum_{j=2}^{k} \int_{C_j} a_j^* = \sum_{j=2}^{k} \int_{P_j} a_j^* && \text{by Stokes's theorem} \\
&\leq \sum_i \left| \int_{S_i} \sum_{j\in J_i} a_j^* \right| \\
&\leq \sum_i \Phi(S_i) && \text{by (2)} \\
&= \Phi(N).
\end{aligned}$$

Therefore, C is Φ-minimizing. □

The following two lemmas pave the way for the main theorem 4.5 of this section.

4.3 Lemma. Suppose we have a set of vectors in R^n *labelled as* $a_1, \ldots, a_k, a_1^*, \ldots, a_k^*$, *with* $k \geq 3$, *and suppose that* $a_j \cdot a_j^* = 1$ *for each* j, *and* $\sum a_j^* = 0$. *Then there is a differentiable norm* Φ *on* R^n *with dual norm* Φ^* *such that*

$$\Phi(a_j) = \Phi^*(a_j^*) = 1 \tag{1}$$

and

$$\Phi^*\left(\sum_{j\in J} a_j^*\right) \leq 1 \qquad \textit{for all} \quad J \subset \{1, \ldots, k\} \tag{2}$$

if and only if

$$(3) \qquad -1 < a_i \cdot a_j^* < 0$$

for all $i \neq j$. Φ may be chosen to be C^∞ and uniformly convex.

Proof. Suppose there is such a norm. Since Φ is differentiable, the unit Φ^* ball B^* is strictly convex (3.1 (2)). If $\xi \in B^*$, then

$$(4) \qquad -1 \leq a_i \cdot \xi \leq 1,$$

with equality only if $\xi = \pm a_i^*$ (since unit duals are unique for a differentiable norm). Now suppose that $i \neq j$, and let $\xi = a_i^* + a_j^*$. By (2), $\xi \in B^*$. Using this ξ in (4) gives

$$1 + a_i \cdot a_j^* < 1,$$

so that

$$a_i \cdot a_j^* < 0.$$

Similarly, (4) with $\xi = a_j^*$ yields

$$-1 \leq a_i \cdot a_j^*,$$

with equality only if $a_j^* = -a_i^*$. But if $a_j^* = -a_i^*$ then $k < 3$ (since $a_3 \cdot a_1^*$ and $a_3 \cdot a_2^*$ cannot both be negative if $a_1^* = -a_2^*$).

Conversely, suppose (3) holds. Consider the symmetric polytope

$$C^* = \{\xi : |a_i \cdot \xi| \leq 1 \qquad \text{for all } i\}.$$

Since $a_i \cdot a_i^* = 1$ and (3) holds, each a_i^* lies on the interior of a distinct face of C^*. Also since $a_i \cdot a_j^* < 0$ for $i \neq j$ and $\sum a_j^* = 0$, each sum $\sum_{j \in J} a_j^*$ satisfies

$$-1 \quad \leq \quad -a_i \cdot \sum_{j \in J^C} a_j^* \quad = \quad a_i \cdot \sum_{j \in J} a_j^* \quad \leq \quad 1,$$

with equality only if $\sum_{j \in J} a_j^* = \pm a_i^*$. Hence each sum $\sum_{j \in J} a_j^* \neq \pm a_i^*$ lies in the interior of C^*. Since there are only a finite number of these sums, we can smooth C^* and obtain a C^∞, compact, symmetric, uniformly convex body $B^* \subset C^*$ having these same properties. Take Φ^* to be the norm with unit ball B^*, and Φ to be the dual norm. Then Φ is C^∞ and uniformly convex, and (1) and (2) hold. □

4.4 Lemma. Let $a_1, \dots, a_k, a_1^*, \dots, a_k^*$ be vectors in R^n such that

$$a_1^* + \cdots + a_k^* = 0 \tag{1}$$

and

$$a_i \cdot a_j^* < 0 \quad \textit{for} \quad i \neq j. \tag{2}$$

Then $Card\{a_i\} \leq n+1$. Indeed, either the a_i are linearly independent or the a_i constitute the k vertices of a $(k-1)$-simplex with the origin in its interior.

Remark. Z. Furedi, J. Lagarias, and F. Morgan [**FLM**, Thm. 3.2] show that if equality is allowed in (2), but still $a_i \cdot a_i^* > 0$, then $Card\{a_i\} \leq 2n$.

Proof of Lemma 4.4. If $a_1, \dots, a_k$ are linearly independent, we are done. Otherwise we may assume

$$a_k = \sum_{i=1}^{k-1} \lambda_i a_i$$

with

$$\lambda_1, \dots, \lambda_p \geq 0 > \lambda_{p+1}, \dots, \lambda_{k-1}.$$

Suppose $p \geq 1$. For fixed j such that $1 \leq j \leq p$,

$$0 > a_k \cdot a_j^* = \sum_{i=1}^{k-1} \lambda_i a_i \cdot a_j^* \geq \sum_{i=1}^{p} \lambda_i a_i \cdot a_j^*$$

because for $i > p$, $\lambda_i a_i \cdot a_j^* > 0$. Summing over j yields

$$0 > \sum_{i,j=1}^{p} \lambda_i a_i \cdot a_j^* \geq \sum_{i=1}^{p} \lambda_i a_i \cdot \left(\sum_{j=1}^{k-1} a_j^* \right),$$

because for $i \leq p < j$, $\lambda_i a_i \cdot a_j^* \leq 0$. Thus

$$0 > \sum_{i=1}^{p} \lambda_i a_i \cdot (-a_k^*) \geq 0,$$

because for $i \le p$, $\lambda_i a_i \cdot a_k^* \le 0$. This contradiction implies that $p = 0$, i.e., each λ_i must be negative. Hence the $k-1$ points $a_1, \dots, a_{k-1}$ must be linearly independent and the k points $a_1, \dots, a_k$ are at the vertices of a $(k-1)$-simplex with 0 in its interior. □

4.5 Theorem. Let Φ be a differentiable norm on R^n. Then at most $n+1$ segments come together in a Φ-minimizing network. Indeed, a 1-dimensional cone C consisting of at least three rays emanating from the origin is Φ-minimizing for some Φ if and only the rays are linearly independent or pass through the k vertices of a $(k-1)$-simplex with the origin in its interior.

Φ may be chosen to be C^∞ and uniformly convex.

Remarks. If C is the cone over the regular tetrahedron centered at 0 in R^3 with vertices a_i, the unit ball of Φ can be taken to be a smoothing of the cube with vertices $\pm a_i$; in R^n, a smoothing of the polytope with vertices $\pm a_i$.

The differentiability hypothesis is necessary. See the remarks after 4.2.

The planar case was treated in [**Coc**], and with more careful attention to differentiability in [**L**] and [**A2**].

M. Alfaro, T. Campbell, J. Sher, and A. Soto (written up in [**A3**]) have considered the related problem for *directed* length-minimizing planar networks. They proved that segments sometimes meet in sixes, but never in sevens.

Proof of Theorem 4.5. First suppose that the rays pass through the vertices $a_1, \dots, a_k$ of a $(k-1)$-simplex with the origin in its interior. Because the origin is in the interior, we can choose the vertices on the rays in such a way that their sum is zero. Using a nonsingular linear transformation we may also assume that the simplex is regular, with $|a_j| = 1$. Let $a_j^* = a_j$. Then $a_j \cdot a_j^* = 1$, $\sum a_j^* = 0$, and $-1 < a_i \cdot a_j^* < 0$ for $i \ne j$. It follows by Lemma 4.3 and Theorem 4.2 that the cone C over $a_1, \dots, a_k$ is Φ-minimizing for some Φ as asserted.

Second, let $a_1, \dots, a_k$ be linearly independent. Using a nonsingular linear transformation, we may assume that the a_j are orthonormal. Let $a_j^* = a_j - \frac{1}{k-1}\sum_{i \ne j} a_i$. Then $a_j \cdot a_j^* = 1$, $\sum a_j^* = 0$, and $-1 < a_i \cdot a_j^* = -\frac{1}{k-1} < 0$ for $i \ne j$ $(k \ge 3)$. Again by Lemma 4.3

and Theorem 4.2, the cone C over $a_1, \dots, a_k$ is Φ-minimizing for some Φ as asserted.

(An alternative argument for part two would have been to perturb part one.)

Now let C be any Φ-minimizing cone. By Theorem 4.2 and Lemma 4.3, C is the cone over points $a_1, \dots, a_k$ in R^n, with associated points a_j^* such that $\sum a_j^* = 0$ and $a_i \cdot a_j^* < 0$ for all $i \neq j$. The conclusion now follows by Lemma 4.4. □

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