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v_1-PERIODIC HOMOTOPY GROUPS OF $Sp(n)$

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In this paper we calculate the 2-primary v_1-periodic homotopy groups of the symplectic groups $Sp(n)$. The proof utilizes new methods of calculating the unstable Novikov spectral sequence. One corollary is that some homotopy group of $Sp(n)$ contains an element of order 2^{2n-1}.

1. Main theorem.

In this paper we calculate the 2-primary v_1-periodic homotopy groups of the symplectic groups $Sp(n)$. The odd-primary v_1-periodic homotopy groups of $Sp(n)$ were determined in [**13**]; they are an immediate consequence of the groups of $SU(n)$. In Section 5, we apply our results to James numbers and exponents of actual homotopy groups.

The v_1-periodic homotopy groups of a space, whose definition we recall in the next paragraph, are a localization of the actual homotopy groups. Very roughly, they are the portion of the homotopy groups detectable by real and complex K-theory and their operations. A first attempt at computing the v_1-periodic homotopy groups of $Sp(n)$ was made in [**16**]. The method there was to study the exact sequence of v_1-periodic homotopy groups associated to the fibration

$$Sp(n-1) \to Sp(n) \to S^{4n-1}. \tag{1.1}$$

Difficulties in determining the homomorphisms in this exact sequence occurred in [**16**] as early as $n = 3$; indeed, there is a minor mistake in the calculation of $v_1^{-1}\pi_*(Sp(3))$ in [**16**].[1] We employ a similar method, but we use the unstable Novikov spectral sequence (UNSS) to lend precision to the calculations.

The p-primary v_1-periodic homotopy groups of any space X were defined in [**15**] by

$$v_1^{-1}\pi_n(X) = \operatorname*{dirlim}_e v_1^{-1}\pi_{n+1}(X;\mathbb{Z}/p^e), \tag{1.2}$$

[1] The d_2-differential in the chart on [**16**, p. 228] should not be present. The image of Mimura and Toda's boundary morphism cited there is in fact not v_1-periodic.

where

$$v_1^{-1}\pi_i(X;\mathbb{Z}/p^e) = \operatorname*{dirlim}_N\,[M^{i+Nqp^{e-1}}(p^e),X],$$

with the latter direct system being taken over Adams maps of the mod p^e Moore spaces. We say that X has an H-space exponent at p if, for some e and L, $\Omega^L X \xrightarrow{p^e} \Omega^L X$ is null-homotopic. For such spaces, which include spheres and compact Lie groups, $v_1^{-1}\pi_n(X)$ is a direct summand of some $\pi_L(X)$, provided $v_1^{-1}\pi_n(X)$ is a finitely-generated abelian group, which is true in the cases just mentioned. The importance of $v_1^{-1}\pi_*(X)$ is that it is often computable and yet gives significant information about actual homotopy groups. Much of the importance of our calculation of $v_1^{-1}\pi_*(Sp(n))$ here is just that we were able to do it. It was an extremely challenging problem, requiring new methods of analysis of the UNSS.

Moreover, our result is new in a fundamental sense. The results for $v_1^{-1}\pi_*(SU(n))$ proved in [**13**] and [**9**] yielded the same numbers that had come up in the work of Crabb and Knapp ([**11**]), although it was important to learn that all v_1-periodic homotopy classes were of this standard type. The same could be said for the large summands which we obtain in $v_1^{-1}\pi_*(Sp(n))$; closely related groups have come up in the work of Morisugi ([**26**]) and Walker ([**32**]). But the way in which the $\mathbb{Z}_2$'s from the various spheres contribute to $Sp(n)$, as described in Proposition 1.8, is completely new, and should have further ramifications.

Let ϵ_k be defined by $0 \le \epsilon_k \le 1$ and $\epsilon_k \equiv k \bmod 2$. Our results involve numbers defined for $k > n$ by

$$(1.3)\; e(k,n) = \epsilon_{k-1} + \min_{j>n}\left(\nu\left(\operatorname{coef}\left(\frac{t^{2k}}{(2k)!},\frac{(e^t+e^{-t}-2)^j}{2j}\right)\right) - \epsilon_{k-j}\right).$$

As usual, $\nu(-) = \nu_2(-)$ denotes the exponent of 2. The numbers $e(k,n)$ were called $d_2^A(k,n)$ in [**26**]. They are periodic in k, and are extended to $k \le n$ using this periodicity. This periodicity is proved similarly to [**12**, 3.15], using the following alternate form for the coefficients in (1.3).

$$(1.4) \qquad \tfrac{1}{j}\sum_{i=0}^{j-1}(-1)^i\tbinom{2j}{i}(j-i)^{2k}.$$

In Section 5, we will tabulate some values of the numbers $e(k,n)$, which, along with the explicit calculations in [**26**], may shed some light on an unilluminating definition.

We now state our main result. We will denote by $G(n)$ some abelian group satisfying $|G(n)| = n$. We make no claim about the precise structure of $G(n)$, and, moreover, if $G(n)$ occurs in two different formulas with the same n, we do not claim that the two groups are isomorphic. We denote by $\mathbb{Z}_2$ the cyclic group of order 2, and by nG the direct sum of n copies of the abelian group G.

Theorem 1.5. *For $1 \le t \le 8$ and any integer h, the 2-primary v_1-periodic homotopy group*

$$v_1^{-1}\pi_{8h+t}(Sp(n)) = \begin{cases} G(2^{e(2h+1,n)}) & \text{if } t=1 \\ \mathbb{Z}/2^{e(2h+1,n)} & \text{if } t=2 \\ \mathbb{Z}_2 & \text{if } t=3 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } t=4 \\ G(2^{1+e(2h+2,n)}) & \text{if } t=5 \\ \mathbb{Z}/2^{e(2h+2,n)} \oplus [\log_2(\frac{4n}{3})]\mathbb{Z}_2 & \text{if } t=6 \\ G(2^{2[\log_2(4n/3)]}) & \text{if } t=7 \\ [\log_2(\frac{4n}{3})]\mathbb{Z}_2 & \text{if } t=8. \end{cases}$$

The $G(-)$-group when $t = 1$, 5, or 7 has at least $[\log_2(4n/3)]$ direct summands. The group $v_1^{-1}\pi_{8h+7}(Sp(n))$ is an extension (possibly the trivial extension) of two $\mathbb{Z}_2$-vector spaces of dimension $[\log_2(4n/3)]$.

One novel aspect of the result is the occurrence of arbitrarily many $\mathbb{Z}_2$'s in a single v_1-periodic homotopy group, which had not occurred explicitly[2] in previous examples such as $SU(n)$. For the reader who wishes to obtain some feeling for these groups without studying the entire proof, we recommend reading the rest of this section, together with the portion of Section 5 which tabulates values of the numbers $e(k,n)$. You might also take a look at Figure 4.4, which displays the final stage of the UNSS converging to $v_1^{-1}\pi_*(Sp(n))$, and the recapitulation of the proof which precedes that figure.

The heart of the proof is the calculation of the E_2-term of the UNSS converging to $v_1^{-1}\pi_*(Sp(n))$. This is done using the exact sequences of v_1-periodic UNSS's induced by (1.1) to give a spectral sequence (SS) converging to $v_1^{-1}E_2(Sp(n))$ whose initial term is

$$\bigoplus_{1\le i\le n} v_1^{-1}E_2^{s,t}(S^{4i-1}), \tag{1.6}$$

[2]But see Proposition 1.17.

where $v_1^{-1}E_2$ refers to the E_2-term of the v_1-periodic UNSS of [**4**]. We call this the cellular SS, or CSS. Strictly speaking, this isn't quite accurate, since $Sp(n)$ has more cells than the S^{4i-1}, but the name "cellular" seems to have the right flavor.

A complete description of $v_1^{-1}E_2(S^{4i-1})$ is given in [**4**, p. 58] and reproduced in our Figure 3.24. In later sections, we will make use of some of the technical names of classes given there, but to describe the result, we prefer the simpler form which we give in Figure 1. Note that the horizontal component in Figure 1 refers to the homotopy group, and not the stem, as it does in Figure 3.24. The letter i is part of the notation of all of these classes in $v_1^{-1}E_2(S^{4i-1})$ because we shall be combining the charts for various values of i. The letters i, i', i_u, and i'_u each represent an η-tower; this is an infinite collection of elements $\{\eta^j x : j \geq 0\}$ of order 2, where η is the UNSS representative of the Hopf map. The action of the element η, which we will also denote by h_1, moves one position to the right, and one position up; the diagonal arrows in Figure 1 suggest this. The boxes which contain the elements U_i and i'_u also contain a second element, which is part of the η-tower labeled $i-1$, and is implicit by the periodicity of the chart. The symbols S_i and U_i represent cyclic 2-groups of the same order. The letters suggest "stable" and "unstable." The groups S_i and U_i are $\mathbb{Z}/8$ if $i \equiv k \bmod 2$, while if $i \not\equiv k \bmod 2$, they are $\mathbb{Z}/2^m$ with $m = \min(\nu(k-i+1)+3, n)$.

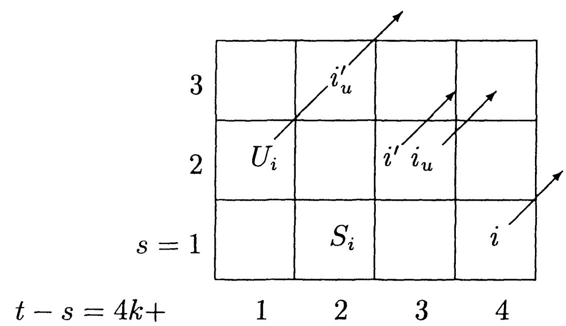

Figure 1.7: $v_1^{-1}E_2^{s,t}(S^{4i-1})$

These charts are superimposed as in (1.6), with d_r-differentials always going from $E^{s,t}$ to $E^{s+1,t}$, and decreasing i by r. Thus they always go to the group one unit above and one to the left of the source class. There are d_r-differentials from S_i-groups to U_{i-r}-groups which cause $v_1^{-1}E_2^{1,4k+3}(Sp(n))$ to be cyclic of order $2^{e_{Sp}(k+1,n)}$, as proved in [**10**]. Here $e_{Sp}(k+1, n)$ is as defined in (1.15). Since $|U_i| = |S_i|$ and there are no differentials in the CSS going out from the groups U_i, the portion of the group $v_1^{-1}E_2^{2,4k+3}(Sp(n))$ built from the U_i's will also have order $2^{e_{Sp}(k+1,n)}$, but it will not necessarily be cyclic.

In Section 3, we will prove the following result, which establishes all differentials among the η-towers in the CSS.

Proposition 1.8. *In the CSS of* (1.6), *the differentials among the η-towers depicted in Figure* 1.7 *satisfy, for all values of k,*

$$\begin{array}{llll}
(1.9) & d_i(2i) = & i_u & \text{if } i = 2^e \text{ or } 3\cdot 2^e,\ e \geq 0 \\
(1.10) & d_i((2i)') = & i'_u & \text{if } i = 2^e,\ e \geq 0 \\
(1.11) & d_{2^{e+1}}((2i+3)2^e) = & ((2i+1)2^e)' & \text{if } i \geq 1,\ e \geq 0 \text{ or } i = e = 0 \\
(1.12) & d_{2^{e+1}}(((2i+3)2^e)_u) = & ((2i+1)2^e)'_u & \text{if } i \geq 1,\ e \geq 0,
\end{array}$$

provided $2i \leq n$ in (1.9) *and* (1.10), *and $(2i+3)2^e \leq n$ in* (1.11) *and* (1.12).

The restrictions are present because in other cases one of the classes will have been involved in an earlier differential. For example, $d_5(10) = 5_u$ is masked by $d_4(10) = 6'$ and $d_2(5_u) = 3'_u$.

The reader can easily verify that every class except "1" is involved in one of the differentials if $n = \infty$. This is consistent with Bott periodicity for Sp. However, if n satisfies $i - r \leq n < i$, then a d_r-differential from a class numbered i to one numbered $i - r$ cannot occur in $Sp(n)$, and so it frees the $(i-r)$-class to survive the CSS. In Proposition 1.14, we enumerate the number of unstable η-towers that survive all the differentials in $Sp(n)$.

We illustrate in Figure 1.13 the pattern of differentials among the η-towers in the CSS converging to $v_1^{-1}E_2(Sp(10))$. Classes in Figure 1.13 which are not involved in differentials survive to η-towers in $v_1^{-1}E_2(Sp(10))$. In Figure 1.16, we will show how the surviving classes in $v_1^{-1}E_2(Sp(n))$ are situated, and in Theorem 4.1 and Corollary 4.3 we will establish the d_3-differentials among them. Figure 4.4 shows a final picture of the UNSS for $Sp(n)$. Theorem 1.5 is a consequence of this analysis.

The following result shows how the $[\log_2(4n/3)]$ in Theorem 1.5 arises, as an enumeration of the number of η-towers in $v_1^{-1}E_2(Sp(n))$.

Proposition 1.14. *The number of classes of the form i'_u in $Sp(n)$ which are not hit by differentials in Proposition* 1.8 *is* $[\log_2(4n/3)]$.

There is only one class, namely "1," of the form "i" which does not support a differential. Consequently, there will also be $1 + [\log_2(4n/3)]$ η-towers beginning in the box at height 2 and $t - s = 4k + 3$ which survive the CSS.

1	2	3		4
	$1'_u \leftarrow$	$2'$	$1_u \leftarrow$	2
	$2'_u \leftarrow$	$4'$	$2_u \leftarrow$	4
	$3'_u \leftarrow$	5_u	$3_u \leftarrow$	6
	$4'_u \leftarrow$	$8'$	$4_u \leftarrow$	8
	$5'_u \leftarrow$	7_u	$6' \leftarrow$	10
	$6'_u \leftarrow$	10_u		1
	$7'_u \leftarrow$	9_u	$1' \leftarrow$	3
	$8'_u$	$10'$	$3' \leftarrow$	5
	$9'_u$	6_u	$5' \leftarrow$	7
	$10'_u$	8_u	$7' \leftarrow$	9
			$9'$	

Figure 1.13: Cellular SS for $v_1^{-1}E_2(Sp(10))$

To understand this, the reader may find it helpful to refer to Figure 1.13, where $[\log_2(40/3)] = 3$.

Proof of Proposition 1.14. We must count the number of integers $j \leq n$ for which $2j > n$ if j is a 2-power, and $j + 2^{\nu(j)+1} > n$ if j is not a 2-power. There is one such 2-power.

Define

$$f(x, y) = \sum_{j \in S} x^j \sum_{n=j}^{j+2^{\nu(j)+1}-1} y^n,$$

where S is the set of positive integers which are not 2-powers. Our desired answer is 1 plus the coefficient of y^n in $f(1, y)$. Collecting terms over common values of $\nu(j)$ yields

$$f(x, y) = \sum_{e \geq 0} x^{2^e} y^{2^e} \left(\frac{x^{2^{e+1}} y^{2^{e+1}}}{1 - x^{2^{e+1}} y^{2^{e+1}}} \frac{1 - y^{2^{e+1}}}{1 - y} \right).$$

Setting $x = 1$ allows some cancellation in the latter product, from which we obtain

$$f(1, y) = \frac{\sum y^{3 \cdot 2^e}}{1 - y}.$$

The coefficient of y^n in this is the number of $3 \cdot 2^e$'s which are equal to or less than n. This is 1 plus the number of 2^e's with $e \geq 1$ which are less than or equal to $n/3$. This number of 2^e's is $[\log_2(n/3)]$, from which we obtain the answer claimed. □

The result of the differentials of Proposition 1.8 is the following chart, Figure 1.16, for $v_1^{-1}E_2(Sp(n))$. Here a • represents a $\mathbb{Z}_2$ (of the type "1" when $t-2s \equiv 3 \bmod 4$, and n' or $(n-1)'$ when $t-2s \equiv 1 \bmod 4$), while "log" represents a $\mathbb{Z}_2$-vector space of dimension $[\log_2(4n/3)]$. For $j=1$ and 2, we abbreviate $e_{Sp}(2h+j,n)$ to e_j in the chart. Here and elsewhere $e_{Sp}(k,n)$ is as in **[10]**; it is defined by

$$(1.15) \qquad e_{Sp}(k,n) = \min_{j>n}\left(\nu_2\left(\operatorname{coef}\left(\frac{t^{2k}}{(2k)!}, \frac{(e^t+e^{-t}-2)^j}{2j}\right)\right)\right).$$

These numbers are obtained from the numbers $e(k,n)$ defined in (1.3) by removing the numbers ϵ_{k-1} and ϵ_{k-j}. As already noted, $e_{Sp}(k,n)$ determines the size of the 1-line group of the UNSS. We shall show in Section 4 that the e_{k-1} which is added in forming $e(k,n)$ corresponds to an extension to the 3-line of the SS, and the e_{k-j} which might be subtracted corresponds to a d_3-differential in the SS.

As in Theorem 1.5, $G(2^e)$ indicates an abelian group of order 2^e. These 2-line groups are built from the U_i-groups of Figure 1. In Section 4, we will discuss how Theorem 1.5 is deduced from this chart, by inserting d_3-differentials and extensions.

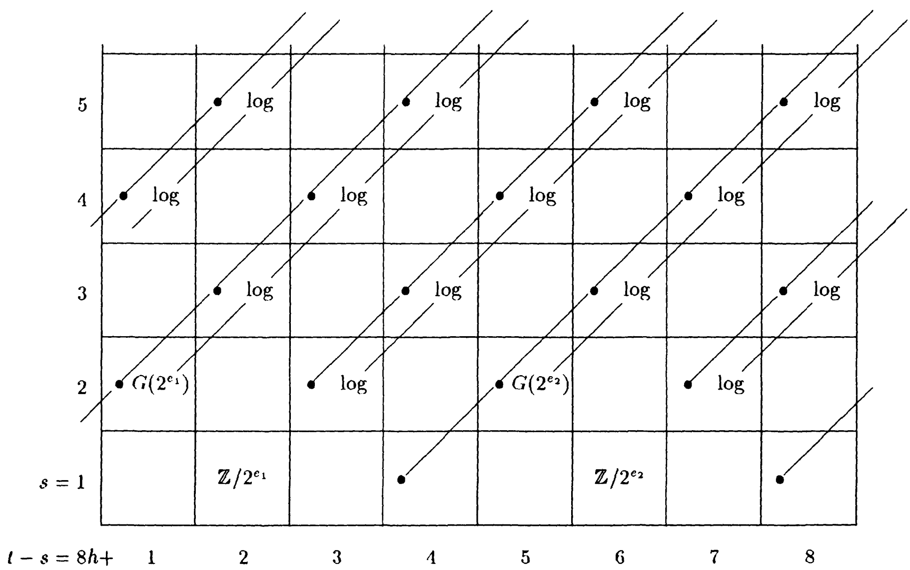

Figure 1.16: $v_1^{-1}E_2^{s,t}(Sp(n))$

Except for changes in values of e_i, this $v_1^{-1}E_2(Sp(n))$ has horizontal period 4; the period 8 is used in Figure 1.16 in order to display the η-towers more clearly, and for later use in inserting d_3-differentials, most of which have period 8. The only additional justification required for the chart is for the η-extension from the G in filtration 2 to the log in filtration 3. This is a consequence of [**4**, 5.4], which says that $\eta : v_1^{-1}E_2^{2,t}(Sp(n)) \to v_1^{-1}E_2^{3,t+2}(Sp(n))$ is surjective. This η-extension implies that $G(2^{e(k,n)})$ has at least $[\log_2(4n/3)]$ summands for all values of k.

A similar analysis yields new and surprising information about

$$v_1^{-1}\pi_*(SU(2n+1)).$$

We recall from [**9**, 1.1] that $v_1^{-1}\pi_{2k}(SU(2n+1))$ is a cyclic 2-group of exponent $\min\{\nu(\mathrm{coef}(x^k/k!, (e^x-1)^j)) : 2n+1 \le j \le k\}$, while $v_1^{-1}\pi_{2k-1}(SU(2n+1))$ is an abelian group of the same order. We can now show that $v_1^{-1}\pi_{2k-1}(SU(2n+1))$ will often have many summands.

Proposition 1.17. *For every integer j, the abelian group*

$$v_1^{-1}\pi_{8j-1}(SU(2n+1))$$

can be written as the direct sum of at least $[\log_2(4n/3)]$ summands.

Proof. Similarly to [**10**, §2], there is an exact sequence

$$\begin{aligned}(1.18)\qquad \to v_1^{-1}\mathrm{Ext}_{\mathcal{U}}^{1,8j+1}(BP_*(\Sigma HP^n)) \to v_1^{-1}E_2^{2,8j+1}(Sp(n))\\ \to v_1^{-1}E_2^{2,8j+1}(SU(2n+1)) \to .\end{aligned}$$

In the case $n = \infty$ of this sequence, $v_1^{-1}\mathrm{Ext}_{\mathcal{U}}^{1,8j+1}(BP_*(\Sigma HP^\infty))$ sits between two 0 groups (using [**10**, 1.1]) and hence is 0. Thus the exact sequence

$$\begin{aligned}v_1^{-1}\mathrm{Ext}_{\mathcal{U}}^{0,8j+1}(BP_*(S^{4n+5})) \to v_1^{-1}\mathrm{Ext}_{\mathcal{U}}^{1,8j+1}(BP_*(\Sigma HP^n))\\ \to v_1^{-1}\mathrm{Ext}_{\mathcal{U}}^{1,8j+1}(BP_*(\Sigma HP^{n+1}))\end{aligned}$$

implies that $v_1^{-1}\mathrm{Ext}_{\mathcal{U}}^{1,8j+1}(BP_*(\Sigma HP^n))$ is 0 for $n \ge 2j$, is a quotient of $\mathbb{Z}$ (and hence cyclic) for $n = 2j-1$, and is a subgroup of this cyclic group, and hence cyclic, for $n < 2j-1$. By the calculation of $v_1^{-1}E_2(Sp(n))$, whose result is illustrated in Figure 1.16, $v_1^{-1}E_2^{2,8j+1}(Sp(n))$ is a direct sum of $1+[\log_2(4n/3)]$ copies of $\mathbb{Z}_2$. Thus by (1.18) $v_1^{-1}E_2^{2,8j+1}(SU(2n+1))$ must have at least $[\log_2(4n/3)]$ direct summands.

Now we refer to the analysis in [**9**] beginning on page 485 which involves the space $R_m = SU(2m+1)/SU(2m-1)$. This space was called Q_m in [**9**], but we have changed the name to avoid confusion with the quaternionic quasi-projective space Q^n, which will occur frequently in this paper. It was shown in [**9**] that $v_1^{-1}\pi_*(R_m) \to v_1^{-1}E_2^{2,*+2}(R_m)$ is bijective when $*$ is odd. Since by [**9**, 2.3a] $v_1^{-1}\pi_*(R_m) \to v_1^{-1}E_2^{1,*+1}(R_m)$ is bijective when $*$ is even, the 5-lemma and induction imply that

$$v_1^{-1}\pi_*(SU(2n+1)) \to v_1^{-1}E_2^{2,2+*}(SU(2n+1))$$

is bijective when $*$ is odd. Here we have also used injectivity of

$$v_1^{-1}\pi_{2j}(SU(2n-1)) \to v_1^{-1}\pi_{2j}(SU(2n+1))$$

and the concentration of the v_1-periodic E_2-term of R_m, and hence $SU(2n-1)$, in filtrations 1 and 2.

The conclusions of the above two paragraphs imply the proposition. □

In order to handle some of the d_3-differentials in the periodic UNSS for $Sp(n)$, we will need a comparison with the periodic (stable) NSS for the quaternionic quasi-projective space Q^n. These spaces, discussed in [**26**] and [**32**], have one cell in each dimension $4i-1$ for $1 \le i \le n$, and embed naturally in $Sp(n)$. The analysis of this periodic NSS, which may be of independent interest, will be carried out in Section 2. Some of the techniques employed there will be used in the more delicate UNSS calculations which follow.

We close the introduction by presenting, for comparison, the easier calculation of $v_1^{-1}\pi_*(Sp)$. Note the shift of dimensions: $\mathbb{Z}/2^\infty \subset v_1^{-1}\pi_i(Sp)$ if and only if $\mathbb{Z} \subset \pi_{i+1}(Sp)$.

Theorem 1.19. *For any prime p,*

$$v_1^{-1}\pi_i(SU) = \begin{cases} \mathbb{Z}/p^\infty & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

If $p=2$, then

$$v_1^{-1}\pi_i(Sp) = \begin{cases} \mathbb{Z}/2^\infty & \text{if } i \equiv 2 \bmod 4 \\ \mathbb{Z}_2 & \text{if } i \equiv 4 \text{ or } 5 \bmod 8 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. As SU is similar and easier, we just prove it for Sp. The exact sequence

$$\pi_{n+1}(Sp) \xrightarrow{2^e} \pi_{n+1}(Sp) \to [M^{n+1}(2^e), Sp] \to \pi_n(Sp) \xrightarrow{2^e} \pi_n(Sp)$$

and Bott's results for $\pi_*(Sp)$ show that $[M^{n+1}(2^e), Sp]$ has summands, for $n>0$, of $\mathbb{Z}/2^e$ if $n \equiv 2 \bmod 4$, $\mathbb{Z}_2$'s coming from the bottom cell of the

Moore space if $n \equiv 4$ or 5 mod 8, and $\mathbb{Z}_2$'s coming from the top cell of the Moore space if $n \equiv 3$ or 4 mod 8. The Adams map induces an isomorphism

$$[M^{n+1}(2^e), Sp] \xrightarrow{A^*} [M^{n+2^{e+1}+1}(2^e), Sp],$$

and so $v_1^{-1}\pi_n(Sp; \mathbb{Z}/2^e)$ is the sum of the above summands. We consider the morphism

$$v_1^{-1}\pi_n(Sp; \mathbb{Z}/2^e) \xrightarrow{\rho^*} v_1^{-1}\pi_n(Sp; \mathbb{Z}/2^{e+1})$$

induced by the map ρ which has degree 1 on the bottom cell and degree 2 on the top cell. This sends the summand $\mathbb{Z}/2^e$ injectively, is 0 on the $\mathbb{Z}_2$'s due to the top cell of the Moore space, and sends the $\mathbb{Z}_2$'s due to the bottom cell of the Moore space injectively. Clearly, the direct limit of (1.2) for $X = Sp$ is as claimed in this theorem. $\square$

2. The periodic stable Novikov spectral sequence.

The 2-primary v_1-periodic homotopy groups of a spectrum X are defined in [**15**] by

$$v_1^{-1}\pi_i(X) = \operatorname*{dirlim}_{e,L} \pi_i(X \wedge M^{-L2^e}(2^e)).$$

Here $M^n(k)$ is a mod k Moore spectrum with top cell in dimension n, and the direct system is over Adams maps of Moore spaces (for increasing L) and maps ρ' defined below (for increasing e). This is compatible with the definition, (1.2), of v_1-periodic unstable homotopy groups since the Moore spectrum is self-dual. The v_1-periodic stable homotopy groups of a space X, denoted $v_1^{-1}\pi_*^s(X)$, are defined to be the v_1-periodic homotopy groups of the suspension spectrum of X.

Analogously, we define the v_1-periodic Novikov SS (NSS) of a space or spectrum X by

$$v_1^{-1}E_r^{s,t}(X) = \operatorname*{dirlim}_{L,e} E_r^{s-1,t-1}(X \wedge M^{-L2^e}(2^e)), \tag{2.1}$$

where the SS on the right is the ordinary NSS. As described in [**27**, 4.4.1], this ordinary NSS satisfies

$$E_2^{s,t}(Y) = \operatorname{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*(Y)).$$

The reader should be warned that the same symbol, E_2, is used for each of the spectral sequences in this paper. These include the cellular SS, the UNSS, and the (stable) NSS. Which SS is intended should be clear from the context. For the v_1-periodic SS's, we use $v_1^{-1}E_2$ instead of E_2.

There are compatible Adams maps $M^n(2^e) \to M^{n-2^{e-1}}(2^e)$ and canonical maps $M^n(2^e)\xrightarrow{\rho'}M^n(2^{e+1})$ among the Moore spaces which define the direct system. The maps ρ' are dual to those of [**15**, 2.1] and those used in the proof of Theorem 1.19. The reason for the shift in s-filtration in (2.1) is that in the direct system of exact sequences based on the cellular decomposition of the Moore spaces, such as

$$\to E_2^{s-1,t-1}(X \wedge S^{-1})\xrightarrow{\alpha}E_2^{s-1,t-1}(X \wedge M^0)\xrightarrow{\beta}E_2^{s,t}(X) \to,$$

it is the part mapped by β which persists in the direct system.

We sketch the computation of $v_1^{-1}E_r^{s,t}(S^0)$. We begin by noting that $\eta : E_2^{s,t} \to E_2^{s+1,t+2}$ acts on the SS and satisfies $2\eta = 0$. We first describe $\operatorname{dirlim}_L E_2^{s,t}(M^{-L2^e}(2^e))$, using results and methods from [**27**, pp. 199-200]. For all odd integers i, $\operatorname{dirlim}_L E_2^{s,t}(M^{-L2^e}(2^e))$ has cyclic summands in $(s,t) = (0,i)$ and $(1,i)$ of order 2 if $i \equiv 1$ mod 4, and of order $2^{\min(\nu(i+1)+1,e)}$ if $i \equiv 3$ mod 4. These summands are acted on freely by η, subject to $2\eta = 0$. There are nonzero d_3-differentials from the generators in $(0,i)$ and $(1,i)$ with $i \equiv 3$ or 5 mod 8. They hit elements divisible by η^3, except that the differential from $(0, k2^{e-1} + 3)$ hits the sum of the two elements in the box. These d_3-differentials act naturally on the entire η-tower. There is a nontrivial extension from the $\mathbb{Z}/4$ in $E_\infty^{\epsilon,i}$ for $\epsilon = 0$ or 1, and $i \equiv 3$ mod 8, to the $\mathbb{Z}_2$ in $E_\infty^{\epsilon+2,i+2}$ which is in the η-tower which begins in filtration ϵ.

Part of this $\operatorname{dirlim}_L E_2^{s,t}(M^{-L2^e}(2^e))$ is depicted in Figure 2.2. In this chart a $\bullet$ represents a $\mathbb{Z}_2$, while a letter or number represents a cyclic 2-group with that symbol as exponent. For example, a 3 denotes a $\mathbb{Z}/8$.

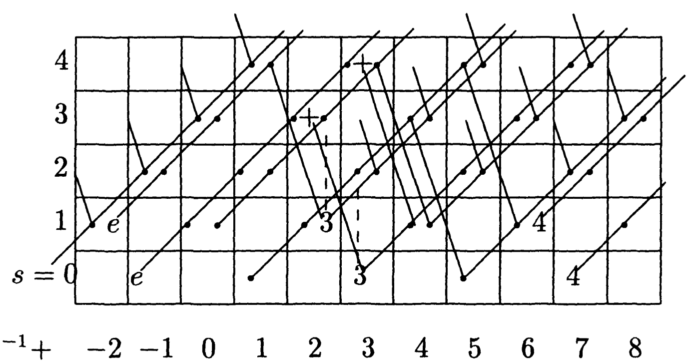

Figure 2.2: Part of $\lim_L E_2^{s,t}(M^{-L2^e}(2^e))$

The map $M(2^e)\xrightarrow{\rho'}M(2^{e+1})$ which has degree 1 on the top cell and degree 2 on the bottom cell induces a morphism of the SS's just described. The direct limit of these morphisms is, after a shift on one s-filtration, the v_1-periodic NSS for the sphere spectrum, according to (2.1). For $i+1 \neq 0$, the direct limit is determined by the morphisms for $e \geq \nu(i+1)+1$. These morphisms send the group in $(0,i)$ and its η-tower isomorphically, while (unless $i=1$) the second summand in $(1,i)$ (the one not divisible by η) and its η-tower are mapped by multiplication by 2. The $\mathbb{Z}/2^e$'s in $(0,-1)$ and $(1,-1)$ are both mapped injectively to $\mathbb{Z}/2^{e+1}$. The morphism sends η times the generators in both $(0,-1)$ and $(1,-1)$ to 0, and it sends the other $\mathbb{Z}_2$ in $(1,1)$ to the sum of the two $\mathbb{Z}_2$'s in the box. We remind the reader that $(-,-)$ here refers to (s,t) before the shift of filtration.

From this information we can read off $\operatorname{dirlim}_{L,e} E_2^{s,t}(M^{-L2^e}(2^e))$, and then use (2.1) to yield the following result, a portion of which is displayed in Figure 2.4.

Theorem 2.3. *The v_1-periodic NSS for S^0 satisfies*

a. $v_1^{-1}E_2^{s,t}(S^0)$ has

- *for every even integer $i \neq 0$ a cyclic summand in $(s,t)=(1,i)$ of order 2 if $i \equiv 2$ mod 4, and of order $2^{\nu(i)+1}$ if $i \equiv 0$ mod 4. Each of these summands supports an η-tower.*
- *a $\mathbb{Z}/2^\infty$ summand in $(s,t)=(1,0)$ and $(2,0)$.*
- *a $\mathbb{Z}_2$ in $(2,2)$ supporting an η-tower.*

b. There are nonzero d_3-differentials on the classes in $(1,i)$ for $i \equiv 4$ and 6 mod 8, and on the η-towers arising from them.

c. There is a nontrivial extension (multiplication by 2) from $(1,i)$ to $(3,i+2)$ if $i \equiv 4$ mod 8.

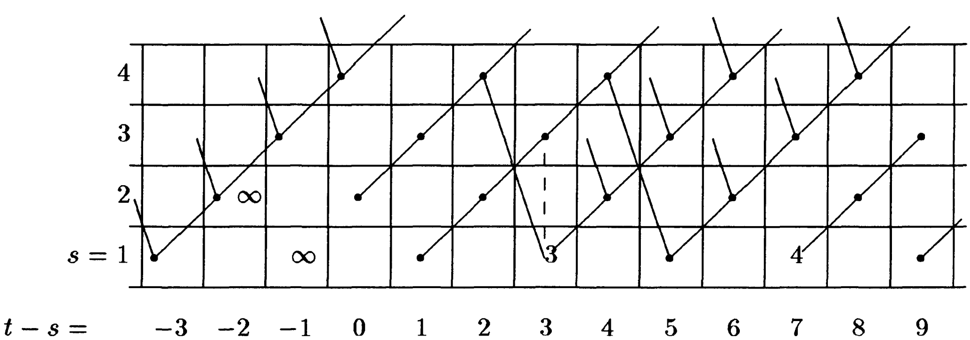

Figure 2.4: Part of periodic NSS for S^0

Let Q^n denote the quaternionic quasi-projective space with top cell of dimension $4n-1$. The standard cofibrations induce exact sequences

$$(2.5) \qquad \to v_1^{-1}E_2^{s,t}(Q^{n-1}) \to v_1^{-1}E_2^{s,t}(Q^n) \to v_1^{-1}E_2^{s,t}(S^{4n-1}) \to .$$

These can be spliced to give a cellular SS

$$\bigoplus_{i\leq n} v_1^{-1}E_2^{s,t}(S^{4i-1}) \Longrightarrow v_1^{-1}E_2^{s,t}(Q^n). \tag{2.6}$$

Differentials in the SS (2.6) correspond to boundary morphisms in the exact sequences (2.5). We describe now the way in which they are related to the coaction

$$BP_*(Q^n) \xrightarrow{\psi} \Gamma \otimes_{BP_*} BP_*(Q^n)$$

and the reduced coaction $\overline{\psi}(x) = \psi(x) - x \otimes 1 - 1 \otimes x$. Here we introduce the common notation $\Gamma = BP_*BP$.

The cobar complex for calculating the NSS is described, for example, in [**27**]. It satisfies

$$C^s(X) = \Gamma \otimes \cdots \otimes \Gamma \otimes BP_*X,$$

with s factors of Γ. Here and throughout, tensor products are always over BP_*. The formula for $d : C^s \to C^{s+1}$ is the usual alternating sum of comultiplications and coactions. It is a derivation and satisfies $d(v) = \eta_R(v) - v$ for $v \in BP_*$. This is illustrated as follows. Suppose $g \in BP_*(X)$ and $v \in BP_*$. Suppose that the coaction satisfies $\psi(g) = 1 \otimes g + \sum \gamma_i \otimes g_i$. Then

$$\begin{aligned} d([\,]vg) &= [1]vg - [v]g - \sum [v\gamma_i]g_i \\ &= [\eta_R(v) - v]g - vd(g). \end{aligned}$$

The classes in the v_1-periodic NSS are all present in this cobar complex. To calculate

$$\partial : E_2^s(S^{4n-1}) \to E_2^{s+1}(Q^{n-1})$$

on a class z represented by a cycle $A \otimes \iota_{4n-1}$, we first note that in the cobar complex for Q^n, $A \otimes \overline{\psi}(g_n)$ pulls back to an element $\gamma_{n-1} \in C^{s+1}(Q^{n-1})$. Here we have introduced generators $g_i \in BP_{4i-1}(Q^n)$ which are compatible under inclusion maps, and reduce to the generator of $BP_{4i-1}(S^{4i-1})$ under the collapse map. More about the selection of these classes g_i will be prescribed in the proof of Proposition 2.7.

Assume that by varying it by boundaries you have pulled the class γ_{n-1} back to $\gamma_i \in C^{s+1}(Q^i)$. Project γ_i into $C^{s+1}(S^{4i-1})$. If this class is not a boundary, then it represents $\partial(z)$ in the SS. Otherwise, this class is $d(B \otimes \iota_{4i-1})$ in $C^{s+1}(S^{4i-1})$ for some $B \in \Gamma^{\otimes s}$. Then $\gamma_i - d(B \otimes g_i)$ pulls back to $\gamma_{i-1} \in C^{s+1}(Q^{i-1})$. This procedure of pulling back to smaller Q^i's is continued as long as possible.

Explicit calculation of the coaction in $BP_*(Q^n)$ can be performed by a method similar to that of [**10**], but the formulas are extremely complicated.

Instead we opt for calculating a homomorphic image of the coaction which contains the information relevant to our application.

The formula for the coaction is obtained using the Adams operation Ψ^3 in $bu_*(-)$, but the route between them is somewhat tricky. We begin with the easy part, the evaluation of the operations. Recall that bu_* is a polynomial algebra over $\mathbb{Z}_{(2)}$ on a single generator v of degree 2. Then $bu_*(Q^n)$ is a free bu_*-module on classes g_i of degree $4i-1$, for $1 \le i \le n$. An explicit choice of these generators will be made in the proof of Proposition 2.7.

Proposition 2.7. *Generators g_i for $bu_*(Q^n)$ can be chosen so that*

$$\Psi^3(g_n) = \sum_{i=0}^{n-1}(-3)^{-i}\tbinom{2n+i-1}{i}v^{2i}g_{n-i}.$$

Since the inclusion maps $Q^i \to Q^n$ send g_j to g_j, this proposition can be used to read off $\Psi^3(g_i)$ in $bu_*(Q^n)$ for all $i \le n$.

Proof. The spectra Q^n and ΣHP_{-n}^{-2} are S-dual. Here HP_{-n}^{-2} is a spectrum which we think of as a shifted version of the suspension spectrum of the stunted quaternionic projective space HP_{A-n}^{A-2}, with A highly divisible. (A must be a multiple of the appropriate number of **[31]**.)

It is standard that $KU(HP^n) = \mathbb{Z}[\gamma]/\gamma^{n+1}$ with $\Psi^3(\gamma) = \gamma(\gamma+3)^2$. Let $v \in bu^{-2}(S^0)$ denote the Bott element, which acts on any $bu^*(-)$. It satisfies $\Psi^3(v) = 3v$. Let $x_i \in bu^{4A-4n}(HP_{A-n}^{A-2})$ be chosen to satisfy

$$v^{2A-2n}x_i = \gamma^{A-i} \in bu^0(HP_{A-n}^{A-2}) \approx KU(HP_{A-n}^{A-2}).$$

Then

$$\begin{aligned}
3^{2A-2n}v^{2A-2n}\Psi^3(x_n) &= \Psi^3(v^{2A-2n}x_n) = \Psi^3(\gamma^{A-n}) = \gamma^{A-n}(\gamma+3)^{2(A-n)}\\
&= \sum \tbinom{2(A-n)}{i}3^{2(A-n)-i}\gamma^{A-n+i}\\
&= \sum \tbinom{2A-2n}{i}3^{2A-2n-i}v^{2A-2n}x_{n-i}
\end{aligned}$$

We cancel $3^{2A-2n}v^{2A-2n}$, and note that since A is highly divisible, $\binom{2A-2n}{i}$ should be considered as $\binom{-2n}{i} = (-1)^i\binom{2n+i-1}{i}$. This yields the following formula in $bu^{-4n}(HP_{-n}^{-2})$.

$$\Psi^3(x_n) = \sum_{i=0}^{n-1}(-3)^{-i}\tbinom{2n+i-1}{i}x_{n-i}.$$

Letting $v^{2(n-i)}g_i \in bu_{4n-1}(Q^n)$ be S-dual to $x_i \in bu^{-4n}(HP_{-n}^{-2})$, we obtain the desired formula in $bu_{4n-1}(Q^n)$. □

Using the $\binom{-2n}{i}$-form for the binomial coefficient immediately yields the following corollary.

Corollary 2.8. *In $bu/(2)_*(Q^n)$,*

$$\Psi^3(g_n) = g_n + v^{2^{\nu(n)+2}} g_{n-2^{\nu(n)+1}} + T,$$

where $T \in \langle v^{2i} g_{n-i} : i > 2^{\nu(n)+1} \rangle$.

We will use Corollary 2.8 to obtain information about the BP-coaction of Q^n. The following result, well-known to experts, will be useful. In it, BP and bu denote the 2-local spectra, as they have throughout this paper.

Proposition 2.9. *There is a multiplicative map $\Psi^3_{BP} : BP \to BP$ which is compatible with $\Psi^3_{bu} : bu \to bu$ under the usual reduction $\rho : BP \to bu$.*

The operation Ψ^3_{bu} here is the same as Ψ^3 of 2.7 and 2.8; we just use the subscript to distinguish it from the BP-operation.

Proof. The map Ψ^3_{BP} was first defined in [**2**], and is also discussed in [**33**, §11]. It is defined by

$$\Psi^3(x) = \tfrac{1}{3}[3](x), \tag{2.10}$$

where $x \in BP^2(CP^\infty)$ denotes the standard generator, and $[3](x)$ is the usual 3-series, defined from the formal group law (FGL). The map ρ induces a map of FGL's to the 2-typical FGL of bu. Hence, if $\Psi^3 : bu \to bu$ is defined from (2.10) using the 2-typical FGL for bu, it will be compatible with Ψ^3_{BP}. The usual formula for Ψ^3 in bu, which was used in Proposition 2.7, is also derived from (2.10), but using the FGL defined by $F(x,y) = x+y-vxy$. This FGL is well-known to be equivalent to the 2-typical FGL (see [**20**, 2.3.1]), and hence the Ψ^3 derived from it is the same map $bu \to bu$, but just using a different choice of generator in $bu^2(CP^\infty)$. This establishes the desired compatibility. □

The proof of the following result shows how the Adams operations in bu are used to give information about the BP-coaction. We recall that $BP_* = \mathbb{Z}_{(2)}[v_1, v_2, \dots]$ and $\Gamma = BP_*BP = BP_*[t_1, t_2, \dots]$, with $|v_i| = |t_i| = 2(2^i - 1)$. In much of the paper, we will use the alternate generating set $\{h_1, h_2, \dots\}$, where $h_i = c(t_i)$, with c denoting the conjugation.

Proposition 2.11. *Let $\phi : BP_*BP \to \mathbb{Z}_2[v]$ be the ring homomorphism defined by*

$$\phi(t_i) = \phi(v_i) = \begin{cases} 0 & \text{if } i > 1 \\ v & \text{if } i = 1. \end{cases} \tag{2.12}$$

Let $g_i \in BP_{4i-1}(Q^n)$ be the generator defined as in the proof of 2.7, *using S-duality and the standard generators of $BP^*(HP^m)$. Suppose the coaction*

$$BP_*(Q^n) \xrightarrow{\psi} \Gamma \otimes_{BP_*} BP_*(Q^n)$$

satisfies

$$\psi(g_n) = \sum z_i \otimes g_i. \tag{2.13}$$

Then

$$\phi(z_i) = \begin{cases} 0 & \text{if } n - 2^{\nu(n)+1} < i < n \\ v^{2^{\nu(n)+2}} & \text{if } i = n - 2^{\nu(n)+1}. \end{cases}$$

The reader should be careful to distinguish between the coaction ψ and the Adams operation Ψ^3.

Proof. The relationship between the coaction ψ and the Adams operation Ψ^3_{BP} is given by the following result of [**1**, p. 72]: If the BP-coaction is given by (2.13), then

$$\Psi^3_{BP}(g_n) = \sum \widehat{\Psi}^3_{BP}(c(z_i))g_i. \tag{2.14}$$

Here, for $z \in BP_*BP$, $\widehat{\Psi}^3_{BP}(z) = \langle \Psi^3_{BP}, z\rangle \in BP_*$ denotes the Kronecker pairing. We will prove in the next paragraph that, for $i \geq 1$,

$$\widehat{\Psi}^3_{BP}(h_i) \equiv \widehat{\Psi}^3_{BP}(v_i) \equiv \widehat{\Psi}^3_{BP}(\eta_R v_i) \equiv v_i \ mod\ 2. \tag{2.15}$$

Since $\widehat{\Psi}^3_{BP}$ is multiplicative, this will imply that $\rho_* \circ \widehat{\Psi}^3_{BP} \circ c = \phi$, where $\rho : BP \to bu/2$ is the usual reduction. Thus we obtain

$$\begin{aligned} \sum \phi(z_i)g_i &= \sum \rho_* \left(\widehat{\Psi}^3_{BP}(c(z_i))\right) g_i \\ &= \rho_* \left(\Psi^3_{BP}(g_n)\right) \\ &= \Psi^3_{bu}(g_n), \end{aligned} \tag{2.16}$$

and this last expression was evaluated in Corollary 2.8, yielding the claim of this proposition.

Now we prove (2.15). It is elementary that

$$\widehat{\Psi}^3_{BP}(v_i) = 3^{2^i-1}v_i \equiv v_i \ mod\ 2. \tag{2.17}$$

See, e.g., [**33**, p. 142], from which we also take the formula

$$\widehat{\Psi}^3_{BP}\left(\sum\nolimits^{F^*}_{i\geq 0} h_i\right) = \tfrac{1}{3}[3](1). \tag{2.18}$$

Here we have also used ([**33**, 3.14]) that the conjugate formal sum on the left hand side of (2.18), defined by $\sum^{F^*} h_i = c\sum^F c(h_i)$, is the BP-analogue of $b = \sum b_i \in MU_*MU$. We next use that if $x +_F y = \sum a_{i,j}x^iy^j$, then $x +_{F^*} y = \sum \eta_R(a_{i,j})x^iy^j$. Using also (2.17) and the formula $\widehat{\Psi}^3_{BP}(\eta_R(-)) = \widehat{\Psi}^3_{BP}(-)$ of [**33**, 11.48iii], the left hand side of (2.18) equals

$$1 +_F \sum\nolimits^F_{i>0} \widehat{\Psi}^3_{BP}(h_i). \tag{2.19}$$

On the other hand, the right hand side of (2.18) is congruent mod 2 to

$$[3](1) = 1 +_F \sum\nolimits^F_{i>0} v_i, \tag{2.20}$$

using [**33**, 3.17] and $[3](x) = x +_F [2](x)$. Comparing terms of (2.19) and (2.20) one degree at a time yields $\widehat{\Psi}^3_{BP}(h_i) \equiv v_i$, as desired. □

The following result will be useful in applying Proposition 2.11, since $d(a) = \eta_R(a) - a$ for $a \in BP_*$.

Proposition 2.21. *The morphism ϕ defined in* (2.12) *sends $\eta_R(a) - a$ to 0, for every $a \in BP_*$.*

Proof. By multiplicativity of η_R, it suffices to show $\phi(\eta_R(v_i))$ is nonzero if and only if $i = 1$. We modify the formal group law of BP by applying ϕ to the coefficients. Thus if $x +_F y = \sum a_{i,j}x^iy^j$, then $x +_\phi y = \sum \phi(a_{i,j})x^iy^j$. Applying ϕ to [**28**, Theorem 1], deleting terms which involve $\phi(v_i)$ or $\phi(t_i)$ for $i > 1$, and letting $\phi(\eta_R v_j) = \delta_j v^{2^j-1}$ yields

$$\sum\nolimits^\phi_{j>0} \delta_j v^{2^j-1} +_\phi \sum\nolimits^\phi_{j>0} v(\delta_j v^{2^j-1})^2 \equiv v +_\phi v^3 \ mod\ 2.$$

By induction on j, this implies $\delta_1 = 1$ and $\delta_j = 0$ when $j > 1$. □

With these preliminaries out of the way, we return to the detemination of the differentials in the SS (2.6). In this SS, shifted copies of Figure 2.4 (without d_3's) are summed to give the initial term. There will be n copies of $\mathbb{Z}_2$ in each box $[t-s, s] = [4j, 1]$ for all $j \in \mathbb{Z}$, with η acting freely on them. We use square brackets to represent chart position to avoid confusion with (s, t) notation used earlier. Think of the *square* bracket as *box* position. This unusual notation will be followed throughout the remainder of the paper. It will usually be the case that an element in $E_r^{s,t}$ will appear in position $[t-s, s]$. We label these $\mathbb{Z}_2$ summands 1, 2, ... , n according to the value of i in the initial sum in (2.6). The value of j (in $[4j, 1]$) will not be incorporated into the notation, as its role is minimal. There will also be n $\mathbb{Z}_2$'s in each box

$[4j-1,2]$ for $j \in \mathbb{Z}$, with η-towers arising from each; these will be labeled i' according to the summand of (2.6). In each box $[4j-2,1]$ there are summands of order $2^{\nu(j-i)+3}$ for all values of i satisfying $1 \le i \le n$. Finally, there is a $\mathbb{Z}/2^\infty$ summand in each $[4j-3,2]$ for $1 \le j \le n$.

We will prove the following result about the differentials in the SS (2.6).

Proposition 2.22. *In the CSS* (2.6), *there are differentials from the η-tower labeled i to the η-tower labeled $(i-2^{\nu(i)+1})'$, whenever i is not a 2-power and $i \le n$. These are the only nonzero differentials in this SS. Thus $v_1^{-1}E_2^{*,*}(Q^n)$, the E_2-term of the v_1-periodic NSS converging to $v_1^{-1}\pi_*^s(Q^n)$, consists of*

- $[\log_2(2n)]$ *η-towers emanating from each position $[4j,1]$. The labels of these are all 2^e satisfying $1 \le 2^e \le n$.*
- $[\log_2(2n)]$ *η-towers emanating from each position $[4j-1,2]$. The labels of these will be primed integers. The only odd integer among these equals n or $n-1$.*
- *An abelian group in $[4j-2,1]$ of order 2^S, with $S = 3n + \sum_{i=1}^n \nu(j-i)$.*
- *A summand of $\mathbb{Z}/2^\infty$ in $[4j-3,2]$ for $1 \le j \le n$.*

If $n=\infty$, the analogous result holds, except that there are no elements of the second type, i.e., no η-towers arising from $[4j-1,2]$.

We illustrate for Q^{10}. The differentials go $3 \to 1'$, $5 \to 3'$, $6 \to 2'$, $7 \to 5'$, $9 \to 7'$, and $10 \to 6'$. Thus 1, 2, 4, 8, $4'$, $8'$, $9'$, and $10'$ survive. The resulting $v_1^{-1}E_2^{*,*}(Q^{10})$ is pictured in Figure 2.23. The orders of the groups G_1 and G_2 are as in Proposition 2.22. They depend on h. This chart is assumed to have $2h \ge n$ so that there are no $\mathbb{Z}/2^\infty$'s in filtration 2.

Before discussing the differentials and extensions in the v_1-periodic NSS for Q^n, we prove Proposition 2.22.

Proof of Proposition 2.22. To prove the differential on the η-tower, it suffices to prove it on the bottom element, since differentials commute with the action of η. It also suffices to prove it on the element labeled n in Q^n. The first main step is to show that g_n can be varied so that $d([\ ]g_n)$ has a particularly nice form, where d is the boundary in the cobar complex. Indeed, we show that there exist elements $V_i \in BP_*$ and $A_i \in BP_*BP$ such that if

$$g_n' = g_n + \sum_{i=1}^{2^{\nu(n)+1}-1} V_i g_{n-i}, \tag{2.24}$$

then

$$d([\ ]g_n') \equiv \sum_{i \ge 2^{\nu(n)+1}} [A_i] g_{n-i} \pmod 2. \tag{2.25}$$

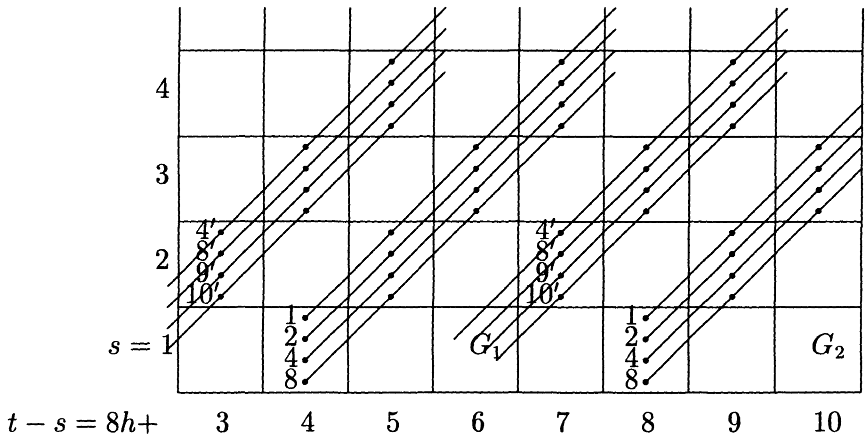

Figure 2.23: $v_1^{-1}E_2^{s,t}(Q^{10})$ if $2h > n$

The point here is that, mod 2, the first $2^{\nu(n)+1}$ terms of $d([\ ]g_n')$ are 0. It also follows from Proposition 2.21 that $\phi(g_n') = \phi(g_n)$.

If M is a Γ-comodule, we abbreviate $\mathrm{Ext}_\Gamma(BP_*, M)$ as $\mathrm{Ext}(M)$. Recall that $\mathrm{Ext}^1(BP_*)$ is the homology of the sequence

$$BP_* \xrightarrow{d} \Gamma \xrightarrow{\overline{\Delta}} \Gamma \otimes \Gamma. \tag{2.26}$$

Recall that unadorned tensor products are always over BP_*. Let $Z = \ker(\overline{\Delta})$ denote the cycles. Then Proposition 2.21 implies that the restriction to Z of the homomorphism ϕ defined in (2.12) factors through a homomorphism

$$\mathrm{Ext}^1(BP_*) \to \mathbb{Z}_2[v].$$

If all groups in (2.26) are reduced mod 2, the homology becomes

$$\mathrm{Ext}^1(BP_*/(2)),$$

and $\phi|Z$ can be factored as

$$Z \to \mathrm{Ext}^1(BP_*) \xrightarrow{\rho} \mathrm{Ext}^1(BP_*/(2)) \xrightarrow{\overline{\phi}} \mathbb{Z}_2[v].$$

By [**27**, 5.3.13], $\overline{\phi}$ sends the image of ρ injectively.

Suppose that g_n' has been chosen similarly to (2.24) except that $2^{\nu(n)+1}$ is replaced by some smaller number J, and that it satisfies

$$d([\ ]g_n') \equiv \sum_{i \geq J} [A_i] g_{n-i} \quad (\text{mod } 2)$$

with $\phi(A_i) = 0$ if $i < 2^{\nu(n)+1}$. Since $\phi(A_J) = 0$, A_J is a boundary mod 2. That is, there exists $V_J \in BP_*$ with $d(V_J) \equiv A_J$ mod 2 in (2.26). Moreover, V_J can be chosen to satisfy $\phi(V_J) = 0$, since the only monomials of BP_* with nonzero ϕ are v_1^j, and they have the property that $d(v_1^j) = (v_1 - 2h_1)^j - v_1^j$ is divisible by 2, so that they may be removed from this mod 2 calculation. Subtracting $V_J g_{n-J}$ from g_n' yields a new g_n' such that

$$d([\ ]g_n') \equiv \sum_{i>J} [A_i'] g_{n-i} \pmod 2$$

with $\phi(A_i') = 0$ if $i < 2^{\nu(n)+1}$. To see $\phi(A_i') = 0$ we use the derivation property of d, Proposition 2.21, and $\phi(V_J) = 0$. This extends the induction, establishing the existence of a generator as in (2.24) satisfying (2.25).

The differential which we wish to establish amounts to computing the homomorphism

$$(2.27)\quad v_1^{-1}E_2^{1,4k+4n+1}(S^{4n-1}) \xrightarrow{\partial} v_1^{-1}E_2^{2,4k+4n+1}(Q^{n-1}) \xrightarrow{\rho} v_1^{-1}E_2^{2,4k+4n+1}(Q^{n-1};\mathbb{Z}_2),$$

where ρ is reduction mod 2. It suffices to consider this reduction since the homomorphism ρ here is injective. The generator of $v_1^{-1}E_2^{1,4k+4n+1}(S^{4n-1})$ is the element usually called $\alpha_{2k+1}\iota_{4n-1}$. The element α_{2k+1} was called a_{2k+1} in [**7**, pp. 245-6], where it was shown that it is represented by $v_1^{2k}h_1$. The image $\rho(\partial(\alpha_{2k+1}\iota_{4n-1}))$ is determined from the mod 2 reduction of $d([\alpha_{2k+1}]g_n')$ by the procedure outlined earlier. Since α_{2k+1} is a mod 2 cycle, this image equals $v_1^{2k}h_1$ times the reduction of a class, $d(g_n')$, which pulls back to an element of $v_1^{-1}E_2^{1,4n-1}(Q^{n-2^{\nu(n)+1}};\mathbb{Z}_2)$ whose projection, A, into $v_1^{-1}E_2^{1,4n-1}(S^{4n-2^{\nu(n)+3}-1};\mathbb{Z}_2)$ maps nontrivially under $\overline{\phi}$. By the calculation of $v_1^{-1}E_2(S^{2n+1})$ sketched earlier, $v_1^{-1}E_2^{1,4n-1}(S^{4n-4d-1}) \approx \mathbb{Z}_2$, and the mod 2 reduction of the nonzero element equals $v_1^{2d-4}u$ by [**27**, 5.3.13c]. Here $u = t_1^4 + v_1t_1^3 + v_1^2t_1^2 + v_1t_2 + v_2t_1$, and so $\overline{\phi}(v_1^{2d-4}u) \neq 0$. Since $\overline{\phi}$ is an injection on the 1-line group, this implies that $A \equiv v_1^{2^{\nu(n)+2}-4}u$ mod boundaries. Thus the image of (2.27) is

$$v_1^{2k}h_1 \otimes v_1^{2^{\nu(n)+2}-4}u \otimes g_{n-2^{\nu(n)+1}}.$$

The v_1's can be shifted across the $\otimes$ mod 2, since $\eta_R(v_1) - v_1 = 2h_1$, and so this element is the desired element $(n - 2^{\nu(n)+1})'$, again using [**27**, 5.3.13c] to see that $(n - 2^{\nu(n)+1})'$ is represented by $v_1^{pwr}h_1 \otimes u \otimes g$.

The final ingredient in the proof of Proposition 2.22 is the following result, which implies that there are no differentials from the $[4k+2,1]$-box to the

$[4k+1,2]$-box. The proof of this lemma involves methods foreign to the NSS, and is relegated to the end of Section 6.

Lemma 2.28. *The boundary morphism*

$$v_1^{-1}E_2^{1,4k+3}(S^{4n-1}) \xrightarrow{\partial} v_1^{-1}E_2^{2,4k+3}(Q^{n-1})$$

is zero.

This completes the proof of Proposition 2.22, the determination of the E_2-term of the v_1-periodic NSS for Q^n. The analysis of the higher differentials in the SS is easier. In fact, the differentials are immediate from those in the spheres.

Proposition 2.29. *In the v_1-periodic NSS for Q^n, whose E_2-term was described in Proposition 2.22, there are d_3-differentials from certain η-towers to the preceding η-tower with the same number as follows:*

- *from the η-tower labeled 1 beginning in $[8h+8,1]$;*
- *from each η-tower labeled 2^e with $2 \le 2^e \le n$ beginning in $[8h+4,1]$;*
- *from the η-tower labeled m' with m odd beginning in $[8h+7,2]$;*
- *from each η-tower labeled m' with m even beginning in $[8h+3,2]$.*

There is also a d_3-differential from the group in $[8h+6,1]$ to the only odd primed class in $[8h+5,4]$, and there are $[\log_2(n)]$ d_3-differentials from the group in $[8h+10,1]$ to the classes labeled m' with m even in $[8h+9,4]$. Thus $v_1^{-1}E_\infty(Q^n)$ consists of

- *$[\log_2(n)]$ η-towers truncated at $\eta^3=0$ emanating from each position $[8h+8,1]$. The labels of these are all 2^e satisfying $2 \le 2^e \le n$.*
- *An η-tower truncated at $\eta^3=0$ emanating from each position $[8h+4,1]$, labeled 1.*
- *$[\log_2(n)]$ η-towers truncated at $\eta^2=0$ emanating from each position $[8h+7,2]$. The labels of these will be primed even integers.*
- *An η-tower truncated at $\eta^2=0$ emanating from each position $[8h+3,2]$, labeled m', where $m=n-1$ or n and is odd.*
- *An abelian group in $[4k-2,1]$ of order 2^S, with $S=3n-D+\sum_{i=1}^{n}\nu(k-i)$, with $D=1$ if k is even, and $D=[\log_2(n)]$ if k is odd.*
- *A summand of $\mathbb{Z}/2^\infty$ in $[4k-3,2]$ for $1\le k\le n$.*

All elements in $[4k+2,3]$ are divisible by 2 in $v_1^{-1}\pi_*^s(Q^n)$. This is accomplished by nontrivial extensions from the 1-line.

If $n=\infty$, a similar description holds without the primed classes, and with $D=0$.

Proof. The differentials are all induced by naturality from the maps $Q^i \to S^{4i-1}$. There can be no other differentials since all remaining classes are in filtration 1, 2, and 3, and t odd. The extensions can be seen from their presence in the spheres, or from the observation that if $x \in \pi_*^s(X)$ satisfies $2x = 0$, then $\eta^2 x$ must be divisible by 2. This is true because there is a map from the mod 2 Moore spectrum M to X sending the bottom class of $\pi_*^s(M)$ to x, and the relation is true in $\pi_*^s(M)$. □

We illustrate with a chart, Figure 2.30, for the d_3-differentials in Q^{10}, and then the much simpler chart, Figure 2.31, for $v_1^{-1}E_\infty(Q^{10})$. The groups G_1, G_2, G_1', and G_2' in Figures 2.30 and 2.31 are the appropriate abelian groups described in the propositions. The primed groups are smaller than the unprimed ones, as they are the kernels of d_3. The chart for $v_1^{-1}E_\infty(Q^\infty)$ is like Figure 2.31, with the classes i' and the short η-towers on them omitted, infinitely many short η-towers going up from $[8h+8,1]$, namely one for each even 2-power, and the G_i'-groups replaced by G_i. These groups will be of infinite order.

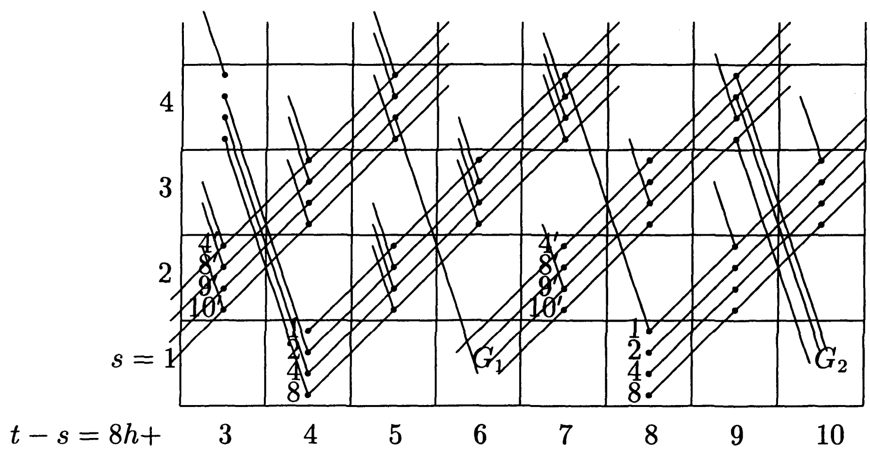

Figure 2.30: Periodic NSS for Q^{10}

3. Determination of $v_1^{-1}E_2(Sp(n))$.

In this section, we prove Proposition 1.8. That is, we compute the differentials among the η-towers in (1.6). This yields $v_1^{-1}E_2(Sp(n))$, which we have depicted in Figure 1.16.

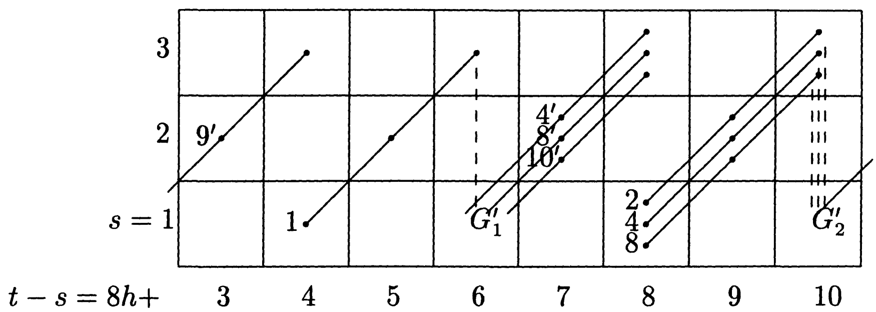

Figure 2.31: $v_1^{-1}E_\infty^{s,t}(Q^{10})$

We discussed in the paragraph containing (1.6) the use of the exact sequence of UNSS's and v_1-periodic UNSS's associated to (1.1). For the UNSS, the existence of these exact sequences is proved similarly to the proof for $SU(n)$ in [**3**], while for the periodic UNSS it follows since the exact sequences commute with the morphisms A'' of [**4**, pp. 50-51] which define the direct systems whose direct limit is $v_1^{-1}E_2(-)$.

We expand upon this a bit, in order to set the stage for the more complicated situation which follows. Indeed we review the definition and basic properties of the UNSS. We follow the exposition of [**3**, §2]. If X is a space, a space $BP(X)$ is defined as $\lim_n \Omega^n(\mathbf{BP}_n \wedge X)$, where $\mathbf{BP}_n$ denotes the nth space in the Ω-spectrum for BP. Define $D^1(X)$ to be the fibre of the unit map $X \to BP(X)$, and inductively define $D^s(X)$ to be the fibre of $D^{s-1}(X) \to D^{s-1}(BP(X))$. This gives rise to a tower of fibrations

$$\cdots \to D^2(X) \to D^1(X) \to X.$$

The homotopy exact couple of this tower is the UNSS of X.

In general, computing this SS can be next to impossible, but if $BP_*(X)$ is free as a BP_*-module and cofree as a coalgebra, then it becomes somewhat tractable. Indeed, in such a case

$$E_2^{s,t}(X) \approx \mathrm{Ext}^s_{\mathcal{U}}(M_t, P(BP_*(X))),$$

where M_t denotes a free BP_*-module on a generator of dimension t, $P(-)$ denotes the primitives in a coalgebra, and $\mathcal{U}$ denotes the category of unstable Γ-comodules. We sketch a definition of the category $\mathcal{U}$, referring the reader to [**3**, p. 744] or [**7**, §7] for more details. Recall that $\Gamma = BP_*BP = BP_*[h_1, h_2, \dots]$. If M is a free BP_*-module, then $U(M)$ is the BP_*-submodule of $\Gamma \otimes_{BP_*} M$ spanned by all elements of the form $h^I \otimes m$ satisfying $2(i_1 + i_2 + \cdots) < |m|$, where $h^I = h_1^{i_1} h_2^{i_2} \cdots$. We shall not need the definition of $U(M)$ when M is not BP_*-free. We define $U^s(M)$ by iterating $U(-)$. The category $\mathcal{U}$ consists of BP_*-modules M equipped with morphisms $M \xrightarrow{\psi} U(M)$, $U(M) \xrightarrow{\delta} U^2(M)$, and $U(M) \xrightarrow{\epsilon} M$ satisfying certain properties.

The category $\mathcal{U}$ is abelian. We abbreviate $\mathrm{Ext}^s_{\mathcal{U}}(M_t, N)$ to $\mathrm{Ext}^{s,t}_{\mathcal{U}}(N)$. These groups may be calculated as the homology groups of the unstable cobar complex $C^{*,*}(N)$, defined by $C^{s,t}(N) = U^s(N)_t$, with boundary $C^s \to C^{s+1}$ defined as an alternating sum of $\Gamma \xrightarrow{\Delta} \Gamma \otimes \Gamma$ and $M \xrightarrow{\psi} U(M)$.

The coalgebra $BP_*(Sp(n))$ is cofree on the primitive elements e_i with $|e_i| = 4i - 1$ for $1 \le i \le n$ which are in the image from $BP_{4i-1}(Q^n)$. Letting N_n denote the free BP_*-module on $\{e_1, \dots, e_n\}$, it follows that the E_2-term of the UNSS, $E_2^{s,t}(Sp(n))$, is isomorphic to $\mathrm{Ext}^{s,t}_{\mathcal{U}}(N_n)$. From the short exact sequence in $\mathcal{U}$

$$0 \to N_{n-1} \to N_n \to M_{4n-1} \to 0,$$

we obtain the desired long exact sequence

$$\to E_2^{s,t}(Sp(n-1)) \to E_2^{s,t}(Sp(n)) \to E_2^{s,t}(S^{4n-1}) \to . \tag{3.1}$$

We could in principle obtain all differentials in the cellular SS which results from these exact sequences, or their v_1-periodic analogues, by using the coaction, similarly to the method used in the stable SS for $E_2(Q^n)$ in Section 2. Indeed, for differentials from stable classes on the 1-line to stable classes on the 2-line, this method works fine. However, in order to more easily relate stable and unstable classes, we prefer to work with $\Omega Sp(n)$, and this complicates the theory.

The following known result will be useful.

Proposition 3.2. ([**24**], [**21**]) *There exist $z_{2i-1} \in H_{4i-2}(\Omega Sp(n); \mathbb{Z})$ for $1 \le i \le n$ such that there is an isomorphism of algebras*

$$H_*(\Omega Sp(n); \mathbb{Z}) \approx \mathbb{Z}[z_1,\ z_3, \dots, z_{2n-1}].$$

The coalgebra structure is given by

$$\widetilde{\Delta}(z_i) = 2 \sum_{j=1}^{i-1} z_j \otimes z_{i-j},$$

where z_{2j} is defined inductively by

$$z_{2j} + \sum_{k=1}^{2j-1} (-1)^k z_k z_{2j-k} = 0.$$

The mod 2 reduction $\overline{z}_i$ *satisfies* $\overline{z}_{2^e(2j+1)} = \overline{z}_{2j+1}^{2^e}$. *The mod 2 dual Steenrod operations are given by*

$$(\overline{z}_{2i-1})\mathrm{Sq} = \sum_{j=1}^{2i-1} \tbinom{j-1}{2i-1-j}\overline{z}_j.$$

For $h = BP$ *or* bu, *there are elements* $g_i \in h_{4i-2}(\Omega Sp(n))$ *for* $1 \le i \le n$ *such that there is an isomorphism of algebras*

$$h_*(\Omega Sp(n)) \approx h_*[g_1,\ g_2, \dots, g_n].$$

The reduction from $h_*(-)$ *to* $H\mathbb{Z}_{2*}(-)$ *sends* g_i *to* z_{2i-1}.

Note that $BP_*(\Omega Sp(n))$ is not a cofree coalgebra, and so the E_2-term of its UNSS cannot be handled by the methods which were used for $Sp(n)$.

We will need the exact sequences of the UNSS and v_1-periodic UNSS of these loop spaces.

Proposition 3.3. *There is an isomorphism of* E_2*-terms of the UNSS*

$$E_2^{s,t-1}(\Omega Sp(n)) \approx E_2^{s,t}(Sp(n)). \tag{3.4}$$

There are exact sequences

$$\begin{aligned}\to E_2^{s,t}(\Omega Sp(n-1)) \to E_2^{s,t}(\Omega Sp(n))&\\ \to E_2^{s,t}(\Omega S^{4n-1}) \to E_2^{s+1,t}&(\Omega Sp(n-1)) \to\end{aligned}$$

and

$$\begin{aligned}\to v_1^{-1}E_2^{s,t}(\Omega Sp(n-1)) \to v_1^{-1}E_2^{s,t}(\Omega Sp(n))&\\ \to v_1^{-1}E_2^{s,t}(\Omega S^{4n-1}) \to v_1^{-1}E_2^{s+1,t}&(\Omega Sp(n-1)).\end{aligned} \tag{3.5}$$

Proof. The exact sequences follow from (3.4) and (3.1). The proof of (3.4) is completely analogous to that for S^{2n+1} in [**8**, §6], but we review it, since the methods will be required later. Indeed, we will provide a rather comprehensive review of the Hopf ring structure of $BP_*(\mathbf{BP}_n)$, since this plays an essential role in the proof.

There are two products on the spaces in the Ω-spectrum for BP. One is a map $\mathbf{BP}_k \times \mathbf{BP}_k \to \mathbf{BP}_k$ induced by loop multiplication. It induces the $*$-product in BP-homology. The other is a map $\mathbf{BP}_k \times \mathbf{BP}_r \to \mathbf{BP}_{k+r}$ induced by the ring spectrum structure. It induces the $\circ$-product in BP-homology.

There is a map $x : \mathbb{C}P^\infty \to \mathbf{BP}_2$ representing a generator of $BP^2(\mathbb{C}P^\infty)$. Dual to x^n is a class $\beta_n \in BP_{2n}(\mathbb{C}P^\infty)$. We let $b_{(i)} = x_*(\beta_{2^i}) \in BP_{2^{i+1}}(\mathbf{BP}_2)$.

If $I = (i_0, i_1, \dots)$, then $b^{\circ I} \in BP_*(\mathbf{BP}_{2|I|})$ denotes the class $b_{(0)}^{\circ i_0} \circ b_{(1)}^{\circ i_1} \circ \cdots$. Here $|I| = \sum i_j$.

If $k \leq 0$, then $\pi_0(\mathbf{BP}_k) = BP^k = BP_{-k}$, and so we have for each element $v \in BP^k$ an element $[v] \in BP_0(\mathbf{BP}_k)$. It follows from [**29**] that the set of all $b_{(i)}$ and all $[v]$ generate $BP_*(\mathbf{BP}_*)$ under $*$ and $\circ$ products. Moreover, $BP_*(\mathbf{BP}_k)$ is a polynomial algebra with respect to the $*$-product on a family of indecomposable elements which are identified in [**29**]. The stabilization map $BP_*(\mathbf{BP}_k) \xrightarrow{\sigma_*} \Gamma \otimes M_k$ sends $*$-products to zero, and is injective on the indecomposables. We will frequently identify elements of $BP_*(\mathbf{BP}_*)$ with their stable images under σ_*.

We first note that the class $b_{(0)}$ is special. It corresponds to the suspension operator. Indeed, $b_{(0)}^{\circ n}$ stabilizes to $1 \otimes \iota_{2n}$ in $\Gamma \otimes M_{2n}$. The class $b_{(1)}$ lives in $BP_4(\mathbf{BP}_2)$ and stabilizes to $h_1 \otimes \iota_2$. In general, $b_{(i)}^{\circ j}$ stabilizes to $h_i^j \otimes \iota_{2j}$ mod $(v_1, v_2, \dots)\Gamma$, by [**7**, 8.3]. The class $[v]$ stabilizes to $\eta_R(v)$. Thus, for example, the class $b_{(1)}^{\circ 2} \circ b_{(0)} \circ [v_1]$ stabilizes to $\eta_R(v_1) h_1^2 \otimes \iota_4 = h_1^2 \otimes v_1 \iota_4$. Note that both classes have degree 10, and the Hopf ring class is defined on $\mathbf{BP}_4$ since $[v_1]$ is defined on $\mathbf{BP}_{-2}$. Also this class is in the image of the double suspension from $BP_8(\mathbf{BP}_2)$ since there is a factor of $b_{(0)}$.

Since the $b_{(n)}$'s and $[v]$'s generate $BP_*(\mathbf{BP}_k)$ as a polynomial algebra, the $*$-indecomposables in $BP_*(\mathbf{BP}_k)$ stabilize to the BP_*-span of $h^I \otimes \iota_k$ with $2|I| \leq k$. (The above examples justify this description for even k. It is shown in [**29**, 5.1] that the same description works for odd k.) By [**29**, 5.3], the primitives in $BP_*(\mathbf{BP}_k)$ are the image of the suspension. Hence, since $b_{(0)}$ is suppressed in the stable notation, we see that the stable image of the primitives is the BP_*-span of $h^I \otimes \iota_k$ with $2|I| < k$. Whenever there is no chance of confusion, we prefer to use the stable names. So, for example, we will write ι_{2k}^{*2} for the class $(b_{(0)}^{\circ k})^{*2}$, even though $(b_{(0)}^{\circ k})^{*2}$ stabilizes to 0.

Having described the stable image of the primitives, we want to understand their names back in $BP_*(\mathbf{BP}_k)$. For this, we use the fact ([**29**]) that $BP_*(\mathbf{BP}_k)$ is a bipolynomial Hopf algebra. By [**30**], such Hopf algebras over $\mathbb{Z}_{(2)}$ can be written as a tensor product of the universal Hopf algebras $B(x_{2n})$ constructed in [**22**]. As an algebra, $B(x_{2n})$ is a polynomial algebra generated by classes $x_{2^s n}$ with $s \geq 1$. The primitives in the coalgebra are of the form

$$x_{2n}^{2^s} - 2(x_{4n})^{2^{s-1}} + \cdots + (-2)^s x_{2^{s+1}n} \tag{3.6}$$

with $s \geq 0$. For example, we have the primitive in $BP_{4k}(\mathbf{BP}_{2k})$ given by $2h_1^k \otimes \iota_{2k} - 1 \otimes \iota_{2k}^{*2}$, or in Hopf ring notation $2b_{(1)}^{\circ k} - (b_{(0)}^{\circ k})^{*2}$. We explain how this class is a primitive in the sense of the earlier description, namely that it is in the image of $\circ b_{(0)}$. The answer is given by a relation in the Hopf ring,

$$2b_{(1)} - (b_{(0)})^{*2} = v_1 b_{(0)} + [v_1] \circ b_{(0)}^{\circ 2},$$

proved in [**29**, p. 261]. This suspends to the familiar formula, $\eta_R(v_1) = v_1 - 2h_1$, for the right action on v_1 and inductively gives the double desuspension of $2h_1^k \otimes \iota_{2k}$. Up in $BP_{4k}(\mathbf{BP}_{2k})$, $2b_{(1)}^{\circ k}$ does not desuspend. This is prevented by the $*$-decomposable class in the formula.

We now generalize the above discussion to the case of a BP_*-module M which is free on a set of generators $\{m_i\}$. Let $BP(M)$ be the zeroth space of the Ω-spectrum associated to the homology theory $BP_*(-) \otimes M$. Then $G(M)$ is defined to be $BP_*(BP(M))$. In particular, $BP(M_k) = \mathbf{BP}_k$, and $G(M_k) = BP_*(\mathbf{BP}_k)$. In terms of the basis, a non-canonical description of $G(M)$ is given by $G(M) = BP_*(\prod \mathbf{BP}_{|m_i|})$.

The stabilization map

$$G(M) \to \Gamma \otimes M$$

is injective on the $*$-indecomposables, and $U(M)$ and $V(M)$ are defined to be the images of the primitives and indecomposables, respectively. As above, $U(M)$ is the BP_*-span of all $h^I \otimes m$ satisfying $2|I| < |m|$, while $V(M)$ is the BP_*-span of all $h^I \otimes m$ satisfying $2|I| \leq |m|$. We will sometimes write $\overline{U}(M) \subset G(M)$ for the primitives when we need the specific representations of elements in $G(M)$ rather than their more familiar stable images.

Thus $U(M)$ agrees with that which we considered earlier this section. Note that if M is concentrated in even degrees, then $U(M)$ is properly contained in $V(M)$, while $U(\sigma M) \approx \sigma V(M)$, where the suspension σ increases degrees by 1. As we did with U, we define $V^p(M)$ by iterating the functor V.

The modules $G(M)$ lie in the nonabelian category $\mathcal{G}$ of unstable BP_*-coalgebras discussed in [**7**, §6]. It was shown there that if X is a simply-connected space with $BP_*(X)$ a free BP_*-module, then the E_2-term of the UNSS of X is given by $\operatorname{Ext}_{\mathcal{G}}(BP_*, BP_*(X))$. As we will need some understanding of how this Ext is computed, we go into some detail about G-structure.

For a space X for which $BP_*(X)$ is a free BP_*-module, we abbreviate $G(BP_*(X))$ to $G(X)$. We note that $G(X) = BP_*(BP(X))$, for the space $BP(X)$ defined earlier. As in the case of $G(S^k)$, we use a mix of stable and unstable notation. We will be primarily interested in $X = \Omega Sp(n)$, for which $BP_*(\Omega Sp(n))$ is given by Proposition 3.2. In $G(\Omega Sp(n))$, the BP_*-module generators act as an indexing set for the factor of $BP_*(\mathbf{BP}_k)$. For example, $1 \otimes g_2^2$ is the bottom class in the factor of $G(\Omega Sp(n))$ corresponding to the generator g_2^2 of $BP_*(\Omega Sp(n))$, while $1 \otimes g_2^{*2}$ is the $*$-square of the bottom class in the factor corresponding to the generator g_2. The class $1 \otimes g_2^2$ is primitive in the G-coproduct, while $1 \otimes g_2^{*2}$ is not. One has to keep in mind that $G(X)$ does not see any of the structure of $BP_*(X)$.

The G-structure map $\eta_X : BP_*(X) \to G(X)$ is obtained by applying $BP_*(-)$ to the Hurewicz map $\eta : X \to BP(X)$. The coaction ψ of $BP_*(X)$

is obtained from this as

$$BP_*(X) \xrightarrow{\eta_X} G(X) = BP_*(BP(X)) \xrightarrow{\sigma_*} \Gamma \otimes BP_*(X), \tag{3.7}$$

while the coproduct Δ is obtained as

$$\begin{aligned} BP_*(X) \xrightarrow{\eta_X} G(X) = BP_*(BP(X)) &\xrightarrow{\Delta_G} G(X) \otimes G(X) \\ &\xrightarrow{\epsilon\otimes\epsilon} BP_*(X) \otimes BP_*(X). \end{aligned} \tag{3.8}$$

Here ϵ satisfies $\epsilon(1 \otimes g_i) = g_i$, while ϵ applied to any other monomial is 0. The coaction and coproduct are related by the following commutative diagram.

$$\begin{array}{ccc} BP_*(X) & \xrightarrow{\psi} & \Gamma \otimes BP_*(X) \\ \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle 1\otimes\Delta} \\ BP_*(X) \otimes BP_*(X) & \xrightarrow{\psi\otimes\psi} \Gamma \otimes BP_*X \otimes \Gamma \otimes BP_*X \xrightarrow{m_\Gamma} & \Gamma \otimes BP_*X \otimes BP_*X \end{array} \tag{3.9}$$

Now we proceed with the proof of (3.4). Let $N = QBP_*(\Omega Sp(n))$, with Q denoting the indecomposable quotient. Thus σN is the module N_n considered earlier. Similarly to [**5**, 4.2] or [**8**, p. 387], there is a cosimplicial resolution

$$BP_*(\Omega Sp(n)) \xrightarrow{\xi} G(N) \rightrightarrows G(V(N)) \rightrightarrows G(V^2(N)) \cdots, \tag{3.10}$$

where the co-augmentation ξ is the composite

$$BP_*(\Omega Sp(n)) \xrightarrow{\eta_{\Omega Sp(n)}} G(BP_*(\Omega Sp(n))) \to G(N). \tag{3.11}$$

The acyclicity follows just as it did for ΩS^{2n+1} in [**8**, p. 387].

We apply $\mathrm{Hom}_{\mathcal{G}}(BP_*, -)$ to (3.10), and taking the alternating sum of the coface maps yields a chain complex whose homology is

$$\mathrm{Ext}_{\mathcal{G}}(BP_*(\Omega Sp(n))),$$

which by [**7**, 6.17] is $E_2(\Omega Sp(n))$. Since $\mathrm{Hom}_{\mathcal{G}}(BP_*, G(T)) \approx T$, this complex, whose homology equals $E_2(\Omega Sp(n))$, is isomorphic to the complex

$$N \to V(N) \to V^2(N) \to \cdots. \tag{3.12}$$

Since N is concentrated in even degrees, (3.12) is isomorphic to the complex

$$N \to \sigma^{-1}U(\sigma N) \to \sigma^{-1}U^2(\sigma N) \to \cdots,$$

and the homology of this complex is $\sigma^{-1}\mathrm{Ext}_{\mathcal{U}}(P(BP_*(Sp(n))))$, where $P(-)$ denotes the primitives. By [**7**, p. 240], this homology is

$$\sigma^{-1}E_2(Sp(n)).$$

Thus

$$E_2^{s,t}(\Omega Sp(n)) \approx E_2^{s,t+1}(Sp(n)),$$

as desired. □

The following result is instructive.

Proposition 3.13. *The morphism ξ of* (3.10) *is a ring homomorphism.*

Proof. There is a commutative diagram

$$\begin{array}{ccccc} BP_*(\Omega Sp(n)) & \xrightarrow{\eta_{\Omega Sp(n)}} & G(\Omega Sp(n)) & \xrightarrow{\sigma_*} & \Gamma \otimes BP_*(\Omega Sp(n)) \\ & \searrow \xi & \downarrow G(\rho) & & \downarrow \Gamma\otimes\rho \\ & & G(N) & \xrightarrow{\sigma_*} & \Gamma \otimes N \end{array} \tag{3.14}$$

If $x_1x_2 \in BP_*(\Omega Sp(n))$ is decomposable, then, using (3.7),

$$\sigma_*\xi(x_1x_2) = (\Gamma\otimes\rho)\sigma_*\eta_{\Omega Sp(n)}(x_1x_2) = (\Gamma\otimes\rho)\psi(x_1)\psi(x_2) = 0,$$

since ρ annihilates decomposables. Hence

$$\xi(x_1x_2) \in \ker(\sigma_*) = I^{*2}, \tag{3.15}$$

where I is the augmentation ideal.

Next we observe that both maps in the composite (3.11) which defines ξ are morphisms in the category $\mathcal{G}$, and hence so is ξ. In particular there is a commutative diagram

$$\begin{array}{ccc} BP_*(\Omega Sp(n)) & \xrightarrow{\xi} & G(N) \\ \downarrow \overline{\Delta} & & \downarrow \overline{\Delta} \\ BP_*(\Omega Sp(n)) \otimes BP_*(\Omega Sp(n)) & \xrightarrow{\xi\otimes\xi} & G(N)\otimes G(N), \end{array} \tag{3.16}$$

where $\overline{\Delta}$ is the reduced coproduct. This diagram implies that

$$\overline{\Delta}(\xi(g_i^2)) = 2g_i \otimes g_i.$$

There are only two classes in $G(N)$ satisfying $\overline{\Delta}(-) = 2g_i \otimes g_i$, namely g_i^{*2} and $2h_1^{2i-1} \otimes g_i$. (To see that $\overline{\Delta}(2h_1^{2i-1} \otimes g_i) = 2g_i \otimes g_i$, note that

$$0 = \overline{\Delta}(2h_1^{2i-1} \otimes g_i - 1 \otimes g_i^{*2}),$$

since this difference is a primitive.) By (3.15), $\xi(g_i^2)$ is decomposable, which $2h_1^{2i-1} \otimes g_i$ is not, and so $\xi(g_i^2)$ must equal g_i^{*2}.

We can now prove that $\xi(g_i^k) = g_i^{*k}$ by induction on k. The diagram (3.16) and induction hypothesis imply

$$\overline{\Delta}(\xi(g_i^k)) = \sum \tbinom{k}{l} g_i^{*l} \otimes g_i^{*(k-l)},$$

which leaves g_i^{*k} as the only candidate for $\xi(g_i^k)$. We deduce similarly that $\xi(g_i^k g_j^l) = g_i^{*k} g_j^{*l}$, and similarly for larger products. □

The other result which we need before calculating the differentials concerns the interlocking EHP and double suspension sequences. Here H is the obstruction to desuspension, and H_2 is the obstruction to double desuspension, both of which are described in [**5**].

Theorem 3.17. *There is a commutative diagram.*

$$\begin{array}{ccc} v_1^{-1}E_2^s(S^{2n+1}) & \xrightarrow{H_2} & v_1^{-1}\mathrm{Ext}_{\mathcal{U}}^{s-1}(BP_*/(2)\{x_{4n-1}\}) \\ \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle \delta} \\ v_1^{-1}E_2^s(S^{4n+1}) & \xleftarrow{\sigma^2} & v_1^{-1}E_2^s(S^{4n-1}), \end{array} \tag{3.18}$$

where δ is the Bockstein homomorphism, H is the ordinary James-Hopf invariant, and H_2 is a factorization of the double suspension Hopf invariant.

Before we begin the proof, we make a few remarks. Let $W(n)$ denote, as in [**5**] and [**8**], the free $BP_*/(2)$-module with basis $\{x_{2^in-1} : i \geq 2\}$. The morphism

$$v_1^{-1}\mathrm{Ext}_{\mathcal{U}}(BP_*/(2)\{x_{4n-1}\}) \xrightarrow{i_*} v_1^{-1}\mathrm{Ext}_{\mathcal{U}}(W(n))$$

is shown on [**4**, p. 57] to send the stable classes bijectively, and the unstable classes to 0. Moreover, it is shown there that the morphism

$$v_1^{-1}E_2(S^{2n+1}) \xrightarrow{H_2} v_1^{-1}\mathrm{Ext}_{\mathcal{U}}(W(n))$$

of [**4**, (2.5)] factors through i_*, yielding the morphism, also called H_2, of our (3.18).

The analogue of Theorem 3.17 before inverting v_1 is claimed in the metastable range in [**5**, 5.7]. The metastable condition was required in order to guarantee that the image of H_2 lies on the bottom cell, which is true without the metastability condition after localization.

Implicit in Theorem 3.17 is the statement that the image of the James-Hopf invariant double desuspends. This is not always true in periodic homotopy or in the unlocalized UNSS, but is true in the localized E_2-term.

Now we begin the proof of Theorem 3.17. Throughout this proof, let $M = M_{2n}$ denote the free BP_*-module on a generator of degree $2n$. We consider the cosimplicial resolution (of coalgebras, by [**7**, 6.10])

$$(3.19)\qquad BP_*(\Omega S^{2n+1}) \xrightarrow{\xi} G(M) \rightrightarrows G(V(M)) \rightrightarrows G(V^2(M)) \cdots,$$

analogous to (3.10). Taking primitives in (3.19), we obtain, as in [**5**, §4], a chain complex

$$(3.20)\qquad \overline{U}(M) \xrightarrow{d_0} \overline{U}(V(M)) \xrightarrow{d_1} \overline{U}(V^2(M)) \to \cdots$$

where d_i is the alternating sum of the coface operators. Here $\overline{U}(M)$ denotes the primitives in $G(M)$. As noted earlier, this $\overline{U}(M)$ maps isomorphically to $U(M) \subset \Gamma \otimes M$. We use $\overline{U}(-)$ when we are interested in the specific representations of elements in $G(-)$ rather than their image in $\Gamma \otimes -$. Since $G(M)$ is a free coalgebra, the homology of (3.20) in homological degree i is the ith derived functor of $P(BP_*(\Omega S^{2n+1}))$. In particular, it has no homology in homological degree greater than 1, by [**5**, 4.1].

We now embed (3.20) into a double complex whose homology is $\sigma^{-1}E_2(S^{2n+1})$, but from which we can identify the James-Hopf map. This double complex has, for $p, q \geq 0$, (p,q)th group $U^p(\overline{U}(V^q(M))$, with vertical differential that of (3.20). The homology of the qth row is $\operatorname{Ext}^p_{\mathcal{U}}(UV^qM) = 0$ if $p > 0$, while if $p = 0$ it is $V^qM \approx \sigma^{-1}U^q(\sigma M)$. (This is true since, for any N, $U(N)$ is injective in $\mathcal{U}$, satisfying $\operatorname{Ext}^0_{\mathcal{U}}(U(N)) \approx N$.) Hence the homology of the total complex is the homology of the complex whose qth term is $V^q(M) \approx \sigma^{-1}U^q(\sigma M)$, and this homology can be interpreted as either $E_2(\Omega S^{2n+1})$ or $\sigma^{-1}E_2(S^{2n+1})$, by (3.4).

We let $T(\Omega S^{2n+1})$ denote this double complex. A double complex analogous to this can be constructed for $\Omega Sp(n)$ using $N = BP_*\{g_1, \ldots, g_n\}$ as M. Thus $T^s(\Omega Sp(n)) = \bigoplus U^{s-q}(\overline{U}(V^q(N)))$. It is the $\overline{U}$ here which causes $*$-products of g_i-classes to come into play here.

Returning to the case $M = M_{2n}$, we now take homology in the opposite direction. The homology of the pth column is obtained by applying U^p to the homology of (3.20). This is possibly nonzero only in $q = 0$ and $q = 1$, where it is $U^p(R^qP)$, since U commutes with homology. Here and throughout the remainder of this proof, we let $P = P(BP_*(\Omega S^{2n+1}))$, and R^q denotes the qth right derived functor. Taking horizontal homology of this yields a 2-line spectral sequence which reduces to the double suspension exact sequences as in [**8**, §8].

The homology of the total complex in homological dimension s is carried by

$$(3.21)\qquad U^s(\overline{U}(M)) \oplus U^{s-1}(\overline{U}(V(M))).$$

Recall that $\overline{U}(M) = P(BP_*(BP_{2n}))$. By [**29**, 5.3], the first two indecomposables in $BP_*(BP_{2n})$ are $b_0^{\circ n}$ and $b_1^{\circ n}$, and, as discussed earlier, the images of these in $\Gamma \otimes M$ are $1 \otimes \iota_{2n}$ and $h_1^n \otimes \iota_{2n}$. By [**5**, 3.6], $2h_1^n \otimes \iota_{2n}$ is the image of a primitive of $\overline{U}(M)$, and by 3.6 this primitive must be $2h_1^n \otimes \iota_{2n} - 1 \otimes \iota_{2n}^{*2}$. (Here we are using the stable/unstable notation introduced earlier.) Thus an element in the first term of (3.21) can be written as $\alpha_1(1\otimes\iota_{2n})+\alpha_2(2h_1^n\otimes\iota_{2n}-1\otimes\iota_{2n}^{*2})$ mod later primitives, with $\alpha_i \in BP_*BP^{\otimes s}$. As for the second term of (3.21): By [**5**, 4.3] and the paragraph which precedes it, the vertical homology of our total complex at $\overline{U}(V(M))$ is R^1P, and the bottom class of this equals $1\otimes h_1^n \otimes \iota_{2n}$, mod terms that desuspend.[3] Under the isomorphism $R^1P \approx W(n)$, this bottom class equals the class called x_{4n-1} in the paragraph following the statement of Theorem 3.17.

Now we note, by [**4**] or, more explicitly, [**5**, 5.3ii], that any class $y \in v_1^{-1}E_2^s(S^{2n+1})$ may be represented, mod terms that desuspend, as $y = \alpha \otimes h_1^n \otimes \iota_{2n}$ with $\alpha \in \Gamma^{\otimes(s-1)}$. The projection of this y to $v_1^{-1}\mathrm{Ext}_{\mathcal{U}}^{s-1}(R^1P) = v_1^{-1}\mathrm{Ext}_{\mathcal{U}}^{s-1}(W(n))$ is just $H_2(y) = \alpha\otimes x_{4n-1}$. In terms of the double complex, y is represented by

$$\begin{aligned}\overline{y} = [\alpha_1 \otimes (1 \otimes \iota_{2n}) \\ + \alpha_2 \otimes (2h_1^n \otimes \iota_{2n} - 1 \otimes \iota_{2n}^{*2})] + [\alpha \otimes 1 \otimes h_1^n \otimes \iota_{2n}] \\ \in [U^s\overline{U}(M)] + [U^{s-1}R^1P],\end{aligned}$$

mod terms of higher vertical filtration in the double complex and terms with higher powers of ι_{2n}.

Since $\overline{y}$ is a cycle, and α is a mod 2 cycle, evaluating the coefficient of $1 \otimes 2h_1^n \otimes \iota_{2n}$ in $d(\overline{y})$ yields

$$0 = \alpha_2(1 \otimes 2h_1^n \otimes \iota_{2n}) + \frac{d(\alpha)}{2} 1 \otimes 2h_1^n \otimes \iota_{2n}.$$

Thus $\alpha_2 = \frac{d(\alpha)}{2} = \delta(\alpha)$, where δ is the Bockstein. Now the James-Hopf map yields a map of double complexes $T(\Omega S^{2n+1}) \xrightarrow{H} T(\Omega S^{4n+1})$ which sends ι_{2n}^{*2} to ι_{4n}. Hence $H(\overline{y}) = \alpha_2 = \delta(\alpha)$, establishing the following proposition, since $\alpha = H_2(y)$.

Proposition 3.22. *An element $x \in v_1^{-1}E_2^s(\Omega S^{2n+1})$ can be represented in $T(\Omega S^{2n+1})$ as $\alpha \otimes h_1^n \otimes \iota_{2n} + \beta \otimes 1 \otimes \iota_{2n}^{*2}$ mod higher powers of ι_{2n} and terms which desuspend. Then $\alpha = H_2(x)$ and $\beta = \delta(\alpha)$. The James-Hopf map H sends this to $\beta \otimes \iota_{4n} \in v_1^{-1}E_2^s(\Omega S^{4n+1})$.*

[3]There is a minor misprint in [**5**, 4.3], which should say $P(BP_*(\Omega S^{2n+1}))$ instead of $P(A(2n))$.

Theorem 3.17 is just a restatement of this proposition. Note that the thing that makes Theorem 3.17 work in the v_1-periodic setting but not for unlocalized E_2 is the fact that $v_1^{-1}\mathrm{Ext}_{\mathcal{U}}(W(n))$ is built entirely on the bottom class of $W(n)$. If we had to worry about elements of $\mathrm{Ext}_{\mathcal{U}}(W(n))$ built on x_{8n-1}, then the next Husemoller primitive,

$$4h_2^n \otimes \iota_{2n} - 2h_1^{2n} \otimes \iota_{2n}^{*2} + 1 \otimes \iota_{2n}^{*4},$$

would have to be taken into account.

Proposition 3.22 links $h_1^n \otimes \iota_{2n}$ with ι^{*2}, while the following result links it with ι^2.

Proposition 3.23. *If the mod 2 reduction of $\psi(g_n) \in BP_*(\Omega Sp(n))$ contains the term $h_1 \otimes g_{n/2}^2$, then it also contains the term $v_1 h_1^{n-1} \otimes g_{n/2}$.*

Proof. We first note that if $\Delta(g_n)$ contains the term $\epsilon v_1 g_{n/2} \otimes g_{n/2}$, then, assuming the hypothesis of the proposition, $(1 \otimes \Delta)(\psi(g_n))$ contains the terms $2h_1 \otimes g_{n/2} \otimes g_{n/2}$ and $1 \otimes \epsilon v_1 g_{n/2} \otimes g_{n/2}$, while $m_\Gamma(\psi \otimes \psi)\Delta(g_n)$ contains $\epsilon v_1 \otimes g_{n/2} \otimes g_{n/2}$. Using $\eta_R(v_1) = v_1 - 2h_1$, the equating of these which is implied by diagram (3.9) yields $2h_1 + \epsilon(v_1 - 2h_1) = \epsilon v_1$, and hence $\epsilon = 1$. Thus $\Delta(g_n)$ contains the term $v_1 g_{n/2} \otimes g_{n/2}$.

In the proof of Proposition 3.13, it was shown that the map Δ_G of (3.8) sends only $v_1 h_1^{n-1} \otimes g_{n/2}$ to $v_1 g_{n/2} \otimes g_{n/2}$. Thus the composite (3.8) implies that $\eta_{\Omega Sp(n)}(g_n)$ goes to $v_1 h_1^{n-1} \otimes g_{n/2}$ plus other monomials. Now the composite (3.7) implies the proposition. □

We will need to make frequent reference to the chart of the periodic UNSS for S^{2n+1} given on page 58 of [**4**]. We reprint it in Figure 3.24 for the convenience of the reader. We have included a few classes which are involved in d_3-differentials which were not included in [**4**]. The notation is explained on page 53 of [**4**]. For us, the most important part is that an element y denoted $\{Ah^i\}$ means that y pulls back to S^{2i+1}, where it satisfies $H_2(y) = A$. The horizontal component of the chart is the mod 8 value of the stem, not the homotopy group.

We provide one other bit of background before proceeding with the proof of Proposition 1.8: the explicit manner in which the remaining differentials are computed. This is similar to the method in Section 2 for the stable NSS. It involves the chain complex $T(X)$ described above, whose homology equals the E_2-term of the UNSS for $X = \Omega S^{2n+1}$ or $\Omega Sp(n)$. If $\overline{U}(-)$ is identified with $U(-)$, then $T^s(X)$ becomes a subset of $\Gamma \otimes \cdots \otimes \Gamma \otimes QBP_*(X)$, with $s+1$ factors of Γ. However, the only terms that we will ever have need to consider are the third and fourth terms in (3.21), both of which involve a

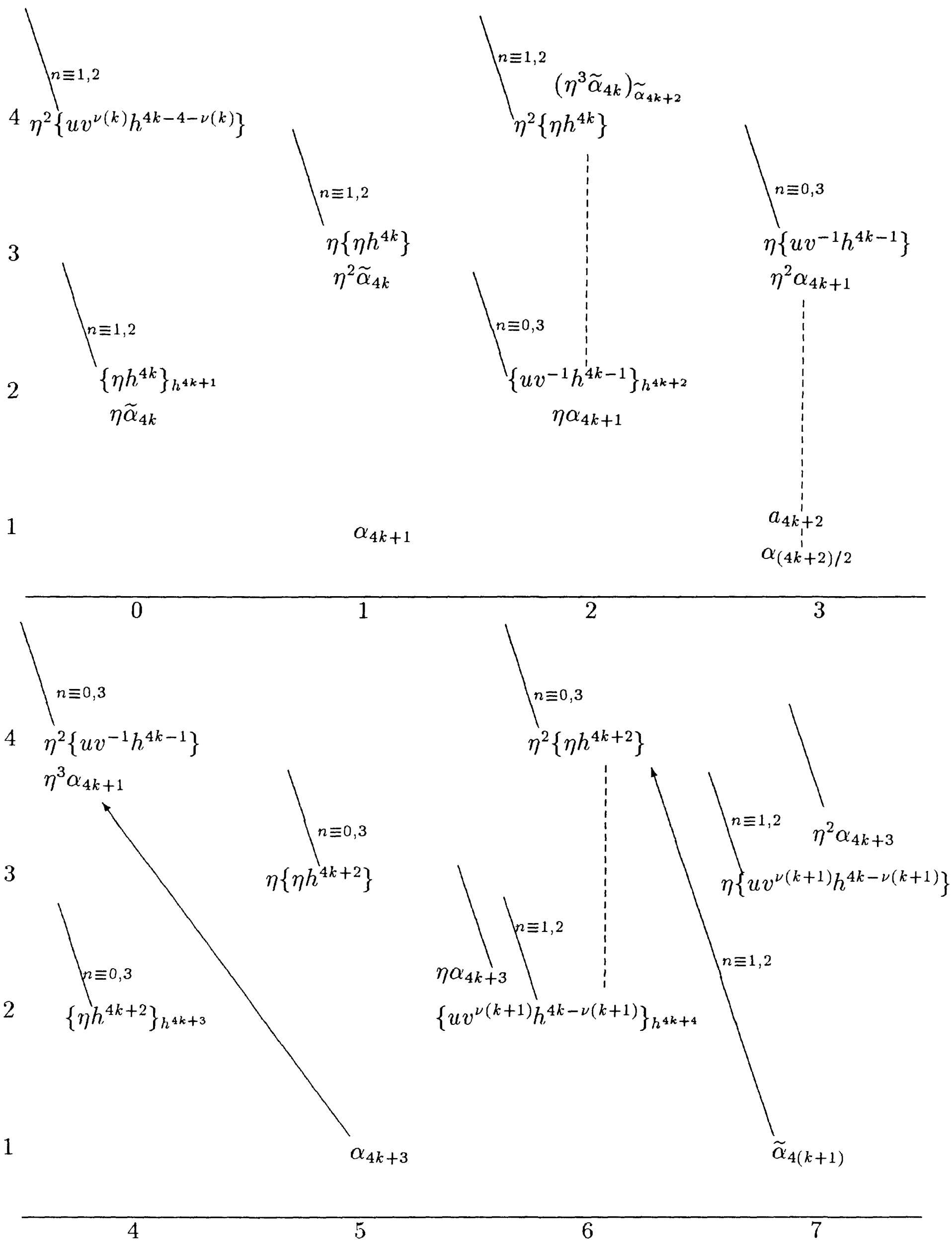

Figure 3.24: E_2 of periodic UNSS for spheres

$1\otimes$ which can be omitted, so that we actually view the elements as involving the s-fold tensor product of Γ.

As in the stable situation,

$$\partial : E_2^s(\Omega S^{4n-1}) \to E_2^{s+1}(\Omega Sp(n-1))$$

is calculated on a class z by finding a representative cycle c for z for which the reduced coaction $\overline{\psi}(c)$ pulls back to a class in $T^{s+1}(\Omega Sp(i))$ for i as small as possible. The projection of $\overline{\psi}(c)$ into $T^{s+1}(\Omega S^{4i-1})$ yields $\partial(z)$. This can be done for the complex $T(-)$ just as well as for the complex $C(-)$ used in Section 2, because the fibration

$$\Omega Sp(n-1) \to \Omega Sp(n) \to \Omega S^{4n-1}$$

induces a short exact sequence of $T(-)$, and the boundary morphism in the associated long exact homology sequence is as described above.

Explicit calculation of the coaction in $BP_*(\Omega Sp(n))$ can be performed by a method similar to that of **[10]**, but the formulas are extremely complicated. Instead we opt for calculating a homomorphic image of the coaction which contains the information relevant to our application, exactly as we did for Q^n in Section 2.

This formula is obtained similarly to that given earlier in Q^n. We use the description of $bu_*(\Omega Sp(n))$ given in 3.2. Since Ψ^3 is a ring homomorphism, it suffices to determine Ψ^3 on the generators g_i. Although we do not have complete information about the action of Ψ^3 in $bu_*(\Omega Sp(n))$, the following will suffice for our applications.

Proposition 3.25. *In $bu_*(\Omega Sp(n)/Sp([n/2]))$,*

$$\Psi^3(g_n) = \sum_{i=0}^{[(n-1)/2]} (-3)^{-i}\binom{2n+i-1}{i} v^{2i} g_{n-i}.$$

Since the inclusion maps $\Omega Sp(i) \to \Omega Sp(n)$ send g_j to g_j, this proposition can be used to read off $\Psi^3(g_i)$ in $bu_*(\Omega Sp(n)/Sp([n/2]))$ for all $i \leq n$.

Proof. Let Q^n denote the quaternionic quasi-projective space which embeds naturally into $Sp(n)$, and let $Q_k^n = Q^n/Q^{k-1}$. Then $Q^n_{[n/2]+1}$ embeds in $Sp(n)/Sp([n/2])$, and is a suspension for dimensional reasons. Thus we shall work with $Q^n_{[n/2]+1}$, and let Y satisfy $\Sigma Y = Q^n_{[n/2]+1}$. By adjointing, we obtain a map $Y \xrightarrow{f} \Omega Sp(n)/Sp([n/2])$. Letting g_i denote the image under f_* of the generators of $bu_*(Y)$, the result follows from Proposition 2.7.

Using the $\binom{-2n}{i}$-form for the binomial coefficient immediately yields the following corollary.

Corollary 3.26. *In $bu/(2)_*(\Omega Sp(n)/Sp([n/2]))$,*

$$\Psi^3(g_n) = g_n + v^{2^{\nu(n)+2}} g_{n-2^{\nu(n)+1}} + T,$$

where $T \in \langle v^{2i} g_{n-i} : i > 2^{\nu(n)+1} \rangle$.

Of course, if $n - 2^{\nu(n)+1} \leq [(n-1)/2]$, then this corollary must be interpreted as saying $\Psi^3(g_n) \equiv g_n \pmod 2$. This will be the case iff $n = 2^e$ or $3 \cdot 2^e$ for some e.

Exactly analogous to Proposition 2.11, we have the following result.

Proposition 3.27. *Let* $\phi : BP_*BP \to \mathbb{Z}_2[v]$ *be the homomorphism defined in Proposition* 2.11. *Let* $g_i \in BP_{4i-2}(\Omega Sp(n)/Sp([n/2]))$ *be the generator defined, similarly to the proof of* 2.7, *using* S*-duality and the standard generators of* $BP^*(HP^m)$. *Suppose the coaction*

$$BP_*(\Omega Sp(n)/Sp([n/2])) \xrightarrow{\psi} \Gamma \otimes BP_*(\Omega Sp(n)/Sp([n/2]))$$

satisfies

$$\psi(g_n) = \sum z_i \otimes g_i.$$

Then

$$\phi(z_i) = \begin{cases} 0 & \text{if } n - 2^{\nu(n)+1} < i < n \\ v^{2^{\nu(n)+2}} & \text{if } i = n - 2^{\nu(n)+1}. \end{cases}$$

Note that $\psi(g_n)$ cannot contain any g_i^2-terms for dimensional reasons.

Now we are ready to calculate the differentials in Proposition 1.8. We begin with the stable differentials (1.11). These follow by the same argument that was used to prove Proposition 2.22. Or they can be deduced from it, using the map from the UNSS of $Sp(n)$ to the stable NSS of Q^n discussed in Proposition 4.6. The reason that the differentials of (1.11) are not given on classes 2^e or $3 \cdot 2^e$ with $e > 0$ is that differentials (1.9) are seen first on these classes.

Next we will prove the differentials (1.9) and (1.10). This proof will be by induction on i. We assume that it has been proved for all values of i less than n. This, along with (1.11), implies that there are no stable classes (i or i') for the η-tower $2n$ to hit. Thus we want to determine the boundary homomorphism

$$v_1^{-1} E_2^{1,4k}(\Omega S^{8n-1}) \xrightarrow{\partial} v_1^{-1} E_2^{2,4k}(\Omega Sp(2n-1)) \tag{3.28}$$

in (3.5), and the target group consists only of unstable classes. By Figure 3.24, the source of the morphism (3.28) is the element $\alpha_{2k-4n+1} g_{2n}$, where, as above, α_j is the element of order 2 on the 1-line in the $(2j-1)$-stem for spheres, as described, for example, in [**4**, p. 52]. By [**7**, p. 246], $\alpha_{2k-4n+1} g_{2n}$ is represented by $v_1^{2k-4n} h_1 \otimes g_{2n}$.

We work now in the complex $T^*(\Omega Sp(n))$, which involves $\overline{U}(N) \subset G(N)$. Recall that $N = QBP_*(\Omega Sp(n))$. This involves classes g_i^{*2} but no classes g_i^2. Note that $\overline{\psi}(g_{2n}) \in \Gamma \otimes G(N)$ cannot contain any terms $B \otimes g_i^{*2}$ with $i > n$ for dimensional reasons. It does contain the term $h_1 \otimes g_n^{*2}$ since Proposition 3.2 implies that $(z_{4n-1})\mathrm{Sq}^2 = z_{2n-1}^2$. Here we use that h_1 is detected by Sq^2, and that the $*$-product reduces to the homology product. This observation about the effect of the nonzero Sq^2 is the key to this family of differentials. This sort of information about ψ was not seen in Proposition 3.27, which was only capable of seeing the stable range.

Thus there is a representative γ_{2n-1} for $\partial(\alpha_{2k-4n+1} \otimes g_{2n})$ which lies in $T^2(\Omega Sp(2n-1))$ and has $v_1^{2k-4n} h_1 \otimes h_1 \otimes g_n^{*2}$ as its leading g_i^{*2}-term. That is, it contains no terms of the form $B \otimes g_i^{*2}$ with $i > n$.

We will inductively pull our class γ_{2n-1} back to a class $\gamma_n \in T^2(\Omega Sp(n))$. Assume that for some j satisfying $n < j < 2n$, there is a representative γ_j for $\partial(\alpha_{2k-4n+1} g_{2n})$ which lies in $T^2(\Omega Sp(j))$ and has $v_1^{2k-4n} h_1 \otimes h_1 \otimes g_n^{*2}$ as its leading g_i^{*2}-term. We project this to $\overline{\gamma}_j \in T^2(\Omega S^{4j-1})$. As observed earlier, this cannot equal the nonzero stable class here, because that already supported a differential. If $\overline{\gamma}_j$ is to be the nonzero unstable class, then it must contain the following terms in $T^2(\Omega S^{4j-1})$:

$$v_1^{2k-4j+1} h_1 \otimes h_1^{2j-1} \otimes g_j + v_1^{2k-4j} h_1 \otimes h_1 \otimes g_j^{*2}. \tag{3.29}$$

To see this, we use Figure 3.24 to see that the unstable element w at height 2 and stem $\equiv 0 \bmod 4$ satisfies $H_2(w) = v_1^{2k-4j+1} h_1$. Then we use Proposition 3.22 to see that this must be accompanied by $\delta(H_2(w)) g_j^{*2}$, and use [**27**, 5.3.13] to evaluate the Bockstein $\delta(v_1^{2k-4j+1} h_1) = v_1^{2k-4j} h_1 \otimes h_1$.

But our induction hypothesis is that $\overline{\gamma}_j$ does not contain this g_j^{*2}-term. Thus $\overline{\gamma}_j$ must be a boundary in $T^2(\Omega S^{4j-1})$. We can vary γ_j by the boundary of an element of $T^*(\Omega Sp(j))$ which corresponds to the bounding element in $T^*(\Omega S^{4j-1})$, obtaining a class γ_{j-1} which extends the induction hypothesis. The boundary can be chosen to be of the form $d(A \otimes v g_j)$ with $\phi(v) = 0$ as in the stable situation considered in proving 2.25, and hence cannot produce detecting terms. Consequently by induction we obtain a representative γ_n which pushes to an element of $T^2(\Omega S^{4n-1})$ containing the term $v_1^{2k-4n} h_1 \otimes h_1 \otimes \iota_{4n-2}^{*2}$. By Proposition 3.22 and Figure 3.24, this must be the nonzero unstable class n_u.

Next we will use the $2n \to n_u$ differential just established to prove the differential $(2n)' \to n_u'$ by precomposing the first differential with the E_2-element corresponding to the Hopf map σ in the 7-stem. We will use the notation of [**27**, p. 199], which denotes this element of $\mathrm{Ext}(BP_*)$ by y_4, and its reduction to $\mathrm{Ext}(BP_*/(2))$ by u. Since we have shown that (3.28), which represents the $(2n \to n_u)$-differential, sends the generator of $v_1^{-1} E_2^{1,4k}(\Omega S^{8n-1})$

into an element in the image of

$$v_1^{-1}E_2^{2,4k}(\Omega Sp(n))\xrightarrow{i_*}v_1^{-1}E_2^{2,4k}(\Omega Sp(2n-1)),$$

the same will be true when this generator is precomposed by y_4. The element of $v_1^{-1}E_2^{2,4k+8}(\Omega S^{8n-1})$ obtained as this composite is $y_4 \otimes v_1^{2k-4n}h_1 \otimes g_{2n}$. When this is reduced mod 2 (which we may do since we are looking at mod 2 classes), one obtains as the coefficient the stable class $u \otimes v_1^{2k-4n}h_1 = h_1 \otimes v_1^{2k-4n}u$, since stable Ext is commutative. By [**27**, 5.3.13c], this equals the mod 2 reduction of what is called $\eta\widetilde{\alpha}_{4k-4n+4}$ in Figure 3.24, which is the coefficient of the element $(2n)'$. Thus the source of our proposed differential is the desired σ-composition.

The element in $E_2^{3,4k+8}(\Omega S^{4n-1})$ which is n_u precomposed with σ has leading term $y_4 \otimes h_1 \otimes v_1^{2k-4n+1}h_1^{2n-1}g_n$. By [**5**, 5.3], this unstable class x satisfies

$$H_2(x) = y_4 \otimes h_1 v_1^{2k-4n+1} = h_1 \otimes u v_1^{2k-4n+1}.$$

Here H_2 denotes the double suspension Hopf invariant, which yields a stable mod 2 class. But this is exactly what Figure 3.24 tells us is H_2 of our proposed target class n_u', namely the element at height 3 in the chart with horizontal coordinate 3 or 7 depending upon the parity of n. This establishes the differential $(2n)' \to n_u'$.

Now we are ready to derive (1.12). By naturality, it suffices to prove it on the top class n_u in the spectral sequence for $v_1^{-1}\pi_*(Sp(n)/Sp([n/2]))$. By 3.22, the unstable classes involved in the proposed differential are detected by their g_i^{*2}-terms, and so we begin by studying the reduced coaction $\overline{\psi}(g_n^{*2})$. As we are looking at elements of order 2, and $d \circ d = 0$, this will consist of terms of the form $z_i \otimes g_i^{*2}$, where z_i is a mod 2 cycle. Here we are also using the multiplicativity of ψ with respect to the $*$-product. It is shown in [**27**, 5.3.13a] that any such cycle can be written, mod boundaries, as

$$z_i = \epsilon v_1^{k+3}h_1 + \epsilon' v_1^k u, \tag{3.30}$$

where ϵ and $\epsilon' = 0$ or 1, and $u = t_1^4 + v_1t_1^3 + v_1^2t_1^2 + v_1t_2 + v_2t_1$. The homomorphism ϕ defined in (2.12) sends the class in (3.30) to $(\epsilon + \epsilon')v^4$. Proposition 2.21 shows that it sends any boundary to 0. Combining these remarks with Proposition 3.27 and the multiplicativity of ψ on $*$-products shows that

$$\overline{\psi}(g_n^{*2}) \equiv \sum_{i<n}(\epsilon_i v_1^{4(n-i)-1}h_1 + \epsilon_i' v_1^{4(n-i)-4}u + d(a_i)) \otimes g_i^{*2} \ mod\ 2, \tag{3.31}$$

with $a_i \in BP_*$ and $\epsilon_i+\epsilon_i' = 0$ if $i > n-2^{\nu(n)+1}$, and $\epsilon_i+\epsilon_i' = 1$ if $i = n-2^{\nu(n)+1}$.

The source x of our proposed differential, also denoted n_u, is the element of $E_2^{2,4k+8n}(\Omega S^{4n-1})$ corresponding to the unstable element at height 2 and horizontal component 0 or 4 in Figure 3.24. The notation of that chart says that it satisfies $H_2(x) = v_1^{2k+1}h_1$, and thus by Proposition 3.22

$$x \equiv v_1^{2k+1}h_1 \otimes h_1^{2n-1} \otimes g_n + h_1 \otimes v_1^{2k}h_1 \otimes g_n^{*2} \tag{3.32}$$

mod terms that desuspend or contain higher powers of g_n. Here we have used [**27**, 5.3.13d] to evaluate $\delta(H_2(x))$. In the same chart, a target element y_i, labeled i'_u in (1.12), would be the element at height 3 and horizontal component 3 or 7, and so would satisfy

$$H_2(y_i) = h_1 \otimes uv_1^{2m+1} \tag{3.33}$$

for appropriate m. The differential, or boundary in the exact sequence, is calculated by applying $\overline{\psi}$ to g_n and g_n^{*2} in (3.32), and then modifying by boundaries. By Proposition 2.21, these boundaries cannot produce detecting terms. Thus the differential on n_u hits i'_u for largest possible i such that $h_1 \otimes v_1^{2k}h_1 \otimes (\epsilon_i v_1^{4l-1}h_1 + \epsilon'_i v_1^{4l-4}u) \otimes g_i^{*2}$ equals $h_1 \otimes u \otimes v_1^{2m}h_1 \otimes g_i^{*2}$. Here we have applied (3.31) to the second term of (3.32) to get the first expression, and have applied 3.22 and [**27**, 5.3.13d] to (3.33) to get the second expression. Since these coefficients are stable mod 2 classes, their factors commute. Thus the differential hits i'_u if and only if $\epsilon_i = 0$ and $\epsilon'_i = 1$. By (3.31), this can first happen when $i = n - 2^{\nu(n)+1}$. Moreover, it must happen for this value of i, since we cannot have $\epsilon_i = 1$ and $\epsilon'_i = 0$ since that would imply the differential hits a class z whose James-Hopf invariant $H(z)$ reduces mod 2 to $v_1^{pwr}h_1 \otimes h_1 \otimes h_1$. By Theorem 3.17 such a class would have $\delta(H_2(z)) = v_1^{pwr}h_1 \otimes h_1 \otimes h_1$. But the only possible value of $H_2(z)$ is given by (3.33), and so this possibility cannot occur since $\delta(h_1 \otimes uv_1^{2m+1}) = h_1 \otimes u \otimes v_1^{2m}h_1$. □

4. Differentials and extensions.

In this section we establish the differentials in the UNSS converging to $v_1^{-1}\pi_*(Sp(n))$, whose E_2-term was depicted in Figure 1.16. We will also establish the exotic group extensions, and from this deduce Theorem 1.5.

The following theorem gives most of the higher differentials among the elements of Figure 1.16. Positions are the $[x, y] = [t - s, s]$ coordinates of that chart. We use the phrase "log classes" for the elements of the $\mathbb{Z}_2$-vector space of dimension $[\log_2(4n/3)]$ which is written "log" in Figure 1.16.

Theorem 4.1. *In the UNSS for $v_1^{-1}\pi_*(Sp(n))$, there are the following families of d_3-differentials, referring to Figure 1.16 for the description of*

$v_1^{-1}E_2(Sp(n))$.

1. *d_3 sends the η-tower of* log *classes starting in* $[8h+3,2]$ *isomorphically to the η-tower of* log *classes starting in* $[8h+2,5]$.
2. *d_3 sends the η-tower of* log *classes starting in* $[8h+2,3]$ *isomorphically to the η-tower of* log *classes starting in* $[8h+1,6]$. *(These latter elements are not actually represented in Figure 1.16, except as a shifted version of the η-tower going up from* $[8h+6,3]$.*) It also sends the η-tower of $\bullet$-classes beginning in* $[8h,1]$ *to the one beginning in* $[8h-1,4]$
3. *d_3 sends the group in* $[8h+1,2]$ ***onto*** *the group in* $[8h,5]$ *(which can be envisioned as a shifted version of the group in* $[8h+8,5]$*). Thus the group $G(2^{e_1})$ has* $[\log_2(4n/3)]$ *summands which support differentials (and perhaps also some which do not).*
4. *The η-tower on the $\bullet$ in* $[8h+7,2]$ *is mapped isomorphically by d_3 to the η-tower on the $\bullet$ in* $[8h+6,5]$.

Proof. Parts 1 and 2 follow immediately from the corresponding d_3-differentials in S^{4i-1}. Indeed, part 1 is the differential in Figure 3.24 emanating from filtration 2 in horizontal position 0 or 4, while the log-part of part 2 is the differential in Figure 3.24 going from filtration 3 and horizontal position 3 or 7. Remember that in Figure 3.24 horizontal position is stem, while in Figure 1.16 it is the number of the homotopy group. The classes involved in the $\bullet$-differential in part 2 come from S^3. They appear in 3.24 as the d_3-differential on the element α_{4k+3}.

The differential in part 3 is immediate from the η-connection from G to log, which was already established, and the d_3's on the log-part established in part 2. The classes involved in part 4 are stable classes coming from S^{4n-1} if n is odd, and S^{4n-5} if n is even. For example, if $n=10$, they correspond to the $9'$ in Figure 1.13. They should appear in Figure 3.24 as $\eta\widetilde{\alpha}_{4k+2}$ in $[4,2]$ with a d_3 going to an element which is η times the top element listed in column 2 there. These differentials (with elements unlabeled) can be seen in [**9**, p. 488]. □

What remains from Figure 1.16 after the differentials of Theorem 4.1 are taken into account is pictured in Figure 4.2. Here $H(2^{e_1})$ is a group which fits into a short exact sequence

$$0 \to H(2^{e_1}) \to G(2^{e_1}) \xrightarrow{d_3} [\log_2(4n/3)]\mathbb{Z}_2 \to 0.$$

Thus the order of $H(2^{e_1})$ is $2^{e_1-[\log_2(4n/3)]}$.

It was proved in [**10**, 1.3] that the groups $E_2^{1,4k-1}$ are cyclic of order $2^{e_{Sp}(k,n)}$, where $e_{Sp}(k,n)$ is defined as in (1.15). It is similar to the number $e(k,n)$ which appears in Theorem 1.5 except that $e_{Sp}(k,n)$ does not involve

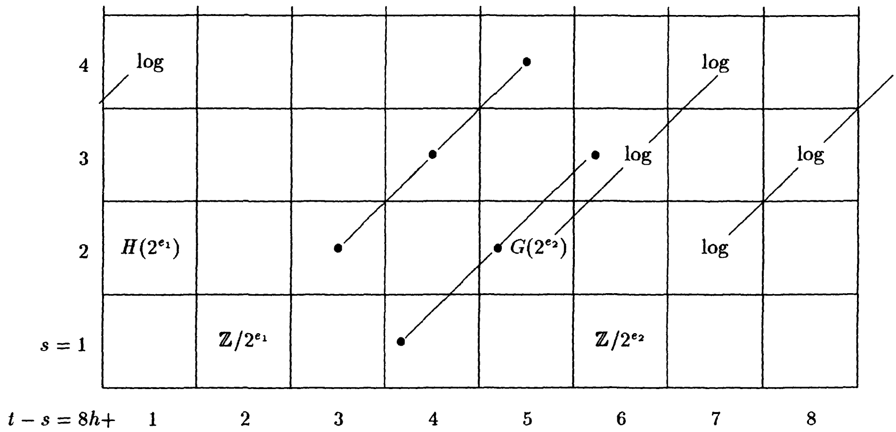

Figure 4.2: A stage of the periodic UNSS for $Sp(n)$

the terms ϵ_{k-1} and ϵ_{k-j} which appear in the definition, (1.3), of $e(k,n)$. Recall that in our figures $e_j = e_{Sp}(2h+j,n)$. In order to pass from Figure 4.2 to Theorem 1.5, we must consider extensions in the SS and additional d_3-differentials from the 1-line.

First, the extensions: There is an extension from the cyclic group $v_1^{-1}E_2^{1,8h+7}(Sp(n))$ to the element in $v_1^{-1}E_2^{3,8h+9}(Sp(n))$ which is in the image from $E_2^{3,8h+9}(Sp(1))$. This element is the one indicated by a $\bullet$ in $[8h+6,3]$ in Figure 4.2. By "extension," we mean a nontrivial extension in the short exact sequence

$$0 \to v_1^{-1}E_\infty^{3,8h+9}(Sp(n)) \to v_1^{-1}\pi_{8h+6}(Sp(n)) \xrightarrow{\partial} v_1^{-1}E_\infty^{1,8h+7}(Sp(n)) \to 0.$$

This accounts for the ϵ_{k-1} in $e(k,n)$ in (1.3). To see this extension, one needs merely to observe that the element of order 2 in $v_1^{-1}E_\infty^{1,8h+7}(Sp(n))$ is in the image from $Sp(1)$, and the extension is present in the UNSS for $v_1^{-1}\pi_*(Sp(1))$.

As for other extension questions: In $8h+1$, there could very well be nontrivial extensions. The two groups together, namely $H(2^{e_1})$ at height 2 and log at height 4, yield a group of order 2^{e_1} or 2^{e_1-1}, the latter being the case if there is a nonzero d_3-differential into $v_1^{-1}E_2^{4,8h+5}(Sp(n))$. Theorem 1.5 makes no claim about the structure of the group $v_1^{-1}\pi_{8h+1}(Sp(n))$. In $8h+4$, the extension is ruled out by the fact that the class in filtration 1 pulls back to $Sp(1)$, while the class in filtration 3 does not. In $8h+5$, there is a possible extension from $G(2^{e_2})$ to the $\bullet$. This is incorporated into the $G(2^{1+e(2h+2,n)})$ in Theorem 1.5. If there is no d_3-differential into $v_1^{-1}E_2^{4,8h+9}(Sp(n))$, then $e(2h+2,n) = e_2+1$. In $8h+7$, we cannot solve the extension question.

The remainder of the work in this section is devoted toward proving the following result, which will complete the proof of Theorem 1.5.

Theorem 4.3. *The differential*

$$d_3 : v_1^{-1}E_2^{1,4k-1}(Sp(n)) \to v_1^{-1}E_2^{4,4k+1}(Sp(n))$$

is nonzero if and only if $e(k,n) < \epsilon_{k-1} + e_{Sp}(k,n)$.

Note that the hypothesis on this proposition says exactly that the ϵ_{k-j} in Definition 1.3 affects the value of $e(k,n)$.

Before embarking upon the lengthy proof of Theorem 4.3, we recapitulate the proof of Theorem 1.5.

- There is a spectral sequence converging to $v_1^{-1}E_2(Sp(n))$ with initial term given by summing Figure 1 over $i \le n$. The differentials among the η-towers are given by Proposition 1.8, while the S_i-groups build a cyclic 2-group of exponent $e_{Sp}(k,n)$, and the U_i-groups build a group of the same order. The number of unstable $\mathbb{Z}_2$'s in each bigrading surviving these differentials is $[\log_2(4n/3)]$, and so $v_1^{-1}E_2(Sp(n))$ is as in Figure 1.16.
- All the η-towers in $v_1^{-1}E_2(Sp(n))$ are involved in d_3-differentials implied by differentials in the spheres. What remains after these are accounted for is pictured in Figure 4.2; it includes the 1-line cyclic groups, part of the 2-line groups, and a few initial elements of target η-towers.
- There is a nonzero d_3-differential on the group in $v_1^{-1}E_2^{1,4k-1}(Sp(n))$ if and only if $e(k,n) < \epsilon_{k-1} + e_{Sp}(k,n)$. There is an extension in $v_1^{-1}\pi_{4k-2}(Sp(n))$ if k is even; this accounts for the ϵ_{k-1}. In Figure 4.4, we show how Figure 4.2 is modified by these considerations. The dotted differentials are present only when $e(k,n) < \epsilon_{k-1} + e_{Sp}(k,n)$.

The proof of Theorem 4.3 will proceed in a sequence of propositions. The first one handles the nonzero differentials.

Proposition 4.5. *If* $e(k,n) < \epsilon_{k-1} + e_{Sp}(k,n)$, *then*

$$d_3 \neq 0 : v_1^{-1}E_2^{1,4k-1}(Sp(n)) \to v_1^{-1}E_2^{4,4k+1}(Sp(n)).$$

Proof. In the exact sequence

$$E_2^{0,4k-1}(Sp(k)) \xrightarrow{p_*} E_2^{0,4k-1}(Sp(k)/Sp(n)) \to E_2^{1,4k-1}(Sp(n)) \to 0,$$

the first two groups are infinite cyclic, and the third is $\mathbb{Z}/2^{e_{Sp}(k,n)}$. In [**32**, 0.2] or [**26**, p. 868], it is shown that the mod torsion index of $p_*(\pi_{4k-1}(Sp(k)))$ in $\pi_{4k-1}(Sp(k)/Sp(n))$ has 2-exponent equal to or less than $e(k,n)$. Assume, for

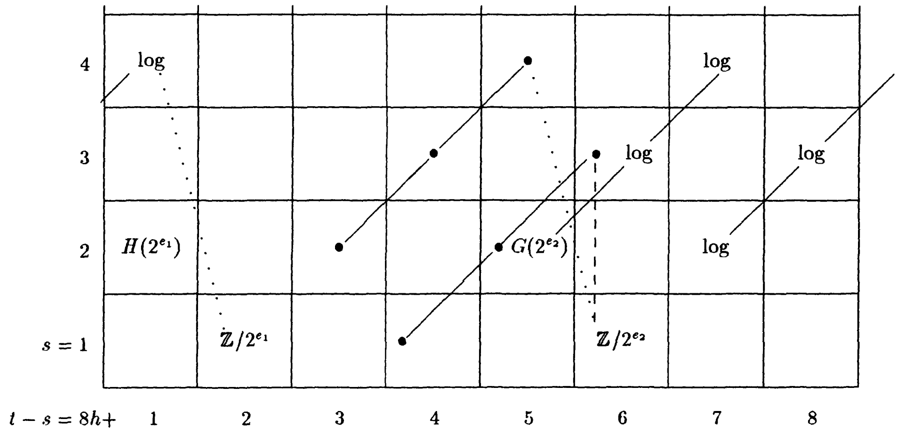

Figure 4.4: The final stage of the periodic UNSS for $Sp(n)$

starters, that $\epsilon_{k-1} = 0$, or, equivalently, that k is odd. Then the hypothesis implies that the mod torsion index of the homotopy homomorphism p_* is less than that of the $E_2^{0,4k-1}$-morphism p_*. This implies that there must be a nonzero differential from the group $E_2^{0,4k-1}(Sp(k)/Sp(n))$.

Next we claim that this implies that $E_2^{1,4k-1}(Sp(n))$ must also support a nonzero differential. If not, then the element of $E_2^{s,s+4k-2}(Sp(k)/Sp(n))$ ($s \geq 2$) which is the target of the differential in $Sp(k)/Sp(n)$ must be in the image from $E_2^{s,s+4k-2}(Sp(k))$. This group is isomorphic to $E_2^{s,s+4k-2}(Sp)$, which was shown in **[10]** to be 0 unless $s = 3$, in which case it has a single nonzero element which supports a nonzero d_3-differential in the UNSS of Sp. This differential must also take place in $Sp(k)$. The target in $E_2^{6,4k+3}(Sp(k))$ of this differential must pull back to an element in $E_2^{6,4k+3}(Sp(n))$ which is hit by a d_5 from the generator of $E_2^{1,4k-1}(Sp(n))$, contrary to our supposition that this group did not support a nonzero differential.

The proof of **[12**, 3.12] shows that there will be a periodic family of values of k for which $e(k,n)$ and $e_{Sp}(k,n)$ have the same values as they do for the value of k specified in the hypothesis of the proposition. The same argument as above applies to each of them, and so we conclude that the differential must be v_1-periodic. Hence it must be a d_3, since, after taking into account the differentials of Proposition 4.1, $v_1^{-1}E_3^{s,s+4k-3}(Sp(n))$ is nonzero only for $s = 2$ and 4.

A similar argument works when $\epsilon_{k-1} = 1$, so that k is even. The hypothesis is that the mod torsion index of

$$\pi_{4k-1}(Sp(k)) \to \pi_{4k-1}(Sp(k)/Sp(n))$$

is equal to or less than that of $E_2^{0,4k-1}(Sp(k)) \to E_2^{0,4k-1}(Sp(k)/Sp(n))$.

This time there is a d_3-differential on $E_2^{0,4k-1}(Sp(k))$ depicted in [**10**, p. 74]. This would make impossible the inequality relating the mod torsion indexes unless there is a nonzero differential on $E_2^{0,4k-1}(Sp(k)/Sp(n))$. The target of that differential (and hence the differential itself) must map across to $E_2^{s+1,s+4k-2}(Sp(n))$, since there is nothing else in $E_2^{s,s+4k-2}(Sp(k))$ which can map to it, by [**10**, p. 74]. Thus we have deduced the asserted differential in $Sp(n)$, and, as in the case when k is odd, it must be a v_1-periodic d_3. $\square$

In order to show that differentials are 0, we use the stable splitting map $\Sigma^\infty Sp(n) \to \Sigma^\infty(Q^n)$. See, for example, [**23**, p. 50].

Proposition 4.6. *The stable splitting map induces a morphism from the UNSS of $Sp(n)$ to the NSS of Q^n which sends $E_2^{1,4k-1}(Sp(n))$ isomorphically to*

$$\ker(E_2^{1,4k-1}(Q^n) \to E_2^{1,4k-1}(Q^\infty)).$$

Proof. The morphism can be thought of as the composite of the stabilization morphism from the UNSS of $Sp(n)$ to the stable NSS of $Sp(n)$ followed by the map of NSS's induced by the stable map from $Sp(n)$ to Q^n. On E_2-terms, this is

$$\mathrm{Ext}_{\mathcal{U}}(P(BP_*(Sp(n)))) \to \mathrm{Ext}_{\mathcal{G}}(BP_*(Sp(n))) \to \mathrm{Ext}_{\mathcal{G}}(BP_*(Q^n)),$$

and $P(BP_*(Sp(n)))$ and $BP_*(Q^n)$ are isomorphic comodules.

Proposition 3.9 and Theorem 3.10 of [**10**] were proved for the UNSS based on MU; however, they also apply to spectral sequences based on BP, and to the stable NSS. The generators and coaction of $BP_*(Q^n)$ correspond precisely to those of $P(BP_*(Sp(n)))$. Hence Proposition 4.6 follows from the results of [**10**], which say that the groups are cyclic with order determined by $\bar{e}$ applied to the coaction. For $Sp(n)$, we did not need to write explicitly that we are computing the kernel into $E_2^{1,4k-1}(Sp)$, since this group is 0. $\square$

The stable splitting map is useful because of the following result.

Proposition 4.7. *The differential*

$$d_3 : v_1^{-1}E_3^{1,4k-1}(Q^n) \to v_1^{-1}E_3^{4,4k+1}(Q^n)$$

is nonzero if and only if $e(k,n) < \epsilon_{k-1} + e_{Sp}(k,n)$.

Proof. The mod torsion index of

$$\pi_{4k-1}^s(Q^k) \to \pi_{4k-1}^s(Q^k/Q^n)$$

is $2^{e(k,n)}$ by [26, p. 872], and it equals the order of the cokernel of

$$E_\infty^{0,4k-1}(Q^k) \to E_\infty^{0,4k-1}(Q^k/Q^n). \tag{4.8}$$

By [10, 1.1], d_3 is nonzero on $E_3^{0,4k-1}(Q^k)$ if and only if $\epsilon_{k-1} = 1$. Also from [10] is the result that the order of the cokernel of

$$E_3^{0,4k-1}(Q^k) \to E_3^{0,4k-1}(Q^k/Q^n)$$

is $2^{e_{Sp}(k,n)}$. Thus the order of the cokernel of (4.8) has 2-exponent equal to $e_{Sp}(k,n) + \epsilon_{k-1} - m$, where m is the number of nontrivial differentials on $E_3^{0,4k-1}(Q^k/Q^n)$. The proposition now follows by comparing with the earlier description of this exponent as $e(k,n)$. □

We can now easily obtain our desired deduction regarding $d_3 = 0$ when k is even. Throughout the remainder of this section, E_r refers to the UNSS for $Sp(m)$ and to the stable NSS for Q^m.

Proposition 4.9. *If k is even and $e(k,n) = 1 + e_{Sp}(k,n)$, then*

$$d_3 = 0 : v_1^{-1}E_2^{1,4k-1}(Sp(n)) \to v_1^{-1}E_2^{4,4k+1}(Sp(n)). \tag{4.10}$$

Proof. We prove the contrapositive. Suppose that d_3 in (4.10) is nonzero. By Figure 4.2, there is only one nonzero element in $v_1^{-1}E_2^{4,4k+1}(Sp(n))$ after the d_3-differentials of Proposition 4.1 are taken into account. If k is sufficiently large, this stable element pulls back to an element of $E_2^{4,4k+1}(Sp(n))$ which maps nontrivially to $E_2^{4,4k+1}(Q^n)$. Thus

$$d_3 : E_2^{1,4k-1}(Q^n) \to E_2^{4,4k+1}(Q^n)$$

must also be nonzero, and hence by Proposition 4.7 $e(k,n) < 1 + e_{Sp}(k,n)$, counter to the hypothesis of this proposition. □

The argument when k is odd is a little bit more delicate because $v_1^{-1}E_2^{4,4k+1}(Sp(n))$ has more than one nonzero element.

Proposition 4.11. *Assume k is odd and $e(k,n) = e_{Sp}(k,n)$. Then*
i. *If d_3 is nonzero on $v_1^{-1}E_2^{1,4k-1}(Sp(n))$, then d_3 is nonzero on*

$$v_1^{-1}E_2^{1,4k-1}(Sp(n+1))$$

and $e(k,n+1) = e_{Sp}(k,n+1)$.
ii. *$d_3 = 0$ on $v_1^{-1}E_2^{1,4k-1}(Sp(n))$.*

Proof. Using induction on n, part i implies that if d_3 is nonzero on $v_1^{-1}E_2^{1,4k-1}(Sp(n))$, then d_3 is nonzero on $v_1^{-1}E_2^{1,4k-1}(Sp(N))$ for all $N \geq n$. This is certainly false for $N > k$, and hence d_3 must have been zero on $v_1^{-1}E_2^{1,4k-1}(Sp(n))$. Thus part i implies part ii, and so we concentrate on proving part i.

We begin with the case when n is even. The target of the differential in $Sp(n)$ must map nontrivially to $v_1^{-1}E_2^{4,4k+1}(Sp(n+1))$. To see this, we note that if it mapped trivially, then it must be in the image from $v_1^{-1}E_2^{3,4k+1}(S^{4n+3})$, which is the group pictured in position $[7,3]$ in Figure 3.24. Each nonzero element in this group supports a d_3-differential in the UNSS of S^{4n+3}. The exact sequences in $v_1^{-1}E_2(-)$ and $v_1^{-1}\pi_*(-)$ would then imply, similarly to the proof of Proposition 4.5, a nonzero d_5-differential in $Sp(n+1)$, contradicting [**4**, 5.5].

The map $Sp(n) \to Sp(n+1)$ implies that d_3 must be nonzero between the image classes in $Sp(n+1)$. Thus $v_1^{-1}E_2^{1,4k-1}(Sp(n))$ must map isomorphically to $v_1^{-1}E_2^{1,4k-1}(Sp(n+1))$ since $v_1^{-1}E_2^{4,4k+1}(Sp(n+1))$ is a $\mathbb{Z}_2$-vector space. Thus $e_{Sp}(k,n) = e_{Sp}(k,n+1)$. Since $k+n$ is odd, Proposition 5.2c says that $e(k,n) = e(k,n+1)$. Hence $e(k,n+1) = e_{Sp}(k,n+1)$, completing the argument when n is even.

If n is odd, the above argument fails because $v_1^{-1}E_3^{3,4k+1}(S^{4n+3})$ has a class on which d_3 is zero. Instead, we argue as follows.

Suppose that there is a nonzero differential on $v_1^{-1}E_2^{1,4k-1}(Sp(n))$ but not on $v_1^{-1}E_2^{1,4k-1}(Sp(n+1))$. Then the target of the differential must be in the image of the morphism

$$v_1^{-1}E_2^{3,4k+1}(S^{4n+3}) \xrightarrow{\partial} v_1^{-1}E_2^{4,4k+1}(Sp(n)).$$

Now $v_1^{-1}E_2^{3,4k+1}(S^{4n+3})$ is spanned by $\eta(n+1)'_u$ and $\eta^2(n+1)$. Of these, only $\eta^2(n+1)$ is sent nontrivially by ∂. We will show that under the stable splitting map

$$v_1^{-1}E_2^{4,4k+1}(Sp(n)) \xrightarrow{\sigma} v_1^{-1}E_2^{4,4k+1}(Q^n)$$

$\partial(\eta^2(n+1))$ is sent nontrivially. This implies that there is a nonzero differential into $v_1^{-1}E_2^{4,4k+1}(Q^n)$ and hence by Proposition 4.7 $e(k,n) < e_{Sp}(k,n)$, contrary to our hypothesis.

To show that $\sigma(\partial(\eta^2(n+1))) \neq 0$, we argue as follows. Consider n to be a fixed odd integer. It follows from Propositions 4.13 and 5.2a that there exist odd values of k such that $e(k,n) < \epsilon_{k-1} + e_{Sp}(k,n)$, and hence by Proposition 4.7 $d_3 \neq 0$ on $E_3^{1,4k-1}(Q^n)$. For such a value of k, the morphism

$$v_1^{-1}E_3^{4,4k+1}(Sp(n)) \xrightarrow{\sigma} v_1^{-1}E_3^{4,4k+1}(Q^n) \tag{4.12}$$

must satisfy $\sigma(\partial(\eta^2(n+1))) \neq 0$. But the graded $\mathbb{Z}_2$-vector spaces $E_3^{4,4*+1}(X)$ are acted on by period 8 Adams periodicity, which induces isomorphisms

$$v_1^{-1}E_3^{4,4k+1}(X) \xrightarrow{A} v_1^{-1}E_3^{4,4k+9}(X)$$

for $X = Sp(n)$ or Q^n, and A commutes with σ. Thus the morphism σ of (4.12) is nonzero on $\partial(\eta^2(n+1))$ for all integers k, and so in particular it is nonzero for the value of k involved in the proposition.

Now we have shown that d_3 is nonzero on $v_1^{-1}E_3^{1,4k-1}(Sp(n))$. We must also prove that $e(k,n+1) = e_{Sp}(k,n+1)$. If this were not true, then by Proposition 4.7, d_3 would be nonzero on $v_1^{-1}E_3^{1,4k-1}(Q^{n+1})$. Since, as we just showed, the nonzero d_3-differential mapped nontrivially from $Sp(n)$ to $Sp(n+1)$, we must have $e_{Sp}(k,n) = e_{Sp}(k,n+1)$ and hence the nonzero differential on Q^{n+1} is the image of a nonzero differential on Q^n, which, with Proposition 4.7, contradicts the hypothesis that $e(k,n) = e_{Sp}(k,n)$. □

The following result was used in the above proof. It implies by Propositions 5.2a and 4.7 that for each odd n, there are some odd values of k for which there are nonzero d_3-differentials on $v_1^{-1}E_3^{1,4k-1}(Q^n)$.

Proposition 4.13. *If $k \equiv 1 \bmod 2^{e-1}$, and n is odd and satisfies $n \leq 2^{e-1}$, then $e_{Sp}(k,n) = 2n-1$ and $e_{Sp}(k,n+1) = 2n+1$.*

Proof. Let $v_U(K,J) = \nu(\sum_{i=1}^{J}(-1)^i\binom{J}{i}i^K)$. We will prove that if $K \equiv 1$ mod 2^e, and $J \leq 2^e + 2$, then $v_U(K,J) = J-2$ if J is odd, while if J is even, then $v_U(K,J) \geq J-1$. This implies that the numbers $e_U(K,N)$, defined as $\min\{v_U(K,J) : J \geq N\}$, satisfy $e_U(K,N) = N-2$ (resp. $N-1$) if N is odd (resp. even). Letting $K = 2k-1$ and $N = 2n+1$ and $2n+2$, and using the result of [**10**, 1.4] that

$$e_U(2k-1,2n) = e_{Sp}(k,n) = e_U(2k-1,2n+1) \tag{4.14}$$

yields the desired result.

Now we prove the claim about $v_U(K,J)$. Mod 2^{e+1}, the terms in the sum with i even are 0, while if i is odd, we have $i^k \equiv i$. Thus the result follows from

$$\sum_{i\ odd}\binom{J}{i}i = J\sum_{i\ even}\binom{J-1}{i} = J\sum_{i=0}^{J-2}\binom{J-2}{i} = J\cdot 2^{J-2}.$$

□

5. Applications to James numbers and exponents.

In this section we discuss specific computations of the numbers $e(k,n)$ and $e_{Sp}(k,n)$, and give applications of our results to exponents of actual homotopy groups of $Sp(n)$ and quaternionic James numbers.

The numbers $e(k,n)$ and $e_{Sp}(k,n)$ share many of the interesting properties of the slightly more basic numbers $e_U(k,n)$ studied in [**13**] and [**11**].

Computer algebra packages allow extensive computation of these numbers, uncovering patterns that cry out for generalization and proof. We have used Mathematica to perform many calculations. The program consists of just three statements.

```
nutwo[x_]:=If[OddQ[x],0,1+nutwo[x/2]]
v[k_,j_,e_]:=nutwo[Sum[(-1)^i Binomial[2j,i] PowerMod[j-i,2k,2^e],
           {i,0,j-1}]]-nutwo[j]
vt[k_,jo_,je_,e_]:=Table[v[k,j,e],{j,jo,je}]
```

The first line defines a function that calculates the exponent of 2. The second line calculates the basic numbers $v(k,j)$ which are the exponents of 2 of the coefficients involved in (1.15), working mod 2^e so as to take advantage of the `PowerMod` function. Answers which it gives that are equal to or greater than e imply only that the correct number here is $\geq e$. Here we have used the alternate form for the coefficients of (1.15) given in (1.4). The third line tabulates numbers for a range of values of j. In practice, a fourth line which performs `vt[-]` over an arithmetic progression of values of k is useful.

For example, if we run `vt[20,3,12,20]`, the output

`{4, 4, 10, 15, 13, 11, 15, 16, 19, 19}`

is almost immediate. Recalling that $e_{Sp}(20,n)$ is the minimum of these numbers $v(20,j)$ for $j > n$, this yields $e_{Sp}(20,2) = e_{Sp}(20,3) = 4$, $e_{Sp}(20,4) = 10$, $e_{Sp}(20,5) = e_{Sp}(20,6) = e_{Sp}(20,7) = 11$, etc. One can be assured that there will not be a smaller value of $v(20,j)$ for some $j > 12$ since $v(k,j) \geq \nu_2((2j-1)!)$ by [**26**, 1.2,1.3]. As for the numbers $e(20,n)$, we subtract 1 from each of the numbers $v(20,j)$ with j odd, do a similar minimizing, and then add 1 (since $\epsilon_{19} = 1$) to all numbers, obtaining $e(20,2) = 4$, $e(20,3) = 5$, $e(20,4) = 10$, $e(20,5) = e(20,6) = e(20,7) = 12$, etc.

The numbers $e_{Sp}(k,n)$ and $e(k,n)$ are periodic in k by [**26**, 1.7]. This makes it feasible to give complete formulas for these numbers for fixed n. The following proposition was obtained in a few hours of work using the above Mathematica program.

Proposition 5.1. a. *If k is even, then*

$$e_{Sp}(k,n) = \begin{cases} 1 & n=1 \\ 4 & n=2 \\ 4 & n=3 \\ \min(11, 6+\nu(k-4)) & n=4 \\ \min(11, 7+\nu(k-4)) & n=5 \\ \min(15, 10+\nu(k-6)) & n=6 \\ \min(18, 10+\nu(k-230)) & n=7 \end{cases}$$

and

$$e(k,n) = \begin{cases} 2 & n=1 \\ \min(5, 3+\nu(k-2)) & n=2 \\ 5 & n=3 \\ \min(12, 6+\nu(k-36)) & n=4 \\ \min(12, 8+\nu(k-4)) & n=5 \\ 12 & n=6, k \equiv 0 \ mod\ 4 \\ \min(16, 10+\nu(k-6)) & n=6, k\equiv 2\ mod\ 4 \\ \min(18, 11+\nu(k-102)) & n=7 \end{cases}$$

b. *If k is odd, then*

$$e_{Sp}(k,n) = \begin{cases} 1 & n=1 \\ 3 & n=2 \\ \min(8, 4+\nu(k-7)) & n=3 \\ \min(8, 6+\nu(k-3)) & n=4 \\ \min(14, 7+\nu(k-37)) & n=5 \\ \min(14, 9+\nu(k-37)) & n=6 \\ \min(16, 11+\nu(k-13)) & n=7, k\equiv 1\ mod\ 4 \\ \min(21, 11+\nu(k-7)) & n=7, k\equiv 3\ mod\ 4 \end{cases}$$

and

$$e(k,n) = \begin{cases} 0 & n=1 \\ 3 & n=2 \\ \min(7, 3+\nu(k-7)) & n=3 \\ 7 & n=4 \\ \min(13, 6+\nu(k-37)) & n=5 \\ \min(13, 9+\nu(k-5)) & n=6 \\ \min(15, 10+\nu(k-13)) & n=7, k\equiv 1\ mod\ 4 \\ \min(20, 10+\nu(k-7)) & n=7, k\equiv 3\ mod\ 4 \end{cases}$$

In the range of this proposition, we have $e(k,n) = e_{Sp}(k,n) - 1$ if k and n are odd. For awhile, it seemed important to know whether this was true for all odd values of k and n. We extended Proposition 5.1 through $n = 18$ for odd values of k, and found that the only case in this range in which $e(k,n) = e_{Sp}(k,n)$ (with n odd) is when $n = 13$ and $k \equiv 13 + 5 \cdot 2^{15} \bmod 2^{19}$. For such values of k, we have $v(k,13) = 22$, $v(k,14) = 41$, $v(k,15) = 40$, and $v(k,j) \geq 41$ for $j \geq 16$. Thus $e(k,13) = e_{Sp}(k,13) = 40$.

To prove things about the numbers $e_{Sp}(k,n)$, it might be useful to use (4.14), which relates them directly to the simpler numbers $e_U(k',n')$. However, the numbers $e(k,n)$ contain information not derivable from the e_U-numbers. We can easily derive some simple relationships between $e_{Sp}(k,n)$, $e_{Sp}(k,n+1)$, and $e(k,n)$.

Proposition 5.2. a. *If $k+n$ is even and $e(k,n) = \epsilon_{k-1} + e_{Sp}(k,n)$, then $e_{Sp}(k,n) = e_{Sp}(k,n+1)$.*

b. *If $k+n$ is odd and $e_{Sp}(k,n) < \epsilon_{k-1} + e_{Sp}(k,n)$, then $e_{Sp}(k,n) = e_{Sp}(k,n+1)$.*

c. *If $k+n$ is odd and $e_{Sp}(k,n) = e_{Sp}(k,n+1)$, then $e(k,n) = e(k,n+1)$.*

Proof. As in the computer program above, let $v(k,j)$ equal the exponent of 2 in the coefficient relevant to the e_{Sp}-numbers.

a. The hypothesis implies that

$$\begin{aligned}&\min(v(k,n+1)-1,\ v(k,n+2),\ v(k,n+3)-1,\dots)\\ &=\min(v(k,n+1),\ v(k,n+2),\ v(k,n+3),\dots).\end{aligned}$$

Thus the second minimum must equal $\min(v(k,n+2),\ v(k,n+3),\dots)$, which implies the desired result.

b. The hypothesis implies that

$$\begin{aligned}&\min(v(k,n+1),\ v(k,n+2)-1,\ v(k,n+3),\dots)\\ &=\min(v(k,n+1),\ v(k,n+2),\ v(k,n+3),\dots)-1,\end{aligned}$$

and hence $v(k,n+1)$ cannot be less than $\min(v(k,n+2),\ v(k,n+3),\dots)$.

c. The hypothesis implies that

$$\begin{aligned}&\min(v(k,n+1),\ v(k,n+2),\ v(k,n+3),\dots)\\ &=\min(v(k,n+2),\ v(k,n+3),\dots).\end{aligned}$$

Hence

$$\begin{aligned}&\min(v(k,n+1),\ v(k,n+2)-1,\ v(k,n+3),\dots)\\ &=\min(v(k,n+2)-1,\ v(k,n+3),\dots).\end{aligned}$$

□

We can apply our results to obtain a lower bound for the 2-exponent of $Sp(n)$, similar to the applications made for $SU(n)$ in [**13**] and [**9**]. This is new information about actual homotopy groups. Recall that the 2-exponent of a space X, denoted $\exp_2(X)$, is the minimal e such that 2^e annihilates the 2-torsion of $\pi_*(X)$. Since the v_1-periodic homotopy groups occur as direct summands of actual homotopy groups, the following results are immediate from Theorem 1.5 and Proposition 5.1.

Proposition 5.3. *Let $e_2^{Sp}(n) = \max\{e(k,n) : k \ge n\}$. Then $e_2^{Sp}(n) \le \exp_2(Sp(n))$.*

Corollary 5.4. *For* $n = 1, \ldots, 7$, *the values of* $\exp_2(Sp(n))$ *are equal to or greater than* 2, 5, 7, 12, 13, 16, *and* 20, *respectively.*

This says, for example, that some homotopy group of $Sp(7)$ contains an element of order 2^{20}. Corollary 5.4 is sharp when $n = 1$. An optimistic conjecture would be that $\exp_2(Sp(n)) = e_2^{Sp}(n)$, although the situation of $SU(3)$ noted in [**15**] dampens one's confidence.

The following theorem, similar to the result which was felt to be important enough to be highlighted as Theorem 1.1 in [**13**], gives a simple and fairly sharp lower bound for $\exp_2(Sp(n))$. It says that, for every n, some homotopy group of $Sp(n)$ has an element of order 2^{2n-1}.

Theorem 5.5. $2n - 1 \le e_2^{Sp}(n) \le \exp_2(Sp(n))$.

Proof. The proof of (5.6) of [**13**] given on page 539 of that paper shows that $\max\{e_U(k, m) : k \not\equiv m \textit{ mod } 2\} \ge m - 1$. In other words, the lower bound for $e_2^U(m)$ given there is obtained using values opposite in parity to m. Since, by (4.14), $e_{Sp}(k, n) = e_U(2k - 1, 2n)$, we obtain $e_2^{Sp}(n) \ge 2n - 1$. The result now follows from Proposition 5.3. □

Finally we can deduce results about quaternionic James numbers similar to those proved for complex James numbers in [**13**] and [**9**]. Letting $X_{n,k}$ denote the quaternionic Stiefel manifold, the quaternionic James number $c_{n,k}$ is defined as the index of $p_*\pi_{4n-1}(X_{n,k})$ in $\pi_{4n-1}(S^{4n-1})$. See [**32**] or [**26**] for earlier work on these numbers. The stable James numbers $c^s_{n,k}$ are defined similarly using stable homotopy groups instead of ordinary homotopy groups. Our main result about quaternionic James numbers is that, if the numbers n and k are sufficiently large compared to their difference, then the stable and unstable James numbers are equal, and they equal their conjectured values. Here ϵ and $e(-, -)$ are as in 1.5.

Theorem 5.6. *If, for fixed* n, k *is sufficiently large, then*

$$\nu(c_{k,k-n}) = \nu(c^s_{k,k-n}) = \nu((2k - 1)!) + \epsilon_{k-1} - e(k, n).$$

Proof. We prove

$$\begin{aligned}\nu((2k - 1)!) + \epsilon_{k-1} - e(k, n) \le \nu(c^s_{k,k-n}) &\le \nu(c_{k,k-n}) \\ &\le \nu((2k - 1)!) + \epsilon_{k-1} - e(k, n).\end{aligned}$$

The first inequality is from [**32**, 0.2] or [**26**, pp. 867-8], and the second is immediate from the definitions. The third follows similarly to [**13**, 4.3]. Indeed, we deduce from Theorem 1.5 that, if k is sufficiently large, then there is a summand of $\pi_{4k-2}(Sp(n))$ of order $2^{e(k,n)}$ which injects into $\pi_{4k-2}(Sp(k-1)) \approx \mathbb{Z}/(2^{\epsilon_{k-1}}(2k-1)!)$. The following commutative diagram then implies the inequality.

$$\begin{array}{ccccccc}
\pi_{4k-1}(Sp(k)) & \to & \pi_{4k-1}(Sp(k)/Sp(n)) & \to & \pi_{4k-2}(Sp(n)) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\pi_{4k-1}(Sp(k)) & \to & \pi_{4k-1}(Sp(k)/Sp(k-1)) & \to & \pi_{4k-2}(Sp(k-1)) & \to & 0
\end{array}$$

6. Interpretation involving J-homology.

When we began this project in [**16**], we intended to use charts of J-homology groups to determine $v_1^{-1}\pi_*(Sp(n))$. For various reasons, this turned out to be infeasible. Nevertheless, it was important in lending insight toward the answer, and might lend insight to others. Therefore, we present a sketch of this point of view. In this section, we also use J-methods to prove Lemma 2.28, and introduce a new spectrum J_1, which has certain advantages over J.

Let $J_*(-)$ denote (stable) connective J-homology, as used, for example, in [**25**], where the crucial result

$$v_1^{-1}\pi_*(S^{2n+1}) \approx v_1^{-1}J_*(\Sigma^{2n+1}P^{2n}) \tag{6.1}$$

is proved.[4] The determination of $v_1^{-1}\pi_*(Sp(n))$ from the exact sequences in $v_1^{-1}\pi_*(-)$ associated to the fibrations

$$Sp(i-1) \to Sp(i) \to S^{4i-1}, \quad 2 \le i \le n,$$

may be viewed as the problem of inserting differentials and extensions in the chart for

$$\bigoplus_{i=1}^{n} v_1^{-1}\pi_*(S^{4i-1}) \approx \bigoplus_{i=1}^{n} v_1^{-1}J_*(\Sigma^{4i-1}P^{4i-2}). \tag{6.2}$$

We sketch how $v_1^{-1}J_*(P^{2m})$ is determined, and refer the reader to [**25**] or [**17**] for more details. It is formed from

$$v_1^{-1}bo_*(P^{2m}) \oplus \phi^{-2}\sigma^{-1}v_1^{-1}bo_*(P^{2m}), \tag{6.3}$$

where ϕ^i increases filtration by i, and σ^j increases stem by j. The second summand is actually $v_1^{-1}(\Sigma^4 bsp)_*(P^{2m})$, suitably positioned to reflect its appearance in the exact sequence associated to the fibration

$$J \to bo \xrightarrow{\psi^3-1} \Sigma^4 bsp. \tag{6.4}$$

[4]It was not until [**19**] that it was stated in this form.

All these spectra are localized at 2. To form $v_1^{-1}J_*(P^{2m})$ from (6.3), one inserts d_r-differentials (i.e., back 1, up r) from a tower in horizontal position $4j-1$ of the first summand to the corresponding tower from the second summand in position $4j-2$ if $r=\nu_2(j)+1$.

We illustrate with the computation of $v_1^{-1}J_*(P^{14})$. In the left half of Figure 6.5 we depict the standard calculation of $bo_*(P^{14})$. See, for example, [**17**] or [**25**]. In the right half is $v_1^{-1}J_*(P^{14})$.

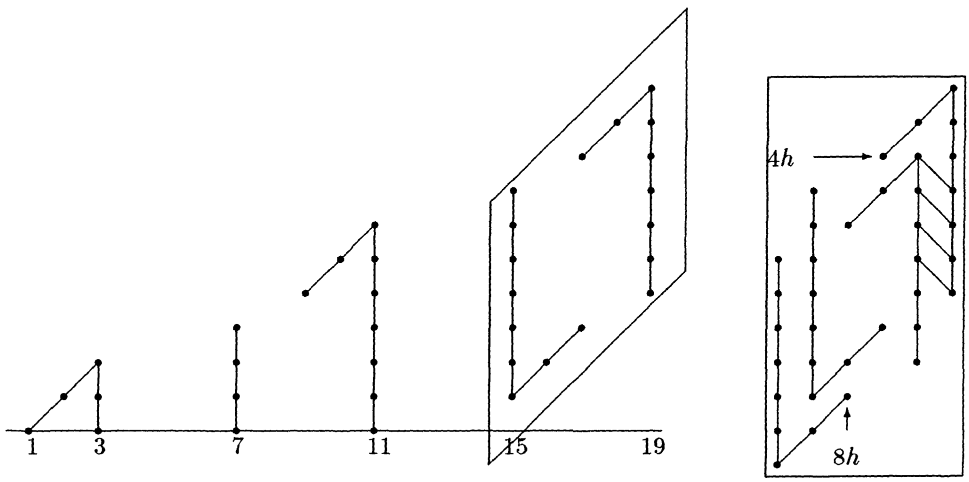

Figure 6.5: $bo_*(P^{14})$ and $v_1^{-1}J_*(P^{14})$

To form $v_1^{-1}bo_*(P^{14})$, the part of Figure 6.5 in the parallelogram is continued in both directions with period $[8,4]$. This involves extending the initial parts of the chart into negative filtration. The rectangle on the right side of Figure 6.5 displays a portion of $v_1^{-1}J_*(P^{14})$ which appears for every integer h. There is a $d_{2+\nu(h)}$-differential from the tower in $8h-1$ provided $\nu(h)\le 2$. If $\nu(h)>2$, then the differential from $8h-1$ to $8h-2$ is zero.

These charts are summed as in (6.2) to obtain the initial term of a cellular SS converging to $v_1^{-1}\pi_*(Sp(n))$. As we shall discuss later, the problem here is that we cannot prove that the s-filtrations are meaningful. In the left half of Figure 6.6, we have listed $\bigoplus v_1^{-1}\pi_j(\Sigma^{4i-1}P^{4i-2})$ in the range $8k+3\le j\le 8k+10$, filtration $s\ge 4k-13$. Elements of the first type in (6.3) are indicated by $\bullet$, and those of the second type by $\circ$. The label on a connected component of this chart is the integer i such that it comes from $\Sigma^{4i-1}P^{4i-2}$. We have omitted elements involved in d_1-differentials.

Every $v_1^{-1}\pi_*(S^{4i-1})$ with $1\le i\le n$ contributes four pairs of $\mathbb{Z}_2$'s to the chart for $v_1^{-1}\pi_*(Sp(n))$. The elements in each pair are connected by $\eta\in\pi_{j+1}(S^j)$, which is represented in the charts by a line of length 1 and slope 1. On the right side of Figure 6.6, we indicate each of these pairs of $\mathbb{Z}_2$'s by a single number, the i such that the pair comes from S^{4i-1}. The four pairs are characterized by whether they come from the bo-part or bsp-part, and whether they are stable or unstable. The bsp-parts, which always lie 1 unit to the left and 2 down from their bo-counterpart, were o's in the left part of Figure 6.6, and are indicated by primes ($'$s) in the right part of the chart. The unstable parts lie below their stable counterparts (except on S^3); they are indicated by a subscript u in the right side of Figure 6.6.

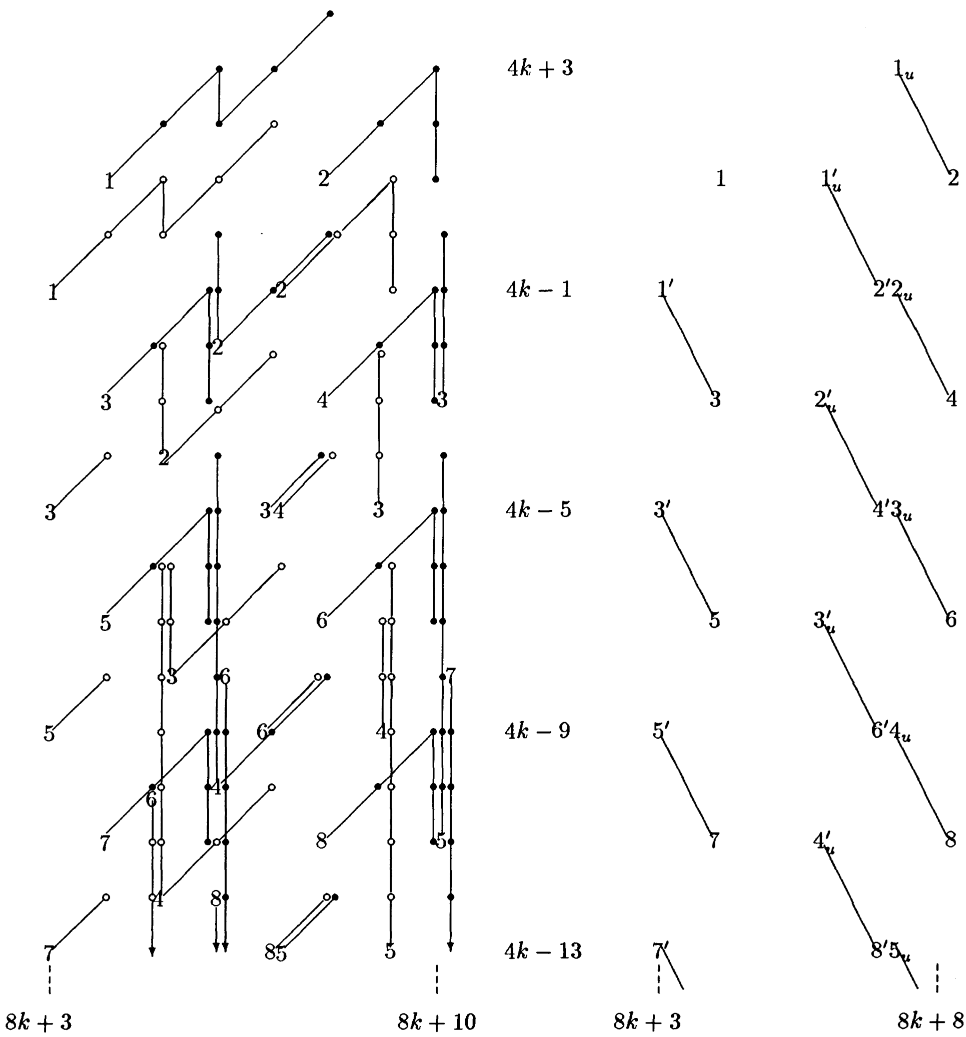

Figure 6.6: J-type chart for $v_1^{-1}\pi_*(Sp(n))$

The pattern of differentials in the right side of Figure 6.6 is the way in which the $\mathbb{Z}_2$'s cancel out in $v_1^{-1}\pi_*(Sp)$ if the s-filtrations are meaningful.

These differentials are not quite the same as those given by Proposition 1.8. The differentials of that proposition involved minimal change of the number of the sphere, while those in Figure 6.6 involve minimal increase of s. On $Sp(n)$ only the $\mathbb{Z}_2$-pairs with number $\leq n$ will exist. This opens up other classes to a more complicated pattern of differentials, similar to that of Proposition 1.8.

When the $\mathbb{Z}_2$'s are removed from consideration, what remains are the pairs of towers which would like to contribute to $v_1^{-1}\pi_{4j+1}$ and $v_1^{-1}\pi_{4j+2}$. These have a pattern of differentials and extensions which is very complicated from this point of view. The Novikov point of view is much more convenient for handling these elements.

The biggest problem with using Figure 6.6 is that it is not clear that its s-filtrations are meaningful. The chart certainly depicts the initial term of a cellular SS converging to $v_1^{-1}\pi_*(Sp(n))$; however, if it is to be useful, we need to know that the differentials in the SS increase s, and that they may be filtered by the amount by which they increase s.

In **[18]**, it was proved that there is a finite spectrum X_n such that $v_1^{-1}\pi_*(X_n) \approx v_1^{-1}\pi_*(Sp(n))$. The nice thing about this is that for a spectrum X there is an isomorphism $v_1^{-1}\pi_*(X) \approx v_1^{-1}J_*(X)$, and there is a SS converging to $v_1^{-1}J_*(X)$ with initial term

$$v_1^{-1}bo_*(X) \oplus \phi^{-2}\sigma^{-1}v_1^{-1}bo_*(X),$$

similarly to (6.3), and with meaningful s-filtrations. The spectrum X_n will be built by cofibrations from suspensions Σ^{k_i} of $\Sigma^{4i-1}P^{4i-2}$. It is the suspensions that are the problem. They cause $bo_*(X_n)$ to be spread out compared to Figure 6.6. That is, if $k_i >> k_{i-1}$, the contribution from $\Sigma^{4i-1+k_i}P^{4i-2}$ will appear far to the right and below the contribution from $\Sigma^{4i-5+k_{i-1}}P^{4i-6}$. The chart for $v_1^{-1}J_*(X_n)$ formed in this way would have possible differentials that we would like to rule out. There is a possible solution to this problem. It involves controlling the Adams filtration of the attaching maps. If we know that the attaching map from $\Sigma^{4i-2+k_i}P^{4i-2}$ to X_{i-1} has filtration s, then we can form a chart for $J_*(X_i)$ from $J_*(X_{i-1})$ and $\phi^{s-1}J_*(\Sigma^{4i-1+k}P^{4i-2})$. (Recall that ϕ increases filtration.) The hope was (is) that these filtrations can be controlled so that the desired chart is obtained, but some difficulties in doing this have not been overcome.

This seems not to be terribly important anyway, since the Novikov methods seem to offer a cleaner way of handling the differentials among the $\mathbb{Z}_2$'s as well as the larger summands. Figure 6.6 can be used for insight even if we don't know that it represents anything real.

One of the unpleasant features of the spectrum J is that the usual J-charts are not actual ASS charts. They do not depict $\mathrm{Ext}_A(H^*(X \wedge J), \mathbb{Z}_2)$, as was discussed in [**14**]. However, in filtration ≥ 2, the usual J-chart is an Ext chart, as we can see by utilizing the spectrum J_1, defined to be the fibre of the nontrivial map $J \to H\mathbb{Z}_2$. The principal properties of J_1 are established in the following result, in which A_2 denotes the subalgebra of the mod 2 Steenrod algebra A generated by Sq^1, Sq^2, and Sq^4.

Proposition 6.7. *As an A-module,*

$$(6.8)\quad H^*J_1 \approx \langle g_0, g_1, g_3 : \mathrm{Sq}^1 g_0, \mathrm{Sq}^2 g_1 + \mathrm{Sq}^3 g_0, \mathrm{Sq}^1 g_3 + \mathrm{Sq}^2\mathrm{Sq}^1 g_1 + \mathrm{Sq}^4 g_0, \mathrm{Sq}^5 g_3 + \mathrm{Sq}^7 g_1\rangle,$$

where $|g_i| = i$. For any connected spectrum X, the ASS converging to $\pi_*(X \wedge J_1)$ has

$$(6.9)\qquad E_2^{s,t}(X \wedge J_1) = \mathrm{Ext}_{A_2}^{s,t}(H^*(X) \otimes T_2, \mathbb{Z}_2),$$

where T_2 is the A_2-module with generators and relations as in (6.8). *The usual chart for $J_*(X)$, formed, for example, as in [**25**, 7.1], has its $E_1^{s,t}$ isomorphic to $E_2^{s-1,t-1}(X \wedge J_1)$ for $s \geq 2$, with differentials agreeing under this isomorphism.*

Proof. Let bo_1 denote the fibre of the nontrivial map $bo \to H\mathbb{Z}_2$. Then there is a fibration

$$J_1 \to bo_1 \xrightarrow{\theta'} \Sigma^4 bsp,$$

and θ' induces the 0-homomorphism in mod 2 cohomology. This is the advantage of J_1 over J, for it implies a short exact sequence in cohomology and hence a long exact sequence

$$(6.10) \to \mathrm{Ext}_A^{s,t}(H^*(\Sigma^3 bsp)) \to \mathrm{Ext}_A^{s,t}(H^*(J_1)) \to \mathrm{Ext}_A^{s,t}(H^*(bo_1)) \to .$$

This sequence holds with any H^*Y in the second variable, although we shall always have $\mathbb{Z}_2$ in the second variable. The exact sequence also holds after smashing the spectra with a spectrum X. In a turnaround from the usual order of business, we can use the Ext-sequence to tell us the A-extensions in the short exact sequence

$$0 \to H^*(bo_1) \to H^*(J_1) \to H^*(\Sigma^3 bsp) \to 0.$$

The chart for $\mathrm{Ext}_A(H^*(J_1))$ begins as in Figure 6.11. Here $\bullet$'s represent classes from bo_1, and $\circ$'s from $\Sigma^3 bsp$. The d_1-differentials represent the boundary homomorphism in (6.10), which is known by comparison with the exact sequence of E_1-terms of (6.4). The h_1-extension from 2 to 3 also follows from this comparison.

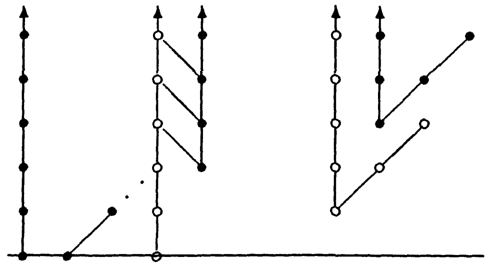

Figure 6.11: Chart for $\mathrm{Ext}_A(H^*(J_1))$

From Figure 6.11, we see that $H^*(J_1)$ has generators g_0, g_1, and g_3, with relations $\mathrm{Sq}^1 g_0$, $\mathrm{Sq}^2 g_1 + \mathrm{Sq}^3 g_0$, $\mathrm{Sq}^1 g_3 + a_1 g_1 + a_2 g_0$, and $\mathrm{Sq}^5 g_3 + a_3 g_1 + a_4 g_0$, with $a_i \in A$. The relations correspond to the elements at height 1 in Figure 6.11, and the presence of Sq^1 and Sq^2 in these relations can be read off from the action of h_0 (vertical lines) and h_1 (diagonal lines) in the chart. The presence of $\mathrm{Sq}^5 g_3$ is a consequence of the known structure of $H^*(bsp) = A/A(\mathrm{Sq}^1, \mathrm{Sq}^5)$. From the element in position [3,2], we deduce that there must be an element b in A such that

$$\mathrm{Sq}^1(\mathrm{Sq}^1 g_3 + a_1 g_1 + a_2 g_0) + \mathrm{Sq}^2(\mathrm{Sq}^2 g_1 + \mathrm{Sq}^3 g_0) + b(\mathrm{Sq}^1 g_0)$$

is identically 0. This implies that $a_1 = \mathrm{Sq}^2\mathrm{Sq}^1$, $a_2 = \mathrm{Sq}^4$, and $b = \mathrm{Sq}^4$. The term $\mathrm{Sq}^3\mathrm{Sq}^1$ could also be included as a summand of a_2, but since we already know that $\mathrm{Sq}^1 g_0 = 0$, there is no need to include it. Similarly, we omit an optional Sq^3 summand from a_1.

The Ext chart tells us that $\mathrm{Sq}^1(\mathrm{Sq}^5 g_3 + a_3 g_1 + a_4 g_0)$ is expandable to an identity. This implies that $\mathrm{Sq}^6\mathrm{Sq}^1$ cannot be a summand of a_3, for there is no way to cancel the $\mathrm{Sq}^7\mathrm{Sq}^1 g_1$ using the earlier relations. Moreover, it implies that this fourth relation in H^*J_1 must be of the form

$$\mathrm{Sq}^5 g_3 + \epsilon \mathrm{Sq}^7 g_1 + \epsilon'(\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1 g_1 + (\mathrm{Sq}^8 + \mathrm{Sq}^6\mathrm{Sq}^2) g_0), \tag{6.12}$$

as Sq^1 times this equals

$$\epsilon'((\mathrm{Sq}^8 + \mathrm{Sq}^6\mathrm{Sq}^2)(\mathrm{Sq}^1 g_0) + \mathrm{Sq}^4\mathrm{Sq}^2(\mathrm{Sq}^2 g_1 + \mathrm{Sq}^3 g_0)).$$

We need not consider $\mathrm{Sq}^5\mathrm{Sq}^2$ as a summand of a_3, or $\mathrm{Sq}^7\mathrm{Sq}^1$ or $\mathrm{Sq}^5\mathrm{Sq}^2\mathrm{Sq}^1$ as summands of a_4, because the resulting elements could be obtained from the first two relations. Finally the fact that $\mathrm{Sq}^2(\mathrm{Sq}^5 g_3 + a_3 g_1 + a_4 g_0)$ is expandable to an identity implies that $\epsilon' = 0$ in (6.12) since there would be no way to cancel the Sq^{10} which it would create, while $\epsilon = 1$, since

$$\mathrm{Sq}^2(\mathrm{Sq}^5 g_3 + \mathrm{Sq}^7 g_1) = \mathrm{Sq}^6(\mathrm{Sq}^1 g_3 + \mathrm{Sq}^2\mathrm{Sq}^1 g_1 + \mathrm{Sq}^4 g_0) \\ + (\mathrm{Sq}^7 + \mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1)(\mathrm{Sq}^2 g_1 + \mathrm{Sq}^3 g_0).$$

Since the coefficients in the relations for $H^*(J_1)$ are all in A_2, we deduce that $H^*(J_1)$ can be written as $A \otimes_{A_2} T_2$, where T_2 is an A_2-module with generators and relations corresponding exactly to those of the A-module presentation of $H^*(J_1)$. The standard change-of-rings theorem now implies the Ext result.

It is standard that, for any A-module M,

$$\mathrm{Ext}_A^{s,t}(M \otimes H^*(bo_1), \mathbb{Z}_2) \approx \mathrm{Ext}_A^{s+1,t+1}(M \otimes H^*(bo), \mathbb{Z}_2)$$

if $s > 0$. The third part of the proposition then follows from a comparison of the "$X\wedge$"-version of (6.10) with the long exact sequence

$$\to \mathrm{Ext}_A^{s-1,t}(H^*(X \wedge \Sigma^4 bsp)) \to E_1^{s,t}(J \wedge X) \to \mathrm{Ext}_A^{s,t}(H^*(X \wedge bo)) \to .$$

□

Now we prove Lemma 2.28. We use the isomorphism

$$v_1^{-1}\pi_*(Q^n) \approx v_1^{-1}J_*(Q^n),$$

and we pull the classes back under the map $J_{1*}(Q^n) \to v_1^{-1}J_*(Q^n)$. We calculate $J_{1*}(Q^n)$ by the ASS, for which an E_1-term is formed by summing charts like Figure 6.11, suspended $4i-1$ times, for $1 \le i \le n$. The boundary homomorphism of Lemma 2.28 has source classes which appear as $\circ$'s in horizontal component congruent to 3 mod 4 in Figure 6.11, and target class of the type of the right-most $\bullet$ in Figure 6.11. Differentials (boundary homomorphisms) cannot go from $\circ$'s to $\bullet$'s in that chart; this is implied by the exact sequence (6.10). □

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