ON A TAUBERIAN THEOREM FOR ABEL SUMMABILITY

OTTO SZÁSZ

1. Introduction. In 1928 the author proved the following theorem [2, Section 2]:

THEOREM A. *If* $p > 1$ *and*

$$(1.1)\qquad \sum_{\nu=1}^{n} \nu^p |a_\nu|^p = O(n), \qquad n \longrightarrow \infty,$$

then Abel summability of the series $\sum_{n=0}^{\infty} a_n$ *to* s *implies its convergence to* s.

The theorem is the more general the smaller p is; it does not hold for $p = 1$ [2, Section 1; 1, pp.119, 122]. However, for this case Rényi proved the following theorem:

THEOREM B. *If*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\nu=1}^{n} \nu |a_\nu| = l < \infty$$

exists, then Abel summability of $\sum_{n=0}^{\infty} a_n$ *to* s *implies convergence of the series to* s.

2. Generalization. We give a simpler proof and at the same time a slight generalization of Theorem B.

THEOREM 1. *Assume that*

$$(2.1)\qquad V_n = \sum_{\nu=1}^{n} \nu |a_\nu| = O(n),$$

and that

$$(2.2)\qquad \frac{1}{m} V_m - \frac{1}{n} V_n \longrightarrow 0,$$

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for every sequence $m = m_n$, *such that* $m_n/n \longrightarrow 1$ *as* $n \longrightarrow \infty$. *Then Abel summability to s of* $\sum_{n=0}^{\infty} a_n$ *implies its convergence to s.*

Property (2.2) is called *slow oscillation* of the sequence V_n/n.

Proof of Theorem 1. We write

$$\sum_{\nu=0}^{n} a_\nu = s_n , \qquad \sum_{\nu=0}^{n} s_\nu = (n+1)\,\sigma_n .$$

It is easy to verify that, for $k = 0, 1, 2, \cdots$, we have

$$(2.3) \qquad s_{n-1} - \sigma_{n+k} = \frac{n}{k+1}\left(\sigma_{n+k} - \sigma_{n-1}\right) - \frac{1}{k+1}\sum_{\nu=0}^{k}(k+1-\nu)\,a_{n+\nu} .$$

It is known [see 2, Section 2] that if for a finite s we have

$$\lim_{x\to 1}\sum_{n=0}^{\infty} a_n x^n = s ,$$

then (2.1) implies $\sigma_n \longrightarrow s$; thus, if

$$(2.4) \qquad \underset{k\geq 0}{\text{l.u.b.}}\ \left|\sigma_{n-1} - \sigma_{n+k}\right| = \epsilon_n ,$$

then $\epsilon_n \longrightarrow 0$.

We now choose

$$(2.5) \qquad k = k_n = \left[n\,\epsilon_n^{1/2}\right], \qquad \text{so that} \quad k \leq n\,\epsilon_n^{1/2} < k+1 ;$$

it follows, in view of (2.4), that

$$\frac{n}{k+1}\left|\sigma_{n-1} - \sigma_{n+k}\right| < \epsilon_n^{1/2} .$$

In view of (2.3) our theorem will be proved if we show that

$$\frac{1}{k+1}\sum_{\nu=0}^{k}(k+1-\nu)\,a_{n+\nu} \longrightarrow 0 , \qquad n \longrightarrow \infty.$$

Now

$$\frac{1}{k+1}\left|\sum_{\nu=0}^{k}(k+1-\nu)\,a_{n+\nu}\right|$$

$$\leq \frac{1}{k+1}\sum_{\nu=0}^{k}(n+\nu)\,|a_{n+\nu}|\;\frac{k+1-\nu}{n+\nu} \leq \frac{1}{n}\,(V_{n+k}-V_{n-1})\,,$$

and

$$\tag{2.6} \frac{1}{n}\,(V_{n+k}-V_{n-1}) = \frac{V_{n+k}}{n+k}\cdot\frac{n+k}{n} - \frac{V_{n-1}}{n-1}\cdot\frac{n-1}{n}$$

$$= \frac{V_{n+k}}{n+k} - \frac{V_{n-1}}{n-1} + \frac{k}{n}\,\frac{V_{n+k}}{n+k} + \frac{1}{n}\,\frac{V_{n-1}}{n-1}\;;$$

using (2.2) and (2.5), we see that

$$\tag{2.7} \frac{1}{n}\,(V_{n+k}-V_{n-1}) \longrightarrow 0 \qquad \text{as} \quad \frac{k}{n}\longrightarrow 0 \quad \text{and} \quad n\longrightarrow\infty\,,$$

and thus Theorem 1 is proved.

Rényi observed that the Theorems A and B are overlapping. We now show that Theorem 1 includes not only Theorem B, but also Theorem A. Clearly (2.1) follows from (1.1) by Hölder's inequality. Furthermore,

$$V_{n+k}-V_n = \sum_{\nu=n+1}^{n+k}\nu\,|a_\nu| \leq k^{(p-1)/p}\left(\sum_{\nu=n+1}^{n+k}\nu^p\,|a_\nu|^p\right)^{1/p}$$

$$= k^{(p-1)/p}\,O[(n+k)^{1/p}]\;;$$

hence,

$$\frac{1}{n}\,(V_{n+k}-V_n) = \frac{k}{n}\,O\left[\left(\frac{n}{k}\right)^{1/p}\right] = O\left[\left(\frac{k}{n}\right)^{(p-1)/p}\right] \longrightarrow 0 \qquad \text{as} \quad \frac{k}{n}\longrightarrow 0\,.$$

It now follows from (2.6) that (2.2) holds; thus (1.1) implies (2.1) and (2.2), which proves our assertion.

An example of a sequence $V_n > 0$, and increasing, for which (2.2) holds, while $n^{-1} V_n \uparrow \infty$, is

$$V_n = n \log n , \qquad n \geq 2 ,$$

because

$$\frac{V_{n+k}}{n+k} - \frac{V_n}{n} = \log\left(1 + \frac{k}{n}\right) \longrightarrow 0 , \qquad \textit{as} \ \frac{k}{n} \longrightarrow 0, \qquad n \longrightarrow \infty .$$

3. A more general result. A generalization of Theorem A is the following [see 5, p.56]:

THEOREM A′. *If for some* $p > 1$, *we have*

$$\text{(3.1)} \qquad \sum_{\nu=1}^{n} \nu^p (|a_\nu| - a_\nu)^p = O(n) , \qquad n \longrightarrow \infty ,$$

then the Abel summability of $\Sigma_{n=0}^{\infty} a_n$ *implies its convergence to the same value.*

An analogue to Theorem 1 is the theorem:

THEOREM 2. *Assume that*

$$\text{(3.2)} \qquad U_n = \sum_{\nu=1}^{n} \nu(|a_\nu| - a_\nu) = O(n) ,$$

and that

$$\text{(3.3)} \qquad \frac{1}{m} U_m - \frac{1}{n} U_n \longrightarrow 0 \qquad \text{as} \ \frac{m}{n} \longrightarrow 1 , \qquad n \longrightarrow \infty .$$

If now $\Sigma_{n=0}^{\infty} a_n$ *is Abel summable to s, then it converges to s.*

Proof of Theorem 2. We have

$$-\sum_{\nu=1}^{n} \nu a_\nu \leq \sum_{\nu=1}^{n} \nu (|a_\nu| - a_\nu) = O(n) ;$$

hence [see 5, the Lemma on p.52] Abel summability of $\Sigma_{n=0}^{\infty} a_n$ implies its summability $(C, 1)$. From (2.3) we have

$$s_{n-1} - \sigma_{n+k} \leq \frac{n}{k+1} (\sigma_{n+k} - \sigma_{n-1})$$

$$+ \frac{1}{k+1} \sum_{\nu=0}^{k} (k+1-\nu)(|a_{n+\nu}| - a_{n+\nu}) ;$$

from (2.4) and (2.5) we obtain

$$\frac{n}{k+1} (\sigma_{n+k} - \sigma_{n-1}) < \epsilon_n^{1/2} .$$

Using the same argument as in the proof of Theorem 1, replacing V_n by U_n, we find that

$$\limsup_{n\to\infty} s_n \leq s . \tag{3.4}$$

We next employ the identity, similar to (2.3),

$$s_n - \sigma_{n-k-1} = \frac{n+1}{k+1} (\sigma_n - \sigma_{n-k-1})$$

$$+ \frac{1}{k+1} \sum_{\nu=0}^{k} (k-\nu) a_{n-\nu} , \qquad k = 0, 1, 2, \cdots,$$

and the inequality

$$a_\nu \geq a_\nu - |a_\nu| .$$

The same reasoning as before now yields

$$\liminf_{n\to\infty} s_n \geq s . \tag{3.5}$$

Finally (3.4) and (3.5) prove Theorem 2.

It is clear from the proof that condition (3.3) can be replaced by

$$\frac{1}{n} (U_m - U_n) \longrightarrow 0 , \qquad \text{as } \frac{m}{n} \longrightarrow 1 , \qquad n \longrightarrow \infty .$$

4. An equivalent result. A glance at the proof of Theorem 1 shows that the following lemma holds:

LEMMA 1. *If V_n is positive and monotone increasing, and if*

$$V_n = O(n), \qquad \text{as} \quad n \longrightarrow \infty, \tag{4.1}$$

and (2.2) *holds, then*

$$\frac{1}{n}(V_m - V_n) \longrightarrow 0, \qquad \text{as} \quad \frac{m}{n} \longrightarrow 1, \quad n \longrightarrow \infty. \tag{4.2}$$

We now prove the inverse:

LEMMA 2. *If $V_n > 0$, and increasing, and if* (4.2) *holds, then* (4.1) *and* (2.2) *hold.*

Proof. We write

$$V_n = n\,\omega_n, \qquad \omega_n \geq 0,$$

and

$$\frac{1}{n}(V_m - V_n) = \omega_m - \omega_n + \left(\frac{m}{n} - 1\right)\omega_m. \tag{4.3}$$

Let

$$\max_{\nu \leq n} \omega_\nu = \rho_n;$$

then $\rho_n \uparrow \rho \leq \infty$. If $\rho < \infty$, then $V_n = O(n)$. Suppose now that $\rho = \infty$; then there are infinitely many indices $m = m_\nu$, so that $\omega_m = \rho_m$ for $m = m_\nu, \nu = 1,2,3,\cdots$. For these m and for $n < m$, from (4.3) we get

$$\frac{1}{n}(V_m - V_n) > \left(\frac{m}{n} - 1\right)\rho_m. \tag{4.4}$$

We now choose

$$n = \frac{m\rho_m^{1/2}}{1 + \rho_m^{1/2}} < m,$$

so that

$$\frac{m}{n} = \frac{1 + \rho_m^{1/2}}{\rho_m^{1/2}} \longrightarrow 1;$$

then, using (4.4), we have

$$\frac{1}{n}\,(V_m - V_n) > \rho_m^{1/2} \longrightarrow \infty\,,$$

in contradiction to the assumption (4.2). It follows that (4.1) holds; finally (2.2) follows from (4.1), (4.2), and (4.3). This proves Lemma 2.

We now prove the following theorem:

THEOREM 3. *Let* $U_n = \sum_{\nu=1}^{n} \nu(|a_\nu| - a_\nu)$; *if*

$$\text{(4.5)} \qquad \frac{1}{n}\,(U_m - U_n) \longrightarrow 0\,, \qquad \text{as } \frac{m}{n} \longrightarrow 1\,, \quad n \longrightarrow \infty\,,$$

and if $\sum_{n=0}^{\infty} a_n$ *is Abel summable, then* $\sum_{n=0}^{\infty} a_n$ *is convergent to the same value.*

Proof of Theorem 3. In view of Lemma 2, Theorem 3 includes Theorem 2; it also includes Theorem 1, because of Lemma 2, and of the inequality

$$U_m - U_n \le 2(V_m - V_n)\,, \qquad m > n\,.$$

Conversely, by Lemma 2, (4.5) implies (3.2) and (3.3), so that Theorem 3 is equivalent to Theorem 2, and is thus valid.

To show that Theorem 1 is actually more general than Theorem B we give an example of a sequence ω_n so that $n\omega_n$ is increasing, ω_n is slowly oscillating and $\omega_n = O(1)$, but $\lim \omega_n$ does not exist. Let

$$\omega_n = \sum_{\nu=1}^{n} \nu^{-1}\epsilon_\nu\,, \qquad \text{where } \epsilon_\nu = \pm 1\,;$$

choose $\epsilon_\nu = +1$ as long as $\omega_n \le 3$; $\nu = 1, 2, \cdots, n_1$, say. Choose $\epsilon_\nu = -1$ as long as $\omega_n \ge 2$; $\nu = 1 + n_1, 2 + n_1, \cdots, n_2$, say; and so on. It is clear that $\omega_n = O(1)$, and that $\lim \omega_n$ does not exist. Furthermore, for $n \le n_1$, $\omega_n \uparrow$, for $n_1 < n \le n_2$, $\omega_n \downarrow$, and so on. Now

$$(n+1)\,\omega_{n+1} - n\,\omega_n = n(\omega_{n+1} - \omega_n) + \omega_{n+1} \ge \frac{3}{2} - 1 = \frac{1}{2}\,,$$

hence $n\omega_n \uparrow$. Finally

$$|\omega_m - \omega_n| \le \sum_{\nu=n+1}^{m} \frac{1}{\nu} < \frac{m-n}{n} \longrightarrow 0\,, \qquad \text{for } \frac{m}{n} \longrightarrow 1\,,$$

hence ω_n is slowly oscillating.

5. Another equivalent result. We first establish the following lemma.

LEMMA 3. *Suppose that* $U_n \geq 0$ *and increasing, with* $U_0 = 0$, *and let*

$$b_n = \frac{1}{n}(U_n - U_{n-1}), \qquad n \geq 1, \qquad b_0 = 0; \tag{5.1}$$

$$B_n = \sum_{\nu=0}^{n} b_\nu, \qquad n \geq 0. \tag{5.2}$$

Then whenever $k = k(n)$ *is so chosen that* $k/n \to 0$, *as* $n \to \infty$, *the two statements*

$$\frac{1}{n}(U_{n+k} - U_n) \longrightarrow 0 \tag{5.3}$$

and

$$B_{n+k} - B_n \longrightarrow 0 \tag{5.4}$$

are equivalent.

Proof. From (5.1) we have

$$U_n = \sum_{\nu=0}^{n} \nu b_\nu, \qquad U_{n+k} - U_n = \sum_{\nu=n+1}^{n+k} \nu b_\nu .$$

Now

$$B_{n+k} - B_n = \sum_{\nu=n+1}^{n+k} b_\nu \leq \frac{1}{n} \sum_{\nu=n+1}^{n+k} \nu b_\nu = \frac{1}{n}(U_{n+k} - U_n);$$

thus (5.3) implies (5.4). Furthermore,

$$B_{n+k} - B_n \geq \frac{1}{n+k}(U_{n+k} - U_n);$$

hence (5.4) implies (5.3). This proves the lemma.

We note that

$$B_n = \frac{1}{n} U_n + \sum_{\nu=1}^{n-1} \frac{1}{\nu(\nu+1)} U_\nu ,$$

and

$$U_n = nB_n - \sum_{\nu=0}^{n-1} B_\nu .$$

It is an immediate consequence of Lemma 3 that Theorem 3 is equivalent to the following theorem (for a direct proof see [4, Theorem IV]).

THEOREM 4. *If*

$$\sum_{\nu=n+1}^{n+k} (|a_\nu| - a_\nu) \longrightarrow 0 , \qquad as \quad \frac{k}{n} \longrightarrow 0 , \qquad n \longrightarrow \infty ,$$

then Abel summability of $\sum_{n=0}^{\infty} a_n$ *implies convergence of the series to the same value.*

A generalization of this theorem to Dirichlet series and to Laplace integrals, on different lines, is given in [3].

REFERENCES

1. A. Rényi, *On a Tauberian theorem of O. Szász,* Acta Univ. Szeged. Sect. Sci. Math. **11** (1946), 119-123.

2. O. Szász, *Verallgemeinerung eines Littlewoodschen Satzes über Potenzreihen,* J. London Math. Soc. 3 (1928), 254-262.

3. ————, *Verallgemeinerung und neuer Beweis einiger Sätze Tauberscher Art,* Sitzungsberichte d. Bayer. Akad. d. Wissenchaften *zu* München, 59 (1929), 325-340.

4. ————, *Generalization of two theorems of Hardy and Littlewood on power series,* Duke Math. J., **1** (1935), 105-111.

5. ————, *Introduction to the theory of divergent series,* University of Cincinnati, Cincinnati, 1944; Hafner, New York, 1948.

UNIVERSITY OF CINCINNATI AND
NATIONAL BUREAU OF STANDARDS, LOS ANGELES