JEAN BOURGAIN'S ANALYTIC PARTITION OF UNITY VIA HOLOMORPHIC MARTINGALES

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Using stopping time arguments on holomorphic martingales we present a soft way of constructing J. Bourgain's analytic partition of unity. Further applications to Marcinkiewicz interpolation in weighted Hardy spaces are discussed.

1. Introduction. In his 1984 Acta Mathematica paper Jean Bourgain derives new Banach space properties of H^∞ and the disc algebra from the existence of the following analytic partition of unity:

THEOREM 1 [J. Bourgain]. *Given f, a strictly positive integrable function on $\mathbf{T}$ with $\int f(t)\, dt = 1$ and $0 < \delta < 1$ then, there exist functions $\tau_j, \gamma_j \in H^\infty(T)$ and positive numbers c_j such that:*

1. $\|\gamma_j\|_\infty < C$
2. $\sum |\tau_j| < C$
3. $|\tau_j| f < c_j$
4. $\sum c_j \|\tau_j\|_1 < \delta^{-C}$
5. $\int \left|1 - \sum \gamma_j \tau_j^2\right| f\, dt < \delta.$

Here I wish to present a soft way to this construction which results from using *probabilistic tools* such as holomorphic martingales.

I should like to point out here that a proof for the existence of analytic partitions of unity - much simpler than J. Bourgain's - has been given recently by Serguei Kislyakov. See [K1] and [K2]. In [K3] S. Kislyakov derived J. Bourgain's result on p-summing operators from the following weighted Marcinkiewicz decomposition.

THEOREM 2 [S. Kislyakov]. *For any positive weight b on $\mathbf{T}$ there exists a weight $B \geq b$ and $\int B\, dt < C \int b\, dt$ so that for any $\lambda >$*

0 *and* $f \in H^1(T, B)$ *there exists* $g \in H^\infty(T)$ *and* $h \in H^1(T, B)$ *satisfying:*

1. $f = g + h$
2. $\|g\|_\infty \leq \lambda$
3. $\displaystyle\int hB \, dt \leq C \int_{\{|f|>\lambda\}} |f|B \, dt.$

Up to small perturbations we shall obtain a *stochastic version* of Kislyakov's decomposition which allows us to prove the following:

THEOREM 3 [J. Bourgain]. *For any 2-summing operator* S *on the disc algebra and any* $2 < q < \infty$ *the* q-*summing norm satisfies the interpolation inequality*

$$\pi_q(S) \leq C_q \pi_2(S)^{\frac{2}{q}} \|S\|^{1-\frac{2}{q}}.$$

A very elegant proof of this interpolation inequality has been given by Gilles Pisier who used vectorvalued H^1 spaces. See [P].

2. The main result. Holomorphic martingales were introduced by H. Föllmer in [F] and N. Varopoulos in [V]. They are stable under stopping times, and generalize analytic functions on the unit circle. This connection has led to probabilistic proofs of several results in Analysis, including Carleson's corona theorem [V], the existence of a logmodolar Banach algebra having no analytic structure [C] and P.W. Jones's interpolation theorems between H^1 and H^∞ [M1], [M2].

This paper is not selfcontained! We freely use definitions from [V] without further explanation. (Ω, P) denotes Wiener's measure space governing complex Brownian motion. $H^p(\Omega, P)$ is the subspace of $L^p(\Omega, P)$ consisting of holomorphic random variables. (See [V].)

THEOREM 4.   *Given* Δ, *a strictly positive integrable function on* (Ω, P) *with* $\int \Delta \, dP = 1$ *and* $0 < \delta < 1$, *there exist functions* w_j, $\theta_j \in H^\infty(\Omega)$ *and positive numbers* c_j *such that:*

1. $\|\theta_j\|_\infty < C$
2. $\sum |w_j| < C$
3. $|w_j^2|\Delta < c_j$
4. $\sum c_j \left\|w_j^2\right\|_1 < \delta^{-C}$

5. $\int \left| 1 - \sum \theta_j w_j^4 \right| \Delta \, dP < \delta.$

Probability offers a *soft way of constructing* the functions θ_j so that the verification of (5) becomes much easier than in J. Bourgain's proof. See [**B1**, pp. 11, 12]. The probabilistic concept will be merged with analytic tools, such as Havin's lemma, which we use in the following form, due to J. Bourgain. (See [**B1**].)

THEOREM 5. *For every measurable subset E of Ω and $0 < \epsilon < 1$ there exist functions $\alpha, \beta \in H^\infty(\Omega)$ such that:*
1. $|\alpha| + |\beta| \leq 1$
2. $\left| \alpha - \dfrac{1}{5} \right| < \epsilon$ *on* E.
3. $|\beta| < \epsilon$ *on* E.
4. $\|\alpha\|_1 < C |\log \epsilon|^2 P(E)$
5. $\|1 - \beta\|_2 < |\log \epsilon| P(E)^{\frac{1}{2}}.$

Proof of Theorem 4. We shall first determine a new weight: Let $d = \Delta^{\frac{1}{2}}$ and put

$$A(d) := \sup_t |E(d|\mathcal{F}_t)|$$

then we let

$$\Delta_1 = \sum_{n=0}^{\infty} A^n(d)(C2)^{-n}$$

where C is determined by Doob's inequality: For $d \in L^2(\Omega)$

$$\int \sup_t |E(d|\mathcal{F}_t)|^2 < C\|d\|_2^2.$$

Clearly this construction gives,
1. $A(\Delta_1) < \Delta_1 3C$
2. $\Delta^{\frac{1}{2}} < C\Delta_1$
3. $\int \Delta_1^2 \, dP < C \int \Delta \, dP.$

We next define holomorphic partitions of unity: Let Ψ be the outer function so that $|\Psi| = \Delta_1$. Consider now the stopping times $\tau_0 = 0$ and

$$\tau_i := \inf\{t > \tau_{i-1} : |E(\Psi|\mathcal{F}_t)| > M^i\}$$

to define $\Psi_i := E(\Psi|\mathcal{F}_{\tau_i})$ and $d_i := \Psi_{i+1} - \Psi_i$, elements of $H^\infty(\Omega)$ for which obviously the identity

$$1 = \frac{E(\Psi)}{\Psi} + \sum_{i=0}^\infty \frac{d_i}{\Psi}$$

holds. The summands of the above expression will be our choice of θ_i: Indeed we define $\theta_{-1} := \frac{E(\Psi)}{\Psi}$ and $\theta_i := \frac{d_i}{\Psi}$ for $i = 0, 1, 2, \ldots$. We obtain from

$$|d_i| \le 2A(\Psi) \le 2A(|\Psi|) = A(\Delta_1) \le 3C\Delta_1 = 3C|\Psi|$$

that

$$\|\theta_i\|_\infty \le 3C.$$

Havin's lemma allows us to truncate the above partition of unity: We apply it to sets $E_i := \{\Psi^* > M^i\}$ and denote the resulting functions by α_i, β_i. Then, following Bourgain, we define for $i = -1, 0, 1, \ldots$

$$w_i := 5\alpha_i \prod_{s=8}^\infty \beta_{i+s}.$$

Verification of property (5). We first eliminate the weight Δ:

$$\int \left| 1 - \sum_{i=-1}^\infty \theta_i w_i^4 \right| \Delta \, dP = \int \left| \sum_{i=-1}^\infty d_i(1 - w_i^4) \right| \Delta^{\frac{1}{2}} \, dP$$

$$\le \left(\int \left| \sum_{i=-1}^\infty d_i(1 - w_i^4) \right|^2 dP \right)^{\frac{1}{2}} \left(\int \Delta \right)^{\frac{1}{2}}.$$

The martingale differences d_i are supported on E_i and bounded by M^{i+1}. Moreover, $|w_i| < 5\epsilon$ on E_{i+1} and $E_{i+s} \subset E_i$ for any s. Therefore, we obtain a domination by:

$$C \left(\sum_{i=-1}^\infty M^{(i+1)2} \int_{E_i} |(1 - w_i^4)|^2 \, dP \right)^{\frac{1}{2}}.$$

Using the inequality

$$\left| 1 - \prod z_i \right| \le \sum |1 - z_i|$$

which holds for complex numbers in the closed unit disc, we get the following upper bound for the square of the above sum of integrals:

$$\sum_i \int_{E_i} |(1 - 5\alpha_i)| M^{2(i+1)} + \sum_{s>8} M^{(i+1)2} \int_{E_i} |(1 - \beta_{i+s})| \, dP.$$

Invoking the estimates from Havin's Lemma and applying Cauchy-Schwarz' inequality give the following estimates:

$$\epsilon M C + \log(\epsilon^{-1}) \sum_{s>8} \sum_{i=-1}^{\infty} M^{(i+1)2} P(E_i)^{\frac{1}{2}} P(E_{i+s})^{\frac{1}{2}}.$$

Again by Cauchy-Schwarz we dominate the above sum by:

$$\epsilon M C + \log(\epsilon^{-1}) \sum_{s>8} M^{2-s} \left(\sum_{i=-1}^{\infty} M^{2i} P(E_i) \right)^{\frac{1}{2}} \left(\sum_{i=-1}^{\infty} M^{2(i+s)} P(E_{i+s}) \right)^{\frac{1}{2}}$$
$$\leq \epsilon M C + \log(\epsilon^{-1}) \sum_{s>8} M^{2-s} C.$$

This is what we want if ϵ is chosen of order M^{-2} and $M := \delta^{-1}$. Havin's lemma, repeatedly applied, gives properties (4) and (3), i.e.,

1. $\sum M^{2i} \left\| w_i^2 \right\|_1 \leq \sum M^{2i} \|\alpha_i\|_1 \leq \sum M^{2i} P(E_i) |\log \epsilon|$

2. $|w_i^2| \Delta < M^{2(i+8)}.$

Verification of property (2). As α_i, β_i satisfy $|\alpha_i| + |\beta_i| \leq 1$ we obtain $\sum |w_i| \leq 40$ from the following elementary considerations concerning reals.

Let s_i, t_i be non negative real numbers so that

$$s_i + t_i \leq 1 \quad i \in \mathbb{N}.$$

Then for any $n \in \mathbb{N}$

$$\sum_{i=1}^{n} s_i \prod_{i<j\leq n} t_j \leq 1.$$

Indeed, for $n = 1$ this claim is true. (Assuming that the product over the empty index set equals **1**.)

Assume the claim holds for "n", then consider

$$s_{n+1} + t_{n+1} \leq 1.$$

Multiplying t_{n+1} with the LHS of the inequality gives:

$$s_{n+1} + \left(\sum_{i=1}^{n} s_i \prod_{i<j\leq n} t_j \right) t_{n+1} \leq 1$$

or:

$$\sum_{i=1}^{n+1} s_i \prod_{i<j\leq n+1} t_j \leq 1$$

(which proves the claim).

Passing to the limit we see that

$$\sum_{i=1}^{\infty} s_i \prod_{i<j} t_j \leq 1.$$

Hence for any $K \subseteq \mathbf{N}$

$$\sum_{i\in K} s_i \prod_{\substack{j>i \\ j\in K}} t_j \leq 1.$$

Taking arithmetic progressions we divide $\mathbf{N}$ disjointly into $K_0, \ldots, K_7$ such that $i, j \in K_m$ implies $|i - j| \geq 8$. For such K we find $(0 \leq t_i \leq 1)$

$$\sum_{i\in K} s_i \prod_{j\geq i+8} t_j \leq 1.$$

Therefore

$$\sum_{i\in \mathbf{N}} s_i \prod_{j\geq i+8} t_j \leq 8.$$

As $|w_i| \leq 5 \cdot |\alpha_i| \prod_{j\geq i+8} |\beta_j|$ we obtain

$$\sum |w_j| \leq 40.$$

To finish the proof of Theorem 4 it is now enough to take $c_i = M^{2i}$. □

3. Reduction of J. Bourgain's partition of unity. To obtain J. Bourgain's original result, we lift the density f from T to Ω construct a new weight together with holomorphic partitions of unity there and project the solutions back to T. This is done by using norm-one operators

$$M : H^p(T) \to H^p(\Omega)$$

and

$$N : H^p(\Omega) \to H^p(T)$$

so that $Id = NM$, and $N(M(f)F) = fN(F)$. (For the construction of M and N see [**V**].)

Proof of Theorem 1. Apply Theorem 4 to the density $\Delta := Mf$. Let

$$g_i := N(\theta_i),$$

$$\tau_i := N(w_i^2)$$

Using §2 it is easy to verify conditions (1)...(5) of Bourgain's theorem.

4. Truncating functions in weighted H^p. Here we combine stopping times and holomorphic partitions of unity to obtain a Marcinkiewicz decomposition in weighted Hardy spaces.

Although the next theorem looks terribly complicated, it simply states that up to a (reasonable) change of density and up to a small error, interpolation is possible in weighted Hardy spaces.

THEOREM 6.   *For any density Δ on Ω and $\delta > 0$ there exists $\phi \in H^\infty(\Omega)$ and a density Δ_1 so that the following conditions hold:*

1. $\|\phi\|_\infty < C$

2. $\Delta_1 > \Delta$ *and* $\int \Delta_1 \, dP < \delta^{-C} \int \Delta \, dP$

3. $\int |1 - \phi| \Delta \, dP < \delta \|\Delta\|_1.$

4. *For $q > 2$, $f \in H^q(\Omega, \Delta_1)$ and $\lambda > 0$ there exists $g \in H^\infty(\Omega)$ and $h \in H^2(\Omega, \Delta)$ satisfying*

    (a) $f\phi = g + h$

    (b) $\|g\|_\infty \leq \lambda$

    (c) $\int |h|^2 \Delta \, dP \leq C_q \lambda^{2-q} \int |f^q| \Delta_1 \, dP.$

REMARK.   Condition 4(c) involves the original density Δ in the LHS and the new weight Δ_1 in the RHS. Although weaker than Kisliakov's result, Theorem 6 suffices to deduce Bourgain's interpolation inequality for q-summing operators.

Proof. Let $w_i \in H^\infty(\Omega)$ and $\theta_i \in H^\infty(\Omega)$ be given by Theorem 4. Then we define:

$$\phi := \sum \theta_i w_i^4$$

$$\Delta_1 := \Delta + \sum c_i |w_i^2|$$

$$f_i := w_i^2 f.$$

Now we use the stopping time

$$\tau_j := \inf\{t : |E(f_j|\mathcal{F}_t)| > \lambda\}$$

to define $g_j := E(f_j|\mathcal{F}_{\tau_j})$ and $h_j := f_j - g_j$. By the stability property of holomorphic martingales these functions are certainly holomorphic and satisfy

1. $\|g_j\|_\infty \leq \lambda$

2. $\int |h_j|^2 \, dP \leq 4 \int_{\{|f_j^*|>\lambda\}} |f_j|^2 \, dP.$

Now using partitions of unity we glue these partial solutions together

$$g := \sum g_j w_j^2 \theta_j$$

and

$$h := \sum h_j w_j^2 \theta_j.$$

Then clearly

$$g + h = \sum (g_j + h_j) w_j^2 \theta_j = f\phi$$

and

$$\|g\|_\infty < C \sup_j \|g_j\|_\infty \left\|\sum |w_j|\right\|_\infty < \lambda C'.$$

The estimate for $\int |h|^2 \Delta \, dP$ follows a well established pattern, which has been carefully presented in the central chapter of Wojtaszczyk's book. See [**W**, Ch III.I].

Cauchy-Schwartz inequality (for the sequence space l^2) and property (2) of Theorem 4 imply that:

$$\int |h|^2 \Delta \, dP \leq C \sum \int |h_j|^2 |w_j|^2 \Delta \, dP.$$

The last sum can be estimated, using the interplay between the partitions and the density, by

$$\sum c_j \int |h_j|^2 \, dP \leq \sum c_j \int_{\{|f_j^*|>\lambda\}} |f_j|^2 \, dP.$$

Then we finish the proof as follows:

$$\sum c_j \int_{\{f_j^* > \lambda\}} |f_j|^2 \, dP \leq \sum c_j \lambda^{2-q} \int |f_j^*|^q \, dP$$

$$\leq C\lambda^{2-q} \sum c_j \int |f_j|^q \, dP$$

$$\leq C\lambda^{2-q} \int \left(|f^q| \cdot \sum c_j |w_j|^q \right) \, dP$$

$$\leq C\lambda^{2-q} \int |f|^q \Delta_1 \, dP.$$

$\square$

5. Reduction of J. Bourgain's interpolation inequality.

For a 2-summing operator S there exists a positive probability measure on $\mathbf{T}$ so that

$$\|Sx\| \leq \pi_2(S) \left(\int |x| \, d\mu \right)^{\frac{1}{2}} \quad \text{for } x \in A.$$

Without loss of generality we may assume that μ is absolutely continuous with respect to Lebesgue measure, i.e.,

$$d\mu = f dt.$$

Consequently for $b \in H^\infty(\Omega)$ the operator $U = SN$ satisfies

$$\|Ub\| \leq \pi_2(S) \left(\int |b| \Delta \, dP \right)^{\frac{1}{2}}$$

where $\Delta = Mf$.

Proof of Theorem 3. Let $0 < \delta < 1$ be given. Theorem 6 applied to the density Δ shows that U can be split into $U = U_1 + R_1$ so that

$$\pi_q(U_1) \leq \delta^{-C} \pi_2(S)^{\frac{2}{q}} \|S\|^{1-\frac{2}{q}}$$

and

$$\pi_2(R_1) \leq \delta \pi_2(S),$$

where $U_1 b = U(b\phi)$ and $R_1 b = U(b(1-\phi))$. Indeed fix $b \in H^q(\Omega, \Delta_1)$ of norm one in that space. Then according to Theorem 6 for $\lambda = \|S\|^{\frac{q}{2}} \pi_2(S)^{-\frac{2}{q}}$ we find a Marcinkiewicz decomposition of ϕb into

$$\phi b = g + h.$$

Therefore

$$\|U_1 b\| = \|U b \phi\| \le \|U g\| + \|U h\|$$

$$\le \|S\| \, \|g\|_\infty + \pi_2(S) \left(\int |h|^2 \Delta \, dP \right)^{\frac{1}{2}}$$

$$\le \|S\| \lambda + \pi_2(S) \lambda^{1-\frac{q}{2}} \le \|S\|^{1-\frac{2}{q}} \pi_2(S)^{\frac{2}{q}}.$$

As for the error term we have

$$\|U(b(1-\phi))\| \le \pi_2(S) \left(\int |b|^2 |1-\phi|^2 \Delta \, dP \right)^{\frac{1}{2}}.$$

Using properties (3), (4) and (5) of Theorem 4 give

$$\int |1-\phi|^2 \Delta \, dP \le C \int |1-\phi| \Delta \, dP \le C\delta \int \Delta \, dP$$

and

$$\int \Delta_1 \, dP \le \delta^{-C} \int \Delta \, dP.$$

We therefore obtained the correct estimates for U_1 and R_1. To finish the proof of Theorem 3, we now iterate the above decomposition and observe that $S = UM$.   □

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