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SOME BASIC BILATERAL SUMS AND INTEGRALS

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By splitting the real line into intervals of unit length a doubly infinite integral of the form $\int_{-\infty}^{\infty} F(q^x)\,dx$, $0 < q < 1$, can clearly be expressed as $\int_0^1 \sum_{n=-\infty}^{\infty} F(q^{x+n})\,dx$, provided F satisfies the appropriate conditions. This simple idea is used to prove Ramanujan's integral analogues of his ${}_1\psi_1$ sum and give a new proof of Askey and Roy's extention of it. Integral analogues of the well-poised ${}_2\psi_2$ sum as well as the very-well-poised ${}_6\psi_6$ sum are also found in a straightforward manner. An extension to a very-well-poised and balanced ${}_8\psi_8$ series is also given. A direct proof of a recent q-beta integral of Ismail and Masson is given.

1. Introduction.

The familiar form of the classical beta integral of Euler is

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}\,, \tag{1.1}$$

Re $(a,b) > 0$. A less familiar form, obtained by a simple change of variable, is

$$B(a,b) = \int_0^\infty \frac{t^{a-1}\,dt}{(1+t)^{a+b}}\,. \tag{1.2}$$

There have been many extensions of both these forms, see, for example, Askey [**2**–**5**], Askey and Roy [**6**], Gasper [**9**, **10**], Rahman and Suslov [**18**], and the references therein. A "curious" extension of (1.2) that was given by Ramanujan [**21**] in 1915 is

$$\int_0^\infty t^{a-1}\,\frac{(-tq^{a+b};q)_\infty}{(-t;q)_\infty}\,dt = \frac{\Gamma(a)\,\Gamma(1-a)}{\Gamma_q(a)\,\Gamma_q(1-a)}\,\frac{\Gamma_q(a)\,\Gamma_q(b)}{\Gamma_q(a+b)}\,, \tag{1.3}$$

where Re $(a,b) > 0$, $0 < q < 1$, the q-gamma function $\Gamma_q(x)$ is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}\,(1-q)^{1-x}\,, \quad x \neq 0,-1,-2,\ldots\,, \tag{1.4}$$

and the infinite products by

$$(a;\, q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n)\,. \tag{1.5}$$

The fact that the limit of the formula (1.3) as $q \to 1^-$ is (1.2) follows from the properties

$$\lim_{q\to 1^-} \Gamma_q(x) = \Gamma(x), \quad \lim_{q\to 1^-} \frac{(-tq^c;\, q)_\infty}{(-t;\, q)_\infty} = \frac{1}{(1+t)^c}, \tag{1.6}$$

see [**11**].

Askey and Roy [**6**] introduced a third parameter into the formulas and gave the following extension of (1.3)

$$\int_0^\infty t^{c-1} \frac{(-tq^{b+c}, -q^{a-c+1}/t;\, q)_\infty}{(-t, -q/t;\, q)_\infty}\, dt = \frac{\Gamma(c)\,\Gamma(1-c)}{\Gamma_q(c)\,\Gamma_q(1-c)}\, \frac{\Gamma_q(a)\,\Gamma_q(b)}{\Gamma_q(a+b)}\,, \tag{1.7}$$

which holds for Re $(a, b, c) > 0$. In the limit $c \to 0^+$ this becomes

$$\int_0^\infty \frac{(-tq^b, -q^{a+1}/t;\, q)_\infty}{(-t, -q/t;\, q)_\infty}\, \frac{dt}{t} = \frac{\log q^{-1}}{1-q}\, \frac{\Gamma_q(a)\,\Gamma_q(b)}{\Gamma_q(a+b)}\,, \tag{1.8}$$

which restores the symmetry in a and b that was there in both (1.1) and (1.2), but not in (1.3), see Gasper [**9–10**]. Following [**11**] we have used the contracted notation

$$(a_1, a_2, \dots, a_k;\, q)_\infty = \prod_{j=1}^{k} (a_j;\, q)_\infty\,. \tag{1.9}$$

Hardy [**12**] gave a proof of (1.3) that Ramanujan did not, and discussed Ramanujan's general method of evaluating such integrals in [**13**]. Askey [**2**] gave another proof of (1.3). Askey's method is rather close to the Pearson-type first order difference equation technique that has been used extensively by the Russian school of Nikiforov, Suslov and Uvarov, see for example, [**17, 23**], as well as of Atakishiyev and Suslov [**7**]. It was pointed out in [**18**] and [**20**] that the origin of both Barnes and Ramanujan-type integrals can be traced to a Pearson equation on linear, q-linear, quadratic or q-quadratic lattices with appropriately chosen coefficient functions so that the boundary conditions can be satisfied in the two cases. In [**20**] Rahman and Suslov found what they consider a better way of dealing with the Ramanujan-type integrals, and evaluated extensions of some of Ramanujan-type formulas as well as extensions of the summation formulas of Gauss, and, Pfaff and

Saalschütz. The idea is very simple. Suppose that $f(x)$ is continuous on $[a, \infty)$, has no singularities on the real line, and its integral on $[a, \infty)$ exists. Suppose also that $\sum_{n=-\infty}^{\infty} f(x+n)$ converges uniformly for $x \in [a, a+1]$. Then

$$\int_a^\infty f(x)\, dx = \int_a^{a+1} \sum_{n=0}^{\infty} f(x+n)\, dx\,. \tag{1.10}$$

For integrals on the whole real line the corresponding formula is

$$\int_{-\infty}^\infty f(x)\, dx = \int_0^1 \sum_{n=-\infty}^{\infty} f(x+n)\, dx\,, \tag{1.11}$$

provided, of course, that the bilateral sum $\sum_{-\infty}^{\infty} f(x+n)$ converges uniformly for $x \in [0,1]$. What this method does is to establish a direct correspondence between the integrals on the left side of (1.10) and (1.11), and the infinite series on the right. So for the method to be useful we have to be able to handle the infinite sums so that we can apply this knowledge to compute the infinite integrals. As is well-known in classical analysis, it is often the case that an infinite integral over a function is easier to compute than an infinite sum. So the method described above has a very limited applicability. It is applicable when the series inside the integrals on the right sides of (1.10) and (1.11) are summable (meaning that the sum can be evaluated in closed forms) or at least transformable in a way that the ensuing formulas are simpler. Such is the case for some hypergeometric and basic hypergeometric series, bilateral or otherwise.

A basic bilateral series in base q (assumed throughout this paper to satisfy $0 < q < 1$), with r numerator and r denominator parameters is defined by

$$(1.12)\quad {}_r\psi_r\left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_r \end{matrix}; q, z\right] =$$

$$= {}_r\psi_r(a_1, \cdots, a_r; b_1, \cdots, b_r; q, z) = \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \dots, a_r;\, q)_n}{(b_1, b_2, \dots, b_r;\, q)_n} z^n\,,$$

where

$$(a_1, a_2, \dots, a_r;\, q)_n = \prod_{j=1}^{r} (a_j;\, q)_n\,,$$

$$(a_j;\, q)_n = \begin{cases} 1, & \text{if } n = 0, \\ \prod_{k=0}^{n-1}(1 - a_j q^k), & \text{if } n = 1, 2, \dots\,. \end{cases} \tag{1.13}$$

The series (1.12) is absolutely convergent in the annulus

$$\left|\frac{b_1 b_2 \dots b_r}{a_1 a_2 \dots a_r}\right| < |z| < 1. \tag{1.14}$$

If any one of the denominator parameters equals q, say, $b_r = q$, then the first non-zero term in the series corresponds to $n = 0$, and the series becomes a basic generalized hypergeometric series:

$$(1.15)\quad {}_r\phi_{r-1}\left[\begin{matrix}a_1, a_2, \dots, a_r\\ b_1, b_2, \dots, b_{r-1}\end{matrix}; q, z\right] = {}_r\phi_{r-1}(a_1, \cdots, a_r; b_1, \cdots, b_{r-1}; q, z)$$
$$:= \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_r;\, q)_n}{(q, b_1, b_2, \dots, b_{r-1};\, q)_n} z^n,$$

which is absolutely convergent inside the unit circle $|z| = 1$, for further details see [**11**].

One of the most important evaluations of a basic bilateral hypergeometric series is the one due to Ramanujan [**13**]

$$(1.16)\qquad {}_1\psi_1\left[\begin{matrix}a\\ b\end{matrix}; q, z\right] = \sum_{n=-\infty}^{\infty} \frac{(a;\, q)_n}{b;\, q)_n} z^n = \frac{(q, b/a, az, q/az;\, q)_\infty}{(b, q/a, z, b/az;\, q)_\infty}.$$

Many different proofs of this formula have appeared in the literature, but the ones that are most often quoted and instructive are in [**1**] and [**14**]. However, one runs into trouble with a bilateral series immediately after the ${}_1\psi_1$ level. Instead of a nice compact formula like the q-Gauss formula

$$(1.17)\qquad {}_2\phi_1\left[\begin{matrix}a,\ b\\ c\end{matrix}; q, c/ab\right] = \frac{(c/a, c/b;\, q)_\infty}{(c, c/ab;\, q)_\infty}, \quad |c/ab| < 1,$$

one has a 2-term formula for the corresponding ${}_2\psi_2$ sum:

$$(1.18)\quad {}_2\psi_2\left[\begin{matrix}a, b\\ c,\ d\end{matrix}; q,\ cd/abq\right]$$
$$- \frac{\alpha}{q}\frac{(q/c, q/d, \alpha/a, \alpha/b;\, q)_\infty}{(q/a, q/b, \alpha/c, \alpha/d;\, q)_\infty}\, {}_2\psi_2\left[\begin{matrix}aq/\alpha,\ bq/\alpha\\ cq/\alpha,\ dq/\alpha\end{matrix}; q, cd/abq\right]$$
$$= \frac{(\alpha, q/\alpha, cd/\alpha q, \alpha q^2/cd, q, c/a, c/b, d/a, d/b;\, q)_\infty}{(c/\alpha, \alpha q/c, d/\alpha, \alpha q/d, c, d, q/a, q/b, cd/abq;\, q)_\infty},$$

see [**11**], where α is an arbitary parameter such that no zeros appear in the denominators. It is clear that this formula reduces to (1.17) when $d = q$. This is not a very well-known formula but a special case of it was mentioned in [**3**]. As the number of parameters of the summation formulas increases, one needs to impose more restrictions. The bilateral series that has the most desirable structure is the very well-poised one, namely

$$(1.19)\qquad {}_{r+2}\psi_{r+2}\left[\begin{matrix} qa^{1/2}, -qa^{1/2}, & a_1, & a_2, & \dots, & a_r \\ a^{1/2}, -a^{1/2}, & qa/a_1, & qa/a_2, & \dots, & qa/a_r \end{matrix}; q, z\right] = \sum_{n=-\infty}^{\infty} \frac{(1-aq^{2n})(a_1, a_2, \dots, a_r; q)_n}{(1-a)(qa/a_1, qa/a_2, \dots, qa/a_r; q)_n} z^n .$$

The most general summation formula for a basic bilateral series is Bailey's [**8**] ${}_6\psi_6$ sum:

$$(1.20)\quad {}_6\psi_6\left[\begin{matrix} qa^{1/2}, -qa^{1/2}, & b, & c, & d, & e \\ a^{1/2}, -a^{1/2}, & aq/b, & aq/c, & aq/d, & aq/e \end{matrix}; q, qa^2/bcde\right] = \frac{(q, q/a, aq, aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/de;\ q)_\infty}{(q/b, q/c, q/d, q/e, aq/b, aq/c, aq/d, aq/e, qa^2/bcde\,;\ q)_\infty},$$

provided that $|qa^2/bcde| < 1$, see [**11**]. Clearly, a function $f(x)$ for which $\sum_{n=-\infty}^{\infty} f(x+n)$ corresponds to the sum on the left side of (1.20) has to be of interest as far as the applicability of (1.11) is concerned. Accordingly, we first rewrite this formula in the form:

$$(1.21)\quad \sum_{n=-\infty}^{\infty} (aq^{n+1}/b, aq^{n+1}/c, aq^{n+1}/d, aq^{n+1}/e;\ q)_\infty \cdot (q^{1-n}/b, q^{1-n}/c, q^{1-n}/d, q^{1-n}/e;\ q)_\infty \cdot (1-aq^{2n})a^{2n}q^{n^2-n} = \frac{(q, a, q/a, aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/de;\ q)_\infty}{(qa^2/bcde;\ q)_\infty}.$$

This suggests considering an integral of the form

(1.22)

$$J := \int_{-\infty}^{\infty} (aq^{x+1}/b, aq^{x+1}/c, aq^{x+1}/d, aq^{x+1}/e, q^{1-x}/b, q^{1-x}/c, q^{1-x}/d;\ q)_\infty \times (q^{1-x}/e;\ q)_\infty (1-aq^{2x})a^{2x}q^{x^2-x}\omega(x)\, dx$$

where $\omega(x)$ is a bounded continuous unit-periodic function on $\mathbf{R}$, i.e, $\omega(x \pm 1) = \omega(x)$.

We shall evaluate this integral, (1.22), in §3 by using (1.11) and (1.21), and consider an extension of it in §5. In §2, however, we shall deal with an integral analogue of (1.16) essentially showing that Ramanujan's formula (1.3) is precisely that analogue. For an integral analogue of (1.18) we refer the reader to [**20**]. As a straightforward application of (1.8) we will also show in §2 how to obtain a q-analogue of an integral of Ramanujan involving a product of two Bessel functions where the variable is the order and not the argument. In §4 we shall consider a case where $\omega(x)$ is a unit anti-periodic function in (1.22), i.e $\omega(x \pm 1) = -\omega(x)$, thereby obtaining a q-analogue of yet another formula due to Ramanujan.

Ismail and Masson [**16**] proved that if the q^{-1}-Hermite polynomials are orthogonal with respect to a probability measure $d\psi$ then

$$\int_{-\infty}^{\infty} \prod_{j=1}^{4} (-t_j(x+\sqrt{x^2+1}), t_j(\sqrt{x^2+1}-x); q)_\infty \, d\psi(x) = \left[\prod_{1 \le j < k \le 4} (-t_j t_k/q; q)_\infty \right] / (t_1 t_2 t_3 t_4 q^{-3}; q)_\infty, \tag{1.23}$$

holds, provided that the integral exists. The corresponding moment problem has infinitely many solutions so one expects (1.23) to lead to an evaluation of an infinite family of integrals. Ismail and Masson [**16**] pointed out that Bailey's ${}_6\psi_6$ sum is (1.23) with $d\psi$ a general extremal measure of the q^{-1}-Hermite moment problem. Ismail and Masson also observed that Askey's q-beta integral (3.4) corresponds to an absolutely continuous $d\psi$ and in this sense (3.4) is a continuous analogue of the ${}_6\psi_6$ sum of (1.20). Ismail and Masson [**16**] proved that the q^{-1}-Hermite polynomials are orthogonal with respect to the absolutely continuous measure $d\mu(x;\eta)$, where

$$\frac{d\mu(x;\eta)}{dx} = \frac{e^{2\eta_1} \sin\eta_2 \cosh\eta_1 (q, -qe^{2\eta_1}, -qe^{-2\eta_1}; q)_\infty |(qe^{2i\eta_2}; q)_\infty|^2}{\pi \, |(e^{\xi+\eta}, -e^{\eta-\xi}, -qe^{\xi-\eta}, qe^{-\xi-\eta}; q)_\infty|^2}, \quad x = \sinh\xi, \eta = \eta_1 + i\eta_2. \tag{1.24}$$

This and (1.23) led Ismail and Masson to the q-beta integral

$$\int_{-\infty}^{\infty} \frac{\prod_{j=1}^{4} (-t_j e^{\xi}, t_j e^{-\xi}; q)_\infty}{|(e^{\xi+\eta}, -e^{\eta-\xi}, -qe^{\xi-\eta}, qe^{-\xi-\eta}; q)_\infty|^2} \cosh\xi \, d\xi = \frac{\pi e^{-2\eta_1} \prod_{1 \le j < k \le 4} (-t_j t_k/q; q)_\infty}{\sin\eta_2 \cosh\eta_1 (q, t_1 t_2 t_3 t_4 q^{-3}, -qe^{2\eta_1}, -qe^{-2\eta_1}; q)_\infty |(qe^{2i\eta_2}; q)_\infty|^2}. \tag{1.25}$$

In Section 3 we shall give a direct proof of (1.25) and show that this q-beta integral is another continuous analogue og the ${}_6\psi_6$ sum.

2. The Askey-Roy Integral and an Application.

In this section we give new evaluations of (1.7) and (1.8) and also give some applications of them. Let us first rewrite (1.16) in the form

$$\sum_{n=-\infty}^{\infty} (bq^n, q^{1-n}/a; q)_\infty q^{n(n-1)/2} (az)^n e^{\pi i n} = \frac{(q, b/a, az, q/az; q)_\infty}{(z, b/az; q)_\infty}, \quad |b/a| < |z| < 1. \tag{2.1}$$

This suggests that we consider the integral

$$(2.2)\qquad I := \int_{-\infty}^{\infty} (bq^x, q^{1-x}/a;\, q)_\infty\, q^{x(x-1)/2} (az)^x\, e^{\pi i x}\, \omega(x)\, dx\,,$$

where $\omega(x \pm 1) = \omega(x)$. For $0 < q < 1$, $|b/a| < |z| < 1$ and continuous bounded functions $\omega(x)$, it is clear that the integral exists.

Proof of (1.8). By (1.11) and (1.16) we have

$$(2.3)\; I = \int_0^1 (bq^x, q^{1-x}/a;\, q)_\infty\, q^{x(x-1)/2} (az)^x\, e^{\pi i x}\, \omega(x)\, {}_1\psi_1 \left[\begin{matrix} aq^x \\ bq^x \end{matrix} ;\, q, z \right] dx$$
$$= \frac{(q, b/a;\, q)_\infty}{(z, b/az;\, q)_\infty} \int_0^1 (azq^x, q^{1-x}/az;\, q)_\infty\, q^{x(x-1)/2} (az)^x\, e^{\pi i x}\, \omega(x)\, dx\,.$$

It is easy to see that the integrand in the last line above is unit periodic, so it can be absorbed in ω. Set

$$(2.4)\; \omega(x) = \frac{p(x)}{(azq^x, q^{1-x}/az;\, q)_\infty\, q^{x(x-1)/2} (az)^x\, e^{\pi i x}}\,, \quad p(x \pm 1) = p(x)\,,$$

which when substituted in (2.2) and (2.3), gives the formula

$$(2.5)\qquad \int_{-\infty}^{\infty} \frac{(bq^x, q^{1-x}/a;\, q)_\infty}{(azq^x, q^{1-x}/az;\, q)_\infty} p(x)\, dx = \frac{(q, b/a;\, q)_\infty}{(z, b/az;\, q)_\infty} \int_0^1 p(x)\, dx.$$

Assuming that $p(x)$ is independent of a, b and z, let us now replace a, b, z by $-q^{-a}, -q^b, -q^a$ in (2.5) to get

$$(2.6)\; \int_{-\infty}^{\infty} \frac{(-q^{b+x}, -q^{a+1-x}; q)_\infty}{(-q^x, -q^{1-x}; q)_\infty} p(x) dx$$
$$= \frac{(q, q^{a+b};\, q)_\infty}{(q^a, q^b;\, q)_\infty} \int_0^1 p(x)\, dx = (1-q)^{-1} \frac{\Gamma_q(a)\, \Gamma_q(b)}{\Gamma_q(a+b)} \int_0^1 p(x)\, dx.$$

Setting $p(x) \equiv 1$ and changing the variable by $q^x = t$, we establish (1.8) and the proof is complete.

The proof of (1.7) is based on a different choice of ω. Instead of (2.4), choose

$$(2.7)\qquad \omega(x) = \frac{q^{-x(x-1)/2}\, p(x)}{(-q^x, -q^{1-x};\, q)_\infty}\,, \quad p(x \pm 1) = p(x)\,.$$

Proof of (1.7). The choice (2.7) gives

$$(2.8)\qquad \int_{-\infty}^{\infty} \frac{(bq^x, q^{1-x}/a;\, q)_\infty}{(-q^x, -q^{1-x};\, q)_\infty}\, e^{\pi i x} (az)^x\, p(x)\, dx$$
$$= \frac{(q, b/a;\, q)_\infty}{(z, b/az;\, q)_\infty} \int_0^1 \frac{(azq^x, q^{1-x}/az;\, q)_\infty}{(-q^x, -q^{1-x};\, q)_\infty} (az)^x\, p(x)\, dx,$$

where we may assume that $p(x)$ is independent of a, b and z. Let us now replace a, b, z by $-q^{-a}, -q^{b}, -q^{a+c}$ in (2.5), respectively. Then (2.8) can be written as

$$\tag{2.9} \begin{aligned} &\int_{-\infty}^{\infty} q^{cx} \frac{(-q^{b+x}, -q^{a+1-x};\, q)_\infty}{(-q^{x}, -q^{1-x};\, q)_\infty}\, p(x)\, dx \\ &= \frac{(q, q^{a+b};\, q)_\infty}{(q^{a+c}, q^{b-c};\, q)_\infty} \int_0^1 q^{cx} \frac{(-q^{c+x}, -q^{1-c-x};\, q)_\infty}{(-q^{x}, -q^{1-x};\, q)_\infty}\, p(x)\, dx \end{aligned}$$

with Re $(a+c) > 0$ and Re $(b-c) > 0$. Setting $p(x) \equiv 1$ and denoting the integral on the right side by $g(c)$ we may rewrite (2.9) in the form

$$\tag{2.10} \begin{aligned} g(c) &= \frac{(q^{a+c}, q^{b-c};\, q)_\infty}{(q, q^{a+b};\, q)_\infty} \int_{-\infty}^{\infty} q^{cx} \frac{(-q^{b+x}, -q^{a+1-x};\, q)_\infty}{-q^{x}, -q^{1-x};\, q)_\infty}\, dx \\ &= \frac{1-q}{\log q^{-1}} \frac{\Gamma_q(a+b)}{\Gamma_q(a+c)\,\Gamma_q(b-c)} \int_0^{\infty} t^{c-1} \frac{(-tq^{b}, -q^{a+1}/t;\, q)_\infty}{(-t, -q/t;\, q)_\infty}\, dt, \end{aligned}$$

assuming, without loss of generality, that $0 < \text{Re}\, c < 1$.

Since the left hand side is independent of a, b, we can set whatever values of a, b we wish, subject to the restriction mentioned above, to compute $g(c)$. The simplest choice of a, b is $a = 0$, $b = 1$. Then

$$\tag{2.11} \begin{aligned} g(c) &= \frac{1-q}{\log q^{-1}} \frac{\Gamma_q(1)}{\Gamma_q(c)\Gamma_q(1-c)} \int_0^{\infty} t^{c-1} \frac{(-tq;\, q)_\infty}{(-t;\, q)_\infty}\, dt \\ &= \frac{1-q}{\log q^{-1}} \frac{\Gamma(c)\,\Gamma(1-c)}{\Gamma_q(c)\,\Gamma_q(1-c)}, \end{aligned}$$

since
$$\int_0^{\infty} t^{c-1} \frac{(-tq;\, q)_\infty}{(-t;\, q)_\infty}\, dt = \int_0^{\infty} \frac{t^{c-1}}{1+t}\, dt = \Gamma(c)\Gamma(1-c).$$

So, by (2.9) and (2.11) we have

$$\tag{2.12} \begin{aligned} &(\log q^{-1}) \int_{-\infty}^{\infty} q^{cx} \frac{(-q^{b+x}, -q^{a+1-x};\, q)_\infty}{(-q^{x}, -q^{1-x};\, q)_\infty}\, dx \\ &= \frac{\Gamma(c)\Gamma(1-c)\Gamma_q(a+c)\Gamma_q(b-c)}{\Gamma_q(c)\Gamma_q(1-c)\Gamma_q(a+b)}. \end{aligned}$$

Substituting $q^{x} = t$ on the left and changing a, b to $a-c$ and $b+c$, respectively, we get (1.7).

It is also clear that setting $a = 0$ in (2.12) gives Ramanujan's formula (1.3).

We now explore a special choice for $p(x)$. We set $b = q^{\alpha}$, $a = q^{1-\beta}$, $z = -q^{\beta-1/2}$, $p(x) \equiv 1$ in (2.5), getting

$$\tag{2.13} \int_{-\infty}^{\infty} \frac{(q^{\alpha+x}, q^{\beta-x};\, q)_\infty}{(-q^{1/2+x}, -q^{1/2-x};\, q)_\infty}\, dx = \frac{(q, q^{\alpha+\beta-1};\, q)_\infty}{(-q^{\alpha-1/2}, -q^{\beta-1/2};\, q)_\infty},$$

Re $(\alpha+\beta-1) > 0$, which is a q-analogue of yet another formula of Ramanujan [**22**]:

$$\int_{-\infty}^{\infty} \frac{dx}{\Gamma(\alpha+x)\,\Gamma(\beta-x)} = \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}. \tag{2.14}$$

Using (2.13) one can show in a straightforward manner that

$$\begin{aligned}\int_{-\infty}^{\infty} &\frac{J^{(1)}_{\lambda+x}(a)\,J^{(2)}_{\mu-x}(b)}{a^{\lambda+x}\,b^{\mu-x}}\,\frac{(-q^{\lambda+1/2},-q^{\mu+1/2};\,q)_\infty}{(-q^{1/2+x},-q^{1/2-x};\,q)_\infty}\,dx \\ &= \frac{(q^{\lambda+\mu+1};\,q)_\infty}{(q;\,q)_\infty}\,{}_2\phi_1\left[\begin{matrix}-q^{\lambda+1/2}, & -\frac{b^2}{a^2}q^{\mu+1/2} \\ & q^{\lambda+\mu+1}\end{matrix};\,q,-\frac{a^2}{4}\right],\end{aligned} \tag{2.15}$$

$|a/2| < 1$, where

$$\begin{aligned}J^{(1)}_\nu(x) &= \sum_{m=0}^{\infty} \frac{(-1)^m(x/2)^{\nu+2m}}{(q;\,q)_m}\,\frac{(q^{\nu+1+m};\,q)_\infty}{(q;\,q)_\infty}, \\ J^{(2)}_\nu(x) &= \sum_{m=0}^{\infty} \frac{(-1)^m(x/2)^{\nu+2m}}{(q;\,q)_m}\,\frac{(q^{\nu+1+m};\,q)_\infty}{(q;\,q)_\infty}\,q^{m(\nu+m)},\end{aligned} \tag{2.16}$$

are the two q-Bessel functions of Jackson, see [**15**]. Formula (2.15) is a q-analogue of the following formula

$$\int_{-\infty}^{\infty} \frac{J_{\lambda+x}(a)\,J_{\mu-x}(b)}{a^{\lambda+x}\,b^{\mu-x}}\,dx = \left(\frac{2}{a^2+b^2}\right)^{\frac{\lambda+\mu}{2}} J_{\lambda+\mu}\left(\sqrt{2(a^2+b^2)}\right), \tag{2.17}$$

due to Ramanujan in [**21**].

3. An integral Analogue of Bailey's ${}_6\psi_6$ Sum.

An application of (1.11) to (1.22) gives

(3.1)

$$\begin{aligned}J = \int_0^1 &(1-aq^{2x})(aq^{x+1}/b, aq^{x+1}/c, aq^{x+1}/d, aq^{x+1}/e;\,q)_\infty \\ &\cdot (q^{1-x}/b, q^{1-x}/c, q^{1-x}/d, q^{1-x}/e;\,q)_\infty a^{2x}q^{2x^2-x}\omega(x) \\ &\cdot {}_6\psi_6\left[\begin{matrix}q^{x+1}a^{1/2}, -q^{x+1}a^{1/2}, & bq^x, & cq^x, & dq^x, & eq^x \\ q^xa^{1/2}, -q^xa^{1/2}, & aq^{x+1}/b, & aq^{x+1}/c, & aq^{x+1}/d, & aq^{x+1}/e\end{matrix};\,q,\frac{qa^2}{bcde}\right]dx \\ = &\frac{(aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/de;\,q)_\infty}{(qa^2/bcde;\,q)_\infty} \\ &\cdot \int_0^1 (aq^{2x}, q^{1-2x}/a;\,q)_\infty a^{2x}q^{2x^2-x}\omega(x)\,dx,\end{aligned}$$

after we apply (1.20). Replacing a by α^2 and q/b, q/c, q/d, q/e by a/α, b/α, c/α, d/α, respectively, and taking

$$\omega(s) = \frac{q^{s-2s^2}\alpha^{-4s}}{(\alpha^2 q^{2s}, q^{1-2s}/\alpha^2;\ q)_\infty} p(s), \quad p(s \pm 1) = p(s), \tag{3.2}$$

we establish the relationship

(3.3)
$$\int_{-\infty}^{\infty} \frac{(\alpha a q^x, a q^{-x}/\alpha, \alpha b q^x, b q^{-x}/\alpha, \alpha c q^x, c q^{-x}/\alpha, \alpha d q^x, d q^{-x}/\alpha;\ q)_\infty}{(\alpha^2 q^{2x+1}, q^{1-2x}/\alpha^2;\ q)_\infty} p(x)\, dx$$
$$= \frac{(q, ab/q, ac/q, ad/q, bc/q, bd/q, cd/q;\ q)_\infty}{(abcd/q^3;\ q)_\infty} \int_0^1 p(x)\, dx,$$

where $|abcd/q^3| < 1$. In order to avoid singularities we also need to assume that $\arg \alpha^2 \neq 2k\pi$, $k = 0, \pm 1, \pm 2, \dots$. When $p(s) \equiv 1$ and $\alpha = i$ we obtain Askey's formula [4]:

$$\int_{-\infty}^{\infty} \frac{h(i \sinh u;\ a, b, c, d)}{h(i \sinh u; q^{1/2}, -q^{1/2}, q, -q)}\, du \tag{3.4}$$
$$= (\log q^{-1}) \frac{(ab/q, ac/q, ad/q, bc/q, bd/q, cd/q, q;\ q)_\infty}{(abcd/q^3;\ q)_\infty},$$

where

$$h(x;\ a) = \prod_{n=0}^{\infty} (1 - 2axq^n + a^2 q^{2n}), \tag{3.5}$$
$$h(x;\ a_1, a_2, \dots, a_r) = \prod_{k=1}^{r} h(x;\ a_k).$$

Askey's formula (3.4) was obtained from (3.3) by taking $\alpha = i$ and specializing the unit periodic function $p(x)$ to be equal to 1. But this is not the only case that we can evalute exactly. We will show now that the integral $\int_0^1 p(x)\, dx$ can be evaluated even in the complicated case when $\alpha = i$ and

(3.6)
$$p(x) = \frac{(-q^{2x+1},\ -q^{1-2x}\ q)_\infty\ (q^{-x} + q^x)}{(fq^{-x}, q^{x+1}/f, -fq^x, -q^{1-x}/f, gq^{-x}, q^{x+1}/g, -gq^x, -q^{1-x}/g;\ q)_\infty},$$

where $\operatorname{Im} f$, $\operatorname{Im} g$ and $\operatorname{Im}(f/g)$ are not 0(mod 2π) and $\operatorname{Im}(fg) \not\equiv \pi$(mod 2π). This will lead in a very natural way to the Ismail-Masson q-beta integral (1.25). It can be easily verified that $p(x \pm 1) = p(x)$. Use of (3.3) then gives

$$\eta(f, g) = \frac{(abcd/q^3;\ q)_\infty}{(q, ab/q, ac/q, ad/q, bc/q, bd/q, cd/q;\ q)_\infty} \tag{3.7}$$
$$\cdot \int_{-\infty}^{\infty} \frac{(iaq^x, -iaq^{-x}, ibq^x, -ibq^{-x}, icq^x, -icq^{-x}, idq^x, -idq^{-x};\ q)_\infty}{(fq^{-x}, q^{x+1}/f, -fq^x, -q^{1-x}/f, gq^{-x}, q^{x+1}/g, -gq^x, -q^{1-x}/g;\ q)_\infty}$$
$$\cdot (q^{-x} + q^x)\, dx,$$

where

(3.8) $\eta(f,g) =$

$$\int_0^1 \frac{(-q^{2x+1},\ -q^{1-2x};\ q)_\infty\ (q^{-x}+q^x)\,dx}{(fq^{-x}, q^{x+1}/f, -fq^x, -q^{1-x}/f, gq^{-x}, q^{x+1}/g, -gq^x, -q^{1-x}/g;\ q)_\infty}.$$

Observe that $\eta(f,g)$ is independent of a, b, c, d, so the expression on the right hand side of (3.7) must have the same property. For the purpose of (3.7) we then set

$$ab = q^2 = cd, \qquad a = -iq/f, \quad c = iq \tag{3.9}$$

to get

$$\eta(f,g) = \frac{1}{(q, q, g/f, fq^2/g, -fg, -q^2/fg;\ q)_\infty} \cdot \int_{-\infty}^{\infty} \frac{(q^{-x}+q^x)\,dx}{[1+f(q^x-q^{-x})-f^2][1-q/g(q^x-q^{-x})-q^2/g^2]}. \tag{3.10}$$

Substituting $q^{-x}-q^x = u$, the integral on the right side of (3.10) becomes

$$\begin{aligned} &\frac{1}{\log q^{-1}} \int_{-\infty}^{\infty} \frac{du}{(1-f^2-fu)(1-q^2/g^2+qu/g)} \\ &= \frac{1}{f\ \log q^{-1}}\ \frac{2\pi i}{(1-qf/g)(1+q/fg)}. \end{aligned} \tag{3.11}$$

Formula (3.11) can be proved by either using a partial fraction decomposition or by a simple contour integration. Thus

$$\eta(f,g) = \frac{2\pi i}{f\ \log q^{-1}\,(q, q, g/f, qf/g, -fg, -q/fg;\ q)_\infty}. \tag{3.12}$$

Combining (3.3), (3.7), (3.8) and (3.12) we find that

(3.13)

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{(iaq^x, -iaq^{-x}, ibq^x, -ibq^{-x}, icq^x, -icq^{-x}, idq^x, -idq^{-x}; q)_\infty (q^{-x}+q^x)}{(fq^{-x}, q^{x+1}/f, -fq^x, -q^{1-x}/f, gq^{-x}, q^{x+1}/g, -gq^x, -q^{1-x}/g;\ q)_\infty} dx \\ &= \frac{2\pi i\ (ab/q, qc/q, ad/q, bc/q, bd/q, cd/q;\ q)_\infty}{f\ \log q^{-1}\,(q, g/f, qf/g, -fg, -q/fg, abcd/q^3;\ q)_\infty}. \end{aligned}$$

This is the same as (1.25) which was established in [**16**] by an entirely different method.

We would like to mention that formula (3.3) is valid even when α is real, provided the integral on the left is interpreted as a principal-value integral. For a detailed discussion of this point, see [**18**].

We should like to point out that if we denote

$$f(t)=\frac{(\alpha at,a/\alpha t,\alpha bt,b/\alpha t,\alpha ct,c/\alpha t,\alpha dt,d/\alpha t;\ q)_\infty}{(q\alpha^2t^2,q/\alpha^2t^2;\ q)_\infty\, t\,(\log q^{-1})},\tag{3.14}$$

then (1.20) and (3.3) state that

$$\begin{aligned}\int_0^\infty f(t)\,dt&=\int_0^\infty\left(\frac{\log q^{-1}}{1-q}\right)f(t)\,d_qt\\&=\frac{(q,ab/q,ac/q,ad/q,bc/q,bd/q,cd/q;\ q)_\infty}{(abcd/q^3;\ q)_\infty},\end{aligned}\tag{3.15}$$

where

$$\int_0^\infty g(t)\,d_qt=(1-q)\sum_{n=-\infty}^{\infty}g(q^n)q^n\tag{3.16}$$

is a q-integral defined by Jackson, see [**11**]. This is an example where an absolutely continuous measure and a purely discrete measure on the real line have the same total weight, indicating an indeterminate moment problem. Ismail and Masson [**16**] have recently found two systems of rational functions which are biorthogonal with respect to the weight function given in the integral (1.23). In earlier unpublished work, Rahman proved the biorthogonality of the same rational functions with respect to the weight function in the Askey integral (3.4). Further properties of these rational functions, their Rodrigues formulas, the q-difference equations etc. will be discussed in a subsequent paper. For a different system of biorthogonal rational functions on $[-1,1]$, see [**19**].

4. Case of an anti-unit-periodic function.

We shall now derive a q-analogue of Ramanujan's formula [**22**]:

$$\begin{aligned}&\int_{-\infty}^\infty\frac{p(x)\,dx}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)\Gamma(\delta-x)}\\&\qquad=\frac{\int_0^1 p(t)\cos[\pi(2t+\alpha-\beta)/2]\,dt}{\Gamma(\alpha+\beta/2)\Gamma(\gamma+\delta/2)\Gamma(\alpha+\delta-1)},\end{aligned}\tag{4.1}$$

where $p(x\pm 1)=-p(x)$, $\alpha+\delta=\beta+\gamma$ and $\mathrm{Re}\,(\alpha+\beta+\gamma+\delta)>2$. Let us first write down the ${}_2\psi_2$ summation formula [**11**, (5.3.4)] in the following form:

$$\begin{aligned}&\sum_{n=-\infty}^{\infty}(aq^{n+1}/b,aq^{n+1}/c,q^{1-n}/b,q^{1-n}/c;\ q)_\infty a^ne^{\pi in}q^{n^2}\\&\quad=\frac{(aq/bc;\ q)_\infty}{(-aq/bc;\ q)_\infty}(q^2,aq,q/a,aq^2/b^2,aq^2/c^2;\ q^2)_\infty,\quad |aq/bc|<1.\end{aligned}\tag{4.2}$$

Let $\omega(x)$ be a bounded continuous function on $\mathbf{R}$ such that $\omega(x\pm 1) = -\omega(x)$. Then it can be shown that

$$\begin{aligned}
(4.3)\qquad &\int_{-\infty}^{\infty} (aq^{x+1}/b, aq^{x+1}/c, q^{1-x}/b, q^{1-x}/c;\ q)_\infty a^x q^{x^2}\omega(x)\,dx \\
&= \int_0^1 (aq^{x+1}/b, aq^{x+1}/c, q^{1-x}/b, q^{1-x}/c;\ q)_\infty a^x q^{x^2}\omega(x) \\
&\quad \cdot {}_2\psi_2\left[\begin{matrix} bq^x, & cq^x \\ aq^{x+1}/b, & aq^{x+1}/c \end{matrix};\ q, -aq/bc\right] dx \\
&= \frac{(aq/bc;\ q)_\infty}{(-aq/bc;\ q)_\infty}(q^2, aq^2/b^2, aq^2/c^2;\ q^2)_\infty \\
&\quad \int_0^1 (aq^{2x+1}, q^{1-2x}/a;\ q^2)_\infty a^2 q^{x^2}\,\omega(x)\,dx\,.
\end{aligned}$$

Replacing a, b, c by $q^{\alpha-\beta}, q^{1-\beta}, q^{1-\delta}$, respectively, and assuming the $\alpha+\delta = \beta+\gamma$, this can be written as

$$\begin{aligned}
(4.4)\qquad &\int_{-\infty}^{\infty} (q^{\alpha+x}, q^{\beta-x}, q^{\gamma+x}, q^{\delta-x};\ q)_\infty q^{(\alpha-\beta)x+x^2}\omega(x)\,dx \\
&= \frac{(q^{\alpha+\delta-1};\ q)_\infty}{(-q^{\alpha+\delta-1};\ q)_\infty}(q^{\alpha+\beta}, q^{\gamma+\delta}, q^2;\ q^2)_\infty \\
&\quad \cdot \int_0^1 (q^{\alpha-\beta+1+2x}, q^{\beta-\alpha+1-2x};\ q^2)_\infty q^{(\alpha-\beta)x+x^2}\omega(x)\,dx\,.
\end{aligned}$$

In terms of the q-gamma function this can be written in the form

$$\begin{aligned}
(4.5)\qquad &\int_{-\infty}^{\infty} \frac{q^{(\alpha-\beta)x+x^2}\omega(x)\,dx}{\Gamma_q(\alpha+x)\Gamma_q(\beta-x)\Gamma_q(\gamma+x)\Gamma_q(\delta-x)} \\
&= \frac{(-q;\ q)_{\alpha+\delta-2}}{(1+q)^{\alpha+\delta-2}} \frac{\Gamma_{q^2}^2(\frac{1}{2})}{\Gamma_{q^2}(\frac{\alpha+\beta}{2})\Gamma_{q^2}(\frac{\gamma+\delta}{2})\Gamma_q(\alpha+\delta-1)} \\
&\quad \cdot \int_0^1 \frac{q^{(\alpha-\beta)x+x^2}\omega(x)\,dx}{\Gamma_{q^2}(\frac{\alpha-\beta+1}{2}+x)\,\Gamma_{q^2}(\frac{\beta-\alpha+1}{2}-x)}\,,
\end{aligned}$$

where

$$(4.6)\qquad (a;\ q)_\lambda = \frac{(a;\ q)_\infty}{(aq^\lambda;\ q)_\infty}\,,$$

$\alpha+\delta = \beta+\gamma$ and Re $(\alpha+\beta+\gamma+\delta) > 2$. It is easy to see that in the limit $q \to 1^-$ formula (4.5) approaches (4.1).

5. An integral analogue of the ${}_8\psi_8$ sum.

It is clear from (1.11) that we can associate a general ${}_r\psi_r$ series with a doubly infinite integral of the type considered in §3. However, there are no known summation formulas for a very-well-poised ${}_r\psi_r$ for $r > 6$, only transformation formulas. One could use these transformation formulas, see [**11**, chapter 5], to express such integrals in terms of a string of basic hypergeometric series, but the exercise does not seem to have a purpose, except for the case $r = 8$. The results that we shall obtain in this section will, hopefully, convince the reader that the case of ${}_8\psi_8$ series has some interesting features.

Let us consider the integral

$$(5.1)\ K := \int_{-\infty}^{\infty} (1 - aq^{2x})(aq^{x+1}/b, aq^{x+1}/c, aq^{x+1}/d, aq^{x+1}/e, aq^{x+1}/f;\ q)_\infty \cdot \frac{(q^{1-x}/b, q^{1-x}/c, q^{1-x}/d, q^{1-x}/e, q^{1-x}/f;\ q)_\infty}{(aq^{x+1}/g, q^{1-x}/g;\ q)_\infty} q^{2x^2-x} a^{2x} \omega(x)\, dx\,,$$

where $\omega(x)$ has the same properties as mentioned in §3. Notice that we have taken a pair of infinite products in the denominator also, which makes the structure of K slightly different from that of the integral J defined in (1.22) and (3.1). By exploiting the unit-periodic property of $\omega(x)$ we could avoid this difference but we believe the form of the integrand in (5.1) is more instructive. It is obvious that when g equals any one of the parameters b, c, d, e, f in the numerator then K will reduce to J. We will assume that

$$(5.2)\qquad \left|\frac{qga^2}{bcdef}\right| < 1\,,$$

which ensures the convergence of the integral. An application of (1.11) then gives

$$(5.3)\quad K = \int_0^1 (1 - aq^{2x})(aq^{x+1}/b, aq^{x+1}/c, aq^{x+1}/d, aq^{x+1}/e, aq^{x+1}/f; q)_\infty \cdot \frac{(q^{1-x}/b, q^{1-x}/c, q^{1-x}/d, q^{1-x}/e, q^{1-x}/f;\ q)_\infty}{(aq^{x+1}/g, q^{1-x}/g;\ q)_\infty} q^{2x^2-x}\omega(x) \cdot {}_8\psi_8\left[\begin{matrix} q^{x+1}a^{\frac12}, -q^{x+1}a^{\frac12}, & bq^x, & cq^x, & dq^x, & eq^x, & fq^x, aq^{x+1}/g \\ q^x a^{\frac12}, -q^x a^{\frac12}, aq^{x+1}/b, aq^{x+1}/c, aq^{x+1}/d, aq^{x+1}/e, aq^{x+1}/f, gq^x \end{matrix};\ q, \frac{qga^2}{bcdef}\right] dx\,.$$

Using the transformation formula [**11**, (5.6.2)] we get, for the ${}_8\psi_8$ series above

$$(5.4)\quad {}_8\psi_8[\,] = \frac{(q, aq/bf, aq/cf, aq/df, aq/ef, qf/b, qf/c, qf/d, qf/e;\ q)_\infty}{(qf/g, aq/fg, qf^2/a;\ q)_\infty (aq^{x+1}/b, aq^{x+1}/c, aq^{x+1}/d, aq^{x+1}/e;\ q)_\infty}$$
$$\cdot \frac{(aq^{x+1}/g, q^{1-x}/g, aq^{2x+1}, q^{-2x}/a;\ q)_\infty}{(aq^{x+1}/f, q^{1-x}/b, q^{1-x}/c, q^{1-x}/d, q^{1-x}/e, q^{1-x}/f;\ q)_\infty}$$
$$\cdot {}_8W_7(f^2/a;\ bf/a, cf/a, df/a, ef/a, gf/a;\ q, qga^2/bcdf)$$
$$+ \frac{(q, aq^2/bg, aq^2/cg, aq^2/dg, aq^2/eg, g/b, g/c, g/d, g/e;\ q)_\infty}{(qf/g, fg/aq, aq^3/g^2;\ q)_\infty (aq^{x+1}/b, aq^{x+1}/c, aq^{x+1}/d, aq^{x+1}/e;\ q)_\infty}$$
$$\cdot \frac{(fq^x, fq^{-x}/a, aq^{2x+1}, q^{-2x}/a;\ q)_\infty}{(gq^x, gq^{-x}/a, q^{1-x}/b, q^{1-x}/c, q^{1-x}/d, q^{1-x}/e;\ q)_\infty}$$
$$\cdot {}_8W_7(aq^2/g^2;\ bq/g, cq/g, dq/g, eq/g, fq/g;\ q, qga^2/bcdf),$$

where

$$(5.5)\quad {}_8W_7(a;\ b, c, d, e, f;\ q, z) := {}_8\phi_7\left[\begin{matrix} a, qa^{\frac12}, -qa^{\frac12}, & b, & c, & d, & e, & f \\ a^{\frac12}, & -a^{\frac12}, aq/b, aq/c, aq/d, aq/e, aq/f \end{matrix};\ q, z\right].$$

Choosing

$$(5.6)\qquad \omega(s) = \frac{q^{s-2s^2}a^{-2s}p(s)}{(aq^{2s}, q^{-2s}/a;\ q)_\infty}, \qquad p(s\pm 1) = p(s),$$

in (5.3) we obtain the following identity

$$(5.7)\quad \int_{-\infty}^{\infty} \frac{(aq^{x+1}/b, aq^{x+1}/c, aq^{x+1}/d, aq^{x+1}/e, aq^{x+1}/f;\ q)_\infty}{(aq^{2x+1}, q^{1-2x}/a;\ q)_\infty}$$
$$\cdot \frac{(q^{1-x}/b, q^{1-x}/c, q^{1-x}/d, q^{1-x}/e, q^{1-x}/f;\ q)_\infty}{(aq^{x+1}/g, q^{1-x}/g;\ q)_\infty} p(x)\, dx$$
$$= \frac{(q, aq/bf, aq/cf, aq/df, aq/ef, qf/b, qf/c, qf/d, qf/e;\ q)_\infty}{(aq/fg, qf/g, qf^2/a;\ q)_\infty}$$
$$\cdot {}_8W_7(f^2/a;\ bf/a, cf/a, df/a, ef/a, gf/a;\ q, qga^2/bcdf) \cdot \int_0^1 p(x)\, dx$$
$$+ \frac{(q, g/b, g/c, g/d, g/e, aq^2/bg, aq^2/cg, aq^2/dg, aq^2/eg;\ q)_\infty}{(fg/aq, aq^3/g^2, qf/g;\ q)_\infty}$$
$$\cdot {}_8W_7(aq^2/g^2;\ bq/g, cq/g, dq/g, eq/g, fq/g;\ q, qga^2/bcdf)$$
$$\cdot \int_0^1 \frac{(fq^x, q^{1-x}/f, aq^{x+1}/f, fq^{-x}/a;\ q)_\infty}{(gq^x, q^{1-x}/g, aq^{x+1}/g, gq^{-x}/a;\ q)_\infty} p(x)\, dx,$$

provided $\arg a$ and $\arg g$ are neither 0 nor multiples of 2π.

Replacing a by α^2 and $q/b, q/c, q/d, q/e, q/f, q/g$ by $a/\alpha, b/\alpha, c/\alpha, d/\alpha, e/\alpha$ and f/α, respectively, we obtain

$$
\begin{aligned}
(5.8)\quad &\int_{-\infty}^{\infty} \frac{(a\alpha q^x, aq^{-x}/\alpha, b\alpha q^x, bq^{-x}/\alpha, c\alpha q^x, cq^{-x}/\alpha, d\alpha q^x, dq^{-x}/\alpha;\ q)_\infty}{(\alpha^2 q^{2x+1}, q^{1-2x}/\alpha^2;\ q)_\infty} \\
&\qquad \cdot \frac{(e\alpha q^x, eq^{-x}/\alpha;\ q)_\infty}{(f\alpha q^x, fq^{-x}/\alpha;\ q)_\infty}\, p(x)\, dx \\
&= \frac{(q, aq/e, bq/e, cq/e, dq/e, ae/q, be/q, ce/q, de/q;\ q)_\infty}{(ef/q, qf/e, q^3/e^2;\ q)_\infty} \\
&\cdot {}_8W_7(q^2/e^2;\ q^2/ae, q^2/be, q^2/ce, q^2/de, qf/e;\ q, abcde/fq^3) \int_0^1 p(x)\, dx \\
&+ \frac{(q, af, bf, cf, df, q/f, b/f, c/f, d/f;\ q)_\infty}{(q/ef, qf/e, qf^2;\ q)_\infty} \\
&\cdot {}_8W_7(f^2;\ qf/a, qf/b, qf/c, qf/d, qf/e;\ q, abcde/fq^3) \\
&\cdot \int_0^1 \frac{(\alpha q^{x+1}/e, eq^{-x}/\alpha, \alpha e q^{x+1}, q^{-x}/\alpha e;\ q)_\infty}{(\alpha q^{x+1}/f, fq^{-x}/\alpha, \alpha f q^{x+1}, q^{-x}/\alpha f;\ q)_\infty}\, p(x)\, dx,
\end{aligned}
$$

where the parameters α and f are such that no zeros occur in any denominator. Recall the transformation formula [**11**, (2.11.1)],

$$
\begin{aligned}
(5.9)\quad &{}_8W_7(q^2/e^2;\ qf/e, q^2/ae, q^2/be, q^2/ce, q^2/de;\ q, abcde/fq^3) \\
&= \frac{(q^3/e^2, cd/q, ac/q, ad/q, bq/d, bq/a, ef/q, fq^3/ade;\ q)_\infty}{(aq/e, cq/e, dq/e, be/q, acde/q^3, bq^3/ade, fq/d, fq/a;\ q)_\infty} \\
&\cdot {}_8W_7(bq^2/ade;\ b/f, bc/q, q^2/ad, q^2/de, q^2/ae;\ q, fq/c) \\
&- \frac{(q^3/e^2, ef/q, bf, cf, df, af, q^2/ae, q^2/ce, q^2/de, b/f, acd/fq^2;\ q)_\infty}{(aq/e, bq/e, cq/e, dq/e, be/q, fq/a, fq/c, fq/d, acde/q^3, q^4/acde;\ q)_\infty} \\
&\frac{(fq^3/acd;\ q)_\infty}{(d/ef, qf^2;\ q)_\infty}\, {}_8W_7(f^2;\ qf/a, qf/b, qf/c, qf/d, qf/e;\ q, abcde/fq^3).
\end{aligned}
$$

Note that the last ${}_8W_7$ series on the right is the same as that on the right side of (5.8). This enables us to rewrite (5.8) in the form

$$
(5.10)\quad \int_{-\infty}^{\infty} \frac{(a\alpha q^x, aq^{-x}/\alpha, b\alpha q^x, bq^{-x}/\alpha, c\alpha q^x, cq^{-x}/\alpha,}{(\alpha^2 q^{2x+1}, q^{1-2x}/\alpha^2,}
$$

$$\frac{d\alpha q^x, dq^{-x}/\alpha, e\alpha q^x, eq^{-x}/\alpha; q)_\infty}{f\alpha q^x, fq^{-x}/\alpha; q)_\infty} p(x) dx$$

$$= \frac{(q, ac/q, ad/q, cd/q, ae/q, ce/q, de/q, bq/a, bq/d, bq/e, fq^3/ade;\ q)_\infty}{(qf/a, qf/d, qf/e, bq^3/ade, acde/q^3;\ q)_\infty}$$

$$\cdot {}_8W_7(bq^2/ade;\ b/f, bc/q, q^2/ad, q^2/de, q^2/ae;\ q, fq/c) \int_0^1 p(x)\, dx$$

$$+ {}_8W_7(f^2;\ qf/a, qf/b, qf/c, qf/d, qf/e;\ q, abcde/fq^3)$$

$$\cdot \left\{ \frac{(q, a/f, b/f, c/f, d/f, af, bf, cf, df;\ q)_\infty}{(qf/e, q/ef, qf^2;\ q)_\infty} \lambda(e,f) \right.$$

$$- \frac{(q, af, bf, cf, df, b/f, ae/q, ce/q, de/q, q^2/ae, q^2/ce, q^2/de;\ q)_\infty}{(qf/e, q/ef, qf^2, fq/a, fq/c, fq/d;\ q)_\infty}$$

$$\left. \cdot \frac{(acd/fq^2, fq^3/acd;\ q)_\infty}{(acde/q^3, q^4/acde;\ q)_\infty} \int_0^1 p(x)\, dx \right\},$$

where

$$\text{(5.11)}\quad \lambda(e,f) = \int_0^1 \frac{(\alpha q^{x+1}/e, eq^{-x}/\alpha, \alpha e q^{x+1}, q^{-x}/\alpha e;\ q)_\infty}{(\alpha q^{x+1}/f, fq^{-x}/\alpha, \alpha f q^{x+1}, q^{-x}/\alpha f;\ q)_\infty} p(x)\, dx\,.$$

The special case $f = abcde/q^4$ is of particular interest because it gives an overall balance of the parameters inside the integral on the left side while reducing the first ${}_8W_7$ series on the right to a very-well-poised ${}_6\phi_5$, which is summable by use of [**11**, (2.7.1)]. This leads to the formula

$$\text{(5.12)}\quad \int_{-\infty}^{\infty} \frac{(a\alpha q^x, aq^{-x}/\alpha, b\alpha q^x, bq^{-x}/\alpha, c\alpha q^x, cq^{-x}/\alpha;\ q)_\infty}{(\alpha^2 q^{2x+1}, q^{1-2x}/\alpha^2;\ q)_\infty}$$

$$\cdot \frac{(d\alpha q^x, dq^{-x}/\alpha, e\alpha q^x, eq^{-x}/\alpha, ;\ q)_\infty}{(\alpha abcde q^{x-4}, abcde q^{-x-4}/\alpha;\ q)_\infty} p(x)\, dx$$

$$= \frac{(q, ab/q, ac/q, ad/q, ae/q, bc/q, bd/q, be/q, cd/q, ce/q, de/q; q)_\infty}{(qf/a, qf/b, qf/c, qf/d, qf/e;\ q)_\infty} \int_0^1 p(x)\, dx$$

$$+ \frac{(q, af, bf, cf, df, a/f, b/f, c/f, d/f;\ q)_\infty}{(qf^2, q/ef, qf/e;\ q)_\infty}$$

$$\cdot \left\{ \lambda(e,f) - \frac{(ae/q, be/q, ce/q, de/q, q^2/ae, q^2/be, q^2/ce, q^2/de;\ q)_\infty}{(a/f, b/f, c/f, d/f, qf/a, qf/c, qf/c, qf/d;\ q)_\infty} \right.$$

$$\left. \cdot \int_0^1 p(x)\, dx \right\} {}_8W_7(f^2;\ qf/a, qf/b, qf/c, qf/d, qf/e;\ q, q)\,,$$

with

$$\text{(5.13)}\qquad f = abcde/q^4\,.$$

This formula is essentially the same as the one found in [**18**].

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