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NONEXISTENCE AND INSTABILITY IN THE EXTERIOR DIRICHLET PROBLEM FOR THE MINIMAL SURFACE EQUATION IN THE PLANE

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In this paper we investigate the Dirichlet problem

$$(1) \qquad Mu \equiv (1+|Du|^2)\Delta u - D_i u D_j u D_{ij} u = 0 \text{ in } \Omega$$

$$(2) \qquad u = f \text{ on } \partial\Omega$$

in a smooth domain $\Omega \subset \mathbb{R}^2$ for which $\mathbb{R}^2 \backslash \Omega$ is bounded. We sharpen previous non-existence results for this exterior Dirichlet problem by showing that even the smallness of the α-Hölder norm, $0 \leq \alpha < \frac{1}{2}$ is not enough for the classical solvability of (1) and (2), not imposing any asymptotical conditions at infinity upon possible solutions. In particular, we explicitely exhibit smooth data f of arbitrary small C^α-norm for which (1), (2) is not solvable in the space $C^0(\overline{\Omega}) \cap C^2(\Omega)$. The key idea of our proof is to replace the original problem (1), (2) on a known domain but with unknown boundary conditions at infinity by the corresponding problem on some unknown (bounded) domain, but with fixed boundary data. By the same method we show the instability of the exterior Dirichlet problem with respect to C^α-small perturbations of the boundary data, $0 \leq \alpha < \frac{1}{2}$, provided that Ω is the complement of a strictly convex set.

1. Introduction.

For a *bounded* domain $\Omega \subset \mathbb{R}^n$ it is well known that the mean-convexivity of $\partial\Omega$ is both necessary and sufficient for the unrestricted solvability of the Dirichlet problem to the n-dimensional minimal surface equation [**JS**]. In two dimensions mean-convexivity, of course, agrees with ordinary convexivity. Existence theorems for unbounded convex domains in arbitrary dimensions have been proved by Massari & Miranda [**MaMi**], excluding the case of a halfspace. The half plane was treated in a recent paper of Collin & Krust [**C-Kr**]. For general bounded, not necessarily mean convex domains one has various existence theorems for (1), (2) under a smallness condition on the data f. Already at the beginning of the century, Korn [**Ko**] and Müntz [**Mü**] applied the method of succesive approximations to (1), (2), requiring small $C^{2,\alpha}$-norm for the data f. Jenkins & Serrin proved a perturbation result for (1), (2) saying that the boundary data of a given solution of (1) may be perturbed in the C^0-norm, where, however, the admissible bound for the perturbation also depends on the C^2-norm of the perturbed boundary data. Williams [**W**] refined this result by showing that for every $K \in]0, 1/\sqrt{n-1}[$ there is an $\varepsilon = \varepsilon(K, n, \Omega) > 0$ such that (1), (2) is solvable for all f of oscillation smaller than ε and Lipschitz constant not exceeding K. He also demonstrated that the bound $1/\sqrt{n-1}$ for K is optimal if the domain is not mean-convex. This immediately implies that for such domains the Dirichlet problem is in general unsolvable even for boundary data of arbitrary small C^α-Hölder norm with $0 \leq \alpha < 1$. In the two-dimensional case such a nonexistence result has been proved by Nitsche [**N**, §411] for $0 \leq \alpha < \frac{1}{2}$.

We start the discussion of our actual subject, the *exterior* Dirichlet problem, with the fundamental observation that any solution of (1) which is defined outside some compact subset of the plane has the C^1-convergent asymptotic development

(3)
$$u(x) = c_1x_1 + c_2x_2 + c_0 + c\ln\left(|x|^2 + (c_1x_1 + c_2x_2)^2\right) + +O(|x|^{-1}\ln|x|)$$

when $|x| \to +\infty$ and with appropriate constants c_i, c. We shall consider these constants as additional unknowns of the Dirichlet problem though, in order to obtain uniqueness, c_1, c_2 and c ought to be treated as data at infinity [**C-Kr**]. Krust [**Kr**] has proved some interesting existence results for (1), (2); in particular, he was able to carry over the above mentioned perturbation theorem of Jenkins & Serrin to exterior domains. The nonexistence results of Jenkins & Serrin [**JS**], Williams [**W**], and Nitsche [**N**] for bounded, not everywhere mean-convex domains make it already very plausible that the exterior Dirichlet problem is in general unsolvable since, naturally, every exterior domain fails to be convex somewhere. Their proofs for nonexistence do, however, not apply to unbounded domains and hence, until very recently, it was undecided if the Dirichlet problem, say on the exterior of the unit disc, is always solvable or not. Osserman [**O**] constructed smooth data f on the unit circle which do not admit a *bounded* solution of (1), (2) (i.e. $c_1 = c_2 = c = 0$ in (3)) in the exterior of the disc. From one of his existence theorems Krust [**Kr**] was able to deduce that in Ossermans example there are no solutions with horizontal tangent plane at infinity, i.e. $c_1 = c_2 = 0$, c arbitrary. Kuwert [**Ku2**] recently constructed a sequence of smooth boundary data $f_j, j \in \mathbb{N}$, whose oscillation tends to infinity and showed by an indirect argument that the corresponding Dirichlet problem, - irrespective of the asymptotic behavior (3) - can only be solved for a finite number of indices j.

We do not know if it is the $C^{1/2}$–norm of the boundary data which is critical for perturbations or rather the $C^{0,1}$–norm, as suggested by the result of Williams [**W**].

Finally we want to remark that the exterior Plateau problem can be solved for quite general boundaries, allowing as solutions more general surfaces than graphs [**Ku1**, **T**, **T-Y**].

2. Non-existence.

For our comparison argument below we are going to use a special minimal surface which was already applied by Osserman in his non-existence proof. We shall therefore call it "Osserman's surface". In parametric form it is given by

$$\begin{aligned} x_1 + ix_2 &= \frac{1}{z} + \frac{1}{3}\bar{z}^3 \\ x_3 &= -2\,\mathrm{Re}\, z \end{aligned} \tag{4}$$

where $z = u + iv \in \mathbb{C}$ and (u, v) are conformal coordinates for this surface. We are interested in the part of the surface corresponding to $|z| \leq 1$. It is easily seen from (4) that this part has a non-parametric representation $x_3 = g(x_1, x_2)$ where g is defined in the closure D of the unbounded component of the complement of the Jordan curve $\frac{1}{z} + \frac{1}{3}\bar{z}^3, |z| = 1$.

This curve is shown in Figure 2.1 and is smooth except at the points $p^{\pm} := (0, \pm 2/3)$ where it has spikes.

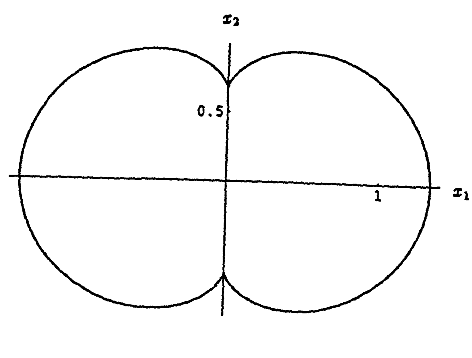

Figure 2.1.

The function g is a solution of (1) with the following properties:

$$g \in C^0(D) \cap C^\omega(\mathring{D}), \tag{5}$$

$$|g| \leq 2,\ \operatorname{sign} g(x) = -\operatorname{sign} x_1,\ g(x) \to 0 \quad \textit{for } |x| \to +\infty, \tag{6}$$

$$\frac{\partial g}{\partial \nu} = +\infty\ (x \in \partial D, x_1 > 0) \tag{7}$$

$$\frac{\partial g}{\partial \nu} = -\infty\ (x \in \partial D, x_1 < 0)$$

where ν denotes the interior (with respect to D) unit normal of $\partial D \backslash \{p^+, p^-\}$. It is essential for our method to use the homothetical images of Osserman's surface, i.e. the minimal graphs

$$x_3 = g_\varepsilon(x_1, x_2),\ g_\varepsilon := \varepsilon g\left(\frac{1}{\varepsilon} x\right),\ \varepsilon > 0.$$

Clearly g_ε is defined on $D_\varepsilon := \varepsilon D$ and has the corresponding properties (5), (6), (7), moreover $|g_\varepsilon| < 2\varepsilon$.

We can prove

Theorem 2.1. *Suppose that $\Omega \subset \mathbb{R}^2$ is a domain of class C^2 which is the complement of a compact set or which is bounded and has a boundary point where $\partial\Omega$ is negatively curved. Then there exist smooth data f with arbitrary small C^α-norm, $0 \leq \alpha < \frac{1}{2}$, for which Dirichlet's problem* (1), (2) *has no classical solution.*

Proof. Let ε be a positive parameter. We may suppose that $p_\varepsilon := (\varepsilon/2, 0) \in \partial\Omega$, that the tangent vector of $\partial\Omega$ at p_ε points in the x_2-direction and that the curvature of $\partial\Omega$ at p_ε is negative. Letting $B_\varepsilon := \mathbb{R}^2 \backslash D_\varepsilon$ we choose $\varepsilon > 0$ so small that $\partial\Omega \cap B_\varepsilon \subset \{x \in B_\varepsilon | x_1 > 0\}$, the curvature $k(x)$ of $\partial\Omega \cap B_\varepsilon$ is negative, $0 < -k(x) < 1/\varepsilon$, and the function $d(x) := \operatorname{dist}(x, \partial\Omega)$ is of class $C^2(\Omega \cap B_\varepsilon)$. Let us now divide $\partial\Omega \cap B_\varepsilon$ into four arcs $S_1, \ldots, S_4$ of equal length and let us choose the following boundary data on $\partial\Omega$, depending on ε (see Figure 2.2):

$$f_\varepsilon \equiv 0 \quad \text{on} \quad \partial\Omega \backslash B_\varepsilon,$$

$$f_\varepsilon < 0 \quad \text{in} \quad \mathring{S}_1 \cup \mathring{S}_3,\ \inf_{S_j} f_\varepsilon = -5\varepsilon - \frac{1}{\kappa} \arccos e^{-\kappa\varepsilon}, (j = 1, 3),$$

$$f_\varepsilon > 0 \quad \text{in} \quad \mathring{S}_2 \cup \mathring{S}_4,\ \sup_{S_j} f_\varepsilon = 5\varepsilon + \frac{1}{\kappa} \arccos e^{-\kappa\varepsilon}, (j = 2, 4)$$

with $\kappa = \frac{1}{2} \inf\{|k(x)| : x \in \partial\Omega \cap B_\varepsilon\}$. Since $\frac{1}{\kappa} \arccos e^{-\kappa\varepsilon} = \sqrt{1/2\kappa}\varepsilon^{1/2} + o(\varepsilon^{1/2})$ for $\varepsilon \to 0$ the data f_ε can be chosen to have arbitrary small C^α-norm,

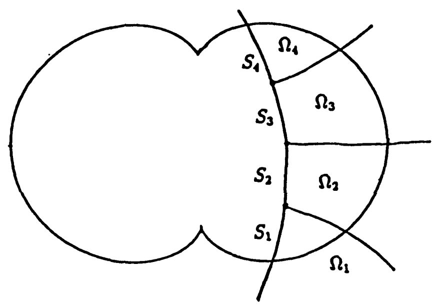

Figure 2.2.

$0 \le \alpha < 1/2$, and bounded $C^{1/2}-$norm when $\varepsilon \to 0$, but the $C^{\beta}-$norm for $\beta \in]1/2, 1]$ will blow up. Let us now suppose that problem (1), (2) with $f = f_\varepsilon$ has a solution $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$. We denote by Ω_j the connected component of the set $\{x \in \overline{\Omega} | u(x) \neq 0\}$ containing $\overset{\circ}{S}_j$, $j = 1, \dots, 4$. Here we do not exclude that some Ω_j coincide. We want to show that there exists some subdomain Ω_0 of Ω such that $\partial\Omega_0$ contains at least one of the arcs S_j, $u|\partial\Omega_0 \backslash (S_1 \cup \ldots \cup S_4) = 0$ and $u(x) \to 0$ when $|x| \to +\infty, x \in \Omega_0$. If all the constants c, c_0, c_1, c_2 in (3) are zero, then we may take $\Omega_0 = \Omega_1$. In all other cases each of the sets $\{x \in \overline{\Omega} | u(x) > 0\}$ and $\{x \in \overline{\Omega} | u(x) < 0\}$ has at most *one* unbounded component, as is readily seen from (3). Therefore, if $\Omega_1 \neq \Omega_3$ then at least *one* of these components will be bounded and we choose a bounded one as our Ω_0. If however $\Omega_1 = \Omega_3$ then there is a Jordan arc in $\Omega_1 = \Omega_3$ joining a point $p_1 \in \overset{\circ}{S}_1$ with a point $p_3 \in \overset{\circ}{S}_3$. This arc together with a suitable subarc of $\partial\Omega$ joining p_1 and p_3 encloses Ω_2 or Ω_4. Therefore, Ω_2 or Ω_4 are bounded and we define $\Omega_0 := \Omega_2$ or $\Omega_0 := \Omega_4$. In all cases u does not change sign in Ω_0 and it is therefore no restriction to assume that $u(x) \le 0$ for $x \in \Omega_0$.

We may thus replace our original problem (1), (2) with unknown data at infinity by the following problem with complete data, but on an unknown domain Ω_0:

$$
\begin{aligned}
(8) \qquad & Mu = 0 \text{ in } \overset{\circ}{\Omega}_0,\ u \le 0, \\
& u = f_\varepsilon \text{ on } \partial\Omega_0 \cap B_\varepsilon, \\
& u = 0 \text{ on } \partial\overset{\circ}{\Omega}_0 \backslash B_\varepsilon, \\
& u(x) \to 0 \ \ (|x| \to \infty, x \in \Omega_0) \text{ if } \Omega_0 \text{ is unbounded.}
\end{aligned}
$$

We shall show that problem (8) has no classical solution so that (1), (2) has no classical solution either.

Let us remark that the arguments of Jenkins & Serrin [**JS**] and Williams [**W**] for non-existence are not applicable in the present situation since we have no information on the geometry of Ω_0 away from $S_1 \cup \ldots \cup S_3$.

Our proof consists of two steps. In the first step we apply the comparison principle [**G-T**, Thm. 14.10] to the functions u and $v = g_\varepsilon - 2\varepsilon$ in the set $\Omega_0 \cap D_\varepsilon$. Since u and v are solutions of (1) in $\overset{\circ}{\Omega}_0 \cap \overset{\circ}{D}_\varepsilon$, $u = 0$ on $\partial\Omega_0 \cap D_\varepsilon$, $v \le 0$ in D_ε, $u(x) \to 0$ for $|x| \to +\infty$, $v(x) \to -2\varepsilon$ for $|x| \to +\infty$, and, finally, $|\frac{\partial u}{\partial \nu}| < +\infty$, $\frac{\partial v}{\partial \nu} = +\infty$ on $\Omega_0 \cap \partial D_\varepsilon$, where ν is the unit normal pointing into D_ε, it follows that $u \ge v$ throughout $\Omega_0 \cap D_\varepsilon$ and hence

$$u \ge -4\varepsilon \text{ on } \Omega_0 \cap D_\varepsilon. \tag{9}$$

In the second step we compare u with the function

$$w(x) = \frac{1}{\kappa} \arccos e^{-\kappa d(x)} - 4\varepsilon - \frac{1}{\kappa} \arccos e^{-\kappa\varepsilon}$$

in $\Omega_0 \cap B_\varepsilon$ where $B_\varepsilon = \mathbb{R}^2 \backslash D_\varepsilon$ as above and $d(x)$ is the distance of x to $\partial\Omega$. Writing $w(x) = \omega(d(x))$, the function ω has the following properties: $\omega' \ge 0$, $\omega'' \le 0$, $\omega'(0) = +\infty$, $\omega'' + \kappa\omega'(1 + \omega'^2) = 0$. Observing that $d(x) \le \varepsilon$ for $x \in \Omega \cap B_\varepsilon$ and sufficiently small ε we therefore see that $w \le -4\varepsilon$ in $\Omega \cap B_\varepsilon$ and $\frac{\partial w}{\partial \nu} = +\infty$ on $\partial\Omega \cap B_\varepsilon$. Furthermore, w satisfies the inequality

$$\begin{aligned} Mw &= (1 + |\omega' Dd|^2)(\omega''|Dd|^2 + \omega'\Delta d) - \\ &\quad - w'^2 D_i d D_j d(\omega'' D_i d D_j d + \omega' D_{ij} d) \\ &= (1 + \omega'^2)\left(\omega'' - \frac{k\omega'}{1 - kd}\right) - \omega'^2 \nu^i \nu^j \left(\omega'' \nu^i \nu^j - \frac{\omega' k}{1 - kd} \lambda^i \lambda^j\right) \\ &= \omega'' - \frac{k}{1 - kd} \cdot \omega'(1 + \omega'^2) \ge \omega'' - \frac{k}{2}\omega'(1 + \omega'^2) \ge 0 = M(u) \end{aligned}$$

where λ denotes a unit tangent vector of $\partial\Omega$ and ν, λ, and k are to be taken at the point $y(x) \in \partial\Omega$ nearest to x (cf. [**G-T**, 14.6]). Since $u = 0 \ge w$ on $\partial\Omega_0 \backslash \partial\Omega$ and $u \ge w$ on $\Omega_0 \cap \partial B_\varepsilon$ by (9) we therefore conclude that $u \ge w$ throughout $\Omega_0 \cap B_\varepsilon$, in particular

$$u \ge -4\varepsilon - \frac{1}{\kappa} \arccos e^{-\kappa\varepsilon} \text{ on } \partial\Omega_0 \cap \partial\Omega, \tag{10}$$

which contradicts the choice of boundary values f_ε above. □

Remark. Since we have not assumed any differentiability of u up to the boundary we must, strictly speaking, apply the comparison principle in $\{x \in \Omega_0 \cap B_\varepsilon | d(x) > \delta\}$ to u and the function $\omega_\delta(x) = \frac{1}{\kappa} \arccos e^{-\kappa(d(x) - \delta)} - \frac{1}{\kappa} \arccos e^{-\kappa(\varepsilon - \delta)} - 4\varepsilon$ and then let $\delta \to 0$.

3. Instability.

By a slight modification of the above arguments we may prove the following instability theorem for exterior domains Ω with connected and strictly concave boundary.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^2$ be a domain of class C^2 such that $\mathbb{R}^2 \backslash \Omega$ is strictly convex and let $u_0 \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ be a solution of the minimal surface equation in Ω with boundary values f_0 which are at least of class $C^{1/2}(\partial\Omega)$. Then there exist boundary data f such that $f - f_0 \in C^2(\partial\Omega)$, $f = f_0$ outside a set of arbitrary small diameter, and the C^α-norm of $f - f_0$ is arbitrary small for $0 \le \alpha < 1/2$, but the Dirichlet problem with data f has no solution in Ω.*

Proof. We may choose our coordinate system in such a way that $p_\varepsilon = (\varepsilon/2, 0) \in \partial\Omega$, the boundary values f_0 attain their minimal value in p_ε, $f_0(p_\varepsilon) = 0$, and the tangent vector of $\partial\Omega$ at p_ε points in x_2-direction. By assumption there is a constant L such that

$$0 \le f_0 \le L\varepsilon^{1/2} \quad \text{on} \quad \partial\Omega \cap B_\varepsilon. \tag{11}$$

We now construct boundary values $f = f_0 + f_\varepsilon$ in the following way, similar to Section 2; we divide $\partial\Omega \cap B_\varepsilon$ into four parts $S_1, \dots, S_4$ of equal length and choose the function f_ε together with suitable points $q_i \in S_i$ such that

$$\begin{aligned}
&f_\varepsilon = 0 \ \text{ on } \ \partial\Omega \backslash B_\varepsilon \\
&f_\varepsilon < 0 \text{ in } \ \overset{\circ}{S}_1 \cup \overset{\circ}{S}_3, \ \inf_{S_i} f_\varepsilon = f_\varepsilon(q_i) = -5\varepsilon - \frac{m}{\kappa} \arccos e^{-\kappa\varepsilon} (i = 1, 3) \\
&f_\varepsilon > 0 \ \text{ in } \ \overset{\circ}{S}_2 \cup \overset{\circ}{S}_4, \ \sup_{S_i} f_\varepsilon = f_\varepsilon(q_i) = 5\varepsilon + \frac{1}{\kappa} \arccos e^{-\kappa\varepsilon} (i = 2, 4)
\end{aligned}$$

where 2κ is the minimal length of the curvature vector of $\partial\Omega$ and the constant m will be chosen large enough (depending only on L in (11) and κ, for example $m \ge 1 + L\sqrt{2\kappa}$) so that

$$f(q_i) \le -5\varepsilon - \frac{1}{\kappa} \arccos e^{-\kappa\varepsilon}, i = 1, 3. \tag{12}$$

The inequality

$$f(q_i) \ge 5\varepsilon + \frac{1}{\kappa} \arccos e^{-\kappa\varepsilon}, i = 2, 4 \tag{13}$$

is trivially satisfied. Assuming the existence of a solution of (1), (2) with the above f we define the sets Ω_i as the connected components of $\{x \in \overline{\Omega} | u(x) \neq$

$0\}$ containing the points $q_i, i = 1, \ldots, 4$. The rest of the proof is the same as before in Section 2: after eventually if necessary replacing u by $-u$ we end up with inequality (10) in one of the components Ω_i which contradicts (12) or (13). $\square$

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