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ON UNIFORM HOMEOMORPHISMS OF THE UNIT SPHERES OF CERTAIN BANACH LATTICES

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We prove that if X is an infinite dimensional Banach lattice with a weak unit then there exists a probability space (Ω, Σ, μ) so that the unit sphere of $(L_1(\Omega, \Sigma, \mu))$ is uniformly homeomorphic to the unit sphere $S(X)$ if and only if X does not contain l_∞^n's uniformly.

1. Introduction. Recently E. Odell and Th. Schlumprecht [**O.S**] proved that if X is an infinite dimensional Banach space with an unconditional basis then the unit sphere of X and the unit sphere of l_1 are uniformly homeomorphic if and only if X does not contain l_∞^n uniformly in n. We extend this result to the setting of Banach lattices. In Theorem 2.1 we obtain that if X is a Banach lattice with a weak unit then there exists a probability space (Ω, Σ, μ) so that the unit sphere $S(L_1(\Omega, \Sigma, \mu))$ is uniformly homeomorphic to the unit sphere $S(X)$ if and only if X does not contain l_∞^n uniformly in n. A consequence of this -Corollary 2.11- is that if X is a separable infinite dimensional Banach lattice then $S(X)$ and $S(l_1)$ are uniformly homeomorphic if and only if X does not contain l_∞^n uniformly in n. Quantitative versions of this corollary are given in Theorem 2.2 and Theorem 2.3. A continuous function $f : [0, \infty) \to [0, \infty)$ with$f(0) = 0$ is a *modulus of continuity* for a function between two metric spaces $F : (A, d_1) \to (B, d_2)$ if $d_2(F(a_1), F(a_2)) \leq f(d_1(a_1, a_2))$ whenever $a_1, a_2 \in A$. Theorem 2.2 says that if X and Y are separable infinite dimensional Banach lattices with $M_q(X) < \infty$ and $M_{q'}(Y) < \infty$ for some $q, q' < \infty$ then there exists a uniform homeomorphism $F : S(X) \to S(Y)$ such that F and F^{-1} have modulus of continuity f where f depends solely on $q, q', M_q(X)$ and $M_{q'}(Y)$. Here $M_q(X)$ is the q-concavity constant of X and will be defined below.

Central in defining these homeomorphisms is the entropy map, considered in [**G**] and [**O.S**]. We refer the reader to [**B**] and its references for a survey of some results concerning uniform homeomorphisms between Banach spaces. In particular it is interesting to note Enflo's result that l_1 and L_1 are not uniformly homeomorphic [**B**] while their unit spheres are. Also we refer to [**L.T**] for facts related to the theory of Banach lattices.

After this work was done, we learned that Professor N. Kalton proved the same result using complex interpolation theory.

Notation. Let us start by recalling some definitions and well known facts. A non negative element e of a Banach lattice X is a *weak unit* if $e \wedge x = 0$ for $x \in X$ implies that $x = 0$. Every separable Banach lattice has a weak unit [**L.T**, p. 9]. A Banach lattice is order continuous if and only if every increasing, order bounded sequence is convergent. By a general representation theorem (see [**L.T**, p. 25]) any order continuous Banach lattice with a weak unit can be represented as a Banach lattice of functions. More precisely:

1. there exist a probability space (Ω, Σ, μ) and an ideal $\widetilde{X}$ of $L_1(\Omega, \Sigma, \mu)$, along with a lattice norm $\|\cdot\|_{\widetilde{X}}$ on $\widetilde{X}$ so that X is order isometric to $(\widetilde{X}, \|\cdot\|_{\widetilde{X}})$.
2. $\widetilde{X}$ is dense in $L_1(\Omega, \Sigma, \mu)$ and $L_\infty(\Omega, \Sigma, \mu)$ is dense in $\widetilde{X}$.
3. $\|f\|_1 \leq \|f\|_{\widetilde{X}} \leq 2\|f\|_\infty$ for all $f \in L_\infty(\Omega, \Sigma, \mu)$.

Moreover $\widetilde{X}^* = \{g : \Omega \longrightarrow \mathbb{R} : \|g\|_{\widetilde{X}^*} < \infty\}$ is isometric to X^*, where

$$\|g\|_{\widetilde{X}^*} = \sup\left\{\int fg d\mu; \|f\|_{\widetilde{X}} \leq 1\right\}$$

and if $g \in \widetilde{X}^*$ and $f \in \widetilde{X}$ then

$$g(f) = \int fg d\mu.$$

If X is a Banach lattice which is not order continuous then X contains c_0 ([**L.T**, pages 6-7]).

A Banach lattice X is *q-concave* if there exists a constant $M_q < \infty$ such that

$$(\star) \qquad \left(\sum_{i=1}^{n} \|x_i\|^q\right)^{\frac{1}{q}} \leq M_q \left\| \left(\sum_{i=1}^{n} |x_i|^q\right)^{\frac{1}{q}} \right\|$$

resp. *p-convex* if there exists $M^p < \infty$ so that

$$(\star\star) \qquad \left\| \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \right\| \leq M^p \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}}$$

for all $n \in \mathbb{N}$ and $x_i \in X$, $1 \leq i \leq n$.

$M_q(X)$ is the smallest constant satisfying $(\star)$ and $M^p(X)$ is the smallest constant that satisfies $(\star\star)$.

Given a Banach lattice of functions X, the p-convexification $X^{(p)}$ of X is given by

$$X^{(p)} = \{f : \Omega \longrightarrow \mathbb{R} : |f|^p \in X\}$$

with

$$|||f||| = \||f|^p\|^{\frac{1}{p}}.$$

The space $X^{(p)}$ is a Banach lattice with $M^p(X^{(p)}) = 1$ ([**L.T**, p. 53]).

We will also need the following result. If X is r-convex and s-concave, for $1 \leq r, s \leq \infty$ then $X^{(p)}$ is pr-convex and ps-concave with

$$M^{pr}(X^{(p)}) \leq (M^r(X))^{\frac{1}{p}}$$

and

$$M_{ps}(X^{(p)}) \leq (M_s(X))^{\frac{1}{p}}.$$

(See [**L.T**, p. 54].)

We will use standard Banach space notations, $BaX = \{x \in X : \|x\| \leq 1\}$ will denote the unit ball of X and $S(X) = \{x \in X : \|x\| = 1\}$ the unit sphere of X. If h is a real function on Ω, then $\operatorname{supp} h = \{\omega \in \Omega : h(\omega) \neq 0\}$ is the support of h. If $B \subset \Omega$, then $Bh(\omega) = h(\omega)\chi_B(\omega)$ where χ_B is the indicator function of B.

2. The main result. We now state the main result of this work.

THEOREM 2.1. *Let X be an infinite dimensional Banach lattice with a weak unit. Then there exists a probability space (Ω, Σ, μ) so that $S(L_1(\Omega, \Sigma, \mu))$ is uniformly homeomorphic to $S(X)$ if and only if X does not contain l_∞^n uniformly in n.*

Our proof of Theorem 2.1 will yield two quantitative results:

THEOREM 2.2. *If X and Y are separable infinite dimensional Banach lattices with $M_q(X) < \infty$ and $M_{q'}(Y) < \infty$ for some $q, q' < \infty$ then there exists a uniform homeomorphism $F : S(X) \longrightarrow S(Y)$ such that F and F^{-1} have modulus of continuity α where α depends solely on $q, q', M_q(X)$ and $M_{q'}(Y)$.*

THEOREM 2.3. *If X and Y are both uniformly convex and uniformly smooth separable infinite dimensional Banach lattices then there exists a uniform homeomorphism $F : S(X) \longrightarrow S(Y)$ such that F has modulus of continuity f where f depends solely on the modulus of uniform convexity of Y and the modulus of uniform smoothness of X, and F^{-1} has a modulus of continuity g depending solely on the modulus of uniform smoothness of Y and the modulus of uniform convexity of X.*

The proofs will involve a sequence of steps similar to those in [**O.S**]. We begin with a simple extension of Proposition 2.8 of [**O.S**]. Recall that $X^{(p)}$ is the p-convexification of X.

PROPOSITION 2.4. *Let X be a Banach lattice of functions on a set Ω and let $1 < p < \infty$. Then the map*

$$G_p : S(X^{(p)}) \longrightarrow S(X)$$

given by $G_p(f) = |f|^p \operatorname{sign} f$ is a uniform homeomorphism. Furthermore the moduli of continuity of G_p and $(G_p)^{-1}$ are functions solely of p.

Proof. Clearly G_p maps $S(X^{(p)})$ one-to-one onto $S(X)$. Let f and g be in $S(X^{(p)})$ with $1 > \delta = \|f - g\|_{X^{(p)}} = \||f - g|^p\|_X^{\frac{1}{p}}$. As in [**O.S**] we shall show that there exist two functions H and F such that

$$H(\delta) \leq \|G_p(f) - G_p(g)\| \leq F(\delta)$$

where $F(\delta) = 2(1 - (1 - \delta^{\frac{1}{p}})^p) + \delta^{p-1} + \delta^p$ and $H(\delta) = \frac{1}{2^{p-1}}\delta^p$. The proposition then follows.
Let

$$\Omega_+ = \{\omega \in \Omega : \operatorname{sign} f(\omega) = \operatorname{sign} g(\omega)\}$$

and

$$\Omega_- = \{\omega \in \Omega : \operatorname{sign} f(\omega) \neq \operatorname{sign} g(\omega)\}.$$

We then have:

$$\begin{aligned}\|G_p(f) - G_p(g)\| &= \| |f|^p \operatorname{sign} f - |g|^p \operatorname{sign} g \| \\ &= \| ||f|^p - |g|^p| \chi_{\Omega_+} + (|f|^p + |g|^p)\chi_{\Omega_-} \|.\end{aligned}$$

But $a^p - b^p \geq (a-b)^p$ and $a^p + b^p \geq 2^{1-p}(a+b)^p$ for $a \geq b \geq 0$. Thus,

$$\begin{aligned}\|G_p(f) - G_p(g)\| &\geq \left\| ||f| - |g||^p \chi_{\Omega_+} + \frac{1}{2^{p-1}}(|f| + |g|)^p \chi_{\Omega_-} \right\| \\ &\geq \left\| \frac{1}{2^{p-1}} ||f| - |g||^p \chi_{\Omega_+} + \frac{1}{2^{p-1}}(|f| + |g|)^p \chi_{\Omega_-} \right\| \\ &= 2^{1-p} \| |f - g|^p \| \\ &= 2^{1-p} \|f - g\|^p_{X^{(p)}}.\end{aligned}$$

So we obtain $H(\delta) = \frac{1}{2^{p-1}}\delta^p$ as a lower estimate. For the upper estimate we have:

$$\begin{aligned}\|G_p(f) - G_p(g)\| &= \left\| ||f|^p - |g|^p| \chi_{\Omega_+} + (|f|^p + |g|^p)\chi_{\Omega_-} \right\| \\ &\leq \left\| ||f|^p - |g|^p|\chi_{\Omega_+} \right\| + \left\| (|f|^p + |g|^p)\chi_{\Omega_-} \right\|.\end{aligned}$$

First we note that since

$$(|f|^p + |g|^p)\chi_{\Omega_-} \leq (|f| + |g|)^p \chi_{\Omega_-} \leq |f - g|^p \chi_{\Omega},$$

we get

$$\left\| (|f|^p + |g|^p)\chi_{\Omega_-} \right\| \leq \|f - g\|^p_{X^{(p)}} = \delta^p.$$

Next we estimate $\| \, ||f|^p - |g|^p| \chi_{\Omega_+} \|$. For this purpose we split Ω_+ into Ω_+^1 and Ω_+^2 where

$$\Omega_+^1 = \{\omega \in \Omega_+ : |f(\omega) \leq q|g(\omega)| \ \text{ or } \ |g(\omega)| \leq q|f(\omega)|\}$$

and

$$\Omega_2^+ = \Omega_+ \sim \Omega_+^1$$

and $q = 1 - \delta^{\frac{1}{p}}$.

Note that if $C = (1-q)^{-p}$ then

$$||f|^p - |g|^p|\chi_{\Omega_+^1} \leq C|f - g|^p.$$

Indeed,

$$C|f-g|^p - |g|^p + |f|^p \geq C|g-qg|^p - |g|^p = 0$$

in case $|f| \leq q|g|$ (the proof is similar if $|g| \leq q|f|$).

Thus

$$\begin{aligned} \|\chi_{\Omega_+^1}||f|^p - |g|^p|\| &\leq C\|\chi_{\Omega_+}|f-g|^p\| \\ &\leq C\||f-g|^p\| \\ &= C\|f-g\|_{X^{(p)}}^p \\ &= C\delta^p \\ &= (1-q)^{-p}\delta^p. \end{aligned}$$

Since $(1-q)^{-p} = \delta^{-1}$, we obtain

$$\|\chi_{\Omega_+^1}||f|^p - |g|^p|\| \leq \delta^{p-1}.$$

Finally we have on Ω_+^2 :

$$\begin{aligned} \|||f|^p - |g|^p|\chi_{\Omega_+^2}\| &\leq (1-q^p)\||f|^p + |g|^p|\| \\ &\leq 2(1-(1-\delta^{\frac{1}{p}})^p). \end{aligned}$$

So

$$F(\delta) = 2(1-(1-\delta^{\frac{1}{p}})^p) + \delta^{p-1} + \delta^p$$

and as $p > 1$, $F(\delta) \longrightarrow 0$ when $\delta \longrightarrow 0$. □

Throughout the rest of the paper, X will denote a Banach lattice with the representation as a lattice of functions on (Ω, Σ, μ) satisfying the conditions mentionned in the introduction. The next step in proving Theorem 2.1 will be to produce a uniform homeomorphism

$$F_X : S(L_1(\Omega, \Sigma, \mu)) \longrightarrow S(X)$$

in the case where our lattice X is uniformly convex and uniformly smooth. In order to do this we need first to define the *entropy function* $E(h, f)$.

Let $h \in (L_\infty(\mu))^+$ and define $E(h, \cdot) : X \longrightarrow [-\infty, \infty)$ by

$$E(h,f) = \int h \log |f| d\mu$$

for $f \in X$, (we use the convention that $0 \log 0 \equiv 0$) and more generally,

$$E(h, f) = E(|h|, |f|)$$

if $h \in L_\infty(\mu)$.

The entropy map was considered in [**G**] and in the sequel we use arguments of both [**O.S**] and [**G**].

PROPOSITION 2.5. *Suppose X is uniformly convex. Let $h \in (L_\infty(\mu))^+$ and set*

$$\lambda \equiv \sup_{f \in BaX} \int h \log |f| d\mu.$$

Then $-\log 2 \leq \lambda \leq \|h\|_\infty$ and if $h \neq 0$ there exists a unique $f \in S(X)^+$ so that $\lambda = E(h, f)$. Moreover $\operatorname{supp} f = \operatorname{supp} h$.

Proof. First we note that $\lambda \leq \|h\|_\infty$. To see this it suffices to observe that

$$\begin{aligned} \lambda &= \sup_{f \in BaX^+} \int h \log |f| d\mu \\ &\leq \sup_{f \in BaX^+} \int h |f| d\mu \\ &\leq \sup_{f \in BaX^+} \|h\|_\infty \|f\|_{L_1} \\ &\leq \sup_{f \in BaX^+} \|h\|_\infty \|f\|_X \\ &\leq \|h\|_\infty. \end{aligned}$$

Also $\lambda \geq -\log 2$ since $\chi_\Omega / 2 \in Ba(X)^+$. Next let $(f_n) \subseteq (BaX)^+$ be such that $E(h, f_n) \geq \lambda - 2^{-n}$. Since X is uniformly convex, by passing to a subsequence, we can suppose that f_n converges weakly to $f \in (BaX)^+$. Let (u_n) be a sequence of "far-out" convex combinations of f_n, such that (u_n) converges to f in norm [**M**], thus $u_n = \sum_{i=p_n+1}^{p_{n+1}} c_i f_i$ where $p_1 < p_2 < \cdots < p_n < \cdots c_i \geq 0, \sum_{i=p_n+1}^{p_{n+1}} c_i = 1$ and $\|u_n - f\|_X \longrightarrow 0$ as $n \longrightarrow \infty$.

We next note that if $(g_i)_{i=1}^n \subseteq BaX$, and $(d_i)_{i=1}^n \subseteq (\mathbb{R})^+$ with $\sum_{i=1}^n d_i = 1$ then

$$E\left(h, \sum_{i=1}^n d_i g_i\right) \geq \sum_{i=1}^n d_i E(h, g_i).$$

Moreover if $B = \operatorname{supp} h$ and $Bg_i \neq Bg_j$ for some i, j then

$$E\left(h, \sum_{i=1}^{n} d_i g_i\right) > \sum_{i=1}^{n} d_i E(h, g_i).$$

This follows from the strict concavity of the logarithm function. Therefore

$$\lim_{n\to\infty} E(h, u_n) = \lambda.$$

CLAIM. $E(h, f) = \lambda$.

Note that

$$\|u_n - f\|_{L_1(\mu)} \leq \|u_n - f\|_X \to 0$$

and so in order to prove the Claim, it suffices to prove the following lemma:

LEMMA 2.6. *Let $\lambda \in \mathbb{R}, h \in L_\infty^+(\mu)$, $(u_n) \subseteq L_1^+(\mu)$ and suppose $u_n \longrightarrow f$ in $L_1(\mu)$. Then*

$$\int h \log u_n d\mu \longrightarrow \lambda \ \textit{ implies } \ \int h \log f d\mu \geq \lambda.$$

Proof. By passing to a subsequence we may assume that $u_n \to f$ a.e. Thus $(\log u_n)^- \to (\log f)^-$ a.e. and so

$$\int h(\log f)^- d\mu \leq \liminf_{n\to\infty} \int h(\log u_n)^- d\mu$$

by Fatou's lemma. Therefore

$$(\star) \qquad \limsup_{n\to\infty} \int -h(\log u_n)^- d\mu \leq \int -h(\log f)^- d\mu.$$

On the other hand, one has also the inequality:

$$(\star\star) \qquad \limsup_{n\to\infty} \int h(\log u_n)^+ d\mu \leq \int h(\log f)^+ d\mu.$$

Indeed, fix $\varepsilon > 0$. Since $0 \leq (\log u_n)^+ \leq u_n$, and (u_n) is uniformly integrable, there exists $\delta > 0$ so that $\mu(A) < \delta$ implies

$$\text{for all } n, \int_A h(\log u_n)^+ d\mu < \varepsilon \ \text{ and } \ \int_A h(\log f)^+ d\mu < \varepsilon.$$

($(\log f)^+$ is integrable since $0 \le (\log f)^+ \le f$.) Now $h(\log u_n)^+ \longrightarrow h(\log f)^+$ a.e: So by Egoroff's theorem, there exists a set C with $\mu(C) < \delta$ such that

$$h(\log u_n)^+ \longrightarrow h(\log f)^+$$

uniformly except perhaps on C. More exactly, for $\varepsilon > 0$, there exist $n(\varepsilon) \in \mathbb{N}$ and a set C with $\mu(C) < \delta$ such that for any $n \ge n(\varepsilon)$ we have

$$\sup_{\omega \in C^c} |h(\log u_n)^+ - h(\log f)^+| < \varepsilon.$$

Thus

$$\begin{aligned}\int h(\log u_n)^+ d\mu &\le \int |h(\log u_n)^+ - h(\log f)^+| d\mu + \int h(\log f)^+ d\mu \\ &= \int_C |h(\log u_n)^+ - h(\log f)^+| d\mu \\ &\quad + \int_{C^c} |h(\log u_n)^+ - h(\log f)^+| d\mu + \int h(\log f)^+ d\mu \\ &< 2\varepsilon + \varepsilon + \int h(\log f)^+ d\mu.\end{aligned}$$

So

$$\limsup_{n\to\infty} \int h(\log u_n)^+ d\mu \le \int h(\log f)^+ d\mu.$$

Now adding ($\star$) and ($\star\star$) yields

$$\lambda \le \int h \log f d\mu,$$

which proves Lemma 2.6. □

Note that since $\lambda \ge E(h,f)$, we get $E(h,f) = \lambda$, proving the Claim. Now we prove that f is unique. Indeed, let $f \neq g$ with $E(h,f) = E(h,g) = \lambda$ and we may assume that $\|f\| = \|g\| = 1$. Thus by uniform convexity $\left\|\frac{f+g}{2}\right\| < 1$ and so $\frac{f+g}{2}$ cannot maximize the entropy, and so

$$\lambda = \frac{1}{2}\left(E(h,f) + E(h,g)\right) \le E\left(h, \frac{f+g}{2}\right) < \lambda,$$

a contradiction.

Let now $B = \operatorname{supp} h$. In order to obtain $\operatorname{supp} f = B$ a.e consider first $g = Bf$ in what preceeds and note that $E(h,g) = E(h,f)$ to get $f = Bf$ a.e. Then observe that trivially $\operatorname{supp} Bf \subset B$ a.e, while if the previous inequality was strict, then there exists a set $A \subset B$ with $\mu(A) > 0$ such that $f_{|A} = 0$. Thus

$$-\infty = E(h,f) \geq E(h, \chi_\Omega/2) = -\log 2;$$

a contradiction. Hence $\operatorname{supp} f = \operatorname{supp} Bf = B$. □

Thus under the assumption that X is uniformly convex we can define

$$F_X : S(L_1(\mu))^+ \bigcap L_\infty(\mu) \longrightarrow S(X)^+$$

by $F_X(h) = f$ where $f \in S(X)^+$ is such that

$$E(h,f) = \max_{g \in (BaX)^+} \int h \log |g| d\mu = E_X(h).$$

We then define

$$F_X : S(L_1(\mu)) \bigcap L_\infty(\mu) \longrightarrow S(X)$$

by $F_X(h) = (\operatorname{sign} h) F_X(|h|)$.

We shall show that F_X is uniformly continuous, and thus extends to a uniformly continuous function on $S(L_1(\mu))$. To do so we will need a proposition similar to Proposition 2.3.C of [**O.S**]. The proof is nearly the same, adapted to function spaces.

PROPOSITION 2.7. *Let h_1, h_2 be in $S(L_1(\mu))^+ \cap L_\infty(\mu)$ with $\|h_1 - h_2\|_1 \leq 1$. Let $x_1 = F_X(h_1)$, and $x_2 = F_X(h_2)$. Then*

$$\left\| \frac{x_1 + x_2}{2} \right\| \geq 1 - \|h_1 - h_2\|_1^{\frac{1}{2}}.$$

Proof. Let $\left\| \frac{x_1+x_2}{2} \right\| = 1 - 2\varepsilon$. We need to show that

$$2\varepsilon \leq \|h_1 - h_2\|^{\frac{1}{2}}.$$

We may assume $\varepsilon > 0$. Define $\widetilde{x_1} = x_1 + \varepsilon x_2$ and $\widetilde{x_2} = x_2 + \varepsilon x_1$. Then

$$\operatorname{supp} \widetilde{x_1} = \operatorname{supp} \widetilde{x_2} = \operatorname{supp} h_1 \cup \operatorname{supp} h_2 \equiv B,$$

and

$$\left\|\frac{\widetilde{x_1}+\widetilde{x_2}}{2}\right\| \leq \left\|\frac{x_1+x_2}{2}\right\| + \varepsilon = 1-\varepsilon.$$

With this we can prove that:

$$(\star) \qquad \varepsilon \leq |\log(1-\varepsilon)| \leq \frac{1}{2}\{E(h_1,\widetilde{x_1}) - E(h_1,\widetilde{x_2})\} \ .$$

Indeed, since $\widetilde{x_1} \geq x_1$, we clearly have:

$$E(h_1,\widetilde{x_1}) \geq E(h_1,x_1) \geq E\left(h_1, \frac{\widetilde{x_1}+\widetilde{x_2}}{2(1-\varepsilon)}\right)$$

since $\frac{\widetilde{x_1}+\widetilde{x_2}}{2(1-\varepsilon)} \in BaX$ and x_1 maximizes the entropy. And

$$\begin{aligned} E\left(h_1, \frac{\widetilde{x_1}+\widetilde{x_2}}{2(1-\varepsilon)}\right) &= E\left(h_1, \frac{\widetilde{x_1}+\widetilde{x_2}}{2}\right) + |\log(1-\varepsilon)| \\ &\geq \frac{1}{2}E(h_1,\widetilde{x_1}) + \frac{1}{2}E(h_1,\widetilde{x_2}) + |\log(1-\varepsilon)|. \end{aligned}$$

Similarly we have

$$(\star\star) \qquad \varepsilon \leq |\log(1-\varepsilon)| \leq \frac{1}{2}\{E(h_2,\widetilde{x_2}) - E(h_2,\widetilde{x_1})\} \ .$$

Then by averaging $(\star)$ and $(\star\star)$ we get

$$\varepsilon \leq \frac{1}{4}[E(h_1,\widetilde{x_1}) - E(h_1,\widetilde{x_2}) + E(h_2,\widetilde{x_2}) - E(h_2,\widetilde{x_1})].$$

So

$$\begin{aligned} \varepsilon &\leq \frac{1}{4}\int_B (h_1-h_2)(\log\widetilde{x_1} - \log\widetilde{x_2})d\mu \\ &\leq \frac{1}{4}\int_B |h_1-h_2| \left|\log\frac{\widetilde{x_1}}{\widetilde{x_2}}\right| d\mu. \end{aligned}$$

But

$$\left|\log\frac{\widetilde{x_1}}{\widetilde{x_2}}\right| \leq \log\frac{1}{\varepsilon} \ \text{ on } \ B$$

for

$$\frac{\widetilde{x_1}}{\widetilde{x_2}} = \frac{x_1+\varepsilon x_2}{x_2+\varepsilon x_1} = \frac{x_1+\varepsilon x_2}{\varepsilon(x_1+\varepsilon^{-1}x_2)} \leq \frac{1}{\varepsilon}$$

and similarly

$$\frac{\widetilde{x_2}}{\widetilde{x_1}} \leq \frac{1}{\varepsilon}.$$

Since $\log \frac{1}{\varepsilon} \leq \frac{1}{\varepsilon}$, we finally get

$$\varepsilon \leq \frac{1}{4}\|h_1 - h_2\|_1 \frac{1}{\varepsilon}.$$

Hence

$$2\varepsilon \leq \|h_1 - h_2\|_1^{\frac{1}{2}}$$

□

PROPOSITION 2.8. *Let X be uniformly convex. Then*

$$F_X : S(L_1(\mu)) \bigcap L_\infty(\mu) \longrightarrow S(X)$$

is uniformly continuous and hence extends to a uniformly continuous map $F_X : S(L_1(\mu)) \longrightarrow S(X)$. Moreover the modulus of continuity of F_X depends only on the modulus of uniform convexity of X.

Proof. Recall that X is uniformly convex if and only if

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1, \|x-y\| \geq \varepsilon\right\} > 0.$$

We first observe that $F_X : S(L_1(\mu))^+ \longrightarrow S(X)$ is uniformly continuous.
Indeed, by Proposition 2.7, if h_1 and h_2 are in $S(L_1(\mu))^+ \bigcap L_\infty(\mu)$ and $\|h_1 - h_2\|_1 \leq 1$ then

$$\left\|\frac{F_X(h_1) + F_X(h_2)}{2}\right\| \geq 1 - \|h_1 - h_2\|_1^{\frac{1}{2}}$$

or

$$1 - \left\|\frac{F_X(h_1) + F_X(h_2)}{2}\right\| \leq \|h_1 - h_2\|_1^{\frac{1}{2}}.$$

So if $\|F_X(h_1) - F_X(h_2)\| \geq \varepsilon$ then $\|h_1 - h_2\| \geq (\delta_X(\varepsilon))^2$. Thus there exists $\eta(\varepsilon) = (\delta_X(\varepsilon))^2$ so that $\|h_1 - h_2\| < \eta(\varepsilon)$ implies

$\|F_X(h_1) - F_X(h_2)\| \leq \varepsilon$. Letting $\eta(0) = 0$, the function η is continuous and strictly increasing on $[0, 2]$. So η has an inverse g depending only on the modulus of uniform convexity of X, and

$$\|F_X(h_1) - F_X(h_2)\| \leq g(\|h_1 - h_2\|).$$

For the general case let h_1, h_2 in $S(L_1(\mu)) \bigcap L_\infty(\mu)$ and set

$$x_i = F_X(h_i) = \operatorname{sign} h_i \cdot F_X(|h_i|)$$

for $i = 1, 2$. Then

$$\|x_1 - x_2\| \leq \|F_X(|h_1|) - F_X(|h_2|)\| + \|\chi_D(F_X(|h_1|) + F_X(|h_2|))\|$$

where

$$D = \{\omega \in \Omega : \operatorname{sign} h_1(\omega) \neq \operatorname{sign} h_2(\omega)\}.$$

By what we observed in the beginning of the proof,

$$\|F_X(|h_1|) - F_X(|h_2|)\| < g(\varepsilon)$$

whenever

$$\||h_1| - |h_2|\| \leq \|h_1 - h_2\| < \varepsilon.$$

Our next step is to estimate $\|\chi_D F_X(|h_i|)\|$, for $i = 1, 2$. To do so, we note that

$$\|\chi_D F_X(|h_1|)\| = \|DF_X(|h_1|)\| \leq \left\|F_X(|h_1|) - F_X\left(\frac{D^c|h_1|}{\|D^c|h_1|\|}\right)\right\|.$$

We are then lead to estimate

$$\left\|h_1 - \frac{D^c h_1}{\|D^c h_1\|}\right\| \leq \left\|D(h_1 - \frac{D^c h_1}{\|D^c h_1\|})\right\| + \left\|D^c(h_1 - \frac{D^c h_1}{\|D^c h_1\|})\right\|$$
$$= \|Dh_1\| + \left\|D^c h_1 - \frac{D^c h_1}{\|D^c h_1\|}\right\|.$$

We first get that

$$\|Dh_1\| = \|D|h_1|\| \leq \|D(|h_1| + |h_2|)\| \leq \|h_1 - h_2\| < \varepsilon;$$

and, since $\|h_1\| = \|Dh_1 + D^c h_1\| = 1$ and $\|Dh_1\| < \varepsilon$, an easy computation yields

$$\left\|D^c h_1 - \frac{D^c h_1}{\|D^c h_1\|}\right\| \leq \|Dh_1\| < \varepsilon.$$

So $\left\| h_1 - \frac{D^c h_1}{\|D^c h_1\|} \right\| < 2\varepsilon$ and thus

$$\begin{aligned}\|DF_X(|h_1|)\| &\leq \left\| F_X(|h_1|) - F_X\left(\frac{D^c|h_1|}{\|D^c|h_1|\|}\right) \right\| \\ &\leq g(2\varepsilon).\end{aligned}$$

Similarly $\|DF_X(|h_2|)\| \leq g(2\varepsilon)$. Hence $\|F_X(h_1) - F_X(|h_2|)\| \leq g(\varepsilon) + 2g(2\varepsilon)$.

Therefore F_X extends uniquely to a uniformly continuous map, that we still denote F_X, from $S(L_1(\mu))$ to $S(X)$, and the modulus of continuity of F_X depends only on the modulus of uniform convexity of X. □

PROPOSITION 2.9. *Let X be uniformly convex and uniformly smooth. Then $F_X : S(L_1(\mu)) \longrightarrow S(X)$ is a uniform homeomorphism. Moreover $(F_X)^{-1} : S(X) \longrightarrow S(L_1(\mu))$ has modulus of continuity depending only on the modulus of uniform smoothness of X. Furthermore $(F_X)^{-1}(x) = |x^*| \cdot x$ where $x^* \in S(X^*)$ is the unique supporting functional of x.*

Proof. Our goal now is to show that the map F_X previously defined is invertible and that $(F_X)^{-1}$ has the described form and is uniformly continuous.

CLAIM 1. *Let $h \in S(L_1(\mu)) \bigcap L_\infty(\mu)$. Then $g = F_X(h)^{-1} \cdot h \in S(X^*)$ where $\cdot$ denotes the pointwise product.*

Note that $\operatorname{supp} F_X(h) = \operatorname{supp} h$ and we define $F_X(h)^{-1} \cdot h$ to be 0 off the support of h. Assume Claim 1 for the moment.

For $x \in S(X)$, define $G(x) = |x^*| \cdot x$, where x^* is the unique supporting functional of x. Let $h \in S(L_1(\mu)) \bigcap L_\infty(\mu)$. Since $\operatorname{sign} F_X(h) = \operatorname{sign} h$,

$$\int \frac{h}{F_X(h)} |F_X(h)| d\mu = \int |h| d\mu = 1.$$

Thus from Claim 1 it follows that

$$\frac{h}{F_X(h)} = |F_X(h)|^* = |F_X(h)^*|.$$

Hence

$$G(F_X(h)) = |F_X(h)|^* \cdot F_X(h) = h \text{ for any } h \in S(L_1(\mu)) \bigcap L_\infty(\mu).$$

Furthermore G is uniformly continuous. Indeed, the support functional $x \mapsto x^*$ is uniformly continuous since X is uniformly smooth, and since $G(x_i) = |x_i^*| \cdot x_i \;\; i = 1, 2$ we have

$$\begin{aligned} \|G(x_1) - G(x_2)\| &= \||x_1^*| \cdot x_1 - |x_2^*| \cdot x_2\| \\ &\le \||x_1^*| \cdot (x_1 - x_2)\| + \|(|x_1^*| - |x_2^*|) \cdot x_2\| \\ &\le \|x_1 - x_2\| + \|x_1^* - x_2^*\|. \end{aligned}$$

Thus G is uniformly continuous. Moreover since the modulus of continuity of $x \mapsto x^*$ depends only on the modulus of uniform smoothness of X, the same is valid for G. Thus $G(F_X(h)) = h$ for all $h \in S(L_1(\mu))$.

CLAIM 2. *G is one-to-one.*

It then follows that $G = (F_X)^{-1}$. We now prove Claim 1.

Proof of Claim 1. We will follow the path of [**G**]. Early work of [**L**] had as an objective to factorize elements of $S(l_1)^+$. Let $h \in S(L_1(\mu)) \bigcap L_\infty(\mu)$ and suppose $x = F_X(h)$. We can assume that $h \in S(L_1(\mu))^+ \bigcap L_\infty(\mu)$. Then $\operatorname{supp} x = \operatorname{supp} h \equiv B$ a.e. and $x \in S(X)^+$. Let $k \in X^+$ be arbitrary, then

$$\infty > E(h, x) \ge \int h \log \frac{x + k}{\|x + k\|} d\mu.$$

So writing $x + k = x(1 + \frac{k}{x})$ on B yields

$$E(h, x) \ge E(h, x) + \int_B h \log(1 + kx^{-1}) d\mu - \log \|x + k\|.$$

This gives:

$$\begin{aligned} \int_B h \log(1 + kx^{-1}) d\mu &\le \log \|x + k\| \\ &\le \log(\|x\| + \|k\|) \\ &= \log(1 + \|k\|). \end{aligned}$$

So

$$(\star)\qquad \int_B h\log(1+kx^{-1})d\mu \le \|k\|.$$

We see that on B, kx^{-1} is finite μ-almost everywhere. Let

$$\sigma_n = \{\omega \in B : k(\omega)x^{-1}(\omega) \le n\}$$

and $\chi_n = \chi_{\sigma_n}$ then $\chi_n \nearrow \chi_B$, pointwise μ-a.e; and since $t \le \log(1+t) + \frac{1}{2}t^2$ holds for all $t \ge 0$ we have for $0 < s < \infty$

$$\begin{aligned} s\int_B & hx^{-1}k\chi_n d\mu \\ &\le \int_B h\log(1+skx^{-1}\chi_n)d\mu + \frac{1}{2}s^2\int_B k^2(x^{-1})^2\chi_n h d\mu \\ &\le \int_B h\log(1+skx^{-1})d\mu + \frac{1}{2}s^2n^2 \\ &\le s\|k\| + \frac{1}{2}s^2n^2 \ \text{ by } (\star). \end{aligned}$$

Thus dividing by s and letting s go to 0, we obtain for all $n \in \mathbf{N}$

$$\int hx^{-1}k\chi_n d\mu \le \|k\|;$$

and therefore by the monotone convergence theorem,

$$\int_B hx^{-1}kd\mu \le \|k\|.$$

Now let $g = hx^{-1}$. The previous equality yields $\|g\|_{X^*} \le 1$. On the other hand

$$1 = \left|\int hd\mu\right| = \left|\int g\cdot xd\mu\right| \le \|x\|_X\|g\|_{X^*}.$$

So $\|g\|_{X^*} = 1$ which proves Claim 1. □

Proof of Claim 2. Let $h = |x_1^*|\cdot x_1 = |x_2^*|\cdot x_2$ be a member of $S(L_1(\mu))$ with $x_i^*(x_i) = 1, x_i \in S(X)$ and $x_i^* \in S(X^*)$ for $i = 1,2$. We first note that supp$h$ = suppx_i a.e for $i = 1,2$. Indeed supp $h \subset$ supp x_i a.e is clear, and in case the inclusion is strict let us consider

$B|x_i|$ where $B = \operatorname{supp} h$. We then note that $\|B|x|\| < 1$ by uniform convexity. Also

$$|x^*|(B|x|) = \int |x^*|B|x|d\mu = \int_B |x^*||x|d\mu$$
$$= \int |h|d\mu = 1, \text{ a contradiction.}$$

Also $\operatorname{supp} x_i^* = B$ since X^* is uniformly convex. Now as in [**G**] we observe that there exists a measurable function θ of modulus one so that $x_2^* = \theta x_1^*$. Indeed define $\theta = \frac{x_2^*}{x_1^*}$ on B and $\theta = 1$ on B^c. Then

$$\int |h||\theta|d\mu = \int |x_1||x_2^*|d\mu \leq \||x_2^*|\|_{X^*}\||x_1|\|_X = 1.$$

Similarly, $\int |h||\theta^{-1}|d\mu \leq 1$. So

$$\int |h|\{|\theta| + |\theta^{-1}|\}d\mu \leq 2.$$

And since $t + t^{-1} \geq 2$ for $t > 0$ we get

$$\int |h|\{|\theta| + |\theta^{-1}|\}d\mu \geq 2\int |h|d\mu = 2.$$

Thus $|\theta| + |\theta^{-1}| = 2$, but this cannot happen unless $|\theta| = 1$. Thus $|x_1^*| = |x_2^*|$. Now $\operatorname{supp} x_i = \operatorname{supp} h$ a.e. and $h = |x_1^*| \cdot x_1 = |x_2^*| \cdot x_2$. yields that $x_1 = x_2$ a.e. □

We are now ready to give a proof of the main result of this work.

Proof of Theorem 2.1. Suppose that$X$ contains l_∞^n uniformly in n. Then $S(X)$ is not homeomorphic to $S(L_1((\Omega, \Sigma, \mu)))$ for any measure space (Ω, Σ, μ). Indeed this follows, as in [**O.S**], from Enflo's result [**E**] that the sets $S(l_\infty^n), n \in \mathbb{N}$ cannot be uniformly embedded into $S(L_1)$.

For the converse assume that X does not contain l_∞^n uniformly in n. Then X must be order continuous since X does not contain c_0 [**L.T**]. Then the proof goes as in [**O.S**]. By a theorem of Maurey and Pisier [**MP**] X must have a finite cotype q'. Thus X is q-concave, in fact for all $q > q'$ ([**L.T**, p.88]). Renorm X by an equivalent norm for which $M_q(X) = 1$ and such that X has the same lattice structure (see [**L.T**, p. 54]). Then the 2-convexification $X^{(2)}$ of X in this norm satisfies

$$M_{2q}(X^{(2)}) = 1 = M^2(X^{(2)})$$

([**L.T**, p. 54]). This implies that $X^{(2)}$ is uniformly convex and uniformly smooth ([**L.T**, p. 80]), and so

$$F_{X^{(2)}} : S(L_1(\mu)) \longrightarrow S(X^{(2)})$$

is a uniform homeomorphism by Proposition 2.9. Therefore

$$G_2 \circ F_{X^{(2)}} : S(L_1(\mu)) \longrightarrow S(X)$$

is a uniform homeomorphism by Proposition 2.4. □

REMARK 2.10. [**O.S**]. If $S(X)$ is uniformly homeomorphic to $S(Y)$ then BaX and BaY are uniformly homeomorphic.

COROLLARY 2.11. *If X is a separable infinite dimensional Banach lattice then $S(X)$ and $S(l_1)$ are uniformly homeomorphic if and only if X does not contain l_∞^n uniformly.*

Proof. By Theorem 2.1, $S(X)$ is uniformly homeomorphic to $S(L_1(\mu))$ for some probability space (Ω, Σ, μ) where $L_1(\mu)$ is separable. By standard representation theorems either $L_1(\mu) \cong l_1$ or $L_1(\mu) \cong (L_1[0,1] \oplus l_1(I))_1$ where I is countable. So $S(X)$ is uniformly homeomorphic to $S((L_1[0,1] \oplus l_1(I))_1)$. Then one can define

$$H : S((L_1[0,1] \oplus l_1(I))_1) \longrightarrow S((l_1 \oplus l_1(I))_1)$$

as follows: Let F be a uniform homeomorphism between $S(L_1)$ and $S(l_1)$. (Such homeomorphism exists by [**O.S**].) If $(g, x) \in S(L_1[0,1] \oplus l_1(I))_1$ then define $H(g,x) = \left(\|g\| F\left(\frac{g}{\|g\|}\right), x\right)$ for $g \neq 0$ and $H(0,x) = (0,x)$. It is easily checked that H is a uniform homeomorphism and now, since I is countable, $l_1 \oplus l_1(I) \equiv l_1$ which proves the Corollary. □

REMARK 2.12. In [**R**], Y.Raynaud already obtained that if the unit ball of a Banach space E, embeds uniformly into a stable Banach space F, then E does not contain c_0. He also proved that if

F is supposed superstable then E does not contain l_∞^n uniformly. Since L_1 is superstable, we could get one direction of Theorem 2.1 in the separable case using the result of [**R**].

REMARK 2.13. If X is q-concave with constant 1, then $X^{(2)}$ satisfies

$$M_{2q}(X^{(2)}) = M^2(X^{(2)}) = 1,$$

([**L.T**, p. 54]) and as we noted before, $X^{(2)}$ is uniformly convex and uniformly smooth ([**L.T**, p. 80]). We then proved that

$$F_{X^{(2)}} : S(L_1(\mu)) \longrightarrow S(X^{(2)})$$

is a uniform homeomorphism with modulus of continuity of $F_{X^{(2)}}$ depending only on the modulus of uniform convexity $\delta_{X^{(2)}}(\varepsilon)$ of $X^{(2)}$ (which in turn is of power type 2, i.e for some constant $0 < K < \infty$, $\delta_{X^{(2)}}(\varepsilon) \geq K\varepsilon^2$ ([**L.T**, p. 80])) and the modulus of continuity of $(F_{X^{(2)}})^{-1}$ depending only on the modulus of uniform smoothness $\rho_{X^{(2)}}(\tau)$ of $X^{(2)}$ (which in turn is of power $2q$ i.e. for some constant $0 < K < \infty$, $\rho_{X^{(2)}}(\tau) \leq K\tau^{2q}$ [**L.T**, p. 80]).

We first observe that X and Y must have weak units, since they are separable [**L.T**, p. 9]; and are order continuous since they both don't contain c_0. In fact, since $q < \infty$ and $q' < \infty, X$ and Y don't contain l_∞^n. So, by Corollary 2.11, $S(X)$ and $S(Y)$ are uniformly homeomorphic to $S(L_1)$. Let $\bar{X}$ be X endowed with an equivalent norm and the same order, for which $M_q(\bar{X}) = 1$, and let $\bar{Y}$ be Y with an equivalent norm and the same order, for which $M_{q'}(\bar{Y}) = 1$. With the previous notations used throughout this work, we have the following diagram:

$$S(X) \xrightarrow{u^{-1}} S(\bar{X}) \xrightarrow{(G_{\bar{X},2})^{-1}} S(\bar{X}^{(2)}) \xrightarrow{(F_{\bar{X}^{(2)}})^{-1}}$$
$$S(L_1) \xrightarrow{F_{\bar{Y}^{(2)}}} S(\bar{Y}^{(2)}) \xrightarrow{G_{\bar{Y},2}} S(\bar{Y}) \xrightarrow{v} S(Y)$$

where v is a uniform homeomorphism from $S(\bar{Y})$ to $S(Y)$ with a modulus of continuity a depending solely on $M_{q'}(Y)$, and u^{-1} is a uniform homeomorphism from $S(X)$ to $S(\bar{X})$ with a modulus of continuity f depending only on $M_q(X)$.

Let

$$F = v \circ G_{\bar{Y},2} \circ F_{\bar{Y}^{(2)}} \circ (F_{\bar{X}^{(2)}})^{-1} \circ (G_{\bar{X},2})^{-1} \circ u^{-1},$$

then F is clearly a homeomorphism and

$$F^{-1} = u \circ G_{\bar{X},2} \circ F_{\bar{X}^{(2)}} \circ (F_{\bar{Y}^{(2)}})^{-1} \circ (G_{\bar{Y},2})^{-1} \circ v^{-1}.$$

Let b, c, d and e be respectively the modulus of continuity of respectively $G_{\bar{Y},2}, F_{\bar{Y}^{(2)}}, (F_{\bar{X}^{(2)}})^{-1}, (G_{\bar{X},2})^{-1}$. b and e are functions solely of 2 by Proposition 2.4 while c and d are functions of q' and q by Proposition 2.9, Proposition 2.8, and Remark 2.13 above. Then the modulus of uniform continuity α of F is of the form $\alpha = a \circ b \circ c \circ d \circ e \circ f$ and is a function solely of $q, q', M_q(X), M_{q'}(Y)$. Note that the modulus of continuity of F^{-1} is also given by $a \circ b \circ c \circ d \circ e \circ f$. □

Proof of Theorem 2.3. The proof is exactly the same as in Theorem 2.2 with the only difference that $F = F_Y \circ (F_X)^{-1}$. Indeed we have now the diagram:

$$S(X) \xrightarrow{(F_X)^{-1}} S(L_1) \xrightarrow{F_Y} S(Y).$$

We then let $F = F_Y \circ (F_X)^{-1}$ and use Proposition 2.9 to get that the modulus of continuity of F depends solely on the modulus of uniform convexity of Y and the modulus of uniform smoothness of X. □

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