THE ASYMPTOTIC EXPANSION OF A RATIO OF GAMMA FUNCTIONS

F. G. TRICOMI AND A. ERDÉLYI

1. Introduction. Many problems in mathematical analysis require a knowledge of the asymptotic behavior of the quotient $\Gamma(z+\alpha)/\Gamma(z+\beta)$ for large values of $|z|$. Examples of such problems are the study of integrals of the Mellin-Barnes type, and the investigation of the asymptotic behavior of confluent hypergeometric functions when the variable and one of the parameters become very large simultaneously.

Stirling's series can be used to find a first approximation for our quotient for very large $|z|$, it being understood that α and β are bounded. Without too much algebra one finds

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = z^{\alpha-\beta}\left[1+\frac{(\alpha-\beta)(\alpha+\beta-1)}{2z}+O(|z|^{-2})\right] \tag{1}$$

as $z \to \infty$, under conditions which will be stated later; but the determination of the coefficients of $z^{-2}, z^{-3}, \cdots$, in the asymptotic expansion of which (1) gives the first two terms, is a very laborious process, and the determination of the general term from Stirling's series is a well-nigh hopeless task.

The present paper originated when the first-named author (F.G. Tricomi) noticed that the asymptotic expansion of $\Gamma(z+\alpha)/\Gamma(z+\beta)$ can be obtained by methods similar to those which he used in a recent investigation of the asymptotic behavior of Laguerre polynomials [3]. The first proof given in this paper, and the detailed investigation of the coefficients A_n and C_n, are entirely due to him. Afterwards, the second named author (A. Erdélyi) pointed out that a shorter proof can be given by using Watson's lemma. His contributions to the present paper are the second proof, the generating function (18) of the coefficients, and their expression in terms of generalized Bernoulli polynomials.

We may mention that the same quotient was recently investigated by J.S. Frame [1]; but there is no overlapping with the results presented here.

Received April 17, 1950 and, with the contribution of A. Erdélyi, July 24, 1950. This research was sponsored by the Office of Naval Research.

Pacific J. Math. 1 (1951), 133-142.

2. The case $\beta = 0$. Let us begin with the particular case $\beta = 0$ (after which the general case will easily be treated), starting from the well-known formula

$$\int_0^\infty \frac{x^{u-1}}{(1+x)^v}\,dx = \frac{\Gamma(u)\,\Gamma(v-u)}{\Gamma(v)}, \qquad (0 < \Re u < \Re v), \tag{2}$$

where each power has its *principal value.*

Putting

$$u = z + \alpha, \quad v = z; \quad \frac{\Gamma(z+\alpha)}{\Gamma(z)} = F(\alpha, z); \quad x = \frac{z}{t}, \quad \zeta = e^{i \arg z},$$

from the previous equality, under the hypothesis

$$0 < \Re(\alpha + z) < \Re z$$

we obtain

$$\Gamma(-\alpha)\, F(\alpha, z) = z^\alpha \int_0^{\zeta\infty} e^{-z\log(1+t/z)}\, t^{-\alpha-1}\, dt .$$

But as long as $|t| < |z|$ we have

$$e^{-z\log(1+t/z)} = e^{-t} \exp\left[\frac{t^2}{z}\left(\frac{1}{2} - \frac{1}{3}\frac{t}{z} + \frac{1}{4}\frac{t^2}{z^2} - \cdots\right)\right]$$

$$= e^{-t} \sum_{m=0}^\infty \frac{t^{2m} z^{-m}}{m!}\left(\frac{1}{2} - \frac{1}{3}\frac{t}{z} + \frac{1}{4}\frac{t^2}{z^2} - \cdots\right)^m .$$

Hence, if we put generally*

$$\left(\frac{1}{2} + \frac{1}{3}w + \frac{1}{4}w^2 + \cdots\right)^m = \sum_{k=0}^\infty c_k^{(m)} z^k, \qquad (m = 0, 1, 2, \cdots), \tag{3}$$

and in particular

$$c_0^{(m)} = \frac{1}{2^m}, \qquad c_1^{(m)} = \frac{m}{2^{m-1}\cdot 3}, \qquad c_2^{(m)} = \frac{m(4m+5)}{2^{m+1}\cdot 3^2}, \cdots; \tag{3'}$$

*The repeated use of the coefficients of the (formal) mth power of a power series is one of the features of the methods of the paper quoted [3].

with the help of the substitution $k + m = n$, we obtain

$$e^{-z \log(1+t/z)} = \sum_{k,m=0}^{\infty} \frac{(-1)^k}{m!} c_k^{(m)} t^{2m+k} z^{-m-k}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{z^n} \sum_{m=0}^{n} \frac{(-1)^m}{m!} c_{n-m}^{(m)} t^{n+m} .$$

This shows that our quotient $F(\alpha, z) = \Gamma(z+\alpha)/\Gamma(z)$ admits *at least formally* the negative-powers expansion

$$\sum_{n=0}^{\infty} A_n(\alpha) z^{\alpha-n} ; \tag{4}$$

where, for the sake of brevity, we put

$$A_n(\alpha) = \frac{(-1)^n}{\Gamma(-\alpha)} \sum_{m=0}^{n} \frac{(-1)^m}{m!} c_{n-m}^{(m)} \int_0^{\zeta\infty} e^{-t} t^{n+m-\alpha-1} dt .$$

Better still, because

$$\frac{1}{\Gamma(-\alpha)} \int_0^{\zeta\infty} e^{-t} t^{n+m-\alpha-1} dt = \frac{\Gamma(n+m-\alpha)}{\Gamma(-\alpha)} = (-1)^{n+m} \binom{\alpha}{n+m} (n+m)!$$

since $\Re\zeta > 0$, we can also write

$$A_n(\alpha) = \sum_{m=0}^{n} \binom{\alpha}{n+m} \frac{(n+m)!}{m!} c_{n-m}^{(m)} . \tag{5}$$

In particular, we have

$$A_0(\alpha) = 1, \quad A_1(\alpha) = \binom{\alpha}{2}, \quad A_2(\alpha) = \frac{3\alpha-1}{4}\binom{\alpha}{3} , \tag{5'}$$

$$A_3(\alpha) = \binom{\alpha}{2}\cdot\binom{\alpha}{4} , \cdots .$$

3. **Relations connecting the coefficients $A_n(\alpha)$.** The infinite series (4) is generally *divergent* because otherwise the function F would be the product of z^{α} by a function regular at infinity, in contradiction with the fact that, as long as α is not

an integer, the function F has an infinite number of poles at $z = 0, -1, -2, \cdots$, with the condensation point $z = \infty$. In spite of its divergence, the series (4) represents the function F asymptotically (in the sense of Poincaré); that is, we have

$$F(\alpha, z) = \frac{\Gamma(z+\alpha)}{\Gamma(z)} \sim \sum_{n=0}^{\infty} A_n(\alpha)\, z^{\alpha-n}, \tag{6}$$

at least as long as

$$0 < -\Re\alpha < \Re z, \tag{7}$$

because for any positive integer N we obviously have

$$e^{-z \log(1+t/z)} = \sum_{n=0}^{N} \frac{(-1)^n}{z^n} \sum_{m=0}^{n} \frac{(-1)^n}{m!}\, c_{n-m}^{(m)}\, t^{n+m} + O(|z|^{-N-1}).$$

Let us now establish some relations connecting the coefficients $A_n(\alpha)$ together; these arise from the unicity theorem for the asymptotic expansions, and from the functional equations

$$F(\alpha+1, z) = (\alpha+z)\, F(\alpha, z), \quad F(\alpha, z+1) = \left(1+\frac{\alpha}{z}\right) F(\alpha, z), \tag{8}$$

which are obviously satisfied by the function F.

Precisely from the first equation (8) it follows immediately that

$$A_n(\alpha+1) = A_n(\alpha) + \alpha A_{n-1}(\alpha), \qquad (n = 1, 2, 3, \cdots), \tag{9}$$

while from the second one it follows that

$$\left(1+\frac{\alpha}{z}\right) \sum_{m=0}^{\infty} A_m(\alpha)\, z^{\alpha-m} \sim \sum_{m=0}^{\infty} A_m(\alpha)\, z^{\alpha-m} \left(1+\frac{1}{z}\right)^{\alpha-m}$$

$$\sim \sum_{m=0}^{\infty} A_m(\alpha)\, z^{\alpha-m} \sum_{k=0}^{\infty} \binom{\alpha-m}{k} z^{-k} \sim \sum_{n=0}^{\infty} z^{\alpha-n} \sum_{m=0}^{n-1} \binom{\alpha-m}{n-m} A_m(\alpha).$$

This shows that

$$A_n(\alpha) + \alpha A_{n-1}(\alpha) = \sum_{m=0}^{n} \binom{\alpha-m}{n-m} A_m(\alpha) = \sum_{m=0}^{n-2} \binom{\alpha-m}{n-m} A_m(\alpha) + (\alpha-n+1);$$

simplifying and changing n into $n+1$, we thus obtain the important recurrence relation

$$(10) \qquad A_n(\alpha) = \frac{1}{n} \sum_{m=0}^{n-1} \binom{\alpha - m}{n - m + 1} A_m(\alpha), \qquad (n = 1, 2, \cdots).$$

From the manner of deduction, it may seem that the validity of (9) and (10) is conditioned by $\Re\alpha < 0$; but since these equalities are equalities between certain analytic functions of α (even polynomials!), there is no doubt that, as a matter of fact, both equations are true for *any* value of α.

4. On the condition (7). By use of the functional equations (8) and the relations (9) and (10) between the coefficients, it would be possible to weaken progressively the conditions (7) by passing successively from α to $\alpha - 1$, $\alpha - 2$, $\cdots$, and from z to $z+1$, $z+2$, $\cdots$. But we do not need to enter into the details of this reasoning because the method of Section 7 will give us directly the end results free of unnecessary restrictions. Nevertheless, we state explicitly that *the asymptotic expansion* (6) *is valid for any* α *(real or complex) on the whole complex z-plane cut along any curve connecting* $z = 0$ *with* $z = \infty$*, *provided that, in going to* ∞, z *avoids the points* $z = 0, -1, -2, \cdots$ *and* $z = -\alpha, -\alpha - 1, -\alpha - 2, \cdots$.

For example, when α is real and positive the expansion (6) is surely valid if

$$-\pi + \epsilon < \arg z < \pi - \epsilon \;,$$

where ϵ is an arbitrarily small positive number.

5. The asymptotic expansion. Now in order to obtain the asymptotic expansion of the quotient indicated at the beginning, it is sufficient to observe that

$$\Phi(z) \equiv \frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = F(\alpha - \beta, \; z + \beta)$$

Precisely, putting

$$\alpha - \beta = \alpha',$$

*This with regard to the many-valuedness of the power z^{α}.

we find thus

$$\Phi(z) \sim \sum_{m=0}^{\infty} A_m(\alpha')(z+\beta)^{\alpha'-m} \sim \sum_{m=0}^{\infty} A_m(\alpha')\, z^{\alpha'-m} \sum_{k=0}^{\infty} \binom{\alpha'-m}{k}\left(\frac{\beta}{z}\right)^k$$

$$\sim \sum_{n=0}^{\infty} z^{\alpha'-n} \sum_{m=0}^{n} \binom{\alpha'-m}{n-m} A_m(\alpha')\, \beta^{n-m} .$$

In other words, if we put

$$C_n(\alpha',\beta) = \sum_{m=0}^{n} \binom{\alpha'-m}{n-m} A_m(\alpha')\, \beta^{n-m} , \qquad (n=0,1,2,\cdots) , \tag{11}$$

on the whole z-plane cut along any curve connecting $z=0$ *with* $z=\infty$, we have

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} \sim \sum_{n=0}^{\infty} C_n(\alpha-\beta,\beta)\, z^{\alpha-\beta-n} , \tag{12}$$

provided that z *avoids the points* $z=-\alpha, -\alpha-1, -\alpha-2, \cdots$ *and* $z=-\beta, -\beta-1, -\beta-2, \cdots$.

The coefficients C_n are given by (11), which shows in particular that

$$C_0 = 1, \quad C_1 = \frac{1}{2}\alpha'(\alpha'+2\beta-1) = \frac{1}{2}(\alpha-\beta)(\alpha+\beta-1) ,$$

$$C_2 = \frac{1}{12}\binom{\alpha'}{2}\left[(\alpha'-2)(3\alpha'-1)+12\beta(\alpha'+\beta-1)\right]$$

$$= \frac{1}{12}\binom{\alpha-\beta}{2}\left[3(\alpha+\beta-1)^2-\alpha+\beta-1\right] , \cdots .$$

6. **The coefficients C_n.** The calculation of the coefficients C_n by means of (11) is quite easy, but in spite of this it may be useful to know that for such coefficients there is also a recursion formula of the kind (10). Precisely, in a similar manner as in Section 3, we notice first that the function $\Phi(z)$ satisfies the functional equation

$$\Phi(z+1) = \frac{z+\alpha}{z+\beta}\Phi(z) = \left(1+\frac{\alpha}{z}\right)\left(1+\frac{\beta}{z}\right)^{-1} \Phi(z) .$$

Consequently, since

$$\left(1+\frac{\beta}{z}\right)^{-1} \sim 1 - \frac{\beta}{z} + \frac{\beta^2}{z^2} - \cdots ,$$

we obtain

$$\left(1+\frac{\beta}{z}\right)^{-1}\Phi(z)\sim\sum_{m=0}^{\infty}C_m\,z^{\alpha'-m}\sum_{k=0}^{\infty}(-1)^k\beta^k z^{-k}$$

$$\sim\sum_{n=0}^{\infty}(-1)^n\,z^{\alpha'-n}\sum_{m=0}^{n}(-1)^m\,\beta^{n-m}C_m\ ,$$

and further

$$\left(1+\frac{\alpha}{z}\right)\left(1+\frac{\beta}{z}\right)^{-1}\Phi(z)$$

$$\sim\sum_{n=0}^{\infty}\left[C_n+(-1)^n(\beta-\alpha)\sum_{m=0}^{n}(-1)^m\,\beta^{n-m-1}\,C_m\right]z^{\alpha'-n}$$

on the other hand,

$$\Phi(z+1)\sim\sum_{m=0}^{\infty}C_m\,z^{\alpha'-m}\sum_{k=0}^{\infty}\binom{\alpha'-m}{k}z^{-k}\sim\sum_{n=0}^{\infty}z^{\alpha'-n}\sum_{m=0}^{n}\binom{\alpha'-m}{n-m}C_m\ .$$

By comparing the two results we thus obtain

$$\sum_{m=0}^{n}\binom{\alpha'-m}{n-m}C_m=C_n-(-1)^n\,\alpha'\sum_{m=0}^{n}(-1)^m\,\beta^{n-m-1}C_m\ ;$$

that is,

$$\text{(13)}\qquad \sum_{m=0}^{n-1}\left[\binom{\alpha'-m}{n-m}+(-1)^{n+m}\,\alpha'\,\beta^{n-m-1}\right]C_m=0\ .$$

In other words, detaching the last term of the sum and changing n into $n+1$, we have the recurrence relation

$$\text{(14)}\quad C_n(\alpha',\beta)=\frac{1}{n}\sum_{m=0}^{n-1}\left[\binom{\alpha'-m}{n-m+1}-(-1)^{n+m}\alpha'\beta^{n-m}\right]C_m(\alpha',\beta),$$

$$(n=1,2,3,\cdots).$$

7. An alternate proof. If we put $u = \exp(-v)$ in Euler's integral of the first kind,

$$\int_0^1 u^{z+\alpha-1}(1-u)^{\beta-\alpha-1}\,du = \frac{\Gamma(z+\alpha)\,\Gamma(\beta-\alpha)}{\Gamma(z+\beta)},$$

we have the integral representation

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = \frac{1}{\Gamma(\beta-\alpha)}\int_0^\infty e^{-(z+\alpha)v}\left(1-e^{-v}\right)^{\beta-\alpha-1}dv\,. \tag{15}$$

We shall now show that an alternative proof of the asymptotic expansion (12) can be obtained by applying the standard technique (Watson's lemma) to this integral representation.

To begin with, (15) holds only if $\Re(\beta-\alpha) > 0$ and $\Re(z+\alpha) > 0$; but its validity can be extended by the introduction of a loop integral. We assume that $z+\alpha$ is not negative real. Then there is a δ such that

$$-\frac{1}{2}\pi < \delta < \frac{1}{2}\pi\,, \qquad \Re\{(z+\alpha)\,e^{i\delta}\} > 0\,.$$

With such a δ, we have

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = \frac{1}{2\pi i}\int_{-\infty\cdot e^{i\delta}}^{(0+)} e^{zt}\,f(t)\,dt\,, \tag{16}$$

where

$$f(t) = \Gamma(1+\alpha-\beta)\,e^{\alpha t}\,(e^t-1)^{\beta-\alpha-1}\,, \tag{17}$$

and for small $|t|$,

$$\delta-\pi \le \arg(e^t-1) \le \delta+\pi$$

on the loop of integration. Now (16) is valid for all α and β, with the trivial exception of $\alpha-\beta = -1, -2, \cdots$, and for all z in the complex plane slit along the line from $-\alpha$ to $-\alpha-\infty$.

Watson's lemma can be applied directly to (16). It is usual to state this lemma for an integral between 0 and ∞, but it is clear that the customary proof [4] goes through for a loop integral like (16) provided that the restriction on the growth of $f(t)$ is imposed along the whole loop, and that the expansion

$$f(t) = \sum_{n=0}^{\infty} a_n\, t^{\beta-\alpha+n-1} \tag{18}$$

is valid in a neighborhood of $t = 0$ on the loop. Both assumptions hold good in our case, and hence a term-by-term integration of (18) leads at once to the asymptotic expansion

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} \sim \sum \frac{a_n \; z^{\alpha-\beta-n}}{\Gamma(\alpha-\beta-n+1)} \qquad \text{as } z \to \infty ,$$

valid for all α, β, $\alpha-\beta \neq -1, -2, \cdots$, and the complex z-plane slit from $-\alpha$ to $-\alpha-\infty$.

Comparing with (12), we see that

$$\Gamma(\alpha-\beta-n+1)\; C_n(\alpha-\beta, \beta) = a_n$$

has the generating function (17). The properties of C_n established in the earlier sections can also be derived from this generating function. It also follows from the generating function that the coefficients can be expressed in terms of generalized Bernoulli polynomials. In Nörlund's [2] notation*, we have

$$a_n = \frac{1}{n!}\Gamma(1+\alpha-\beta)\; B_n^{(\alpha-\beta+1)}(\alpha) \;. \tag{19}$$

8. Particular cases. Finally, we notice that in the particular case $\alpha = n$, for $n = 1, 2, \cdots$, the expansion (6) becomes

$$\frac{\Gamma(z+n)}{\Gamma(z)} = z(z+1) \cdots (z+n-1) = \sum_{m=0}^{n-1} A_m(n)\, z^{n-m} \;; \tag{20}$$

hence, we have

$$A_m(n) = (-1)^m\, S_n^{(m)} \;, \tag{21}$$

where $S_n^{(m)}$ denotes the sum of the products of the negative numbers $-1, -2, \cdots, -(n-1)$ taken m at a time in all the possible manners (*Stirling's numbers of the first kind*).

Another interesting particular case of (6) is the case $\alpha = 1/2$, $z = n+1$, in which we have

$$\frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} \sim \frac{1}{(\pi n)^{1/2}} \left(1 - \frac{1}{8n} + \frac{1}{128n^2} - \cdots \right) . \tag{22}$$

---

*In the first instance, n in $B_\nu^{(n)}(x)$ is an integer, but Nörlund remarks (p.146) that it may be replaced by an arbitrary complex parameter.

Among the other things we can read from (22) is the following approximation formula for π:

$$(23)\ \pi = \frac{1}{n}\left[\frac{2^n n!}{1\cdot 3\cdots(2n-1)}\left(1-\frac{1}{8n}+\frac{1}{128n^2}+\epsilon_n\right)\right]^2, \quad \epsilon_n = O(n^{-3});$$

for instance, taking $n = 20$ and neglecting the remainder ϵ_n, from (23) we obtain the good approximation $\pi = 3.141557$, with an error of only 36 millionths.

Another interesting application concerns the asymptotic evaluation of the binomial coefficient $\binom{x}{n}$ as $n \to \infty$ and x (which is not a positive integer) remains bounded. Since

$$\binom{x}{n} = \frac{\Gamma(x+1)}{\Gamma(x-n+1)n!} = \frac{(-1)^n}{n\Gamma(-x)}\,\frac{\Gamma(n-x)}{\Gamma(n)},$$

we obtain from (6), with $z = n$ and $\alpha = -x$, the relation

$$\binom{x}{n} \sim \frac{(-1)^n}{\Gamma(-x)}\,n^{-(x+1)}\sum_{m=0}^{\infty}\frac{A_m(-x)}{n^m}$$

$$= \frac{(-1)^n}{\Gamma(-x)}\,n^{-(x+1)}\left[1+\binom{x+1}{2}\frac{1}{n}+\binom{x+2}{3}\frac{1+3x}{4n^2}+\cdots\right].$$

This formula gives very good numerical results even for relatively small values of n, for instance for $n = 10$, provided only that x/n is small.

References

1. J. S. Frame, *An approximation to the quotient of gamma functions*, Amer. Math. Monthly 56 (1949), 529-535.

2. N. E. Nörlund, *Vorlesungen über Differenzenrechnung*, Springer, Berlin (1924), 145.

3. F. Tricomi, *Sul comportamento asintotico dei polinomi di Laguerre*, Ann. Mat. Pura Appl. (4) 28 (1949), 263-289.

4. G. N. Watson, *A treatise on the theory of Bessel functions*, University Press, Cambridge, England (1922), 236.

California Institute of Technology