THE CORESTRICTION OF VALUED DIVISION ALGEBRAS OVER HENSELIAN FIELDS I

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When L/F is an unramified extension of Henselian fields, we analyze the underlying division algebra ^cD of the corestriction $\mathrm{cor}_{L/F}(D)$ of a tame division algebra D over L with respect to the unique valuations on ^cD and D extending the valuations on F and L. We show that the value group of ^cD lies in the value group of D and for the center of residue division algebra, $Z(\overline{^cD}) \subseteq \mathcal{N}(Z(\overline{D})/\overline{F})^{1/k}$, where $\mathcal{N}(Z(\overline{D})/\overline{F})$ is the normal closure of $Z(\overline{D})$ over $\overline{F}$ and k is an integer depending on which roots of unity lie in F and L.

Introduction.

For any finite separable extension L/F of fields and any central simple algebra A over L, the corestriction of A is a central simple F-algebra obtained as the fixed point algebra under a Galois group action (cf. [**Ri**]). This induces the map from the Brauer group Br (L) to Br (F) corresponding to the homological corestriction. Though this algebraic corestriction is an important tool in the theory of division algebras, it is actually very hard to work with. To gain a better insight into the behavior of the corestriction, we analyze here the corestriction for valued division algebras over Henselian valued fields, for which there is a well-developed structure theory. We will here concentrate on the case when L is inertial (unramified) over F. In a subsequent paper [**H2**], we will consider the more general case when L is tame over F, i.e. char $(\overline{F}) \nmid [L\colon F]$, where $\overline{F}$ is the residue field of the valuation on F.

For any ring R we write $Z(R)$ and R^* for the center of R and the group of units of R, respectively. We will consider only central simple algebras A *finite-dimensional* over a field F. By Wedderburn's theorem, $A \cong M_n(D)$, a matrix ring over a division algebra D, which is called the *underlying division algebra* of A.

A valued field (F, v) is called *Henselian* if v extends uniquely to each field algebraic over F. For a nice account for several other characterizations of Henselian valuations, see Ribenboim's paper [**Rb**]. Recall (e.g. from [**W1**]) that if D is a central division algebra over a Henselian valued field (F, v), there exists one and only one valuation on D extending v on F.

Let $(L, v) \supseteq (F, v)$ be a finite separable inertial extension of Henselian fields. This means that $[\overline{L} : \overline{F}] = [L : F]$ and $\overline{L}$ is separable over $\overline{F}$. Let D be a central division algebra over L. We denote by ^cD the underlying division algebra of the corestriction $\mathrm{cor}_{L/F}(D)$ of D. If D is inertially split over L, i.e., split by the maximal unramified extension of L, we have good information obtained homologically using [**JW2**, Sec. 5]. Also, we can analyze a tame and totally ramified division algebra since it is isomorphic to a tensor product of symbol algebras (cf. Prop. 1.6). Furthermore, since a tame division algebra over L decomposes in the Brauer group to $S \otimes_L T$ with S inertially split and T tame and totally ramified (cf. Prop. 1.7), we obtain information about a tame division algebra by combining the results about S and T. Our basic results are summarized in the following table. Here Γ_D is the value group of the valuation on D and $\overline{D}$ is the residue division ring of the valuation ring of D. Also $\mathcal{N}(Z(\overline{D}) / \overline{F})$ denotes the normal closure of $Z(\overline{D})$ over $\overline{F}$, $N_{\overline{L}/\overline{F}}$ denotes the norm map from $\overline{L}$ to $\overline{F}$, and θ_D is the map of (1.5) below, so $\ker(\theta_D)$ is a subgroup of Γ_D/Γ_L.

D	$\Gamma_{^cD}$	$Z\left(\overline{^cD}\right)$
inertially split (Th. 2.4)	$\Gamma_{^cD} \subseteq \Gamma_D$	$Z\left(\overline{^cD}\right) \subseteq \mathcal{N}\left(Z\left(\overline{D}\right) / \overline{F}\right)$
tame, totally ramified (Prop. 3.2, Th. 3.9)	$\Gamma_{^cD} \subseteq \Gamma_D$	$Z\left(\overline{^cD}\right) \subseteq \overline{F}\left(N_{\overline{L}/\overline{F}}(\overline{L})^{1/t}\right)$ $t = \exp\left(\Gamma_D/\Gamma_L\right)$ (if $\mu_t \subseteq F$)
tame (Th. 4.5, 4.6)	$\Gamma_{^cD} \subseteq \Gamma_D$	$Z\left(\overline{^cD}\right) \subseteq \mathcal{N}\left(Z\left(\overline{D}\right) / \overline{F}\right)^{1/k}$ $k \mid \exp(\ker \theta_D)$

The integer k in the table above depends not only on Γ_D/Γ_L and $[L : F]$ but also on which roots of unity lie in F. One of the interesting results of the investigation is to see how heavily the corestriction depends on the roots of unity in F and L.

Here is an overview of the paper: After giving some preliminary results in Section 1, we will analyze the corestriction of inertially split division algebras in Section 2. In Section 3, we will consider the corestriction of tame and totally ramified division algebras T over L (i.e. $|\Gamma_T : \Gamma_L| = [T : L]$ and $\mathrm{char}(\overline{L}) \nmid [T : L]$) when $\mu_t \subseteq F$ with $t = \exp(\Gamma_T/\Gamma_L)$ (i.e. F has a t-th primitive root of unity). Finally, in Section 4 we will analyze the corestriction of tame division algebras. (In the Appendix, we will give an explicit formula of the homological corestriction.)

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1. Preliminaries.

Let $F \subseteq K$ be fields with K algebraic over F. We write $\mathrm{Gal}\,(K/F)$ for the profinite group of all F-automorphisms of K. If H is any subset of $\mathrm{Gal}\,(K/F)$, we write $\mathcal{F}(H)$ for the fixed field of H. We say K is *normal* over F if K is the splitting field of some family of polynomials over F; K is *Galois* over F if K is both normal and separable (if and only if $F = \mathcal{F}(\mathrm{Gal}\,(K/F))$). We write $\mathcal{N}(K/F)$ for the normal closure of K over F. Also, if $[K : F] < \infty$, $N_{K/F}$ denotes the norm map from K to F. We write F_{sep} for the separable closure of F.

For a central simple F-algebra A, $[A]$ denotes the class of A in the Brauer group $\mathrm{Br}\,(F)$ of F. If B is another central simple F-algebra with $[B] = [A]$, we write $A \sim B$. For any field $K \supseteq F$, let $\mathrm{Br}\,(K/F)$ denote the relative Brauer group of K/F, which is the kernel of the canonical "restriction" homomorphism $\mathrm{Br}\,(F) \to \mathrm{Br}\,(K)$ given by $[A] \to [A \otimes_F K]$.

Here is our notation for certain central simple F-algebras: $(K/F, G, f)$ is the crossed product algebra determined by a 2-cocycle $f \colon G \times G \to K^*$, where K is a finite dimensional Galois extension field of F with Galois group G;

$(K/F, \sigma, a)_n$ is the cyclic F-algebra generated over K by a single element x with defining relations $xcx^{-1} = \sigma(c)$ for all $c \in K$ and $x^n = a \in F^*$, where K is a Galois extension of F with cyclic Galois group generated by σ and $n = [K : F]$;

$(a, b; F)_n$ is the symbol algebra generated over F by i and j with defining relations: $i^n = a$, $j^n = b$, $ij = \omega ji$, where $\omega \in F^*$ is a primitive n-th root of unity and $a, b \in F^*$. Since a symbol algebra depends on a choice of a primitive n-th root of unity ω, we will sometimes write $(a, b, \omega; F)_n$ for $(a, b; F)_n$. In writing symbol algebra $(a, b; F)_n$ there is an implicit choice made of which primitive n-th root of unity ω to use. When this has been done, then for an integer n' dividing n, $(a, b; F)_{n'}$ denotes the symbol algebra built using the primitive n'-th root of unity $\omega^{n/n'}$.

Recall that the cyclic algebra $(K/F, \sigma, a)_n$ corresponds to the cohomology class of the cup product

$$(1.1) \qquad \left[\delta \left(\inf_{G_0}^{G} (f) \right) \right] \cup (a)$$

in the continuous cohomology group $H_c^2(G, F_{sep}^*)$,

where $G = \mathrm{Gal}\,(F_{sep}/F)$ is the absolute Galois group of F, $G_0 = \mathrm{Gal}\,(K/F)$

$= \langle \sigma \rangle$, $f \in H^1(G_0, \mathbb{Q}/\mathbb{Z}) = \text{Hom}(G_0, \mathbb{Q}/\mathbb{Z})$ is defined by $f(\sigma) = \frac{1}{n} + \mathbb{Z}$, $\delta : H^1(G, \mathbb{Q}/\mathbb{Z}) \to H^2(G, \mathbb{Q}/\mathbb{Z})$ is the connecting homomorphism, and $a \in F^* = H^0(G, F^*_{sep})$.

In fact, the corestriction (or transfer) homomorphism arose from the cohomology of groups, and the algebraic corestriction is compatible with the homological corestriction via the crossed product construction. Let G be a group and let A be a left G module. Let H be a subgroup of G of finite index n. For $m \geq 0$ an integer, $H^m(G, A)$ (resp. $H^m(H, A)$) denotes the m-th cohomology group of G (resp. H) with coefficients in A.

For each m, there is the *corestriction* (or *transfer*) $\text{cor}_H^G : H^m(H, A) \to H^m(G, A)$. (See [**B**, III, Sec. 9].) We give an explicit formula describing this homological corestriction for $m \geq 0$. Let $\mathcal{R} = \{\rho_1, \ldots, \rho_n\}$ be a set of representatives of the left cosets of H in G. For $m = 0$, $H^0(H, A) = A^H :=$ $\{a \in A \mid \sigma a = a \text{ for all } \sigma \in H\}$ and $H^0(G, A) = A^G$, and $\text{cor}_H^G : H^0(H, A) \to$ $H^0(G, A)$ is given by $a \mapsto \sum_{i=1}^{n} \rho_i a$ for $a \in H^0(H, A) = A^H$. Given $\rho_i \in \mathcal{R}$ and $\sigma \in G$, there are uniquely determined elements $\rho_{\sigma_*(i)} \in \mathcal{R}$ and $\delta(\sigma, \rho_i) \in H$ such that

$$(1.2) \qquad\qquad \sigma \rho_i = \rho_{\sigma_*(i)} \delta(\sigma, \rho_i),$$

where the map $\sigma \mapsto \sigma_*$ is a homomorphism from G to the symmetric group S_n. Now take any $f \in Z^m(H, A)$, the group of m cocycles of H with values in A; define a function $\text{cor}_H^G(f) \in Z^m(G, A)$ as follows: for $\sigma_1, \ldots, \sigma_m \in G$

$$(1.3) \quad \text{cor}_H^G(f)(\sigma_1, \ldots, \sigma_m)$$

$$= \sum_{i=1}^{n} \rho_{(\sigma_1 \cdots \sigma_m)_*(i)} \left[f \left(\delta \left(\sigma_1, \rho_{(\sigma_2 \cdots \sigma_m)_*(i)} \right), \right. \right.$$

$$\left. \left. \ldots, \delta \left(\sigma_j, \rho_{(\sigma_{j+1} \cdots \sigma_m)_*(i)} \right), \ldots, \delta(\sigma_m, \rho_i) \right) \right].$$

(We will deduce this formula from [**Ta**, p. 259] in the Appendix.) For $m \geq 1$, the homological corestriction $\text{cor}_H^G : H^m(H, A) \to H^m(G, A)$ is induced by the map $f \mapsto \text{cor}_H^G(f)$ given in (1.3) above. Recall that the (homological) corestriction map commutes with inflation and connecting homomorphisms (cf. [**We**, Prop 2.4.5 and Prop. 2.4.7]).

Recall the following case of the projection formula which will be used often in this paper (cf. [**Ti**, Th. 3.1]).

Proposition 1.4 (Projection Formula). *Let L/F be finite dimensional and separable. Assume F contains a primitive n-th root of unity. If $\alpha \in L^*$ and $b \in F^*$, then $\text{cor}_{L/F} \left((\alpha, b; L)_n \right) \sim \left(N_{L/F}(\alpha), b; F \right)_n$ in Br (F).*

The valued division algebra (D, v) is a division ring D with valuation v, that is, a function $v : D^* \to \Gamma$ (a totally ordered abelian group written

additively) such that for all a, b in D^*, $v(ab) = v(a) + v(b)$ and $v(a + b) \geq \min(v(a), v(b))$ if $a \neq -b$. Associated to v, we have its value group $\Gamma_D = v(D^*)$; the valuation ring $V_D = \{d \in D^* \mid v(d) \geq 0\} \cup \{0\}$; the unique maximal left ideal M_D of V_D, $M_D = \{d \in D^* \mid v(d) > 0\} \cup \{0\}$; the group of v-units of D^*, $U_D = V_D - M_D = V_D^*$; the residue division ring $\overline{D} = V_D/M_D$. If F is the center $Z(D)$ of D, there is a well-defined epimorphism

$$(1.5) \qquad \theta_D \colon \Gamma_D/\Gamma_F \to \mathrm{Gal}\left(Z(\overline{D})\,/\,\overline{F}\right),$$

induced by $\alpha \colon D^* \to \mathrm{Gal}\left(Z(\overline{D})\,/\,\overline{F}\right)$ which is given by $d \mapsto \overline{c_d}$ where $\overline{c_d}$ is the map induced by conjugation by d (cf. [**JW2**, 1.6]).

Let (F, v) be a Henselian valued field. Let D be a central division F-algebra (with a unique valuation extending v on F). We say D is *tame and totally ramified* over F if $\mathrm{char}\,(\overline{F}) \nmid [D \colon F]$ and $|\Gamma_D \colon \Gamma_F| = [D \colon F]$. D is said to be *inertially split* over F if D is split by F_{nr} where F_{nr} is the maximal unramified extension in some algebraic closure of F. Also, D is said to be *tame* if $\mathrm{char}\,(\overline{F}) = 0$ or $\mathrm{char}\,(\overline{F}) = q \neq 0$ and the q-primary component of D is split by F_{nr}. (See [**JW2**, Lemma 5.1] and [**JW2**, Lemma 6.1] for other characterizations of inertially split and tame division algebras.) Recall also that D is said to be inertial over F if $[\overline{D} \colon \overline{F}] = [D \colon F]$ and $Z(\overline{D}) = \overline{F}$. Let

$$\mathcal{D}(F) = \{D \mid D \text{ is a central division } F\text{-algebra with } [D \colon F] < \infty\}$$
$$\mathcal{D}_{ttr} = \{D \in \mathcal{D}(F) \mid D \text{ is tame and totally ramified over } F\}$$
$$\mathcal{D}_i(F) = \{D \in \mathcal{D}(F) \mid D \text{ is inertial over } F\}$$
$$\mathcal{D}_{is}(F) = \{D \in \mathcal{D}(F) \mid D \text{ is inertially split over } F\} \quad \text{and}$$
$$\mathcal{D}_t(F) = \{D \in \mathcal{D}(F) \mid D \text{ is tame over } F\}.$$

It is clear that $\mathcal{D}_i(F) \subseteq \mathcal{D}_{is}(F) \subseteq \mathcal{D}_t(F)$ and $\mathcal{D}_{ttr}(F) \subseteq \mathcal{D}_t(F)$.

Given (D, v) a valued central division F-algebra, one has the "fundamental inequality" $[D \colon F] \geq [\overline{D} \colon \overline{F}] \cdot |\Gamma_D \colon \Gamma_F|$ (cf. [**S**, p. 21]). We say D is *defectless* over F if equality holds.

Throughout this paper, we will assume that L/F is a finite separable inertial (unramified) extension of Henselian valued fields. We now recall some facts about inertial extensions as summarized in [**JW2**, Sec. 1]. Let F_{nr} denote the maximal unramified extension of F in some algebraic closure F_{alg} of F. Recall that $\Gamma_{F_{nr}} = \Gamma_F$, $\overline{F_{nr}}$ is the separable closure $\overline{F}_{sep}$ of $\overline{F}$, and F_{nr} is Galois over F. Also, for any field E with $F \subseteq E \subseteq F_{alg}$ and $[E \colon F] < \infty$, E is inertial over F if and only if $E \subseteq F_{nr}$. Further, the map $L \mapsto \overline{L}$ gives a $1-1$ correspondence between fields L with $F \subseteq L \subseteq F_{nr}$ and fields $\widetilde{L}$ with $\overline{F} \subseteq \widetilde{L} \subseteq \overline{F_{nr}}$. Given a field $\widetilde{L}$ with $\overline{F} \subseteq \widetilde{L} \subseteq \overline{F}_{sep}$ and

$[\widetilde{L} : \overline{F}] < \infty$, the corresponding L (inertial over F with $\overline{L} = \widetilde{L}$) will be called the *inertial lift* of $\widetilde{L}$ over F.

We now finish this section by recalling two propositions which will be very useful for this paper. See [**D2**, Th. 1] and [**JW2**, Lemma 6.2, Th. 6.3] for proofs of Props. 1.6 and 1.7, respectively.

Proposition 1.6 (Draxl's decomposition theorem). *If $D \in \mathcal{D}(F)$ is tame and totally ramified over a Henselian field F, D is isomorphic to a tensor product of symbol algebras.*

Proposition 1.7. *If $D \in \mathcal{D}(F)$ is tame over a Henselian field F, then there exist $S \in \mathcal{D}_{is}(F)$ and $T \in \mathcal{D}_{ttr}(F)$ such that $D \sim S \otimes_F T$ in Br (F). (Such S and T are not unique.) Furthermore, if $D \sim S \otimes_F T$ is such a decomposition,*

$$Z(\overline{D}) = \mathcal{F}(\theta_S((\Gamma_S \cap \Gamma_T) / \Gamma_F)) \subseteq Z(\overline{S}),$$
$$\Gamma_D = \Gamma_S + \Gamma_T \text{ and } \ker(\theta_D) = \Gamma_T/\Gamma_F.$$

2. Inertially split division algebras.

If $D \in \mathcal{D}_{is(F)}$, we have good information about D homologically, which is obtained in [**JW2**, Sec. 5]. We will give a brief summary on this. (See [**JW2**, Th. 5.6]).

We fix a Henselian valued field (F, v). Let Δ be the divisible hull of Γ_F, so $\Delta \simeq \Gamma_F \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $G = \text{Gal}(F_{nr} / F) \cong \text{Gal}(\overline{F}_{sep} / \overline{F})$, a profinite group. Let $U = U_{F_{nr}}$ the group of v-units of F_{nr}. Since $\Gamma_{F_{nr}} = \Gamma_F$ (on which G acts trivially), we have a short exact sequence of discrete G-modules:

$$1 \to U \to F_{nr}^* \overset{v}{\to} \Gamma_F \to 0.$$

From this short exact sequence, we have the following exact sequence, which will be the basic focus of our attention:

$$(2.1) \qquad 0 \to H_c^2(G, U) \to H_c^2(G, F_{nr}^*) \overset{\gamma}{\to} \text{Hom}_c(G, \Delta/\Gamma_F),$$

where H_c^2 and Hom_c denote the second (continuous) cohomology group and the continuous homomorphism group, respectively.

In this exact sequence, $H_c^2(G, F_{nr}^*) \cong Br(F_{nr}/F)$ and $H_c^2(G, U) \cong IBr(F)$, where $IBr(F) = \{[D] \in Br(F) | D \in D_i(F)$ i.e., D is inertial over $F\}$. (See [**JW2**, Th. 5.6].) Also, $IBr(F)$ is isomorphic to $Br(\overline{F})$ via the isomorphism β given by $[D] \to [\overline{D}]$. (See [**JW2**, Th. 2.8].) Further, if $\alpha \colon H_c^2(G, U) \to H_c^2\left(G, \overline{F_{nr}}^*\right) = H_c^2\left(G, \overline{F}_{sep}^*\right)$ is the map induced by a map $U \to \overline{F}_{sep}^*$ given

by $u \mapsto \overline{u}$, the following diagram commutes: (See [**JW2**, Examples 2.4, Prop. 2.5].)

$$(2.2) \qquad \begin{array}{ccc} H_c^2(G, U) & \xrightarrow{\;\alpha\;} & H_c^2(G, \overline{F}_{sep}^*) \\ \cong \downarrow & & \downarrow \cong \\ I \operatorname{Br}(F) & \xrightarrow{\;\beta\;} & \operatorname{Br}(\overline{F}) \end{array}$$

where the vertical isomorphisms are given by crossed product construction. We now identify $H_c^2(G, F_{nr}^*)$ with $\operatorname{Br}(F_{nr}/F)$. For any $D \in \mathcal{D}_{is}(F)$, as $[D] \in \operatorname{Br}(F_{nr}/F)$, set $h_D = \gamma[D] \in \operatorname{Hom}_c(G, \Delta/\Gamma_F)$, where γ is the map in (2.1); further, let $\widetilde{h}_D : G/\ker(h_D) \to \operatorname{im}(h_D)$ be the isomorphism induced by h_D. Then

(2.3)
$$\begin{cases} (1) & \operatorname{im}(h_D) = \Gamma_D/\Gamma_F \qquad \text{and } \mathcal{F}(\ker(h_D)) \\ & \qquad\qquad\qquad\qquad\qquad \text{is the inertial lift of } Z(\overline{D}) \\ (2) & \left(\text{After identifying } G/\ker(h_D) \quad \text{with } \operatorname{Gal}\left(Z\left(\overline{D}\right)/\overline{F} \right) \right) \widetilde{h}_D = \theta_D^{-1}, \end{cases}$$

where $\theta_D : \Gamma_D/\Gamma_F \to \operatorname{Gal}\left(Z\left(\overline{D}\right)/\overline{F} \right)$ is the map of (1.5) induced by conjugation. See [**JW2**, Th. 5.6] for the proof.

Now let $(L, v) \supseteq (F, v)$ be a finite (separable) inertial extension of Henselian fields. Since L is inertial over F, $L \subseteq F_{nr}$ so $L_{nr} = F_{nr}$. Let

$$G = \operatorname{Gal}(F_{nr}/F) \cong \operatorname{Gal}(\overline{F}_{sep}/\overline{F}) \quad \text{and}$$
$$H = \operatorname{Gal}(F_{nr}/L) = \operatorname{Gal}(L_{nr}/L) \cong \operatorname{Gal}(\overline{L}_{sep}/\overline{L}).$$

Then H is a subgroup of finite index $|G : H| = [L : F] = [\overline{L} : \overline{F}]$.

Note that, for $i \geq 0$ the corestriction map cor_H^G from $H^i(H, F_{nr}^*)$ to $H^i(G, F_{nr}^*)$ sends the continuous cohomology group $H_c^i(H, F_{nr}^*)$ into the continuous cohomology group $H_c^i(G, F_{nr}^*)$.

Theorem 2.4. *Let $(L, v) \supseteq (F, v)$ be a finite inertial extension of Henselian fields. Suppose $D \in \mathcal{D}_{is}(L)$. Then,*

(a) $\ ^c D \in \mathcal{D}_{is}(F)$

(b) $\ \Gamma_{^cD} \subseteq \Gamma_D$

(c) $\ Z\left(\overline{^cD}\right) \subseteq \mathcal{N}\left(Z\left(\overline{D}\right)/\overline{F} \right)$, *the normal closure of* $Z\left(\overline{D}\right)$ *over* $\overline{F}$.

(d) *If $D \in \mathcal{D}_i(L)$, then $\ ^cD \in \mathcal{D}_i(F)$ and $\overline{^cD} \sim \operatorname{cor}_{\overline{L}/\overline{F}}\left(\overline{D}\right)$ in $\operatorname{Br}\left(\overline{F}\right)$.*

(*So, if $D \in \mathcal{D}_t(L)$, then $^cD \in \mathcal{D}_t(F)$.*)

Proof. (a) Let $N = \mathcal{N}(L/F)$. As F_{nr} is normal over F, $N \subseteq F_{nr}$, so $N_{nr} = F_{nr}$. Let $\mathcal{R}$ be a set of representatives of the left cosets of $\mathrm{Gal}\,(N/L)$ in $\mathrm{Gal}\,(N/F)$. Then, by [**D2**, p. 52, Cor. 1], $^cD \otimes_F N \sim \mathrm{cor}_{L/F}(D) \otimes_F N \cong \otimes_{\rho \in \mathcal{R}}^{\rho}(D \otimes_L N) \in \mathrm{Br}\,(N_{nr}/N)$ since $D \in \mathcal{D}_{is}(L)$, where $^{\rho}(D \otimes_L N)$ means $D \otimes_L N$ made into an N-algebra by $(d \otimes n) \cdot n' = d \otimes n\rho^{-1}(n')$. So, ^cD is split by $N_{nr} = F_{nr}$, hence $^cD \in \mathcal{D}_{is}(F)$.

From this and definition of tame division algebras, it is clear that if $D \in \mathcal{D}_t(L)$, then $^cD \in \mathcal{D}_t(F)$.

(b) Let $H = \mathrm{Gal}\,(L_{nr}/L) = \mathrm{Gal}\,(F_{nr}/L)$, $G = \mathrm{Gal}\,(F_{nr}/F)$, $\Delta =$ the divisible hull of $\Gamma_F = \Gamma_L$, and $U = U_{F_{nr}} = U_{L_{nr}}$. Then since the corestriction map is functorial and commutes with the connecting homomorphism, the following diagram is commutative,

$$
\begin{array}{ccccccc}
0 & \longrightarrow & H_c^2(H, U) & \longrightarrow & H_c^2(H, F_{nr}^*) & \xrightarrow{\gamma} & \mathrm{Hom}_c\,(H, \Delta/\Gamma_F) \\
 & & {\scriptstyle \mathrm{cor}_H^G}\Big\downarrow & & {\scriptstyle \mathrm{cor}_H^G}\Big\downarrow & & {\scriptstyle \mathrm{cor}_H^G}\Big\downarrow \\
0 & \longrightarrow & H_c^2(G, U) & \longrightarrow & H_c^2(G, F_{nr}^*) & \xrightarrow{\gamma} & \mathrm{Hom}_c\,(G, \Delta/\Gamma_F)
\end{array}
$$

where the row exact sequences are those of (2.1). As done in (2.3), we identify $H_c^2(H, F_{nr}^*)$ and $H_c^2(G, F_{nr}^*)$ with $\mathrm{Br}\,(L_{nr}/L)$ and $\mathrm{Br}\,(F_{nr}/F)$, respectively. As $D \in \mathcal{D}_{is}(L)$ and by (a), $^cD \in \mathcal{D}_{is}(F)$. Set $h_D = \gamma[D] \in \mathrm{Hom}_c\,(H, \Delta/\Gamma_F)$ and $h_{^cD} = \gamma[^cD] \in \mathrm{Hom}_c\,(G, \Delta/\Gamma_F)$. Since the diagram above is commutative, $h_{^cD} = \gamma(\mathrm{cor}_H^G\,([D])) = \mathrm{cor}_H^G\,(\gamma([D])) = \mathrm{cor}_H^G\,(h_D)$. Now, by (2.3), $\Gamma_D/\Gamma_F = \mathrm{im}\,(h_D)$ and $\Gamma_{^cD}/\Gamma_F = \mathrm{im}\,(h_{^cD})$. But $\mathrm{im}\,(h_{^cD}) = \mathrm{im}\,(\mathrm{cor}_H^G\,(h_D)) \subseteq \mathrm{im}\,(h_D)$ by (1.3), as G acts trivially on Δ/Γ_F. Hence $\Gamma_{^cD} \subseteq \Gamma_D$.

(c) As $[D] \in \mathrm{Br}\,(L_{nr}/L) = \mathrm{Br}\,(F_{nr}/L) = H_c^2(H, F_{nr}^*)$, and $[^cD] \in \mathrm{Br}\,(F_{nr}/F) = H_c^2(G, F_{nr}^*)$, by (2.3), $\mathcal{F}(\ker\,(h_D))$ and $\mathcal{F}(\ker\,(h_{^cD}))$ are the inertial lift of $Z(\overline{D})$ and $Z(\overline{^cD})$ respectively. Let $Z = \mathcal{F}(\ker\,(h_D))$ and $M = \mathcal{N}(Z/F)$. Since $Z \subseteq F_{nr}$ and F_{nr} is normal over F, $M \subseteq F_{nr}$. Let $I = \mathrm{Gal}\,(F_{nr}/M)$ and $K = \mathrm{Gal}\,(F_{nr}/Z) = \ker\,(h_D)$. Then I is normal in $G = \mathrm{Gal}\,(F_{nr}/F)$, and $I \subseteq K \subseteq H = \mathrm{Gal}\,(F_{nr}/L) \subseteq G$. Note that $\mathrm{res}_K^H\,(h_D) = 0$ as $K = \ker\,(h_D)$. Let E be a set of representatives of the double cosets IgH. Then, by [**B**, p. 82, III, Prop. 3.5], $\mathrm{res}_I^G\,(\mathrm{cor}_H^G\,(h_D)) = \sum_{g \in E} \mathrm{cor}_{I \cap gHg^{-1}}^I \circ \mathrm{res}_{I \cap gHg^{-1}}^H\,(h_D)$, where the "restriction map" on the right is with respect to the conjugation map $(i \mapsto g^{-1}ig, a \mapsto g^{-1}a)$, regarded as a map $(I \cap gHg^{-1}, F_{nr}^*) \xrightarrow{\cong} (g^{-1}Ig \cap H, F_{nr}^*) \to (H, F_{nr}^*)$. But as I is normal in G, $g^{-1}Izg \cap H = I \cap H \subseteq K$, so $\mathrm{res}_{I \cap gHg^{-1}}^H\,(h_D) = 0$ for each $g \in E$. Hence, $\mathrm{res}_I^G\,(\mathrm{cor}_H^G\,(h_D)) = 0$, so $\ker\,(h_{^cD}) \supseteq I$. Therefore $Z(\overline{^cD}) = \mathcal{F}(\ker\,(h_{^cD})) \subseteq \mathcal{F}(I) = \overline{M} = \mathcal{N}(Z(\overline{D})/F)$.

(d) Since $D \in \mathcal{D}_i(L)$, $[D] \in I \operatorname{Br}(L) = H_c^2(H, U) = \ker(\gamma)$, where γ is the map of (2.1). Since the diagram above in the proof of (b) is commutative, $[^cD] = \operatorname{cor}_H^G([D]) \in \ker(\gamma) = H_c^2(G, U) = I \operatorname{Br}(F)$. Hence $^cD \in \mathcal{D}_i(F)$. As the corestriction map is functorial, the map α in (2.2) is compatible with corestriction. So, since the algebraic corestriction corresponds to the homological corestriction, and by (2.2), we have $\overline{^cD} \sim \operatorname{cor}_{\overline{L}/\overline{F}}(\overline{D})$ in $\operatorname{Br}(\overline{F})$. $\qquad\square$

3. Tame and totally ramified division algebras.

We begin this section by giving a proposition we will use a number of times.

Proposition 3.1. *Let (F, v) be a Henselian valued field. For $1 \leq i \leq k$, let $n_i \geq 1$ be integers with $(\operatorname{char} \overline{F}, n_i) = 1$, and $a_i, b_i \in F^*$. Assume F has primitive n_i-th roots of unity ω_i. Let $n = n_1 \cdots n_k$ and $\ell = \ell\operatorname{cm}(n_1, \ldots, n_k)$. Then*

(1) The tensor product of symbol algebras, $T = \overset{k}{\underset{i=1}{\otimes}}(a_i, b_i; F)_{n_i}$ is a tame and totally ramified division algebra over F if and only if

$$\{(\ell/n_i)v(a_i), (\ell/n_i)v(b_i) \mid 1 \leq i \leq k\}$$

generates a subgroup of order n^2 in $\Gamma_F/\ell\Gamma_F$; when this occurs,

$$\Gamma_T = \langle (1/n_i)v(a_i), (1/n_i)v(b_i) \rangle_{i=1}^k + \Gamma_F.$$

(2) If $D \in \mathcal{D}_{ttr}(F)$ and D^m is the underlying division algebra of $D^{\otimes m}$, the m-fold tensor product of D with itself over F, then $D^m \in \mathcal{D}_{ttr}(F)$, and

$$\Gamma_{D^m}/\Gamma_F = m(\Gamma_D/\Gamma_F), \quad so \quad \Gamma_{D^m} = m\Gamma_D + \Gamma_F.$$

Proof. (1) Suppose $\{(\ell/n_i)v(a_i) \ (\ell/n_i)v(b_i)\}_{i=1}^k$ generates a subgroup of order n^2 in $\Gamma_F/\ell\Gamma_F$. Then by [**JW1**, Cor. 2.6], $T \in \mathcal{D}_{ttr}(F)$ and $\Gamma_T = \langle \frac{1}{n_i}v(a_i), \frac{1}{n_i}v(b_i) \rangle_{i=1}^k + \Gamma_F$. Conversely, suppose $T = \overset{k}{\underset{i=1}{\otimes}}(a_i, b_i; F)_{n_i} \in \mathcal{D}_{ttr}(F)$. Let α_i, β_i be the generators of T over F with relations: $\alpha_i^{n_i} = a_i$, $\beta_i^{n_i} = b_i$, and $\alpha_i\beta_i = \omega_i\beta_i\alpha_i$. Then $\mathcal{A} = \langle \alpha_i F^*, \beta_i F^* \rangle_{i=1}^k$ is an armature of T, that is, an abelian subgroup of T^*/F^* of order $\dim_F T$ such that $\{\alpha_i, \beta_i\}_{i=1}^k$ generates T as an F-algebra. Since $T \in \mathcal{D}_{ttr}(F)$, by [**TW**, Prop. 3.3], the map $\overline{v}: \mathcal{A} \to \Gamma_T/\Gamma_F$ induced by v (i.e., the map $xF^* \mapsto v(x) + \Gamma_F$) is an isomorphism. So, $\Gamma_T/\Gamma_F = \langle \frac{1}{n_i}v(a_i) + \Gamma_F, \frac{1}{n_i}v(b_i) + \Gamma_F \rangle_{i=1}^k$ and

$\{(\ell/n_i)v(a_i), (\ell/n_i)v(b_i)\}_{i=1}^{k}$ generates a subgroup of order n^2 in $\Gamma_F / \ell\Gamma_F$ (namely $\ell\Gamma_T / \ell\Gamma_F$), as $|\Gamma_T / \Gamma_F| = [T : F] = n^2$. Also,

$$\Gamma_T = \left\langle \frac{1}{n_i}v(a_i), \frac{1}{n_i}v(b_i) \right\rangle_{i=1}^{k} + \Gamma_F$$

as $\Gamma_T / \Gamma_F = \left\langle \frac{1}{n_i}v(a_i) + \Gamma_F, \frac{1}{n_i}v(b_i) + \Gamma_F \right\rangle_{i=1}^{k}$.

(2) As $D \in \mathcal{D}_{ttr}(F)$ and F is Henselian, by Prop. 1.6 $D = \overset{k}{\underset{i=1}{\otimes}} (a_i, b_i; F)_{n_i}$ for some $a_i, b_i \in F^*$.

Let $d_i = \gcd(n_i, m)$ and $n_i = d_i n_i'$, $m = d_i m_i'$ so that $\gcd(n_i', m_i') = 1$. Since $D = \overset{k}{\underset{i=1}{\otimes}} (a_i, b_i; F)_{n_i} \in \mathcal{D}_{ttr}(F)$, by (1),

$$\{(\ell/n_i)v(a_i), (\ell/n_i)v(b_i) \,|\, 1 \le i \le k\}$$

generates a subgroup of order n^2 in $\Gamma_F / \ell\Gamma_F$. Also, since the map

$$\Gamma_F / \ell\Gamma_F \to \frac{1}{\ell}\Gamma_F / \Gamma_F,$$

induced by multiplication of $\frac{1}{\ell}$, is an isomorphism,

$$\left\{ \frac{1}{n_i}v(a_i), \frac{1}{n_i}v(b_i) \,|\, 1 \le i \le k \right\}$$

generates a subgroup $\left\langle \frac{1}{n_i}v(a_i) + \Gamma_F, \frac{1}{n_i}v(b_i) + \Gamma_F \,|\, 1 \le i \le k \right\rangle$ of order n^2 in $\frac{1}{\ell}\Gamma_F / \Gamma_F$. We want to show $\overset{k}{\underset{i=1}{\otimes}} (a_i, b_i^{m_i'}; F)_{n_i'} \in \mathcal{D}_{ttr}(F)$. Since $n_i' \,|\, n_i$, $1 \le i \le k$, $\left\langle \frac{1}{n_i'}v(a_i) + \Gamma_F, \frac{1}{n_i'}v(b_i) + \Gamma_F \right\rangle_{i=1}^{k}$ is a subgroup of

$$\left\langle \frac{1}{n_i}v(a_i) + \Gamma_F, \frac{1}{n_i}v(b_i) + \Gamma_F \right\rangle_{i=1}^{k}.$$

Since $\left| \left\langle \frac{1}{n_i}v(a_i) + \Gamma_F, \frac{1}{n_i}v(b_i) + \Gamma_F \,|\, 1 \le i \le k \right\rangle \right| = n^2$,

$$\left| \left\langle \frac{1}{n_i'}v(a_i) + \Gamma_F, \frac{1}{n_i'}v(b_i) + \Gamma \right\rangle_{i=1}^{k} \right| = n'^2,$$

where $n' = n_1' \cdots n_k'$. Let $\ell' = \mathrm{lcm}(n_1', \ldots, n_k')$. Then

$$\left\langle \frac{1}{n_i'}v(a_i) + \Gamma_F, \frac{1}{n_i'}v(b_i) + \Gamma_F \right\rangle_{i=1}^{k} \subseteq \frac{1}{\ell'}\Gamma_F / \Gamma_F.$$

Since the map $\frac{1}{\ell'}\Gamma_F/\Gamma_F \rightarrow \Gamma_F/\ell'\Gamma_F$ induced by multiplication of ℓ', is an isomorphism, $\left| \left\langle \frac{\ell'}{n_i'}v(a_i) + \ell'\Gamma_F, \frac{\ell'}{n_i'}v(b_i) + \ell'\Gamma_F \right\rangle_{i=1}^k \right| = n'^2$. Also, since $\gcd(n_i', m_i') = 1$,

$$\left\{ (\ell'/n_i')v(a_i), (\ell'/n_i')v(b_i^{m_i'}) \mid 1 \le i \le k \right\}$$

and

$$\left\{ (\ell'/n_i')v(a_i), (\ell'/n_i')v(b_i) \mid 1 \le i \le k \right\}$$

generate the same subgroup of $\Gamma_F/\ell'\Gamma_F$. So,

$$\left\{ (\ell'/n_i')v(a_i), (\ell'/n_i')v(b_i^{m_i'}) \mid 1 \le i \le k \right\}$$

generates a subgroup of order n'^2 in $\Gamma_F/\ell'\Gamma_F$. By applying (1) we have $\overset{k}{\underset{i=1}{\otimes}}(a_i, b_i^{m_i'}; F)_{n_i'} \in \mathcal{D}_{ttr}(F)$ with value group

$$\left\langle (1/n_i')v(a_i), (1/n_i')v(b_i^{m_i'}) \right\rangle_{i=1}^k + \Gamma_F.$$

Since $D^m \sim \overset{k}{\underset{i=1}{\otimes}}(a_i, b_i^m; F)_{n_i} \sim \overset{k}{\underset{i=1}{\otimes}}(a_i, b_i^{m'}; F)_{n_i'}$, we have

$$D^m \cong \overset{k}{\underset{i=1}{\otimes}}(a_i, b_i^{m_i'}; F)_{n_i'} \in \mathcal{D}_{ttr}(F).$$

Because $\gcd(n_i', m_i') = 1$, we have

$$\langle (1/n_i')v(b_i) + \Gamma_F \rangle = \langle (m_i'/n_i')v(b_i) + \Gamma_F \rangle =$$
$$= \langle (m/n_i)v(b_i) + \Gamma_F \rangle, \text{ so } \Gamma_{D^m}/\Gamma_F =$$
$$= \left\langle (1/n_i')v(a_i) + \Gamma_F, (1/n_i')v(b_i^{m_i'}) + \Gamma_F \right\rangle_{i=1}^k =$$
$$= \langle (1/n_i')v(a_i) + \Gamma_F, (1/n_i')v(b_i) + \Gamma_F \rangle_{i=1}^k = m(\Gamma_D/\Gamma_F) .$$

(In fact, the equality $\Gamma_{D^m} = m\Gamma_D + \Gamma_F$ is a special case of [**JW2**, Prop. 6.9].) $\qquad\square$

Recall that we assume that $(L, v) \supseteq (F, v)$ is a finite (separable) inertial extension of Henselian fields. Given any positive integer n and a field F, we write $\mu_n \subseteq F$ to say that F contains n distinct n-th roots of unity.

Proposition 3.2. *Let* $T \in \mathcal{D}_{ttr}(L)$ *and* $n = \exp(\Gamma_T/\Gamma_L)$. *If* $\mu_n \subseteq F$, *then* $\Gamma_{cT} \subseteq \Gamma_T$.

Remark. Without the assumption $\mu_n \subseteq F$, it is still true $\Gamma_{cT} \subseteq \Gamma_T$. We will prove this in Theorem 4.6 below.

Proof. First we will show that if $T = (\alpha, \beta; L)_n$ is a symbol algebra in $\mathcal{D}_{ttr}(L)$, then $[L : F]\Gamma_T + \Gamma_F \subseteq \Gamma_{cT} \subseteq \Gamma_T$: Since $\Gamma_L = \Gamma_F$, $\alpha = au$ and $\beta = bw$ for some a, $b \in F^*$ and v-units u, $w \in L^*$. Let $T_0 = (a, b; L)_n$, $A = (a, b; F)_n$, and let S be the underlying division algebra of $(a, w; L)_n \otimes_L (u, \beta; L)_n$. Since $T = (\alpha, \beta; L)_n \in \mathcal{D}_{ttr}(L)$, by Prop. 3.1 $\{v(\alpha), v(\beta)\}$ generates a subgroup of order n^2 in $\Gamma_L / n\Gamma_L$, and $\Gamma_T = \langle (1/n)v(\alpha), (1/n)v(\beta) \rangle + \Gamma_L$. As $v(a) = v(\alpha)$, $v(b) = v(\beta)$ and $\Gamma_F = \Gamma_L$, by Prop. 3.1, again, $T_0 \in \mathcal{D}_{ttr}(L)$ and $A \in \mathcal{D}_{ttr}(F)$ with

$$\Gamma_A = \langle (1/n)v(a), (1/n)v(b) \rangle + \Gamma_F = \Gamma_{T_0} = \Gamma_T.$$

Clearly $S \in \mathcal{D}_{is}(L)$. So $T \sim T_0 \otimes_L S$ in Br (L) with $T_0 \in \mathcal{D}_{ttr}(L)$, $S \in \mathcal{D}_{is}(L)$. Then by Prop. 1.7, $\Gamma_T = \Gamma_{T_0} + \Gamma_S$. Note that as $T_0 = A \otimes_F L$, $^cT_0 \sim A^{\otimes [L:F]}$ in Br (F), so $^cT_0$ is the underlying division algebra of $A^{\otimes [L:F]}$, where $A \in \mathcal{D}_{ttr}(F)$. Then by Prop. 3.1 (2), $^cT_0 \in \mathcal{D}_{ttr}(F)$, and $\Gamma_{^cT_0} = [L : F]\Gamma_A + \Gamma_F = [L : F]\Gamma_T + \Gamma_F$ as $\Gamma_T = \Gamma_A$. Also, by Theorem 2.4, $^cS \in \mathcal{D}_{is}(F)$ and $\Gamma_{^cS} \subseteq \Gamma_S$. Since $^cT \sim {}^cT_0 \otimes_F {}^cS$ in Br (F) where $^cT_0 \in \mathcal{D}_{ttr}(F)$ and $^cS \in \mathcal{D}_{is}(F)$, by Prop. 1.7, $\Gamma_{^cT} = \Gamma_{^cT_0} + \Gamma_{^cS}$. Hence $[L : F]\Gamma_T + \Gamma_F \subseteq \Gamma_{^cT_0} + \Gamma_{^cS} = \Gamma_{^cT} \subseteq \Gamma_{T_0} + \Gamma_S = \Gamma_T$, as desired.

Now, if $T \in \mathcal{D}_{ttr}(L)$, then by Prop. 1.6, $T \cong \overset{k}{\underset{i=1}{\otimes}} T_i$ where each T_i is a symbol algebra in $\mathcal{D}_{ttr}(L)$ of index n_i with $n_i \mid n$. Also, $\Gamma_T = \sum_{i=1}^k \Gamma_{T_i}$. Then $^cT \sim \overset{k}{\underset{i=1}{\otimes}} {}^cT_i$ in Br (F). Since each $^cT_i \in \mathcal{D}_t(F)$ by Theorem 2.4, $\Gamma_{^cT} \subseteq \sum_{i=1}^k \Gamma_{^cT_i}$ by [**JW2**, Cor. 6.7]. Since $\Gamma_{^cT_i} \subseteq \Gamma_{T_i}$ by what we have shown above, $\Gamma_{^cT} \subseteq \sum_{i=1}^k \Gamma_{^cT_i} \subseteq \sum_{i=1}^k \Gamma_{T_i} = \Gamma_T$. $\qquad \square$

We will shortly give Proposition 3.4, which will be very useful for this paper. The following purely group-theoretic lemma will be used in the proof of that proposition.

Lemma 3.3. *Let $A_1 \times A_2$ be the direct product of cyclic groups A_i of order r_i, $i = 1, 2$. Let π_i be the projection map of $A_1 \times A_2$ onto A_i for $i = 1, 2$. Let H be a subgroup of $A_1 \times A_2$ such that $\pi_i(H) = A_i$, $i = 1, 2$. Let $B_1 = H \cap A_2$ and $B_2 = H \cap A_1$. If t is any positive integer and T is the t-torsion subgroup of H, then $T = C_1 \cap C_2$, where C_i, $i = 1, 2$, are the unique subgroups of H such that $C_i \supseteq B_i$ and $|C_i : B_i| = \gcd(r_i, t)$.*

Proof. Since $|C_i : B_i|$ divides t, $t(C_i / B_i) = (0)$, so $tC_i \subseteq B_i$. Hence $t(C_1 \cap C_2) \subseteq B_1 \cap B_2 = (0)$, and $C_1 \cap C_2 \subseteq T$. Take $h \in H$ with $th = 0$ (i.e., $h \in T$). Then $0 = \pi_i(th) = t\pi_i(h)$. But $r_i\pi_i(h) = 0$ as $|A_i| = r_i$. So $\gcd(r_i, t) \cdot \pi_i(h) = 0$. Let D_i be the subgroup of A_i of order $\gcd(r_i, t)$. Then

$\pi_i(h) \in D_i$ as A_i is cyclic. Since $B_i = \ker(\pi_i|_H)$ and $\pi_i|_H$ is onto, the map $\tilde{\pi}_i : H/B_i \to A_i$ induced by $\pi_i|_H$ is an isomorphism. Since $\tilde{\pi}_i(h + B_i) = \pi_i(h) \in D_i$, $h + B_i \in \tilde{\pi}_i^{-1}(D_i)$. But $\tilde{\pi}_i^{-1}(D_i) = C_i / B_i$, for $i = 1, 2$. Hence $h \in C_1 \cap C_2$. Therefore, $T \subseteq C_1 \cap C_2$. $\qquad\square$

Proposition 3.4. *Let (F, v) be a Henselian field. Let r_1, r_2 and t be positive integers, and let M_i/F, $i = 1, 2$, be cyclic inertial extensions of degree r_i with $\mathrm{Gal}(M_i/F) = \langle \tau_i \rangle$. Let x_1, $x_2 \in F^*$ such that, for any prime p dividing r_1, r_2 or t, $v(x_1) + p\Gamma_F$, $v(x_2) + p\Gamma_F$ are $\mathbb{Z}_p$-linearly independent in $\Gamma_F / p\Gamma_F$. Suppose D is the underlying division algebra of $(M_1/F, \tau_1, x_1)_{r_1} \otimes_F (M_2/F, \tau_2, x_2)_{r_2} \otimes_F (x_1, x_2; F)_t$. Then $Z(\overline{D}) = \widetilde{E}_1 \widetilde{E}_2$, where $\overline{M_i} \supseteq \widetilde{E}_i \supseteq \overline{F}$, and $\left[\overline{M_i} : \widetilde{E}_i \right] = \gcd(r_i, t)$ for $i = 1, 2$.*

Proof. Let Δ be the divisible hull of Γ_F and let $r_1 = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, $r_2 = p_1^{f_1} p_2^{f_2} \cdots p_k^{f_k}$ and $t = p_1^{t_1} p_2^{t_2} \cdots p_k^{t_k}$, where the p_i are distinct primes and e_i, f_i, $t_i \geq 0$ such that at least one of e_i, f_i and t_i is larger than 0 for all i, $1 \leq i \leq k$. Since $v(x_1) + p_i\Gamma_F$, $v(x_2) + p_i\Gamma_F$ are $\mathbb{Z}_{p_i}$-linearly independent in $\Gamma_F / p_i\Gamma_F$ by assumption, in Δ/Γ_F,

$$\left| \left\langle \frac{1}{p_i^{e_i}} v(x_1) + \Gamma_F, \ \frac{1}{p_i^{f_i}} v(x_2) + \Gamma_F \right\rangle \right| = p_i^{e_i + f_i}$$

and

$$\left| \left\langle \frac{1}{p_i^{t_i}} v(x_1) + \Gamma_F, \ \frac{1}{p_i^{t_i}} v(x_2) + \Gamma_F \right\rangle \right| = p_i^{2t_i}.$$

Then in Δ / Γ_F,

$$\left| \left\langle \frac{1}{r_1} v(x_1) + \Gamma_F, \ \frac{1}{r_2} v(x_2) + \Gamma_F \right\rangle \right| = r_1 r_2$$

and

$$\left| \left\langle \frac{1}{t} v(x_1) + \Gamma_F, \ \frac{1}{t} v(x_2) + \Gamma_F \right\rangle \right| = t^2.$$

Let $T = (x_1, x_2; F)_t$ and $S_i = (M_i/F, \tau_i, x_i)_{r_i}$ for $i = 1, 2$. Then by Prop. 3.1, $T \in \mathcal{D}_{ttr}(F)$. Also, by [**JW1**, Cor.2.9], $S_i \in \mathcal{D}(F)$ with $\overline{S_i} = \overline{M_i}$ and $\Gamma_{S_i} = \left\langle \frac{1}{r_i} v(x_i) \right\rangle + \Gamma_F$ for $i = 1, 2$. Also, S_i is inertially split over F as M_i is a splitting field of S_i. Let S be the underlying division algebra of $S_1 \otimes_F S_2$. Then $D \sim S \otimes_F T$ in $\mathrm{Br}(F)$ where $S \in \mathcal{D}_{is}(F)$ and $T \in \mathcal{D}_{ttr}(F)$. Let h_S, h_{S_1}, $h_{S_2} \in \mathrm{Hom}_c(G_F, \Delta / \Gamma_F)$ as in (2.3). Since $h_S = h_{S_1} + h_{S_2}$, $\mathrm{im}(h_{S_i}) = \Gamma_{S_i} / \Gamma_F = \left\langle \frac{1}{r_i} v(x_i) + \Gamma_F \right\rangle$, $i = 1, 2$, and $\left\langle \frac{1}{r_1} v(x_1) + \Gamma_F \right\rangle \cap \left\langle \frac{1}{r_2} v(x_2) + \Gamma_F \right\rangle = (0)$ as $\left| \left\langle \frac{1}{r_1} v(x_1) + \Gamma_F, \ \frac{1}{r_2} v(x_2) + \Gamma_F \right\rangle \right| = r_1 r_2$, we have $\ker(h_S) = \ker(h_{S_1}) \cap$

$\ker(h_{S_2})$. So by (2.3),

$$Z(\overline{S}) = \overline{\mathcal{F}(\ker h_S)} = \overline{\mathcal{F}(\ker h_{S_1})} \cdot \overline{\mathcal{F}(\ker h_{S_2})}$$
$$= Z(\overline{S_1}) Z(\overline{S_2}) = \overline{M_1} \cdot \overline{M_2}.$$

Also,

$$\Gamma_S / \Gamma_F = \operatorname{im}(h_S) \subseteq \operatorname{im}(h_{S_1}) + \operatorname{im}(h_{S_2}) =$$
$$= \left\langle \frac{1}{r_1} v(x_1) + \Gamma_F \right\rangle \times \left\langle \frac{1}{r_2} v(x_2) + \Gamma_F \right\rangle \cong \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2}.$$

Since $S \in \mathcal{D}_{is}(F)$, by [**JW2**, Lemma 5.1], $\theta_S : \Gamma_S / \Gamma_F \to \operatorname{Gal}(Z(\overline{S})/\overline{F})$ is an isomorphism. Let $H = \Gamma_S / \Gamma_F$ and $A_i = \Gamma_{S_i}/\Gamma_F$ for $i = 1, 2$. Then $H \subseteq A_1 \times A_2$. Let $B_1 = H \cap A_2$ and $B_2 = H \cap A_1$. Since $H = \operatorname{im}(h_S) = \operatorname{im}(h_{S_1} + h_{S_2})$, and $\operatorname{im}(h_{S_1}) \cap \operatorname{im}(h_{S_2}) = (0)$, $\pi_i(H) = A_i$ where $\pi_i : A_1 \times A_2 \to A_i$ is the i-th projection map. Note that

$$(\Gamma_S \cap \Gamma_T)/\Gamma_F = (\Gamma_S / \Gamma_F) \cap \left\langle \frac{1}{t} v(x_1) + \Gamma_F, \frac{1}{t} v(x_2) + \Gamma_F \right\rangle =$$
$$= \{\alpha \in \Gamma_S / \Gamma_F \mid t\alpha = 0\} \text{ as } \Gamma_S / \Gamma_F \subseteq \left\langle \frac{1}{r_1} v(x_1) + \Gamma_F \right\rangle \times \left\langle \frac{1}{r_2} v(x_2) + \Gamma_F \right\rangle.$$

So $(\Gamma_S \cap \Gamma_T)/\Gamma_F$ is the t-torsion subgroup of H. Then by Lemma 3.3, $(\Gamma_S \cap \Gamma_T)/\Gamma_F = C_1 \cap C_2$, where $C_i \supseteq B_i$ and $|C_i : B_i| = \gcd(r_i, t)$ for $i = 1, 2$.

We have the map $h_S : \operatorname{Gal}(F_{nr}/F) \to \Gamma_S / \Gamma_F = H \subseteq A_1 \times A_2$, where $A_i = \operatorname{im}(h_{S_i})$ for $i = 1, 2$. Since $A_1 \cap A_2 = (0)$, for $g \in \operatorname{Gal}(F_{nr}/F)$, $A_2(= \operatorname{im}(h_{S_2}))$ contains $h_S(g) = h_{S_1}(g) + h_{S_2}(g)$ if and only if $h_{S_1}(g) = 0$. So, $h_S^{-1}(B_1) = h_S^{-1}(G \cap A_2) = \ker(h_{S_1}) = \operatorname{Gal}(F_{nr}/M_1)$. Likewise, $h_S^{-1}(B_2) = \ker(h_{S_2}) = \operatorname{Gal}(F_{nr}/M_2)$. Hence, in the isomorphism $\theta_S = \tilde{h}_S^{-1} : \Gamma_S / \Gamma_F \longrightarrow \operatorname{Gal}(\overline{M_1 M_2}/\overline{F})$, $\theta_S(B_i) = \operatorname{Gal}(\overline{M_1 M_2}/\overline{M_i})$ for $i = 1, 2$. Therefore $\theta_S(C_i) = \operatorname{Gal}(\overline{M_1 M_2}/\tilde{E}_i)$ where $\overline{M_i} \supseteq \tilde{E}_i \supseteq \overline{F}$ and $[\overline{M_i} : \tilde{E}_i] = \gcd(r_i, t)$ for $i = 1, 2$. Then by Prop. 1.7, $Z(\overline{D}) = \mathcal{F}(\theta_S((\Gamma_S \cap \Gamma_T)/\Gamma_F)) = \mathcal{F}(C_1 \cap C_2) = \tilde{E}_1 \tilde{E}_2$. $\qquad \square$

Corollary 3.5. *Let (F, v) be a Henselian field. Let r_1, r_2, and t be positive integers and assume μ_t, μ_{r_1}, $\mu_{r_2} \subseteq F$. Let u_1, u_2 be v-units in F, and let $x_1, x_2 \in F^*$ such that for any prime p dividing r_1, r_2 or t, $v(x_1) + p\Gamma_F$, $v(x_2) + p\Gamma_F$ are $\mathbb{Z}_p$-linearly independent in $\Gamma_F / p\Gamma_F$. Let ω_t (resp. ω_{r_i}) be an arbitrary primitive t-th (resp. r_i-th) root of unity in F. Suppose D is the underlying division algebra of $(x_1, u_1, \omega_{r_1}; F)_{r_1} \otimes_F (x_2, u_2, \omega_{r_2}; F)_{r_2} \otimes_F (x_1, x_2, \omega_t; F)_t$. Then $Z(\overline{D}) = \overline{F}(\overline{u}_1^{t/r_1}, \overline{u}_2^{t/r_2})$.*

Proof. Let $s_i = o(u_i F^{*r_i})$ in F^*/F^{*r_i} for $i = 1, 2$. Then $s_i \mid r_i$, $u_i^{s_i} = w_i'^{r_i}$ for some $w_i' \in F^*$, so $u_i = \omega_{s_i}^k w_i'^{r_i/s_i} = (\omega_{r_i}^k w_i')^{r_i/s_i}$ for some $k \in \mathbb{Z}$,

where ω_{r_i} is the given primitive r_i-th root of unity, and $\omega_{s_i} = \omega_{r_i}^{r_i/s_i}$. Let $w_i = \omega_{r_i}^k w_i'$. Then $u_i = w_i^{r_i/s_i}$ and $F\left(u_i^{1/r_i}\right) = F\left(w_i^{1/s_i}\right)$ is a cyclic inertial extension of F of degree s_i for each $i = 1, 2$. Also, $D \sim \left(x_1, w_1^{r_1/s_1}; F\right)_{r_1} \otimes_F$ $\left(x_2, w_2^{r_2/s_2}; F\right)_{r_2} \otimes_F (x_1, x_2; F)_t \sim (x_1, w_1; F)_{s_1} \otimes_F (x_2, w_2; F)_{s_2} \otimes_F (x_1, x_2; F)_t$. But $(x_i, w_i; F)_{s_i} = (M_i/F, \tau_i, x_i)_{s_i}$ where $M_i = F\left(w_i^{1/s_i}\right)$ are cyclic inertial over F of degree s_i, and τ_i are some generators of $\mathrm{Gal}\,(M_i/F)$. So, by Prop. 3.4, $Z\left(\overline{D}\right) = \widetilde{E}_1 \widetilde{E}_2$ where $\overline{M_i} \supseteq \widetilde{E}_i \supseteq \overline{F}$, and $[\overline{M_i} : \widetilde{E}_i] = \gcd\,(s_i, t)$, so $\widetilde{E}_i = \overline{F}\left(\overline{w_i}^{t/s_i}\right) = \overline{F}\left(\overline{u_i}^{t/r_i}\right)$. Hence $Z\left(\overline{D}\right) = \overline{F}\left(\overline{u_1}^{t/r_1}, \overline{u_2}^{t/r_2}\right)$. $\qquad\square$

We will consider $Z\left(\overline{^cT}\right)$ for $T \in \mathcal{D}_{ttr}(L)$ in Theorem 3.9. We first introduce an equivalence relation in $\mathcal{D}_t(F)$ for a Henselian field (F, v). For any $D_1, D_2 \in \mathcal{D}_t(F)$, we write $D_1 \approx D_2$ if $[D_2] \equiv [D_1] \bmod I\mathrm{Br}\,(F)$ in $\mathrm{Br}\,(F)$. This is an equivalence relation since $I\mathrm{Br}\,(F)$ is a subgroup of $\mathrm{Br}\,(F)$ (cf. [**JW2**, Prop. 2.5]).

Remark 3.6. Let (F, v) be Henselian.

(i) If $D_1, D_2 \in \mathcal{D}_t(F)$ and $D_1 \approx D_2$, then $Z(\overline{D_1}) = Z(\overline{D_2})$ by [**JW2**, Cor. 6.8].

(ii) If L/F is inertial, $D_1, D_2 \in \mathcal{D}_t(L)$ and $D_1 \approx D_2$, then $^cD_1 \approx {}^cD_2$ since $^cI \in \mathcal{D}_i(F)$ for any $I \in \mathcal{D}_i(L)$ by Theorem 2.4 (d).

The following lemma will be used in proof of Prop. 3.8.

Lemma 3.7. *Let K/F be a finite extension of Henselian fields. If α is a v-unit in K, then $N_{K/F}(\alpha)$ is a v-unit in F, and $\overline{N_{K/F}(\alpha)} = N_{\overline{K}/\overline{F}}(\overline{\alpha})^e$ where $e = [K : F]/[\overline{K} : \overline{F}]$.*

Proof. Recall that the integral closure of V_F in K is the intersection of all the valuation rings of K extending V_F (cf. [**Bo**, Ch. VI, Sec. 1, Th. 3, p. 378]). As F is Henselian, V_K is the unique valuation ring of K extending V_F. So V_K is the integral closure of V_F in K. Hence α is integral over V_F. Let f be the minimal polynomial of α over F. Then $f \in V_F[x]$ since V_F is integrally closed.

Note that $\overline{f} = \varphi^i$ where $\varphi \in \overline{F}[X]$ is a monic irreducible polynomial, $i \geq 1$ and $i \deg(\varphi) = \deg(f)$ since $f \in V_F[X]$ is a monic irreducible polynomial and F is Henselian (cf. [**Rb**, Th. 3]). Since $\varphi(\overline{\alpha})^i = \overline{f}(\overline{\alpha}) = \overline{f(\alpha)} = 0$, i.e., $\varphi(\overline{\alpha}) = 0$, φ is the minimal polynomial of $\overline{\alpha}$ over $\overline{F}$. So $N_{\overline{K}/\overline{F}}(\overline{\alpha}) = (-1)^{[\overline{K}:\overline{F}]}(\varphi(0))^{[\overline{K}:\overline{F}]\deg}(\varphi)$ (cf. [**L**, Ch. VIII, Sec. 5, Th. 5.1]). On the other hand, $N_{K/F}(\alpha) = (-1)^{[K:F]}(f(0))^{[K:F]\deg}(f)$. But since $\overline{f(0)} = \varphi(0)^i$ and $\deg(f) = i \deg(\varphi)$, $\overline{N_{K/F}(\alpha)} = (-1)^{[K:F]}(\varphi(0))^{i[K:F]/i \deg}(\varphi) = N_{\overline{K}/\overline{F}}(\overline{\alpha})^e$ where $e = [K : F]/[\overline{K} : \overline{F}]$. $\qquad\square$

Proposition 3.8. *Let the symbol algebra* $T = (\alpha_1, \alpha_2; L)_n \in \mathcal{D}_{ttr}(L)$ *where* $n > 1$ *is any integer, (so* $\mu_n \subseteq L$) *and* $\alpha_1, \alpha_2 \in L^*$. *If* $\mu_n \subseteq F$, *then*

$$Z\left({}^c T\right) \subseteq \overline{F}\left(N_{\overline{L}/\overline{F}}\left(\overline{L}\right)^{1/d}\right), \text{ where } d = \gcd\,(n, [L:F]).$$

Proof. Since $\Gamma_L = \Gamma_F$, $\alpha_i = a_i u_i$ for some $a_i \in F^*$, $u_i \in U_L$. Note that since $(\alpha_1, \alpha_2; L)_n \in \mathcal{D}_{ttr}(L)$, by Prop. 3.1, in $\Gamma_L / n\Gamma_L$

$$|\langle v(\alpha_1) + n\Gamma_L, v(\alpha_2) + n\Gamma_L \rangle| = n^2.$$

But as $v(a_i) = v(\alpha_i)$, $i = 1, 2$, and $\Gamma_L / n\Gamma_L = \Gamma_F / n\Gamma_F \cong \frac{1}{n}\Gamma_F / \Gamma_F$, we have in $\frac{1}{n}\Gamma_F / \Gamma_F$, $\left|\langle \frac{1}{n}v(a_1) + \Gamma_F, \frac{1}{n}v(a_2) + \Gamma_F \rangle\right| = n^2$. So for any prime p dividing n, in $\frac{1}{p}\Gamma_F / \Gamma_F$, $\left|\langle \frac{1}{p}v(a_1) + \Gamma_F, \frac{1}{p}v(a_1) + \Gamma_F \rangle\right| = p^2$, or equivalently in $\Gamma_F / p\Gamma_F$, $v(a_1) + p\Gamma_F$, $v(a_2) + p\Gamma_F$ are $\mathbb{Z}_p$-linearly independent.

Note that $[(u_1, u_2; L)_n] \in I\,\mathrm{Br}\,(L)$ by [**JW2**, Prop. 2.5], since the "symbol algebra" $(u_1, u_2; V_L)_n$ over the valuation ring V_L is an Azumaya algebra over V_L and an order in $(u_1, u_2; L)_n$ (cf. [**JW2**, Ex. 2.4 (i)]). Then since $T \sim (u_1, u_2; L)_n \otimes (a_1, u_2; L)_n \otimes (u_1, a_2; L)_n \otimes (a_1, a_2; L)_n$ in $\mathrm{Br}\,(L)$, we have $T \approx D$ where D is the underlying division algebra of $(a_1, u_2; L)_n \otimes_L (a_2, u_1^{-1}; L)_n \otimes_L (a_1, a_2; L)_n$. By Remark 3.6 (ii), and Prop. 1.4, and as $N_{L/F}(a_2) = a_2^{[L:F]}$, ${}^c T \approx {}^c D \sim \left(a_1, N_{L/F}(u_2); F\right)_n \otimes_F \left(a_2, N_{L/F}(u_1^{-1}); F\right)_n \otimes_F \left(a_1, a_2^{[L:F]}; F\right)_n$. But $\left(a_1, a_2^{[L:F]}; F\right)_n \sim (a_1, a_2^e; F)_{n'}$, where $d = \gcd\,(n, [L:F])$, $[L:F] = de$, and $n = dn'$ so that $\gcd\,(e, n') = 1$. Say, let $(a_1, a_2^e; F)_{n'} = (a_1, a_2^e, \omega; F)_{n'}$ for some primitive n'-th root ω of unity. Then $(a_1, a_2^e, \omega; F)_{n'} = (a_1, a_2, \omega^k; F)_{n'}$ for some k with $\gcd\,(k, n') = 1$. So $\left(a_1, a_2^{[L:F]}; F\right)_n$ is similar to a symbol algebra $(a_1, a_2, \omega^k; F)_{n'}$ where $n' = n/\gcd\,(n, [L:F])$. Hence ${}^c T \approx {}^c D \sim (a_1, N_{L/F}(u_2); F)_n \otimes_F (a_2, N_{L/F}(u_1^{-1}); F)_n \otimes (a_1, a_2, \omega^k; F)_{n'}$ in $\mathrm{Br}\,(F)$. Then by Remark 3.6, and Cor. 3.5,

$$Z({}^c T) = Z({}^c D) = \overline{F}\left((\overline{N_{L/F}(u_2)})^{n'/n}, (\overline{N_{L/F}(u_1^{-1})})^{n'/n}\right).$$

Since $[L:F] = [\overline{L}:\overline{F}]$, $\overline{N_{L/F}(u_1^{-1})} = N_{\overline{L}/\overline{F}}(\overline{u_1^{-1}})$ and $\overline{N_{L/F}(u_2)} = N_{\overline{L}/\overline{F}}(\overline{u_2})$ by Lemma 3.7 above. So

$$Z(\overline{{}^c T}) = \overline{F}\left(N_{\overline{L}/\overline{F}}(\overline{u_2})^{1/d}, N_{\overline{L}/\overline{F}}(\overline{u_1^{-1}})^{1/d}\right) \subseteq \overline{F}(N_{\overline{L}/\overline{F}}(\overline{L})^{1/d}),$$

where $d = n/n' = \gcd(n, [L:F])$. $\square$

Theorem 3.9. *Suppose* $T \in \mathcal{D}_{ttr}(L)$ *and* $n = \exp\,(\Gamma_T/\Gamma_L)$. *(So* $\mu_n \subseteq L$.) *If* $\mu_n \subseteq F$, *then*

$$Z(\overline{{}^c T}) \subseteq \overline{F}\left(N_{\overline{L}/\overline{F}}(\overline{L})^{1/d}\right), \quad d = \gcd\,(n, [L:F]),$$

where $N_{\overline{L}/\overline{F}}$ is the norm map from $\overline{L}$ to $\overline{F}$.

Proof. By Prop. 1.6, $T = \overset{k}{\underset{i=1}{\otimes}} T_i$, where each T_i is a symbol division algebra in $\mathcal{D}_{ttr}(L)$ of index n_i with $n_i | n$. Then $^cT \sim \overset{k}{\underset{i=1}{\otimes}} {}^cT_i \sim {}^cT_0 \otimes_F {}^cT_k$ where $T_0 = \overset{k-1}{\underset{i=1}{\otimes}} T_i$. We prove that $Z(\overline{^cT}) = Z(\overline{^cT_1}) \cdots Z(\overline{^cT_k})$. By induction on k, we need only to prove $Z(\overline{^cT}) = Z(\overline{^cT_0}) Z(\overline{^cT_k})$. Since $^cT_0 \in \mathcal{D}_t(F)$ by Theorem 2.4, $^cT_0/F$ is defectless, and $Z(\overline{^cT_0})/\overline{F}$ is Galois by [**JW2**, Lemma 6.1]. So, $\mathcal{L} = Z(\overline{^cT_0}) \cap Z(\overline{^cT_k})$ is separable over $\overline{F}$, and $Z(\overline{^cT_0})$ and $Z(\overline{^cT_k})$ are linearly disjoint over $\mathcal{L}$. Also, by Prop. 3.2, and as $\Gamma_T/\Gamma_L = \overset{k}{\underset{i=1}{\oplus}} (\Gamma_{T_i}/\Gamma_L)$, we have $\Gamma_{^cT_0} \cap \Gamma_{^cT_k} \subseteq \Gamma_{T_0} \cap \Gamma_{T_k} = \Gamma_L = \Gamma_F$. Hence, by [**MW**, Cor. 3.12], $Z(\overline{^cT}) = Z(\overline{^cT_0}) Z(\overline{^cT_k}) = Z(\overline{^cT_1}) \cdots Z(\overline{^cT_{k-1}}) Z(\overline{^cT_k})$. But by Prop. 3.8, for $1 \le i \le k$,

$$Z(\overline{^cT_i}) \subseteq \overline{F}\left(N_{\overline{L}/\overline{F}}(\overline{L})^{1/d_i}\right), \quad d_i = \gcd\left(n_i, [L:F]\right).$$

Therefore, as $n_i | n$,

$$Z(\overline{^cT}) \subseteq \overline{F}\left(N_{\overline{L}/\overline{F}}(\overline{L})^{1/d}\right), \quad d = \gcd\left(n, [L:F]\right).$$

<div align="right">□</div>

4. Tame division algebras.

We will show in Theorems 4.4-4.6 below that $\Gamma_{^cD} \subseteq \Gamma_D$ for $D \in \mathcal{D}_t(L)$, without any restriction on roots of unity in F, and that for $D \in \mathcal{D}_t(L)$, $Z(\overline{^cD}) \subseteq \mathcal{N}(Z(\overline{D})/\overline{F})^{1/k}$, where $k \ge 1$ is some integer dividing $\exp(\ker \theta_D)$ which depends on which roots of unity lie in F. We first give some propositions which will be used in the proofs of the theorems.

Proposition 4.1. *Let $n = p^k$ with p an odd prime, $k \ge 1$, and let ω_k be a primitive n-th root of unity in L. Suppose $a, b \in F^*$ and $T = (a, b; L)_n \in \mathcal{D}_{ttr}(L)$. Then*
   (a) *if $\mu_p \subseteq F$, then $^cT \sim (a, b^\ell; F)_{n/r}$ in $\mathrm{Br}(F)$, where $\ell = [L : F(\omega_k)]$, and $r = [F(\omega_k) : F]$. Hence, $^cT \in \mathcal{D}_{ttr}(F)$ and $\Gamma_{^cT} = [L:F]\Gamma_T + \Gamma_F$;*
   (b) *if $\mu_p \nsubseteq F$, then $^cT = F$.*

Proof. (a) For $1 \le i \le k$, set $\omega_i = (\omega_k)^{p^{k-i}}$, a primitive p^i-th root of unity. Let $F_i = F(\omega_i)$. Let t be maximal with $\omega_t \in F$ so that $F = F_t$ and $[F_{i+1} : F_i] = p$ for $i \ge t$. (Note that $r = [F_k : F] = [F_k : F_t] = p^{k-t}$, so $p^t = n/r$.) Then by [**M**, #15, p. 144],

$$\mathrm{cor}_{F_{i+1}/F_i}\left((a, b, \omega_{i+1}; F_{i+1})_{p^{i+1}}\right) \sim \left((a, b, \omega_i; F_i)_{p^i}\right).$$

So it follows that

$$\mathrm{cor}_{F_k/F}\left((a,b,;F_k)_n\right) \sim (a,b;F_t)_{p^t} = (a,b;F)_{n/r}.$$

Hence, by Prop. 1.4,

$$\begin{aligned}
^cT &\sim \mathrm{cor}_{L/F}((a,b,;L)_n) \sim \mathrm{cor}_{F_k/F}\left(\mathrm{cor}_{L/F}((a,b,;L)_n)\right)\\
&\sim \mathrm{cor}_{F_k/F}\left((a,b^\ell;F_k)_n\right) \sim (a,b^\ell;F_k)_{n/r}.
\end{aligned}$$

The second statement follows easily by Prop. 3.1.

(b) Let $K = F(\omega_1)$ and let s be maximal with $\omega_s \in K$ so that $K = F_s$ and $[F_{i+1} : F_i] = p$ for $i \geq s$. As above, it follows by [M, #15, p. 144] that

$$\mathrm{cor}_{F_k/K}\left((a,b,\omega_k;F_k)_{p^k}\right) \sim (a,b,\omega_s;K)_{p^s}.$$

If $\mathrm{Gal}(K/F) = \langle\sigma\rangle$, then $\sigma(\omega_s) = \omega_s^u$ where $u^m \equiv 1 \pmod{p^s}$ for $m = [K : F]$ and $u \not\equiv 1 \pmod{p}$ since σ moves ω_1. It follows easily that $N_{K/F}(\omega_s) = 1$ and so $\mathrm{cor}_{K/F}((a,b,\omega_s;K)_{p^s}) \sim F$ by [M, #16, p. 145]. So $\mathrm{cor}_{F_k/F}((a,b,\omega_k;F_k)_n) \sim F$. Hence by Prop. 1.4., as $N_{L/F_k}(b) = b^\ell$, $\ell = [L : F_k]$,

$$\begin{aligned}
^cT &\sim \mathrm{cor}_{L/F}((a,b;L)_n) \sim \mathrm{cor}_{F_k/F}(\mathrm{cor}_{L/F_k}((a,b;L)_n)) \sim\\
&\sim \mathrm{cor}_{F_k/F}((a,b^\ell;F_k)_n) \sim F.
\end{aligned}$$

$\square$

Proposition 4.2. *Let* $n = 2^k$, $k \geq 3$, *and let* ω_k *be a primitive n-th root of unity in* L. *Suppose* $a,b \in F^*$ *and* $T = (a,b;L)_n \in \mathcal{D}_{ttr}(L)$. *Then*

(a) *if* $\mu_4 \subseteq F$, *then* $^cT \sim (a,b^\ell;F)_{n/r} \otimes_F (a,b^\ell;F)_2 \sim (a,b^{\ell e};F)_{n/r}$ *in* $\mathrm{Br}\,(F)$, *where* $\ell = [L : F(\omega_k)], r = [F(\omega_k) : F]$ *and* $e = 1 + \frac{n}{2r}$ *is an odd integer. Hence* $^cT \in \mathcal{D}_{ttr}(F)$ *and* $^cT = [L : F]\Gamma_T + \Gamma_F$;

(b) *if* $\mu_4 \not\subseteq F$, *then* $^cT = (a,b;F)_2$ *or* $= F$, *and* $^cT \in \mathcal{D}_{ttr}(F)$.

Proof. (a) For $1 \leq i \leq k$, set $\omega_i = (\omega_k)^{2^{k-i}}$, a primitive 2^i-th root of unity. Let $F_i = F(\omega_i)$. Let t be maximal with $\omega_t \in F$ so that $F = F_t$ and $[F_{i+1} : F_i] = 2$ for $i \geq t$. (Note that $t \geq 2$ as $\mu_4 \subseteq F$.) Then by [M, #15 and #13, p. 144],

$$\begin{aligned}
\mathrm{cor}_{F_{i+1}/F_i}\left((a,b,\omega_{i+1};F_{i+1})_{2^{i+1}}\right) &\sim \left((a,b,-\omega_i;F_i)_{2^i}\right)\\
&\sim \left((a,b,\omega_i;F_i)_{2^i}\right) \otimes_F \left((a,b,-1;F_i)_2\right).
\end{aligned}$$

Since $\mathrm{cor}_{F_i/F_{i-1}}((a,b,-1;F_i)_2) \sim F_{i-1}$ by Prop. 1.4 as $N_{F_i/F_{i-1}}(b) = b^2$, we end up with $\mathrm{cor}_{F_k/F}((a,b;F_k)_n) \sim (a,b;F)_{n/r} \otimes_F (a,b;F)_2$. Hence, by Prop. 1.4,

$$\begin{aligned}
^cT &\sim \mathrm{cor}_{L/F}((a,b,;L)_n) \sim \mathrm{cor}_{F_k/F}\left(\mathrm{cor}_{L/F_k}((a,b,;L)_n)\right)\\
&\sim \mathrm{cor}_{F_k/F}\left((a,b^\ell;F_k)_n\right) \sim (a,b^\ell;F_k)_{n/r} \otimes (a,b^\ell;F)_2.
\end{aligned}$$

The second statement follows easily by Prop. 3.1.

(b) Let $K = F_2 = F(\sqrt{-1})$ and let s be maximal with $\omega_s \in K$ so that $K = F_s$ and $[F_{i+1} : F_i] = 2$ for $i \geq s$. As above, it follows by [M, #15 and #13, p. 144] that

$$\mathrm{cor}_{F_k/K}\left((a, b, \omega_k; F_k)_{2^k}\right) \sim (a, b, \omega_s; K)_{2^s} \otimes_K (a, b, -1; K)_2.$$

Note that $\mathrm{cor}_{K/F}((a, b, -1; K)_2) \sim F$ by Prop. 1.4. If $\mathrm{Gal}(K/F) = \langle \sigma \rangle$, then $\sigma(\omega_s) = \omega_s^u$ where $u^2 \equiv 1 \pmod{2^s}$ and $u \not\equiv 1 \pmod 4$ since σ moves $\sqrt{-1}$. It follows easily that $N_{K/F}(\omega_s) = 1$ or -1, and so $\mathrm{cor}_{K/F}((a, b, \omega_s; K)_{2^s}) \sim F$ or $(a, b, -1; F)_2$ by [M, #16 and #7, pp. 143-145]. So $\mathrm{cor}_{F_k/F}((a, b, \omega_k; F_k)_n) \sim F$ or $(a, b, -1; F)_2$. Hence,

$${}^cT \sim \mathrm{cor}_{L/F}((a, b; L)_n) \sim F \quad \text{or} \quad (a, b; F)_2.$$

$\square$

Lemma 4.3. *Let ω_4 be a primitive fourth root of unity in L. Let $a, b, c, d \in F^*$, and let $T = (a, b; L)_4$ and $T_0 = (a, b; F)_2$. Suppose $T, T_0 \in \mathcal{D}_{ttr}(L)$. Then*

(a) if $\omega_4 \in F$, then ${}^cT \sim (a, b^\ell; F)_4$ in $\mathrm{Br}\,(F)$, where $\ell = [L : F]$. So ${}^cT \in \mathcal{D}_{ttr}(F)$, and $\Gamma_{{}^cT} = [L : F]\Gamma_T + \Gamma_F$. Also, ${}^cT_0 \sim (c, d^\ell; F)$ in $\mathrm{Br}\,(F)$, so ${}^cT_0 \in \mathcal{D}_{ttr}(F)$, and $\Gamma_{{}^cT_0} = [L : F]\Gamma_{T_0} + \Gamma_F$;

(b) if $\omega_4 \notin F$, then ${}^cT = F$.

Proof. (a) follows easily by Prop. 1.4 and Prop. 3.1. (b) can be proved by the exact same arguments in the proof of Prop. 4.2 (b). $\square$

Recall that we are assuming L/F is a finite separable inertial extension of Henselian fields.

Theorem 4.4. *Let $T \in \mathcal{D}_{ttr}(L)$ and $\exp(\Gamma_T/\Gamma_L) = p_0^{e_0} p_1^{e_1} \cdots p_r^{e_r}$, where $p_0 = 2$, $e_0 \geq 0$, and for $1 \leq i \leq r$, the p_i are distinct odd primes, and $e_i > 0$. Let $T = \overset{r}{\underset{i=0}{\otimes}} T_i$ be the primary decomposition of T where T_i is the p_i-primary component of T. (So $\Gamma_T = \sum_{i=0}^r \Gamma_{T_i}$ and Γ_{T_i} / Γ_L is the p_i-primary component of Γ_T/Γ_L.) Suppose $\mu_{p_i} \subseteq F$ for $1 \leq i \leq r_0$ and $\mu_{p_i} \not\subseteq F$ for $r_0 + 1 \leq i \leq r$. Then*

(a) if $4 \nmid \exp(\Gamma_T/\Gamma_L)$ or $\mu_4 \subseteq F$, then

$$[L : F]\left(\Gamma_{T_0} + \Gamma_{T_1} + \cdots + \Gamma_{T_{r_0}}\right) + \Gamma_F \subseteq \Gamma_{{}^cT} \subseteq \Gamma_T, \text{ and}$$

(b) if $4 \mid \exp(\Gamma_T/\Gamma_L)$ and $\mu_4 \not\subseteq F$, then

$$[L : F]\left(\Gamma_{T_1} + \cdots + \Gamma_{T_{r_0}}\right) + \Gamma_F \subseteq \Gamma_{{}^cT} \subseteq \Gamma_T.$$

Proof. As $T_i \in \mathcal{D}_{ttr}(L)$ by Prop. 1.6, $T_i = \overset{\ell_i}{\underset{k=1}{\otimes}} (\alpha_k, \beta_k; L)_{m_k}$ for some $\alpha_k, \beta_k \in L^*$, and some m_k powers of p_i for $1 \le k \le \ell_i$ $\left(\text{so } \Gamma_{T_i}/\Gamma_L = \overset{\ell_i}{\underset{k=1}{\oplus}} \left\langle \frac{1}{m_k} v(\alpha_k) + \Gamma_L, \frac{1}{m_k} v(\beta_k) + \Gamma_L \right\rangle\right)$. Since $\Gamma_L = \Gamma_F$, $T_i \sim \left(\overset{\ell_i}{\underset{k=1}{\otimes}} (a_k, b_k; L)_{m_k}\right) \otimes_F S_i$ for some $S_i \in \mathcal{D}_{is}(L)$ and $a_k, b_k \in F^*$ with $v(a_k) = v(\alpha_k)$, $v(b_k) = v(\beta_k)$ $(1 \le k \le \ell_i)$. Let $T_i' \sim \overset{\ell_i}{\underset{k=1}{\otimes}} (a_k, b_k; L)_{m_k}$. As $T_i \in \mathcal{D}_{ttr}(L)$, Prop. 3.1 shows $T_i' \in \mathcal{D}_{ttr}(L)$, and $\Gamma_{T_i'} = \Gamma_{T_i}$. Since $T_i \sim T_i' \otimes_L S_i$ where $T_i' \in \mathcal{D}_{ttr}(L)$, and $S_i \in \mathcal{D}_{is}(L)$, by Prop. 1.7, $\Gamma_{T_i} = \Gamma_{T_i'} + \Gamma_{S_i}$. Also, as L/F is inertial, $^cS_i \in \mathcal{D}_{is}(F)$ and $\Gamma_{^cS_i} \subseteq \Gamma_{S_i}$ by Theorem 2.4.

For $1 \le i \le r_0$, as $\mu_{p_i} \subseteq F$, by Prop. 4.1 $^cT_i' \sim \overset{\ell_i}{\underset{k=1}{\otimes}} (a_k, b_k^{n_k}; F)_{m_k/r_k}$ where $n_k = [L : F(\omega_{m_k})]$ (ω_{m_k}, a primitive m_k-th root of unity) and $r_k = [F(\omega_{m_k}) : F]$. Let A_k be the underlying division algebra of $(a_k, b_k^{n_k}; F)_{m_k/r_k}$. As $(a_k, b_k; F)_{m_k/r_k} \in \mathcal{D}_{ttr}(F)$, by Prop. 3.1, $A_k \in \mathcal{D}_{ttr}(F)$ and

$$\Gamma_{A_k}/\Gamma_F = n_k \left(\left\langle \frac{r_k}{m_k} v(a_k) + \Gamma_F, \frac{r_k}{m_k} v(b_k) + \Gamma_F \right\rangle\right)$$

$$= [L : F] \left(\left\langle \frac{1}{m_k} v(a_k) + \Gamma_F, \frac{1}{m_k} v(b_k) + \Gamma_F \right\rangle\right)$$

as $n_k r_k = [L : F]$. Then, by Prop. 3.1 again, $A := \overset{\ell_i}{\underset{k=1}{\otimes}} A_k \in \mathcal{D}_{ttr}(F)$ and $\Gamma_A/\Gamma_F = \overset{\ell_i}{\underset{k=1}{\oplus}} (\Gamma_{A_k}/\Gamma_F) = [L : F](\Gamma_{T_i'}/\Gamma_F)$. So, $^cT_i' = A \in \mathcal{D}_{ttr}(F)$ and $\Gamma_{^cT_i'}/\Gamma_F = [L : F](\Gamma_{T_i'}/\Gamma_F) = [L : F](\Gamma_{T_i}/\Gamma_F)$. Since $^cT_i \sim {}^cT_i' \otimes_F {}^cS_i$, $\Gamma_{^cT_i} = \Gamma_{^cT_i'} + \Gamma_{^cS_i}$ by Prop. 1.7. So,

$$[L : F]\Gamma_{T_i} + \Gamma_F \subseteq \Gamma_{^cT_i} = \Gamma_{^cT_i'} + \Gamma_{^cS_i} \subseteq \Gamma_{T_i'} + \Gamma_{S_i} = \Gamma_{T_i}.$$

For $r_0 + 1 \le i \le r$, as $\mu_{p_i} \nsubseteq F$ by assumption, $^cT_i' = F$ by Prop. 4.1. So $^cT_i = {}^cS_i$, and $\Gamma_{^cT_i} = \Gamma_{^cS_i} \subseteq \Gamma_{S_i} \subseteq \Gamma_{T_i}$.

(a) If $4 \nmid \exp(\Gamma_T/\Gamma_L)$ or $\mu_4 \subseteq F$, then the same argument proving of the case when $1 \le i \le r_0$ gives us $[L : F]\Gamma_{T_0} + \Gamma_F \subseteq \Gamma_{^cT_0} \subseteq \Gamma_{T_0}$, using Prop. 4.2 and Lemma 4.3 instead of Prop. 4.1. Thus, since $\Gamma_T = \sum_{i=0}^r \Gamma_{T_i}$ and $\Gamma_{^cT} = \sum_{i=0}^r \Gamma_{^cT_i}$ by [**JW2**, Cor. 1.15], $[L : F](\Gamma_{T_0} + \Gamma_{T_1} + \cdots + \Gamma_{T_{r_0}}) + \Gamma_F \subseteq \Gamma_{^cT} \subseteq \Gamma_T$.

(b) If $4 \mid \exp(\Gamma_T/\Gamma_L)$ and $\mu_4 \nsubseteq F$, then by Prop. 4.2 and Lemma 4.3, $^cT_0' \sim \overset{\ell_0}{\underset{k=1}{\otimes}} A_k$, where for $1 \le k \le \ell_0$, $A_k = (a_k, b_k; F)_2$ or $= F$. Hence $^cT_0' \in \mathcal{D}_{ttr}(F)$ and $\Gamma_{^cT_0'} \subseteq \Gamma_{T_0'}$. As $^cT_0 \sim {}^cT_0' \otimes {}^cS_0$ with $^cS_0 \in \mathcal{D}_{is}(F)$, by Prop. 1.7, $\Gamma_{^cT_0} = \Gamma_{^cT_0'} + \Gamma_{^cS_0} \subseteq \Gamma_{T_0'} + \Gamma_{S_0} = \Gamma_{T_0}$. Therefore, $[L : F](\Gamma_{T_1} + \cdots + \Gamma_{T_{r_0}}) + \Gamma_F \subseteq \Gamma_{^cT} \subseteq \Gamma_T$. $\square$

Theorem 4.5. *If $D \in \mathcal{D}_t(L)$, then $\Gamma_{^cD} \subseteq \Gamma_D$.*

Proof. By Prop. 1.7, there are $S \in \mathcal{D}_{is}(L)$ and $T \in \mathcal{D}_{ttr}(L)$ such that $D \sim S \otimes_L T$ in Br (L), whence $\Gamma_D = \Gamma_S + \Gamma_T$. Then by Theorem 2.4 and Theorem 4.4, $\Gamma_{^cS} \subseteq \Gamma_S$ and $\Gamma_{^cT} \subseteq \Gamma_T$. Also, by Theorem 2.4, ^cS and ^cT are tame over F. As $^cD \sim {}^cS \otimes_F {}^cT$ in Br (F), by [**JW2**, Cor. 6.7], $\Gamma_{^cD} \subseteq \Gamma_{^cS} + \Gamma_{^cT} \subseteq \Gamma_S + \Gamma_T = \Gamma_D$. $\qquad\square$

Now, we will give relations between $Z(\overline{D})$ and $Z(\overline{{}^cD})$ and between $\exp(\ker \theta_D)$ and $\exp(\ker \theta_{^cD})$ for $D \in \mathcal{D}_t(L)$.

Theorem 4.6. *Let $D \in \mathcal{D}_t(L)$ and $t = \exp(\ker \theta_D)$. Let $t = 2^{e_0}p_1^{e_1} \cdots p_r^{e_r}$, where $e_0 \geq 0$, and the p_i are distinct odd primes and $e_i > 0$ for $1 \leq i \leq r$. Suppose $\mu_{p_i} \subseteq F$ for $1 \leq i \leq r_0$, and $\mu_{p_i} \nsubseteq F$ for $r_0 + 1 \leq i \leq r$. Let $s = 2^{e_0}p_1^{e_1} \cdots p_{r_0}^{e_{r_0}}$ and $s' = p_1^{e_1} \cdots p_{r_0}^{e_{r_0}}$. Then*

(a) if $4 \nmid t$ or $\mu_4 \subseteq F$, then $\exp(\ker \theta_{^cD}) = s/\gcd([L:F], s)$ and $Z(\overline{{}^cD}) \subseteq \mathcal{N}(Z(\overline{D})/\overline{F})^{1/n}$ where $n = \gcd([L:F], s) \cdot (t/s)$, and

(b) if $4 \mid t$ and $\mu_4 \nsubseteq F$, then $\exp(\ker \theta_{^cD}) = \varepsilon s'/\gcd([L:F], s')$ where $\varepsilon = 1$ or 2, and $Z(\overline{{}^cD}) \subseteq \mathcal{N}(Z(\overline{D})/\overline{F})^{1/n'}$ where $n' = \gcd([L:F], s') \cdot (t/s')$.

Proof. By Prop. 1.7, there is a noncanonical decomposition in Br (L), $D \sim S' \otimes_L T'$ where $S' \in \mathcal{D}_{is}(L)$ and $T' \in \mathcal{D}_{ttr}(L)$. As $\ker(\theta_D) = \Gamma_{T'}/\Gamma_L$, we have $t = \exp(\ker \theta_D) = \exp(\Gamma_{T'}/\Gamma_L)$. Let $T' = \overset{r}{\underset{i=0}{\otimes}} T_i'$ be the primary decomposition of T'. Since $T_i' \in \mathcal{D}_{ttr}(L)$, by Prop. 1.6, T_i' is a tensor product of tame totally ramified symbol algebras

$$T_i' = \overset{j(i)}{\underset{j=1}{\otimes}} (x_{(i,2j-1)}, x_{(i,2j)}; L)_{t(i,j)},$$

where

$$\Gamma_{T_i'}/\Gamma_L = \overset{j(i)}{\underset{j=1}{\oplus}} \left| \left\langle \frac{1}{t(i,j)} v(x_{(i,2j-1)}) + \Gamma_L, \; \frac{1}{t(i,j)} v(x_{(i,2j)}) + \Gamma_L \right\rangle \right|$$

(cf. Prop. 3.1). Note that as

$$\exp(\Gamma_{T_i'}/\Gamma_L) = \quad \text{the } p_i\text{-component of} \quad \exp(\Gamma_{T'}/\Gamma_L) = p_i^{e_i},$$

each $t(i,j) \mid p_i^{e_i}$. (For $n \in \mathbb{Z}$ and p a prime, if we write $n = n_p \cdot n_p'$ where n_p is a power of p and n_p' is prime to p, then we refer to n_p as the p-component of n.)

Since $\Gamma_L = \Gamma_F$, $x_{(i,j')} = u_{(i,j')}y_{(i,j')}$, $1 \leq j' \leq 2j(i)$, for some $y_{(i,j')} \in F^*$ and $u_{(i,j')}$ v-units in L^*. Note that

$$
\begin{aligned}
&\left(x_{(i,2j-1)}, x_{(i,2j)}; L\right)_{t(i,j)} \\
&\sim \left(u_{(i,2j-1)}, x_{(i,2j)}; L\right)_{t(i,j)} \otimes \left(y_{(i,2j-1)}, u_{(i,2j)}; L\right)_{t(i,j)} \otimes \left(y_{(i,2j-1)}, y_{(i,2j)}; L\right)_{t(i,j)},
\end{aligned}
$$

and

$$
\left[\left(u_{(i,2j-1)}, x_{(i,2j)}; L\right)_{t(i,j)} \otimes \left(y_{(i,2j-1)}, u_{(i,2j)}; L\right)_{t(i,j)}\right] \in \mathrm{Br}\left(L_{nr}/L\right).
$$

Let S_i be the underlying division algebra of

$$
\overset{j(i)}{\underset{j=1}{\otimes}} \left[\left(u_{(i,2j-1)}, x_{(i,2j)}; L\right)_{t(i,j)} \otimes \left(y_{(i,2j-1)}, u_{(i,2j)}; L\right)_{t(i,j)}\right]
$$

and $T_i = \overset{j(i)}{\underset{j=1}{\otimes}} \left(y_{(i,2j-1)}, y_{(i,2j)}; L\right)_{t(i,j)}$. It follows from Prop. 3.1, that $T := \overset{r}{\underset{i=0}{\otimes}} T_i \in \mathcal{D}_{ttr}(L)$. Then by replacing S' and T' with the underlying division algebra of $S' \otimes \left(\overset{r}{\underset{i=0}{\otimes}} S_i\right)$ and T, we have in $\mathrm{Br}(L)$, $D \sim S \otimes_L T$ where $S \in \mathcal{D}_{is}(L)$, $T = \overset{r}{\underset{i=0}{\otimes}} T_i$ and $T_i = \overset{j(i)}{\underset{j=1}{\otimes}} \left(y_{(i,2j-1)}, y_{(i,2j)}; L\right)_{t(i,j)}$ with $y_{(i,j')} \in F^*$. And $^cD \sim {}^cS \otimes_F {}^cT$ where $^cS \in \mathcal{D}_{is}(F)$ by Theorem 2.4.

(a) Suppose $4 \nmid t$ or $\mu_4 \subseteq F$.

By Prop. 4.1 (a) for $1 \leq i \leq r_0$, and by Prop. 4.2 (a) and Lemma 4.3 for $i = 0$, $^cT_i \sim \overset{j(i)}{\underset{j=1}{\otimes}} \left(y_{(i,2j-1)}, y_{(i,2j)}^{\ell(i,j)}; F\right)_{t(i,j)/r(i,j)}$ where $\ell(0,j) = f_j \cdot [L: F(\omega_{t(0,j)})]$ with f_j an odd integer, $\ell(i,j) = [L: F(\omega_{t(i,j)})]$ for $1 \leq i \leq r_0$ and $r(i,j) = [F(\omega_{t(i,j)}) : F]$ for $0 \leq i \leq r_0$ ($\omega_{t(i,j)}$ is a primitive $t(i,j)$-th root of unity). Let $d(i,j) = \gcd\left(\ell(i,j), \frac{t(i,j)}{r(i,j)}\right)$, $\ell'(i,j) = \frac{\ell(i,j)}{d(i,j)}$, and $t'(i,j) = \frac{t(i,j)}{r(i,j)d(i,j)} = \frac{t(i,j)}{\gcd([L:F],t(i,j))}$ so that $\gcd\left(\ell'(i,j), t'(i,j)\right) = 1$. Then $r(i,j)d(i,j) = \gcd\left([L:F], t(i,j)\right)$ as $r(i,j)\ell(i,j) = [L:F]$ for $1 \leq i \leq r_0$ and $r(0,j)\ell(0,j) = [L:F]f_j$, $2 \nmid f_j$, and $t(0,j)$ are powers of 2. Then $^cT_i \sim \overset{j(i)}{\underset{j=1}{\otimes}} \left(y_{(i,2j-1)}, y_{(i,2j)}^{\ell'(i,j)}; F\right)_{t'(i,j)}$.

For $r_0 + 1 \leq i \leq r$, as $\mu_{p_i} \nsubseteq F$, $^cT_i = F$ by Prop. 4.1. So $^cT \sim \overset{r_0}{\underset{i=0}{\otimes}} {}^cT_i$ in $\mathrm{Br}(F)$. Since

$$
\begin{aligned}
\Gamma_T/\Gamma_L = \Gamma_T/\Gamma_F &= \overset{r}{\underset{i=0}{\oplus}} \left(\Gamma_{T_i}/\Gamma_F\right) = \\
&= \overset{r}{\underset{i=0}{\oplus}} \overset{j(i)}{\underset{j=1}{\oplus}} \left\langle \frac{1}{t(i,j)}v(y_{(i,2j-1)}) + \Gamma_F, \frac{1}{t(i,j)}v(y_{(i,2j)}) + \Gamma_F \right\rangle
\end{aligned}
$$

and $\Gamma_{cT_i}/\Gamma_F \subseteq \Gamma_{T_i}/\Gamma_F$, it follows from Prop. 2.4 that $^cT \cong \overset{r_0}{\underset{i=0}{\otimes}} {}^cT_i \in \mathcal{D}_{ttr}(F)$ and

$$\Gamma_{cT}/\Gamma_F = \overset{r_0}{\underset{i=0}{\oplus}} (\Gamma_{cT_i}/\Gamma_F) =$$
$$= \overset{r_0}{\underset{i=0}{\oplus}} \overset{j(i)}{\underset{j=1}{\oplus}} \left\langle \frac{1}{t'(i,j)} v(y_{(i,2j-1)}) + \Gamma_F, \frac{1}{t'(i,j)} v(y_{(i,2j)}) + \Gamma_F \right\rangle,$$

as $\gcd(\ell'(i,j), t'(i,j)) = 1$.

Let $T' = \overset{r_0}{\underset{i=0}{\otimes}} T_i$. Then

$$\Gamma_{T'}/\Gamma_L = \overset{r_0}{\underset{i=0}{\oplus}} (\Gamma_{T_i}/\Gamma_L)$$
$$= \overset{r_0}{\underset{i=0}{\oplus}} \overset{j(i)}{\underset{j=1}{\oplus}} \left\langle \frac{1}{t(i,j)} v(y_{(i,2j-1)}) + \Gamma_L, \frac{1}{t(i,j)} v(y_{(i,2j)}) + \Gamma_L \right\rangle.$$

Since $\Gamma_L = \Gamma_F$ and

$$\frac{t'(i,j) = t(i,j)}{\gcd([L:F], t(i,j)), \Gamma_{cT}/\Gamma_F = [L:F](\Gamma_{T'}/\Gamma_L)}.$$

Recall that $s = 2^{e_0} p_1^{e_1} \cdots p_{r_0}^{e_{r_0}} = \exp(\Gamma_{T'}/\Gamma_L)$, as $t = 2^{e_0} p_1^{e_1} \cdots p_r^{e_r} = \exp(\Gamma_T/\Gamma_L)$. Note that $(t/s)\Gamma_T \subseteq \Gamma_{T'}$, as $\exp(\Gamma_{T_i}/\Gamma_L) \mid (t/s)$ for $i > r_0$. Also, for $0 \le i \le r_0$, as each $t(i,j) \mid s$, each $t(i,j)/t'(i,j) = \gcd([L:F], t(i,j)) \mid \gcd([L:F], s)$. Therefore, $\gcd([L:F], s)\Gamma_{T'} \subseteq \Gamma_{cT}$. Hence, as $n = \gcd([L:F], s) \cdot (t/s)$, we have $n\Gamma_T \subseteq \Gamma_{cT}$.

Note that if A is a finite abelian group (written additively) and ℓ is a positive integer, then

$$(4.7) \qquad\qquad \exp(\ell \cdot A) = \frac{\exp(A)}{\gcd(\ell, \exp(A))}.$$

(This follows from the fact that there is $b \in A$ with $o(b) = \exp(A)$ and the formula $o(\ell \cdot b) = o(b)/\gcd(\ell, o(b))$.) Since $\Gamma_{cT}/\Gamma_F = [L:F](\Gamma_{T'}/\Gamma_L)$ and $\Gamma_{T'}/\Gamma_L$ is a finite abelian group,

$$\exp(\Gamma_{cT}/\Gamma_F) = \frac{\exp(\Gamma_{T'}/\Gamma_L)}{\gcd([L:F], \exp(\Gamma_{T'}/\Gamma_L))} = s/\gcd([L:F], s)$$

by (4.7) above. Since $^cD \sim {}^cS \otimes_F {}^cT$ with $^cS \in \mathcal{D}_{is}(F)$ and $^cT \in \mathcal{D}_{ttr}(F)$, by Prop. 1.7 $\exp(\ker \theta_{cD}) = \exp(\Gamma_{cT}/\Gamma_F) = s/\gcd([L:F], s)$.

Let

$$K = F\left(\left\{ y_{(i,2j-1)}^{1/t'(i,j)}, y_{(i,2j)}^{1/t'(i,j)} \right\}_{i=0 \ j=1}^{r_0 \ j(i)} \right).$$

Then K/F is totally ramified and $\Gamma_K = \Gamma_{^cT}$. Since $^cS \in \mathcal{D}_{is}(F)$ and K/F is totally ramified, by [**JW2**, Cor. 5.13] $Z(\overline{(^cS)_K}) = \mathcal{F}(\theta_{^cS}((\Gamma_{^cS} \cap \Gamma_K)/\Gamma_F))$, where $(^cS)_K$ is the underlying division algebra of $^cS \otimes_F K$. Also since $^cD \sim {}^cS \otimes_F {}^cT$ with $^cS \in \mathcal{D}_{is}(F)$ and $^cT \in \mathcal{D}_{ttr}(F)$, by Prop. 1.7 $Z(\overline{{}^cD}) = \mathcal{F}(\theta_{^cS}((\Gamma_{^cS} \cap \Gamma_{^cT})/\Gamma_F))$. Hence $Z(\overline{{}^cD}) = Z(\overline{(^cS)_K})$ as $\Gamma_{^cT} = \Gamma_K$.

Now, since K/F is Galois and $K \cap L = F$ with K/F totally ramified and L/F inertial, L and K are linearly disjoint over F and $L \otimes_F K$ is the field $L \cdot K$. So by [**D1**, p. 56, Ex. 1], $(^cS)_K \sim \mathrm{cor}_{LK/K}(S_{LK})$ in Br (K), where S_{LK} is the underlying division algebra of $S \otimes_L LK$. Since $S_{LK} \in \mathcal{D}_{is}(LK)$ and LK/K is inertial, by Theorem 2.4,

$$Z(\overline{{}^cD}) = Z(\overline{(^cS)_K}) \subseteq \mathcal{N}(Z(\overline{S_{LK}})/\overline{K}). \tag{4.8}$$

Recall that $n = \gcd([L:F], s) \cdot (t/s)$. We showed just before (4.7) that $n\Gamma_T \subseteq \Gamma_{^cT} = \Gamma_K = \Gamma_{LK}$. So, $n((\Gamma_S \cap \Gamma_T)/(\Gamma_S \cap \Gamma_{LK})) = 0$.

Let
$$\theta_S\colon \Gamma_S/\Gamma_L \to \mathrm{Gal}\,(Z(\overline{S})/\overline{L})$$
be the map of (1.5). Then by [**JW2**, Lemma 5.1], θ_S is an isomorphism and $Z(\overline{S})/\overline{L}$ is abelian Galois as $S \in \mathcal{D}_{is}(L)$. By Prop. 1.7, $\theta_S((\Gamma_S \cap \Gamma_T)/\Gamma_L) = \mathrm{Gal}\,(Z(\overline{S})/Z(\overline{D}))$. Also, since $S \in \mathcal{D}_{is}(L)$ and LK/L is totally ramified, by [**JW2**, Cor. 5.13] again $\theta_S((\Gamma_S \cap \Gamma_{LK})/\Gamma_L) = \mathrm{Gal}\,(Z(\overline{S})/Z(\overline{S_{LK}}))$. So

$$\mathrm{Gal}\,(Z(\overline{S_{LK}})/Z(\overline{D})) \cong \mathrm{Gal}\,(Z(\overline{S})/Z(\overline{D}))/\mathrm{Gal}\,(Z(\overline{S})/Z(\overline{S_{LK}}))$$
$$\cong (\Gamma_S \cap \Gamma_T)/(\Gamma_S \cap \Gamma_{LK}).$$

Hence $\mathrm{Gal}\,(Z(\overline{S_{LK}})/Z(\overline{D}))$ is an abelian n-torsion group by the preceding paragraph. Note that $\mu_n \subseteq Z(\overline{D})$ as $n \mid t = \exp(\Gamma_T/\Gamma_L)$, and $\mu_t \subseteq \overline{L} \subseteq Z(\overline{D})$. Hence by Kummer theory $Z(\overline{S_{LK}})$ is an n-Kummer extension of $Z(\overline{D})$, i.e., $Z(\overline{S_{LK}}) = Z(\overline{D})\,(\{\alpha_m^{1/n}\}_{m=1}^s)$ for some $\alpha_m \in Z(\overline{D})$. Therefore,

$$(4.9) \qquad\qquad Z(\overline{S_{LK}}) \subseteq Z(\overline{D})^{1/n}.$$

Since $\overline{K} = \overline{F}$, by (4.8) and (4.9) above

$$Z(\overline{D}) \subseteq \mathcal{N}(Z(\overline{D})^{1/n}/\overline{F}) \subseteq \mathcal{N}(Z(\overline{D})/\overline{F})^{1/n}.$$

(b) Suppose $4 \mid t$ and $\mu_4 \not\subseteq F$.

By Prop. 4.1, for $1 \leq i \leq r_0$, as shown in proof of (a)

$$^cT_i \sim \overset{j(i)}{\underset{j=1}{\otimes}}\left(y_{(i,2j-1)}, y_{(i,2j)}^{\ell(i,j)}; F\right)_{t(i,j)/r(i,j)}$$

where $\ell(i,j) = [L: F(\omega_{t(i,j)})]$ and $r(i,j) = [F(\omega_{t(i,j)}): F]$, and for $r_0 + 1 \leq i \leq r$, $^cT_i = F$. Also, by Prop. 4.4 and Lemma 4.3, $^cT_0 \sim \overset{j(0)}{\underset{j=1}{\otimes}} A_j$ where

$A_j = (y_{(0,2j-1)}, y_{(0,2j)}; F)_2$ or $= F$. So we have $^cT \sim \left(\overset{j(0)}{\underset{j=1}{\otimes}} A_j \right) \otimes_F \left(\overset{r_0}{\underset{i=1}{\otimes}} {}^cT_i \right)$.

Then by the exact same arguments as in proof of (a), $^cT \cong \left(\overset{j(0)}{\underset{j=1}{\oplus}} A_j \right) \otimes_F$

$\left(\overset{r_0}{\underset{i=1}{\otimes}} {}^cT_i \right) \in \mathcal{D}_{ttr}(F)$ and

$$\Gamma_{^cT} / \Gamma_F = \left(\overset{j(0)}{\underset{j=1}{\oplus}} (\Gamma_{A_j} / \Gamma_F) \right) \oplus \left(\overset{r_0}{\underset{i=1}{\oplus}} (\Gamma_{^cT_i} / \Gamma_F) \right)$$

$$= \left(\overset{j(0)}{\underset{j=0}{\oplus}} (\Gamma_{A_J} / \Gamma_F) \right) \oplus [L:F](\Gamma_{T''} / \Gamma_L)$$

where $T'' = \overset{r_0}{\underset{i=1}{\otimes}} T_i$. So $\exp (\Gamma_{^cT} / \Gamma_F) = \varepsilon \exp ([L:F](\Gamma_{T''} / \Gamma_L))$ where $\varepsilon = 1$ or 2. But by (4.7),

$$\exp ([L:F](\Gamma_{T''} / \Gamma_L)) = \frac{\exp (\Gamma_{T''} / \Gamma_L)}{\gcd ([L:F], \exp (\Gamma_{T''} / \Gamma_L))}$$

$$= \frac{s'}{\gcd ([L:F], s')} \cdot$$

Hence $\exp (\Gamma_{^cT} / \Gamma_F) = \varepsilon s' / \gcd ([L:F], s')$.

As in (a), let K' be a totally ramified extension of F with $\Gamma_{K'} = \Gamma_{^cT}$. Let $n' = \gcd ([L:F], s') \cdot (t/s')$. Since $t(0,j) \mid 2^{e_0}$, we have $t(0,j) \mid n'$. For $i > 0$, we have, just as in (a), $t(i,j) \mid t'(i,j)n'$ if $i \leq r_0$ and $t(i,j) \mid n'$ if $i > r_0$. These divisibility relations show $n'\Gamma_T \subseteq \Gamma_{LK'}$, hence $Z(^c\overline{D}) \subseteq \mathcal{N}(Z(\overline{D})/\overline{F})^{1/n'}$, by just the same argument as for (a). $\square$

Theorem 4.6 gives us the best general relation one can expect between $Z(\overline{D})$ and $Z(^c\overline{D})$ for $D \in \mathcal{D}_t(L)$, L/F inertial, as the following example illustrates.

Example 4.10. Let r_1, r_2, and t be positive integers with $t \mid [L:F]$. Let x_1, $x_2 \in F^*$ such that for any prime p dividing r_1, r_2 or t, we have $v(x_1) + p\Gamma_F$, $v(x_2) + p\Gamma_F$ are $\mathbb{Z}_p$-linearly independent in $\Gamma_F / p\Gamma_F = \Gamma_L / p\Gamma_L$. Suppose $\mu_\ell \subseteq F$, where $\ell = \ell cm(r_1, r_2, t)$. Then if D is the underlying division algebra of $(x_1, u_1; L)_{r_1} \otimes_L (x_2, u_2; L)_{r_2} \otimes (x_1, x_2; L)_t$, where u_1 and u_2 are units in L^*, then by Prop. 1.4 and as $t \mid [L:F]$, in Br (F)

$$^cD \sim \left(x_1, N_{L/F}(u_1); F \right)_{r_1} \otimes_F p \left(x_1, N_{L/F}(u_2); F \right)_{r_2}$$

$$\otimes_F \left(x_1, x_2^{[L:F]}; F \right)_t$$

$$\sim \left(x_1, N_{L/F}(u_1); F \right)_{r_1} \otimes_F \left(x_2, N_{L/F}(u_2); F \right)_{r_2} .$$

So, by Cor. 3.5, $Z(\overline{D}) = \overline{L}(\bar{u}_1^{t/r_1}, \bar{u}_2^{t/r_2})$, and $Z(^c\overline{D}) = \overline{F}((N_{\overline{L}/\overline{F}}(\bar{u}_1))^{1/r_1}$, $(N_{\overline{L}/\overline{F}}(\bar{u}_2))^{1/r_2})$. Hence $Z(^c\overline{D}) \subseteq \mathcal{N}(Z(\overline{D})\,/\,\overline{F})^{1/t}$, where $t = \gcd(t, [L\colon F])$ as $t \mid [L\colon F]$.

Appendix.

In [**Ta**, p. 259], Tate gave a formula describing the homological corestriction using a set of representatives of the right cosets. From this, we will deduce a formula stated in Section 1 describing the homological corestriction with respect to a set of representatives of the left cosets. Presumably, this formula is known, but we have been unable to find it in the literature, except for $m \le 2$. Here we use the notation $\overline{\sigma\rho_i}$ for $\rho_{\sigma_*(i)}$ in (1.3) above.

Let G be a group and A a left G-module. Let H be a subgroup of G of finite index n, and let $\mathcal{R} = \{\alpha_1, \dots, \alpha_n\}$ be a set of representatives of the right cosets of H in G. For $\tau \in G$, we write $\bar{\tau} \in R$ for the representative of $H\tau$. Given $\alpha_i \in \mathcal{R}$ and $\sigma \in G$, there are uniquely determined elements $\varepsilon(\alpha_i, \sigma) \in H$ and $\overline{\alpha_i \sigma} \in \mathcal{R}$ such that

$$(A1) \qquad\qquad \alpha_i \sigma = \varepsilon(\alpha_i, \sigma)\overline{\alpha_i \sigma}\,.$$

Then we have, for any σ, $\tau \in G$ and $\alpha_i \in \mathcal{R}$,

$$(A2) \qquad \varepsilon(\alpha_i, 1) = 1, \quad \text{and} \quad \varepsilon(\alpha_i, \sigma\tau) = \varepsilon(\alpha_i, \sigma)\varepsilon(\overline{\alpha_i \sigma}, \tau)\,.$$

By [**Ta**, p. 259], the homological corestriction from $H^m(H, A)$ to $H^m(G, A)$ ($m \ge 1$) can be described as follows: It is the homomorphism cor_H^G of the cohomology groups induced by the cochain transformation of $C^m(H, A)$ into $C^m(G, A)$, for a cochain $f \in C^m(H, A)$, given by

$$(A3) \quad \mathrm{cor}_H^G(f)(\sigma_1, \dots, \sigma_m) =$$
$$= \sum_{i=1}^n \alpha_i^{-1}[f(\varepsilon(\alpha_i, \sigma_1), \varepsilon(\overline{\alpha_i \sigma_1}, \sigma_2), \dots, \varepsilon(\overline{\alpha_i \sigma_1 \cdots \sigma_{j-1}}, \sigma_j),$$
$$\dots, \varepsilon(\overline{\alpha_i \sigma_1 \cdots \sigma_{m-1}}, \sigma_m))]\,.$$

Let $\rho_i = \alpha_i^{-1}$ for $1 \le i \le n$, so $\mathcal{L} = \{\rho_1, \dots, \rho_n\}$ is a set of representatives of the left cosets of H in G. For $\tau \in G$, we write $\bar{\tau} \in \mathcal{L}$ for the representative of τH in $\mathcal{L}$. (It will be clear from the context when $\bar{\tau}$ stands for a left coset representative and when for a right coset representative.) Given $\rho_i \in \mathcal{L}$ and $\sigma \in G$, there are uniquely determined elements $\overline{\sigma\rho_i} \in \mathcal{L}$ and $\delta(\sigma, \rho_i) \in H$ such that

$$(A4) \qquad\qquad \sigma\rho_i = \overline{\sigma\rho_i}\delta(\sigma, \rho_i)\,.$$

Then, since $\overline{\sigma\tau\rho_i}\delta(\sigma\tau,\rho_i) = \overline{\sigma\overline{\tau\rho_i}}\delta(\sigma,\overline{\tau\rho_i})\delta(\tau,\rho_i)$ for any $\sigma,\tau \in G$, and $\rho_i \in \mathcal{L}$, we have

(A5) $\overline{\sigma\tau\rho_i} = \overline{\sigma\overline{\tau\rho_i}}$,   and $\delta(\sigma\tau,\rho_i) = \delta(\sigma,\overline{\tau\rho_i})\delta(\tau,\rho_i)$.

Since $1 = \delta(\sigma\sigma^{-1},\rho_i) = \delta(\sigma,\overline{\sigma^{-1}\rho_i})\delta(\sigma^{-1},\rho_i)$, we have also

(A6) $\delta(\sigma^{-1},\rho_i)^{-1} = \delta(\sigma,\overline{\sigma^{-1}\rho_i})$ for any $\sigma \in G$ and $\rho_i \in \mathcal{L}$.

By taking the inverses in the both sides of (A1), we have

$$\sigma^{-1}\alpha_i^{-1} = \overline{\alpha_i\sigma}^{-1}\varepsilon(\alpha_i,\sigma)^{-1}$$

where $\overline{\alpha_i\sigma}^{-1} \in \mathcal{L}$ and $\varepsilon(\alpha_i,\sigma)^{-1} \in H$. But $\sigma^{-1}\alpha_i^{-1} = \overline{\sigma^{-1}\rho_i}\delta(\sigma^{-1},\rho_i)$ where $\overline{\sigma^{-1}\rho_i} \in \mathcal{L}$ and $\delta(\sigma^{-1},\rho_i) \in H$ are uniquely determined elements as in (A4). Hence we have

(A7) $\overline{\alpha_i\sigma}^{-1} = \overline{\sigma^{-1}\rho_i}$,

and

(A8) $\varepsilon(\alpha_i,\sigma) = \delta(\sigma^{-1},\rho_i)^{-1} = \delta(\sigma^{-1},\alpha_i^{-1})^{-1}$.

Now by using successively (A3), (A8), (A7), (A5) and (A6), the cochain transformation (A3) becomes;

$$\mathrm{cor}_H^G(f)(\sigma_1,\ldots,\sigma_m) =$$

$$= \sum_{i=1}^{n} \alpha_i^{-1} [f(\varepsilon(\alpha_i,\sigma_1), \varepsilon(\overline{\alpha_i\sigma_1},\sigma_2),$$

$$\ldots, \varepsilon(\overline{\alpha_i\sigma_1\cdots\sigma_{j-1}},\sigma_j), \ldots, \varepsilon(\overline{\alpha_i\sigma_1\cdots\sigma_{m-1}},\sigma_m))]$$

$$= \sum_{i=1}^{n} \alpha_i^{-1} [f(\delta(\sigma_1^{-1},\alpha_i^{-1})^{-1}, \delta(\sigma_2^{-1},\overline{\alpha_i\sigma_1}^{-1})^{-1},$$

$$\ldots \delta(\sigma_j^{-1}, \overline{\alpha_i\sigma_1\cdots\sigma_{j-1}}^{-1})^{-1},$$

$$\ldots, \delta(\sigma_m^{-1}, \overline{\alpha_i\sigma_1\cdots\sigma_{m-1}}^{-1})^{-1})]$$

$$= \sum_{i=1}^{n} \rho_i [f(\delta(\sigma_1^{-1},\rho_i)^{-1}, \delta(\sigma_2^{-1},\overline{\sigma_1^{-1}\rho_i})^{-1},$$

$$\ldots, \delta(\sigma_j^{-1}, \overline{(\sigma_1\cdots\sigma_{j-1})^{-1}\rho_i})^{-1},$$

$$\ldots, \delta(\sigma_m^{-1}, \overline{(\sigma_1\cdots\sigma_{m-1})^{-1}\rho_i})^{-1})]$$

$$= \sum_{i=1}^{n} \rho_i \left[f(\delta(\sigma_1,\overline{\sigma_1^{-1}\rho_i}), \delta(\sigma_2,\overline{\sigma_2^{-1}\sigma_1^{-1}\rho_i}),\right.$$

$$\left.\ldots, \delta(\sigma_j,\overline{\sigma_j^{-1}\sigma_{j-1}^{-1}\cdots\sigma_1^{-1}\rho_i}), \ldots, \delta(\sigma_m,\overline{\sigma_m^{-1}\sigma_{m-1}^{-1}\cdots\sigma_1^{-1}\rho_i}) \right]$$

$$= \sum_{i=1}^{n} \overline{\sigma_1\cdots\sigma_m\rho_i} [f(\delta(\sigma_1,\overline{\sigma_2\cdots\sigma_m\rho_i}), \delta(\sigma_2,\overline{\sigma_3\cdots\sigma_m\rho_i}),$$

$$\ldots, \delta(\sigma_j,\overline{\sigma_{j+1}\cdots\sigma_m\rho_i}), \ldots, \delta(\sigma_m,\rho_i))],$$

by replacing ρ_i with $\overline{\sigma_1\cdots\sigma_m\rho_i}$, since $\{\overline{\sigma_1\cdots\sigma_m\rho_1}, \ldots, \overline{\sigma_1\cdots\sigma_m\rho_n}\}$ is a permutation of $\{\rho_1, \ldots, \rho_n\}$.

The homological corestriction cor_H^G from $H^m(H,A)$ to $H^m(G,A)$ $(m \geq 1)$ is the homomorphism of cohomology groups induced by the cochain transformation $C^m(H,A) \to C^m(G,A)$ just computed.

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