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HARDY SPACES AND OSCILLATORY SINGULAR INTEGRALS: II

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We consider oscillatory singular integral operators with real-analytic phases. The uniform boundedness from $H_E^1 \to L^1$ of such operators is proved, where H_E^1 is a variant of the standard Hardy space H^1. The result is false for general C^∞ phases. This work is a continuation of earlier work by Phong and Stein (on bilinear phases) and the author (on polynomial phases).

1. Introduction. In [**8**], Phong and Stein established an H^1 theory for oscillatory singular integral operators with bilinear phase functions. Their H^1 boundedness result, together with the L^2 estimate for such operators, led to the L^p boundedness of oscillatory singular integral operators with bilinear phases, via interpolation.

Ricci and Stein considered oscillatory singular integral operators with polynomial phases. They showed that such operators are bounded on L^p for $1 < p < \infty$, and the bound for the operator norm depends only on the degree of the polynomial, not its coefficients ([**9**]). In [**1**], Chanillo and Christ proved that such operators are of weak-type (1,1).

In an earlier paper, we extended Phong-Stein's H^1 theory for operators with bilinear phases to operators with polynomial phases. Let $x, y \in \mathbb{R}^n$, $K(x,y)$ be a Calderón-Zygmund kernel, $P(x,y)$ be a real-valued polynomial in x and y. Define T:

$$Tf(x) = \text{p. v.} \int_{\mathbb{R}^n} e^{iP(x,y)} K(x,y) f(y) dy. \tag{1.1}$$

The following theorem is proved in [**4**].

THEOREM A. *The operator T is bounded from H_E^1 to L^1, and the bound for $\|T\|$ depends only on the degree of P, not its coefficients.*

The space H_E^1 in Theorem A depends on P and is an variant of the standard Hardy space H^1. The precise definition for H_E^1 will be given later. As we mentioned above, Theorem A was first proved by Phong and Stein in the case where $P(x,y)$ is assumed to be a bilinear form. The space H^1_E was introduced as a substitute for the ordinary H^1 space, since in general the operators T defined in (1.1) do not map ordinary H^1 to L^1. L^p boundedness can be obtained by using $H^1_E \to L^1$ estimate, $L^2 \to L^2$ estimate, and interpolation (for details, see [**8**]).

The H^1 boundedness for operators with general translation invariant phase functions was considered in [**7**]. But, the problem seems to be considerably harder if the phase functions are not assumed to be of the form $\Phi(x-y)$. In this paper, we consider oscillatory singular integrals with real-analytic (non-convolution type) phase functions and we shall restrict our attention to dimension one. Let $x, y \in \mathbb{R}$, $\phi(x,y) \in C_0^\infty(\mathbb{R}\times\mathbb{R})$, and $\Phi(x,y)$ be real-analytic on $\operatorname{supp}(\phi)$. For $\lambda \in \mathbb{R}$, we define T_λ:

$$(1.2) \qquad T_\lambda f(x) = \text{p.v.} \int_{\mathbb{R}} e^{i\lambda\Phi(x,y)} k(x,y)\phi(x,y)f(y)dy,$$

where $k(x,y)$ is a Calderón-Zygmund kernel, i.e. $k(x,y)$ is C^1 away from $\{(x,y) \mid x = y\}$, and satisfies

$$(1.3) \qquad |k(x,y)| \le A|x-y|^{-1},\ |\nabla k(x,y)| \le A|x-y|^{-2}, \text{ for some } A > 0;$$

$$(1.4) \quad \text{The operator } f \to \int k(x,y)f(y)dy \text{ extends as a bounded operator on } L^2(\mathbb{R}).$$

The uniform boundedness of T_λ on L^p is obtained in [**6**]. The fact that T_λ are uniformly bounded from L^1 to $L^{1,\infty}$ is proved in [**5**].

For fixed λ and Φ, let $E = (\lambda, \Phi)$. A function $a(x)$ is called an H^1_E atom if there is an interval $I \subset \mathbb{R}$, which is centered at x_I, such that

$$(1.5) \qquad \operatorname{supp}(a) \subset I;$$

$$(1.6) \qquad \|a\|_\infty \le \frac{1}{|I|};$$

$$\int_I e^{i\lambda\Phi(x_I,y)}a(y)dy = 0. \tag{1.7}$$

A function f is said to be in H^1_E if $f \in L^1$, and f can be written as

$$f = \sum_j \beta_j a_j \tag{1.8}$$

for some $\beta_j \in \mathbb{R}$ and atoms $a_j(x)$. The H^1_E norm of f is given by

$$\|f\|_{H^1_E} = \inf\left\{\sum_j |\beta_j| \;\middle|\; f = \sum_j \beta_j a_j\right\}. \tag{1.9}$$

We have the following theorem on the uniform boundedness of T_λ from H^1_E to L^1.

THEOREM B. *Let Φ be real-analytic, T_λ be given as above. Then there is a constant $C > 0$ such that*

$$\|T_\lambda f\|_1 \le C\|f\|_{H^1_E}, \tag{1.10}$$

for $f \in H^1_E$. The constant C is independent of λ.

The result in [**4**] implies that Theorem B holds if Φ is a polynomial. It should be pointed out that the theorem becomes false if the phase function is assumed to be merely smooth (see Section 4).

2. Some preliminary estimates. Let $P(x,y)$ be a real-valued polynomial, $k(x,y)$ be given as in Section 1. The following L^2 boundedness result follows from Ricci-Stein's theorem in [**9**].

PROPOSITION 2.1. *Let*

$$Tf(x) = \text{p. v.} \int_{\mathbb{R}} e^{iP(x,y)}k(x,y)f(y)dy. \tag{2.1}$$

Then, T is bounded on $L^2(\mathbb{R})$, and $\|T\|_{2,2}$ is bounded above by a constant which depends only on $\deg(P)$ *and A.*

Next we state a result which deals with L^2 estimates for operators with real-analytic phases.

PROPOSITION 2.2. *Let $\phi \in C_0^\infty(\mathbb{R} \times \mathbb{R})$, $I \subset \mathbb{R}$ be a closed interval. Suppose $\Phi(x, y, t)$ is real-analytic in* $\operatorname{supp}(\phi) \times I$. *Define* $T_{\lambda,t}$:

$$(2.2) \qquad T_{\lambda,t} f(x) = \text{p. v.} \int_{\mathbb{R}} e^{i\lambda\Phi(x,y,t)} k(x,y)\phi(x,y) f(y) dy.$$

Then, there is a constant $C > 0$ such that

$$(2.3) \qquad \|T_{\lambda,t} f\|_2 \leq C\|f\|_2,$$

for $\lambda \in \mathbb{R}$, $t \in I$. The constant C is independent of λ and t.

Proof. Since $k(x, y)$ is smooth away from the diagonal $\Delta = \{x = y\}$, we may assume that the support of $\phi(x, y)$ is contained in a small square which is centered at a certain point in Δ (by a partition of unity, if necessary). Without loss of generality, we may assume that $\phi(x, y)$ is supported in a small square centered at the origin.

For fixed t_0, $t_0 \in I$, the uniform boundedness of $\|T_{\lambda,t_0}\|_{2,2}$ in λ is proved in [**6**] (Corollary 1, p. 210). The proof consists of two parts. One part deals with things that are close to the singularity, where Proposition 2.1 is used. The other part (away from the singularity) uses the fact that $\partial^2\Phi/\partial x\partial y$ does not vanish of infinite order. To prove Proposition 2.2, it suffices to prove that, for given $t_0 \in I$, there are $d > 0$, $C > 0$, such that

$$(2.4) \qquad \|T_{\lambda,t}\|_{2,2} \leq C,$$

for $|t - t_0| < d$, $\lambda \in \mathbb{R}$. There are several cases.

Case I. If there are j_0, $k_0 \geq 1$ such that

$$(2.5) \qquad \frac{\partial^{j_0+k_0}\Phi(0,0,t_0)}{\partial x^{j_0}\partial y^{k_0}} = c_0 > 0,$$

then, we may choose d small such that

$$(2.6) \qquad \frac{\partial^{j_0+k_0}\Phi(0,0,t)}{\partial x^{j_0}\partial y^{k_0}} \geq \frac{c_0}{2},$$

for $|t - t_0| < d$. A quick examination of the proof of Theorem 2 in [**6**] shows that (2.3) holds uniformly in $\lambda \in \mathbb{R}$, $t \in (t_0 - d, t_0 + d)$.

Case II. If $\partial^{j+k}\Phi(0,0,t_0)/\partial x^j\partial y^k = 0$ for all $j \geq 1$, $k \geq 1$, then

$$(2.7) \qquad \frac{\partial^2\Phi(x,y,t_0)}{\partial x\partial y} \equiv 0.$$

II(a): Suppose that the function $\partial^2\Phi(x,y,t)/\partial x\partial y$ is not identically zero. Then, there exists a positive integer m, such that

$$\frac{\partial^2\Phi(x,y,t)}{\partial x\partial y} = (t-t_0)^m F(x,y,t), \tag{2.8}$$

where $F(x,y,t_0)$ is not identically zero. We let

$$\Psi(x,y,t) = \int_0^x \left(\int_0^y F(u,v,t)dv\right) du$$

and write

$$\lambda\Phi(x,y,t) = (\lambda(t-t_0)^m)\Psi(x,y,t) + \lambda(W_1(x,t) + W_2(y,t)), \tag{2.9}$$

for suitable functions W_1 and W_2. The desired result now follows from the arguments in case I.

II(b): Suppose that the function $\partial^2\Phi(x,y,t)/\partial x\partial y$ is identically zero, then we have

$$\Phi(x,y,t) = W_1(x,t) + W_2(y,t),$$

and (2.3) follows from (1.4). □

3. Main estimates. Let $d > 0$, $I = [-d,d]$. Suppose that $\Phi(x,y,t)$ is real-analytic on $I\times I\times I$, $\phi \in C_0^\infty(\mathbb{R}\times\mathbb{R})$ and $\mathrm{supp}(\phi) \subset I\times I$. Let $k(x,y)$ be given as in Section 1, $T_{\lambda,t}$ be defined as in (2.2). We have

LEMMA 3.1. *Suppose $\Phi(0,y,t) = 0$. Then, there is a constant $C > 0$ such that*

$$||T_{\lambda,t}a||_1 \le C, \tag{3.1}$$

for every function $a(x)$ which satisfies

$$\mathrm{supp}(a) \subset [-\delta,\delta]; \tag{3.2}$$

$$||a||_\infty \le (2\delta)^{-1}; \tag{3.3}$$

$$\int_{-\delta}^{\delta} a(y)dy = 0. \tag{3.4}$$

The constant C is independent of λ and t.

We begin by describing and proving several facts, and once this is done we will be ready to prove Lemma 3.1.

LEMMA 3.2 (van der Corput, [**10**, **12**]). *Suppose ϕ and ψ are smooth in $[a, b]$ and ϕ is real-valued. If $|\phi^{(k)}(x)| \geq 1$, then*

$$\left|\int_a^b e^{i\lambda\phi(x)}\psi(x)dx\right| \leq c_k\lambda^{-1/k}\left(|\psi(b)| + \int_a^b |\psi'(x)|dx\right) \tag{3.5}$$

holds when
(i) $k \geq 2$
(ii) *or $k = 1$, if in addition it is assumed that $\phi'(x)$ is monotonic.*

LEMMA 3.3 (Ricci-Stein, [**9**]). *Let $P(x) = \sum_{j=0}^{d} a_j x^j$ be a real-valued polynomial of degree d. Suppose $\varepsilon < 1/d$, then*

$$\int_{|x|\leq 1} |P(x)|^{-\varepsilon}dx \leq A_\varepsilon \left(\sum_{j=0}^{d} |a_j|\right)^{-\varepsilon}. \tag{3.6}$$

The constant A_ε is independent of the coefficients $\{a_j\}$.

LEMMA 3.4 ([**7**]). *Let $\Psi \in C^\infty(\mathbb{R})$, $\psi \in C_0^\infty(\mathbb{R})$ and k be a positive integer. Assume that $|\Psi^{(k)}(x)| \leq B \leq M$ for all $x \in \operatorname{supp}(\psi)$. Define $V = \{x \mid \operatorname{dist}(x, \operatorname{supp}(\psi)) \leq B\}$. Let $A = \sup_{x\in V} |\Psi^{(k+1)}(x)|$. Then, there exists a constant C, which depends only on A, M, k and ψ, such that*

$$\left|\int_{\mathbb{R}} e^{i\lambda\Psi(x)}\psi(x)dx\right| \leq C\lambda^{-\varepsilon/k}\int_V |\Psi^{(k)}(x)|^{-\varepsilon(1+1/k)}dx \tag{3.7}$$

holds for $\varepsilon \in [0, 1]$.

PROPOSITION 3.5. *Let $m \geq 1$, $P(x)$ be a real-valued polynomial such that $P(0) = 0$, and $a(x)$ be given as in Lemma 3.1. Let $R(x, y)$ be a function which satisfies $\partial^m R(x, y)/\partial y^m \equiv 0$. Let*

$$Z_0 f(x) = k(x, 0)\int_{\mathbb{R}} e^{iR(x,y)}\phi(x, y)f(y)dy; \tag{3.8}$$

$$Zf(x) = k(x,0)\int_{\mathbb{R}} e^{i(R(x,y)+P(x)y^m)}\phi(x,y)f(y)dy. \tag{3.9}$$

Assume that

$$\int_{|x|>2\delta} |Z_0 a(x)|dx \le C_0 \tag{3.10}$$

for some constant C_0. *Then there is a constant* $C > 0$ *such that*

$$\int_{|x|>2\delta} |Za(x)|dx \le C. \tag{3.11}$$

The constant C *may depend on* C_0 *and* $\operatorname{dep}(P)$, *but is independent of the coefficients of* P.

Proof. We use induction on $\deg(P)$. Since a similar argument was used in [**4**], we shall present a sketch of the proof only (see also the proof of Proposition 3.6).

For $\deg(P) = 0$, (3.11) follows from (3.10). Assume that (3.11) holds for $\deg(P) \le k-1$, i.e.

$$\int_{|x|>2\delta} \left| k(x,0)\int_{\mathbb{R}} e^{i(R(x,y)+Q(x)y^m)}\phi(x,y)a(y)dy\right| dx \le C \tag{3.12}$$

for all Q with $Q(0) = 0$, $\deg(Q) \le k-1$. Now we prove (3.11) for $\deg(P) = k$. Suppose

$$P(x) = Ax^k + P_{k-1}(x),$$

where $A \ne 0$, $P_{k-1}(0) = 0$ and $\deg(P_{k-1}) \le k-1$. Let $\Delta = \max\{(|A|\delta^m)^{-1/k}, 2\delta\}$. By (3.12), we have

(3.13)

$$\begin{aligned}
&\int_{2\delta<|x|<\Delta} \left| k(x,0)\int_{\mathbb{R}} e^{i(R(x,y)+P(x)y^m)}\phi(x,y)a(y)dy\right| dx \\
&\quad\le \int_{2\delta<|x|<\Delta} \left| k(x,0)\int_{\mathbb{R}} |e^{iAx^k y^m}-1||\phi(x,y)||a(y)|dy\right| dx \\
&\qquad+ \int_{2\delta<|x|<\Delta} \left| k(x,0)\int_{\mathbb{R}} e^{i(R(x,y)+P_{k-1}(x)y^m)}\phi(x,y)a(y)dy\right| dx \\
&\quad\le C|A|\delta^m \int_{|x|<\Delta} |x|^{k-1}dx + C \le C.
\end{aligned}$$

For $j \in Z$, define S_j by

$$S_j f(x) = \chi_{[2^j,2^{j+1}]}(x) \int_{-\delta}^{\delta} e^{i(R(x,y)+P(x)y^m)} \phi(x,y) f(y) dy.$$

By considering $S_j S_j^*$ and using the fact that $\partial^m R(x,y)/\partial y^m \equiv 0$, one can show that (see [**4**])

$$\|S_j\|_{2,2} \leq C\delta^{1/2-1/4k} 2^{j/2} (|A|2^{jk})^{-1/4km}.$$

Hence

(3.14)

$$\begin{aligned} \int_{|x|\geq\Delta} \left| k(x,0) \int_{\mathbb{R}} e^{i(R(x,y)+P(x)y^m)} \phi(x,y) a(y) dy \right| dx \\ &\leq C \sum_{2^j \geq \Delta} \left(\int_{2^j}^{2^{j+1}} \frac{dx}{x^2} \right)^{1/2} \|S_j a\|_2 \\ &\leq C|A|^{-1/4km} \delta^{-1/4k} \sum_{2^j \geq \Delta} (2^{-j/2} 2^{j/2} 2^{-j/4m}) \leq C. \end{aligned}$$

By combining (3.13) and (3.14), we see that Proposition 3.5 is proved. □

PROPOSITION 3.6. *Let $a(x)$ be given by (3.2) – (3.4). Suppose that $q_0(x,t), \dots, q_m(x,t)$ are real-analytic on $I \times I$, and $q_j(0,t) = 0$ for $j = 0, 1, \dots, m$. Then, there is a constant $C > 0$ such that*

$$\left\| \int_{\mathbb{R}} e^{i\lambda\left(\sum_{j=0}^m q_j(x,t) y^j\right)} k(x,y) \phi(x,y) a(y) dy \right\|_1 \leq C, \tag{3.15}$$

for $\lambda \in \mathbb{R}$, $t \in I$.

Proof. Let

(3.16)

$$T_{\lambda,t} f(x) = \text{p.v.} \int_{\mathbb{R}} e^{i\lambda\left(\sum_{j=0}^m q_j(x,t) y^j\right)} k(x,y) \phi(x,y) f(y) dy.$$

By Hölder's inequality and Proposition 2.2, there is a $C > 0$ such that

$$\int_{|x|\leq M\delta} |T_{\lambda,t} a(x)| dx \leq (2M\delta)^{1/2} \|T_{\lambda,t} a\|_2 \leq CM^{1/2}, \tag{3.17}$$

where $M > 0$, C is independent of λ and t.

In view of (3.17) and the fact that $\operatorname{supp}(T_{\lambda,t}a) \subset I$, we may assume that δ is extremely small throughout the proof. Without loss of generality, we will also assume that d is small (see section 4).

We now use induction on m to prove (3.15). For $m = 0$, (3.15) follows from the usual theory of singular integrals. Suppose (3.15) holds for $m-1$, i.e.

$$\left\| \int_{\mathbb{R}} e^{i\lambda\left(\sum_{j=0}^{m-1} q_j(x,t)y^j\right)} k(x,y)\phi(x,y)a(y)dy \right\|_1 \le C. \tag{3.18}$$

To prove (3.15), we assume that $q_m(x,t)$ is not identically zero (otherwise there is nothing to prove). For fixed $t_0 \in I$, there is an integer $s \ge 0$ such that

$$q_m(x,t) = (t-t_0)^s q(x,t), \tag{3.19}$$

where $\partial^i q(0,t_0)/\partial x^i \ne 0$, for some $i \ge 1$. Let l be the smallest such i. Set $\sigma = \max\{(\delta^m|\lambda(t-t_0)^s|)^{-1/l}, 2\delta\}$. For $|x| > 2\delta$, $|y| < \delta$, we have

$$|k(x,y) - k(x,0)| \le C|y||x|^{-2}. \tag{3.20}$$

Hence

(3.21)
$$\int_{|x|\ge 2\delta} \left| \int_{\mathbb{R}} e^{i\lambda\left(\sum_{j=0}^{m} q_j(x,t)y^j\right)} (k(x,y) - k(x,0))\phi(x,y)a(y)dy \right| dx$$
$$\le C\left(\int_{2\delta}^{\infty} \frac{dx}{x^2}\right)\left(\int_{-\delta}^{\delta} |a(y)y|dy\right) \le C.$$

In view of (3.17) and (3.21), to prove (3.15), it suffices to show that

(3.22)
$$\int_{|x|>2\delta} \left| k(x,0) \int_{\mathbb{R}} e^{i\lambda\left(\sum_{j=0}^{m} q_j(x,t)y^j\right)} \phi(x,y)a(y)dy \right| dx \le C.$$

First we treat the integral over $\{2\delta \le |x| \le \sigma\}$. To this end, we write

$$q(x,t) = \sum_{\nu=1}^{l-1} \frac{1}{\nu!} \frac{\partial^\nu q(0,t)}{\partial x^\nu} x^\nu + Q(x,t), \tag{3.23}$$

where $|Q(x,t)| \le C|x|^l$. Then we have

(3.24)

$$\begin{aligned}
&\int_{2\delta\le|x|\le\sigma}\left|k(x,0)\int_{\mathbb{R}} e^{i\lambda\left(\sum_{j=0}^{m} q_j(x,t)y^j\right)}\phi(x,y)a(y)dy\right|dx \\
&\quad\le \int_{2\delta\le|x|\le\sigma}\Bigg|k(x,0) \\
&\qquad\cdot\int_{\mathbb{R}} e^{i\lambda\left(\sum_{j=0}^{m-1} q_j(x,t)y^j+\sum_{\nu=1}^{l-1}(t-t_0)^s q^{(\nu)}(0,t)x^\nu y^m/\nu!\right)}\phi(x,y)a(y)dy\Bigg|dx \\
&\qquad+\int_{2\delta\le|x|\le\sigma}|k(x,0)|\int_{\mathbb{R}}|e^{i\lambda(t-t_0)^s Q(x,t)y^m}-1||a(y)|dy|dx \\
&\quad\le C+C|\lambda(t-t_0)^s|\delta^m\int_0^\sigma x^{l-1}dx\le C,
\end{aligned}$$

where we used (3.18), (3.19), (3.21)(for $m-1$), (3.23) and Proposition 3.5. Next we treat the integral over $\{|x|\ge\sigma\}$.

Choose $\eta_1,\eta_2\in C_0^\infty(\mathbb{R})$ such that

$$\eta_1(x)=1 \quad \text{for} \quad 1\le|x|\le 2$$

and

$$\eta_2(x)=1 \quad \text{for} \quad |x|\le 1.$$

Let $2^{-j}\le d$ and define the operator P_j by

(3.25)

$$P_j f(x)=\eta_1(2^{-j}x)\int_{\mathbb{R}} e^{i\lambda\left(\sum_{j=0}^{m} q_j(x,t)y^j\right)}\phi(x,y)\eta_2(y/\delta)f(y)dy.$$

The kernel of $P_jP_j^*$, denoted by $L_j(x,y)$, can be written as

(3.26)

$$\begin{aligned}
L_j(x,y)=&\eta_1(2^{-j}x)\eta_1(2^{-j}y) \\
&\cdot\int_{\mathbb{R}} e^{i\lambda\sum_{j=0}^{m}(q_j(x,t)-q_j(y,t))z^j}\phi(x,z)\phi(y,z)|\eta_2(z/\delta)|^2dz.
\end{aligned}$$

By van der Corput's lemma, we get

(3.27)

$$\begin{aligned}
&|L_j(x,y)| \\
&\quad\le C|\eta_1(2^{-j}x)\eta_1(2^{-j}y)|(\lambda|q_m(x,t)-q_m(y,t)|)^{-1/m} \\
&\quad= C|\lambda(t-t_0)^s|^{-1/m}|\eta_1(2^{-j}x)\eta_1(2^{-j}y)||q(x,t)-q(y,t)|^{-1/m}.
\end{aligned}$$

On the other hand, we have the following trivial estimate

$$
(3.28)\qquad \begin{aligned} |L_j(x,y)| &\le C|\eta_1(2^{-j}x)\eta_1(2^{-j}y)| \int_{\mathbb{R}} |\eta_2(z/\delta)|^2 dz \\ &\le C\delta|\eta_1(2^{-j}x)\eta_1(2^{-j}y)|. \end{aligned}
$$

By applying the Malgrange Preparation Theorem ([**2**]) to $q(x,t) - q(y,t)$ at the point $(0,0,t_0)$, we can find functions $b_0(y,t), \ldots, b_{l-1}(y,t)$ and $c(x,y,t)$, which are smooth in a neighborhood of $(0,0,t_0)$, such that

$$
(3.29)\qquad q(x,t) - q(y,t) = c(x,y,t)\left(x^l + \sum_{\nu=0}^{l-1} b_\nu(y,t)x^\nu\right),
$$

$b_0(0,t_0) = \cdots = b_{l-1}(0,t_0) = 0$ and $c(0,0,t_0) \neq 0$. Since d is assumed to be small, we may assume that $c(x,y,t) \ge c_0 > 0$, and (3.29) holds for $x, y \in I$, t being close to t_0. From (3.27) and (3.28) we get

$$
(3.30)\quad \begin{aligned} |L_j(x,y)| \le C\delta^{(2l-1)/2l}|\lambda(t-t_0)^s|^{-1/2lm} \\ |\eta_1(2^{-j}x)\eta_1(2^{-j}y)||q(x,t)-q(y,t)|^{-1/2lm}. \end{aligned}
$$

For fixed y, by Lemma 3.3, we find

$$
(3.31)\qquad \begin{aligned} &\int_{\mathbb{R}} |q(x,t)-q(y,t)|^{-1/2ml}|\eta_1(2^{-j}x)|dx \\ &\le C\int_{|u|\le 1}\left|2^{jl}u^l + \sum_{\nu=0}^{l-1} b_\nu(y,t)2^{j\nu}u^\nu\right|^{-1/2ml} 2^j du \\ &\le C2^j 2^{-j/2m}. \end{aligned}
$$

(3.30) and (3.31) implies that

(3.32)
$$
\sup_y \int_{\mathbb{R}} |L_j(x,y)|dx \le C2^j 2^{-j/2m}\delta^{(2l-1)/2l}|\lambda(t-t_0)^s|^{-1/2ml}.
$$

Similar estimate holds for $\sup_x \int_{\mathbb{R}} |L_j(x,y)|dy$. Hence we have

(3.33)
$$
||P_j||_{2,2} \le ||P_jP_j^*||_{2,2}^{1/2} \le C2^{j/2}2^{-j/4m}\delta^{(2l-1)/4l}|\lambda(t-t_0)^s|^{-1/4ml}.
$$

By Hölder's inequality, we find

$$\begin{aligned}(3.34)\quad &\int_{|x|\geq\sigma}\left|k(x,0)\int_{\mathbb{R}} e^{i\lambda\left(\sum_{j=0}^{m} q_j(x,t)y^j\right)}\phi(x,y)a(y)dy\right|dx\\ &\leq C\sum_{2^j\geq\sigma}\left(\int_{2^j}^{2^{j+1}}\frac{dx}{x^2}\right)^{1/2}\|P_j\|_{2,2}\|a\|_2\\ &\leq C\delta^{-1/4l}|\lambda(t-t_0)^s|^{-1/4ml}\sum_{2^j\geq\sigma}2^{-j/4m}\leq C.\end{aligned}$$

By (3.17), (3.24) and (3.34), the proposition is proved. □

We are now ready to prove Lemma 3.1.

Proof. We use ideas that are similar to those used in the proof of Proposition 3.6.

If $\partial^2\Phi(x,y,t)/\partial x\partial y$ is identically zero, we have

$$\Phi(x,y,t)=W_1(x,t)+W_2(y,t).$$

By $\Phi(0,y,t)=0$, we find that $W_2(y,t)=-W_1(0,t)$, and (3.1) follows from standard argument.

Now we assume that $\partial^2\Phi(x,y,t)/\partial x\partial y$ is not identically zero, and write

$$\Phi(x,y,t)=(t-t_0)^s\Psi(x,y,t)+\Phi(x,0,t),$$

where $\partial^2\Psi(x,y,t_0)/\partial x\partial y$ is not identically zero (see (2.8) and (2.9)). Define P_j by

$$(3.35)\quad P_jf(x)=\eta_1(2^{-j}x)\int_{\mathbb{R}} e^{i\lambda\Phi(x,y,t)}\phi(x,y)\eta_2(y/\delta)f(y)dy.$$

Then, the kernel of $P_jP_j^*$ is given by

(3.36)
$$\begin{aligned}L_j(x,y)=&\eta_1(2^{-j}x)\eta_1(2^{-j}y)\\ &\cdot\int_{\mathbb{R}} e^{i\lambda(\Phi(x,z,t)-\Phi(y,z,t))}\phi(x,z)\phi(y,z)|\eta_2(z/\delta)|^2dz.\end{aligned}$$

Let $k \geq 1$ be an integer such that $\partial^{k+1}\Psi(x,0,t_0)/\partial x \partial y^k$ is not identically zero (as a function of x). By Lemma 3.4, we find

(3.37)
$$\begin{aligned}|L_j(x,y)| \leq & C|\lambda(t-t_0)^s|^{-\varepsilon/k}|\eta_1(2^{-j}x)\eta_1(2^{-j}y)| \\ & \cdot \int_{|z|\leq d}\left|\frac{\partial^k\Psi(x,z,t)}{\partial z^k} - \frac{\partial^k\Psi(y,z,t)}{\partial z^k}\right|^{-\varepsilon(1+1/k)} dz,\end{aligned}$$

for $\varepsilon \in [0,1]$. We note that, although the function

$$g_{x,y}(z) = \phi(x,z)\phi(y,z)|\eta_2(z/\delta)|^2$$

depends on δ, the constant C in (3.37) can be taken to be independent of δ, since $||g||_\infty$ and $||g'||_1$ are finite and independent of δ.

Let

$$F(x,y,z,t) = \frac{\partial^k\Psi}{\partial z^k}(x,z,t) - \frac{\partial^k\Psi}{\partial z^k}(y,z,t).$$

It is easy to check that $F(0,0,0,t_0) = 0$ and

$$\frac{\partial^j F}{\partial x^j}(x,y,z,t) = \frac{\partial^{k+j}\Psi}{\partial z^k \partial x^j}(x,z,t) \quad \text{for} \quad j \geq 1.$$

Let $l \geq 1$ be the smallest integer such that $\partial^l F(0,0,0,t_0)/\partial x^l \neq 0$. By using the Malgrange Preparation Theorem, we get

$$F(x,y,z,t) = c(x,y,z,t)\left(x^l + \sum_{\nu=0}^{l-1} b_\nu(y,z,t)x^\nu\right). \tag{3.38}$$

for $x, y \in I$, t being close to t_0, and $|c(x,y,z,t)| \geq c_0 > 0$ (see also the argument in the proof of Proposition 3.6). By taking $\varepsilon = k/2l(k+1)$ in (3.37) and using (3.38) and Lemma 3.3, we find

$$\sup_y \int_{\mathbb{R}} |L_j(x,y)|dx \leq C|\lambda(t-t_0)^s|^{-1/2l(k+1)}2^{j/2}. \tag{3.39}$$

A similar estimate holds for $\sup_x \int_{\mathbb{R}} |L_j(x,y)|dy$. Hence we have

$$||P_j||_{2,2} \leq C|\lambda(t-t_0)^s|^{-1/4l(k+1)}2^{j/4}. \tag{3.40}$$

Let $\sigma = \max\{\delta^{-2}|\lambda(t-t_0)^s|^{-1/l(k+1)}, 2\delta\}$, we find

$$\int_{|x|>\sigma} \left| k(x,0) \int_{\mathbb{R}} e^{i\lambda\Phi(x,y,t)} \phi(x,y)a(y)dy \right| dx \tag{3.41}$$

$$\leq \sum_{2^j \geq \sigma} \left(\int_{2^j}^{2^{j+1}} \frac{dx}{x^2} \right)^{1/2} \|P_j\|_{2,2} \|a\|_2$$

$$\leq C|\lambda(t-t_0)^s|^{-1/4l(k+1)} \delta^{-1/2} \sum_{2^j \geq \sigma} 2^{-j/4} \leq C. \tag{3.42}$$

It remains for us to show that

$$\int_{2\delta<|x|\leq\sigma} \left| k(x,0) \int_{\mathbb{R}} e^{i\lambda\Phi(x,y,t)} \phi(x,y)a(y)dy \right| dx \leq C. \tag{3.43}$$

If $\sigma \leq 2\delta$, (3.43) holds automatically. Now suppose $\sigma > 2\delta$, we have

$$|\lambda(t-t_0)^s|\delta^{3l(k+1)} \leq 1. \tag{3.44}$$

Let $k_0 = 3l(k+1)$ and write

$$\Psi(x,y,t) = \sum_{j=0}^{k_0} q_j(x,t)y^j + Q(x,y,t),$$

where $q_j(x,t) = (1/j!)\partial^j\Psi(x,0,t)/\partial y^j$, $|Q(x,y,t)| \leq C|y|^{k_0+1}$. Since $\Psi(0,y,t) \equiv 0$, we have $q_j(0,t) = 0$ for all j. By (3.21), Proposition 3.6 and (3.44), we find

(3.45)

$$\begin{aligned}
&\int_{2\delta\leq|x|\leq\sigma} \left| k(x,0) \int_{\mathbb{R}} e^{i\lambda\Phi(x,y,t)} \phi(x,y)a(y)dy \right| dx \\
&\quad\leq \int_{2\delta\leq|x|\leq\sigma} \left| k(x,0) \int_{\mathbb{R}} e^{i\lambda(t-t_0)^s \sum_{j=0}^{k_0} q_j(x,t)y^j} \phi(x,y)a(y)dy \right| dx + \\
&\quad\quad + C|\lambda(t-t_0)^s| \int_{2\delta\leq|x|\leq d} \frac{1}{|x|} \int_{-\delta}^{\delta} |y|^{k_0+1}|a(y)|dydx \leq C,
\end{aligned}$$

which completes the proof of Lemma 3.1. □

REMARK. The proofs of Proposition 3.5, 3.6 and Lemma 3.1 are similar in nature. They are proved in the order Proposition 3.5 →

Proposition 3.6 $\to$ Lemma 3.1, to make the approximation part of the proof work. In the L^2 part of the proof of Lemma 3.1, Lemma 3.4 plays an important role, because one cannot use van der Corput's lemma there. The L^2 estimates obtained by using Lemma 3.4 are not as precise as what one gets in the case of polynomial phase (where one may use van der Corput's lemma). This difficulty is overcome by first proving Proposition 3.6.

4. Conclusion. Here we shall prove Theorem B. Let T_λ be given as in (1.2). It suffices to prove that

$$||T_\lambda a||_1 \leq C \tag{4.1}$$

for every function $a(x)$ which satisfies (1.5)-(1.7). Since $k(x,y)$ is integrable away from the diagonal Δ, we may assume that ϕ is supported in a small neighborhood of Δ. By using a partition of unity, we can further assume that $\text{supp}(\phi)$ is contained in $[x_0-d, x_0+d] \times [x_0-d, x_0+d]$, for some x_0 and a certain small number d. We may assume that $x_0 = 0$ (as we did in the proof of Proposition 2.2). We also assume that $\Phi(x,y)$ is real-analytic in $[-4d,4d]\times[-4d,4d]$.

Let $I = [x_I - \delta, x_I + \delta]$, $a(x)$ be a function which satisfies (i) $\text{supp}(a) \subset I$; (ii) $||a||_\infty \leq (2\delta)^{-1}$; (iii) $\int e^{i\lambda\Phi(x_I,y)}a(y)dy = 0$. To prove (4.1), it suffices to consider small $\delta's$, say, $\delta < d$ (see (3.17)). Hence we have either $|x_I| < 2d$ or $T_\lambda a \equiv 0$. Let $I_1 = [-2d, 2d]$, $\Psi(x,y,t) = \Phi(x+t, y+t) - \Phi(t, y+t)$, $k_t(x,y) = k(x+t, y+t)$ and $\phi_t(x,y) = \phi(x+t, y+t)$. If we let $t = x_I$, $a_0(x) = e^{i\lambda\Phi(x_I, y+x_I)}a(x+x_I)$, then

$$T_\lambda a(x+x_I) = \int_{\mathbb{R}} e^{i\lambda\Psi(x,y,t)}k_t(x,y)\phi_t(x,y)a_0(y)dy. \tag{4.2}$$

We observe that $\Psi(0,y,t) \equiv 0$ and $a_0(x)$ satisfies (3.2)-(3.4). By Lemma 3.1, there is a constant $C > 0$, which is independent of λ, x_I and δ such that

$$||T_\lambda||_1 \leq C$$

holds. It should be noted that the dependence of k_t and ϕ_t on t does not cause any trouble here. The proof is now complete.

Remark. Theorem B becomes false if the assumption on the real-analyticity of Φ is dropped. This can be shown by using a C^∞

function constructed by Nagel and Wainger in [**3**]. We also refer the reader to [**7**], where the same issue in the translation invariant case was discussed. The phase function $\Phi(x-y)$ used in [**7**] (where Φ is the function due to Nagel and Wainger) cannot be used in the current situation. But, if we replace $\Phi(x-y)$ by $\Phi'(x)\Phi(x-y)$, then the argument in [**7**, p. 290] can be adapted to show that Theorem B cannot hold for general C^∞ phase functions.

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