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HOLOMORPHY TESTS BASED ON CAUCHY'S INTEGRAL FORMULA

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We prove some holomorphy tests based on the Cauchy integral formula for the unit disk, the upper half-plane and the complex plane.

1. Statement of the problems and results.

1.1. The classical Morera problem consists on studying the closed rectifiable curves Γ on the complex plane $\mathbb{C}$ such that any continuous function f on $\mathbb{C}$ satisfying

$$\int_\Gamma (f \circ \sigma)(z)\, dz = 0, \tag{1.1}$$

for every $\sigma \in M(2)$, is entire. Here $M(2)$ denotes the group of holomorphic rigid motions of $\mathbb{C}$, that is the group of all the mappings of the form $\sigma(z) = \alpha z + \beta$, $\alpha, \beta \in \mathbb{C}$, $|\alpha| = 1$.

If Γ is the boundary of a "regular" domain Ω then the above problem is equivalent to the classical Pompeiu problem, *i.e.* when the only continuous function f on $\mathbb{C}$ such that

$$\int_\Omega (f \circ \sigma)(z)\, dm(z) = 0, \qquad \text{for every } \sigma \in M(2),$$

is $f \equiv 0$.

Note that both problems are invariant under the action of the group $M(2)$, in the sense that if Γ (respectively, Ω) has the Morera (resp., Pompeiu) property then $\sigma(\Gamma)$ (resp., $\sigma(\Omega)$) also does, for every $\sigma \in M(2)$.

A lot of work on those problems has been done by several authors, among them, Brown, Schreiber and Taylor [**8**], Zalcman [**23**], Berenstein [**1**], Berenstein and Yang [**4**], Williams [**21, 22**], Brown and Kahane [**7**], Garofalo and Segala [**12, 13**], and Ebenfelt [**10**].

One of the most general results known about the Morera problem is the following: if Γ is a non real-analytic curve which is the boundary of a Jordan Lipschitz domain then Γ has the Morera property.

For a more complete description of the history and list of references on the above problems see [**23**] and the nice survey [**24**].

The Morera and the Pompeiu problem can be stated in other spaces different from the complex plane. In fact, Berenstein and Zalcman [**5**, **6**], and Berenstein and Shahshahani [**3**] obtained similar results for analogous invariant problems in the context of the symmetric spaces of rank one.

There are also some non-invariant problems (in the above sense) which are natural to consider. One of them is which are the closed rectifiable curves $\Gamma \subset \mathbb{D}$ such that any $f \in C(\mathbb{D})$ satisfying (1.1), for every conformal automorphism σ of $\mathbb{D}$, is holomorphic on $\mathbb{D}$. Some results on this problem have been obtained in [**2**]. For instance, the answer to the above question is affirmative for any non real-analytic curve which is the boundary of a Jordan domain of class $C^{2,\varepsilon}$.

In this paper we study a problem related with the Cauchy integral formula in the same way that the Morera problems are with the Cauchy theorem. That is studied on three settings: the complex plane, the unit disk and the upper half-plane. In the first case the problem is invariant whereas in the remaining ones it is not.

1.2. Let X denote one of the following domains: the unit disk $\mathbb{D}$, the upper-half plane $\mathbb{U}$, or the complex plane $\mathbb{C}$. Let G be either $M(2)$ (the group of holomorphic rigid motions of $\mathbb{C}$) if $X = \mathbb{C}$, or $\operatorname{Aut}(X)$ (the group of holomorphic automorphisms of X), otherwise. Let Γ be a rectifiable closed curve in X, and let $a \in X \setminus \Gamma$. Then the Cauchy integral formula says that

$$\int_\Gamma \frac{f(z)}{z-a}\,\frac{dz}{2\pi i} = \operatorname{Ind}(\Gamma; a) \cdot f(a),$$

for every holomorphic function f on X. In particular,

$$\int_\Gamma \frac{(f\circ\sigma)(z)}{z-a}\,\frac{dz}{2\pi i} = \operatorname{Ind}(\Gamma; a) \cdot (f\circ\sigma)(a), \tag{1.2}$$

for every $\sigma \in G$, since $f \circ \sigma$ is holomorphic on X.

The purpose of this work is to establish in which cases the converse works if Γ is a Jordan curve. Namely, our aim is to study when a continuous function f on X satisfying (1.2), for a Jordan curve Γ in X, is holomorphic.

The simplest case is when Γ is an Euclidean circle. Then it is clear that if a is the Euclidean center of Γ, (1.2) means that $f\circ\sigma$ satisfies the mean value property on Γ. Hence any harmonic non-holomorphic function f satisfies that property. Thus we will always assume that a is not the Euclidean center of Γ. In this case we obtain the following result:

Theorem 1.1. *Let D be an Euclidean closed disk contained in X. Let $a \in X$ be a point different from the Euclidean center of D and not lying on* *the Euclidean circle* $\Gamma = \partial D$. *Assume that* $f \in C(X)$ *satisfies* (1.2), *for every* $\sigma \in G$. *Then* f *is a holomorphic function on* X.

In proving the above result for the upper half-plane we obtain the following mean-value theorem:

Corollary 1.2. *Let* D *be a Euclidean closed disk contained in* $\mathbb{U}$ *with center* $c \in \mathbb{U}$ *and radius* $r > 0$. *Assume that* $f \in C(\mathbb{U})$ *satisfies*

$$\int_{\partial D} (f \circ \sigma)(z) \frac{|dz|}{2\pi r} = (f \circ \sigma)(c) \qquad (\forall \sigma \in \operatorname{Aut}(\mathbb{U})). \tag{1.3}$$

Then f *is harmonic on* $\mathbb{U}$.

It is noteworthy that, while the above result is always a "one radius" theorem, the corresponding result for the unit disk $\mathbb{D}$ is a "two radii" theorem for $c = 0$ (see [**5**]), and a "one radius" theorem when $c \neq 0$ (see [**2**]). That fact is in a certain sense a consequence of the non-invariance (under biholomorphic mappings) of the mean-value problem we are dealing with.

For a general curve Γ in $\mathbb{D}$ or $\mathbb{U}$ we obtain the following result:

Theorem 1.3. *Let* X *be either* $\mathbb{D}$ *or* $\mathbb{U}$. *Consider a* $C^{2,\epsilon}$ *Jordan domain* $\Omega \subset\subset X$, $0 < \epsilon < 1$. *Assume* $\Gamma = \partial\Omega$ *fails to be a real analytic curve, and* $a \in X \setminus \Gamma$. *If* $f \in C(X)$ *satisfies* (1.2), *for any* $\sigma \in \operatorname{Aut}(X)$, *then* f *is holomorphic in* X.

Observe that, while our problem for $X = \mathbb{C}$ is invariant under the action of $M(2)$ (in the above sense), in the upper half-plane and disk cases it is clearly not invariant under the corresponding automorphism groups. That invariance is somehow reflected on the fact that in the planar case there are points where (1.2) is automatically satisfied for a certain non-holomorphic function. Thus the general planar results have to be stated for a class of points which we call *admissible* (for a precise definition see §2). Then the proofs in [**1**], [**2**], [**8**], [**21**] and [**22**] are easily adapted to obtain the following result:

Theorem 1.4. *Let* Γ *be a rectifiable Jordan curve in* $\mathbb{C}$, *and let* a *be an admissible point for* Γ. *Assume* Γ *is the boundary of a Jordan domain* $\Omega \subset\subset \mathbb{C}$ *satisfying one of the following conditions:*

(i) Ω *is convex and it does not have a unique supporting line through some point* $p \in \Gamma$.

(ii) Ω *is a* $C^{2,\epsilon}$ *domain,* $0 < \epsilon < 1$, *but* Γ *is not a real analytic curve.*

(iii) *There is a (connected) real analytic curve* $\widetilde{\Gamma}$ *not completely included in* Γ *but containing an open subarc of* Γ.

(iv) Ω *is a Lipschitz domain, but* Γ *is not real analytic and* $a \in \mathbb{C} \setminus \overline{\Omega}$. *Then any* $f \in C(\mathbb{C})$ *satisfying* (1.2), *for every* $\sigma \in M(2)$, *is entire.*

In Section 3 we obtain examples of couples (Γ, a) for which we can apply the above theorem. For instance, any (Γ, a) where Γ is a regular polygonal curve and a is not its barycenter.

The paper is organized as follows. Next section is devoted to give some technical results which are used in the proofs of the theorems. The main one is the reduction of our problem to one of testing harmonicity. In connection with that, we also introduce the notion of planar admissible points. Our planar theorems are proved in Section 3, while the disk and upper half-plane ones are done in the last section.

The general approach to the proofs of the above results is based on the ideas of Berenstein and Zalcman [**6**] and the methods in [**2**]. Basically, X is considered as an homogeneous space G/K and then condition (1.2) is written as a convolution equation in G, which can be solved using harmonic analysis on G.

The general planar case could also be treated using the results of Brown, Schreiber and Taylor [**8**], but we want to present a common approach for all the cases in the spirit of [**6**]. However we were not able to prove the circular planar case using their methods.

2. Some preliminary results.

2.1. The first results will reduce the problem of testing the holomorphy of a function under the hypothesis of the theorems to the problem of checking its harmonicity.

We need some definitions.

Definition. Let Γ be a rectifiable Jordan curve in $\mathbb{C}$. A point $a \in C \setminus \Gamma$ is *admissible* for Γ if

$$\int_\Gamma \frac{\overline{z}}{z-a} \frac{dz}{2\pi i} \neq Ind(\Gamma, a)\overline{a}. \tag{2.1}$$

If $a \in \mathbb{C} \setminus \Gamma$ is not admissible, it is called *singular* for Γ.

Remark. Let Ω be the interior of Γ. Since the function of a in the left-hand side of (2.1) is holomorphic in $\mathbb{C} \setminus \Gamma$, the set of admissible points in Ω is dense, and there is at most a countable set of singular points in $\mathbb{C} \setminus \overline{\Omega}$.

In the next section we will study more in detail admissible points.

Lemma 2.1. *Let* Γ *be a rectifiable Jordan curve in* $\mathbb{C}$, *and* $a \in \mathbb{C} \setminus \Gamma$. *Then the following statements are equivalent:*

(i) *Any harmonic function f in $\mathbb{C}$ satisfying* (1.2), *for any $\sigma \in M(2)$, is entire.*

(ii) *The point a is admissible for Γ.*

Moreover, if a is a singular point for Γ, then the non-entire function $f(z) = \overline{z}$ satisfies (1.2).

Proof. If f is harmonic then $f = h + g$, where $h, \overline{g}$ are holomorphic in $\mathbb{C}$, and $g(0) = 0$. Now h obviously satisfies (1.2), so does g, and we only need to show that $g \equiv 0$.

Since $g \circ \sigma$ also verifies (1.2), for every $\sigma \in M(2)$, it is enough to see that

$$\frac{\partial g}{\partial \overline{z}}(0) = 0.$$

Consider the Taylor expansion of g at the origin:

$$g(z) = \sum_{n=1}^{\infty} d_n \overline{z}^n.$$

Take as σ any rotation, i.e. $\sigma(z) = e^{-i\theta} z$. Then (1.2) gives

$$\sum_{n=1}^{\infty} d_n \left(\int_\Gamma \frac{\overline{z}^n}{z-a} \frac{dz}{2\pi i} \right) e^{in\theta} = \operatorname{Ind}(\Gamma; a) \sum_{n=1}^{\infty} d_n \overline{a}^n e^{in\theta}.$$

Thus

$$d_n \int_\Gamma \frac{\overline{z}^n}{z-a} \frac{dz}{2\pi i} = \operatorname{Ind}(\Gamma; a) d_n \overline{a}^n \qquad \text{for every } n \geq 1. \tag{2.2}$$

If a is admissible for Γ then $\frac{\partial g}{\partial \overline{z}}(0) = d_1 = 0$.

Conversely, assume a is a singular point. Then any $\sigma \in M(2)$ can be written as $\sigma(z) = e^{i\theta}(z+b)$, $\theta \in \mathbb{R}$, $b \in \mathbb{C}$, so $f(z) = \overline{z}$ satisfies (1.2). ☐

The upper half-plane and unit disk cases will be treated simultaneously making a change of variables in (1.2), when $X = \mathbb{U}$.

Let $\sigma_0 : \mathbb{U} \to \mathbb{D}$ be the conformal mapping given by

$$\sigma_0(z) = \frac{z-i}{z+i}.$$

Then, by making the change of variables $w = \sigma_0(z)$, it is clear that (1.2) is equivalent to

$$\int_{\Gamma_0} \frac{(g \circ \sigma)(w)}{w - a_0} \frac{1 - a_0}{1 - w} \frac{dw}{2\pi i} = \operatorname{Ind}(\Gamma_0; a_0) \cdot (g \circ \sigma)(a_0) \qquad (\forall \sigma \in \operatorname{Aut}(\mathbb{D})),$$

where $\Gamma_0 = \sigma_0(\Gamma)$, $a_0 = \sigma_0(a) \in \mathbb{D} \setminus \Gamma_0$ and $g = f \circ \sigma_0^{-1}$.

When $X = \mathbb{D}$, let $\Gamma_0 = \Gamma$ and $a_0 = a$. With that notation (1.2) has the following common reformulation for both the disk and the upper half-plane:

$$\int_{\Gamma_0} \frac{(f \circ \sigma)(w)}{w - a_0} H(z) \frac{dw}{2\pi i} = \operatorname{Ind}(\Gamma_0; a_0) \cdot (g \circ \sigma)(a_0) \tag{2.3}$$

for every $\sigma \in \operatorname{Aut}(\mathbb{D})$, where $H \equiv 1$ (unit disk case) or $H(z) = \frac{1-a_0}{1-w}$ (upper half-plane). Note that if we also take $\Gamma_0 = \Gamma$, $a_0 = a$ and $H \equiv 1$, (1.2) for $X = \mathbb{C}$ is equivalent to (2.3), for any $\sigma \in M(2)$.

Now assume Γ is the circle $\partial D \subset\subset \mathbb{U}$, where as usual $D = D(c, r) = \{ z \in \mathbb{C} : |z - c| < r \}$. Then Γ_0 is also a circle in $\mathbb{D}$. Let c_0 and r_0 be its (Euclidean) center and radius, respectively. We are going to express the condition $a \neq c$, i.e. $a_0 \neq \sigma_0(c)$, in terms of c_0 and r_0. In order to do that we recall that the Euclidean disk $D = D(c, r)$ coincides with the pseudohyperbolic disk

$$\Delta_{\mathbb{U}}(c', r') = \left\{ z \in \mathbb{U} : \left| \frac{z - c'}{z - \overline{c'}} \right| < r' \right\},$$

where

$$c = \frac{c' - \overline{c'} r'}{1 - (r')^2} \qquad \text{and} \qquad r = \frac{2r' \operatorname{Im} c'}{1 - (r')^2}. \tag{2.4}$$

Now $D_0 = \sigma_0(\Delta_{\mathbb{U}}(c', r')) = \Delta_{\mathbb{D}}(c'_0, r')$, where $c'_0 = \sigma_0(c')$ and

$$\Delta_{\mathbb{D}}(c'_0, r') = \left\{ z \in \mathbb{D} : \left| \frac{z - c'_0}{1 - \overline{c'}_0 z} \right| < r' \right\}. \tag{2.5}$$

By [**11**, p. 3],

$$c_0 = \frac{1 - (r')^2}{1 - (r')^2 |c'_0|^2} c'_0 \qquad \text{and} \qquad r_0 = \frac{1 - |c'_0|^2}{1 - (r')^2 |c'_0|^2} r'. \tag{2.6}$$

(In fact, (2.4) follows easily from (2.6) using conformal invariance.)

A straightforward but tedious calculation using (2.4) and (2.6) shows that

$$\sigma_0(c) = c_0 + \frac{r_0^2}{1 - c_0},$$

so $a \neq c$ means that $a_0 \neq c_0 + \frac{r_0^2}{1 - c_0}$.

Now we state a result similar to Lemma 2.1 for $\mathbb{D}$ and $\mathbb{U}$.

Lemma 2.2. *Let X be either the unit disk or the upper half-plane. Let Γ be a piecewise C^1 Jordan curve in X, and $a \in X \setminus \Gamma$. Then, if Γ is not an*

Euclidean circle, any harmonic function f in X satisfying (1.2), *for every* $\sigma \in \operatorname{Aut}(X)$, *is holomorphic on* X.

Moreover, the same conclusion holds when Γ is an Euclidean circle and a does not coincide with its center.

Proof. We will carry out the proof of the lemma in three steps.

FIRST STEP: Taking into account the observation just above the lemma, we prove the next statement:

Let Γ_0 be a piecewise C^1 Jordan curve in $\mathbb{D}$, and $a_0 \in \mathbb{D} \setminus \Gamma_0$. Then the following items are equivalent:

(i) *Any harmonic function f in $\mathbb{D}$ satisfying* (2.3), *for every* $\sigma \in \operatorname{Aut}(\mathbb{D})$, *is holomorphic on* $\mathbb{D}$.

(ii) *There exists $n \geq 1$ such that*

$$\int_{\Gamma_0} \frac{\overline{z}^n}{z-a_0} H(z) \frac{dz}{2\pi i} \neq \operatorname{Ind}(\Gamma_0, a_0)\overline{a}_0^n. \tag{2.7}$$

(ii)$\Rightarrow$(i): An argument like the one in the planar case shows that we just have to prove that if $f(z) = \sum_{n=1}^{\infty} d_n \overline{z}^n$ is a conjugate holomorphic function on $\mathbb{D}$ satisfying (2.3), then $d_1 = 0$.

Now, since the rotations are automorphisms of $\mathbb{D}$, formula (2.2) works. Then the hypotheses yield

$$\frac{\partial^n f}{\partial \overline{\zeta}^n}(0) = n! d_n = 0.$$

If τ_z denotes the automorphism of $\mathbb{D}$ given by

$$\tau_z(w) = \frac{w+z}{1+\overline{z}w}, \tag{2.8}$$

$f \circ \tau_z$ obviously satisfies (2.3) so

$$\frac{\partial^n (f \circ \tau_z)}{\partial \overline{\zeta}^n}(0) = 0, \qquad \text{for every } z \in \mathbb{D}.$$

When $n = 1$, $\partial f / \partial \overline{\zeta} \equiv 0$, i.e. $f \equiv 0$, so we may assume that $n \geq 2$. We compute the above derivative using Fad di Bruno formula:

$$\begin{aligned}
\frac{\partial^n (f \circ \tau_z)}{\partial \overline{\zeta}^n}(0) &= \sum \frac{n!}{k_1! \cdots k_n!} \frac{\partial^k f}{\partial \overline{\zeta}^k}(\tau_z(0)) \left(\frac{1}{1!} \overline{\frac{\partial \tau_z}{\partial \zeta}(0)} \right)^{k_1} \cdots \left(\frac{1}{n!} \overline{\frac{\partial^n \tau_z}{\partial \zeta^n}(0)} \right)^{k_n} \\
&= \sum \frac{n!}{k_1! \cdots k_n!} \frac{\partial^k f}{\partial \overline{\zeta}^k}(z)(1-|z|^2)^k(-z)^{n-k},
\end{aligned}$$

where the sums are taken over $k_1 + 2k_2 + \cdots + nk_n = n$, and $k = k_1 + k_2 + \cdots + k_n$. Since the previous sum is zero, isolating the highest order derivative we have that

$$\frac{\partial^n f}{\partial \overline{z}^n}(z) = -\sum \frac{n!}{k_1! \cdots k_n!} \frac{\partial^k f}{\partial \overline{z}^k}(z) \left(\frac{-z}{1-|z|^2}\right)^{n-k}, \qquad \text{for every } z \in \mathbb{D},$$

where the sum is as above, except that now $k < n$. But f is conjugate holomorphic, hence

$$0 = -(1-|z|^2)^2 \frac{\partial}{\partial z}\left(\frac{\partial^n f}{\partial \overline{z}^n}(z)\right)(z) = \sum_{k=1}^{n-1} A_k \left(\frac{-z}{1-|z|^2}\right)^{n-1-k} \frac{\partial^k f}{\partial \overline{z}^k}(z),$$

where

$$A_k = \sum \frac{n!(n-k)}{k_1! \cdots k_n!},$$

and the sum is over $k_1 + \cdots + k_n = k$ and $k_1 + 2k_2 + \cdots + nk_n = n$. It is easy to see that for a fixed k, $1 \leq k \leq n-1$, there is at least one possible choice of such $k_1, \ldots, k_n$. Hence any A_k is positive. Therefore it is enough to prove the following:

Let $n \geq 1$ and let f be a conjugate holomorphic function on $\mathbb{D}$. Assume that

$$0 = \sum_{k=1}^{n} a_k \left(\frac{-z}{1-|z|^2}\right)^{n-k} \frac{\partial^k f}{\partial \overline{z}^k}(z), \qquad \textit{for every } z \in \mathbb{D}, \tag{2.9}$$

where $a_k > 0$, $k = 1, \ldots, n$. Then f is constant.

We proceed by induction on $n \geq 1$. For $n = 1$ it is obvious. Assume (2.9) holds for $n \geq 2$. Then

$$\frac{\partial^n f}{\partial \overline{z}^n}(z) = -\sum_{k=1}^{n-1} \frac{a_k}{a_n} \left(\frac{-z}{1-|z|^2}\right)^{n-k} \frac{\partial^k f}{\partial \overline{z}^k}(z),$$

so

$$0 = (1-|z|^2)^2 \frac{\partial}{\partial z}\left(\frac{\partial^n f}{\partial \overline{z}^n}\right)(z) = \sum_{k=1}^{n-1} (n-k) \frac{a_k}{a_n} \left(\frac{-z}{1-|z|^2}\right)^{n-1-k} \frac{\partial^k f}{\partial \overline{z}^k}(z),$$

and by the induction hypothesis we are done.

(i)$\Rightarrow$(ii): Suppose (ii) does not hold. Let $\sigma(z) = e^{-i\theta}\tau_b(z)$, $\theta \in \mathbb{R}$, $b \in \mathbb{D}$. Since

$$\begin{aligned} &\int_{\Gamma_0} \frac{\overline{\sigma(z)}}{z-a_0} H(z) \frac{dz}{2\pi i} \\ &\quad = e^{i\theta}\left(-b \cdot \operatorname{Ind}(\Gamma_0, a_0) + (1-|b|^2) \sum_{n=1}^{\infty} b^{n-1} \int_{\Gamma_0} \frac{\overline{z}^n}{z-a_0} H(z) \frac{dz}{2\pi i}\right), \end{aligned}$$

it is clear that the non holomorphic function $f(z) = \overline{z}$ satisfies (2.3).

SECOND STEP: Now we are going to show that if (ii) does not hold then $a_0 \in \Omega_0$.

By Stokes theorem our assumption is equivalent to:

$$\int_{\Omega_0} \frac{\overline{z}^n}{z - a_0} H(z)\, dm(z) = 0 \qquad \text{for every } n \geq 0. \tag{2.10}$$

If $a_0 \in \mathbb{D} \setminus \overline{\Omega_0}$, the function $H(z)/(z - a_0)$ is holomorphic on $\overline{\Omega}_0$, and Runge's theorem shows that it is the uniform limit on $\overline{\Omega}_0$ of a sequence $\{P_n\}$ of holomorphic polynomials. Thus using (2.10) we obtain

$$\int_{\Omega_0} \frac{|H(z)|^2}{|z - a_0|^2}\, dm(z) = \lim_{n\to\infty} \int_{\Omega_0} \frac{H(z)}{z - a_0} \overline{P_n(z)}\, dm(z) = 0,$$

which clearly is a contradiction.

THIRD STEP: Now assume that $a_0 \in \Omega_0$ and (ii) does not hold.

First we are going to check that

(2.11)
$$\int_{\Gamma_0} \frac{u(z)}{z - a_0} H(z) \frac{dz}{2\pi i} = u(a_0), \qquad \text{for every } u \in C(\overline{\Omega}_0) \text{ harmonic on } \Omega_0.$$

The assumption implies that (2.11) is satisfied by $u(z) = \overline{z}^n$, $n = 0, 1, \dots,$ and also by $u(z) = z^n$, $n = 0, 1, \dots,$ using Cauchy's integral formula. So it is enough to check that any continuous function u on $\overline{\Omega}_0$, which is harmonic on Ω_0, can be uniformly approximated on $\overline{\Omega}_0$ by real parts of holomorphic polynomials.

By a theorem of Keldysh [**19**, Thm. 5.15, p. 33] u is the uniform limit on $\overline{\Omega}_0$ of a sequence $\{u_n\}$ of harmonic functions on $\overline{\Omega}_0$. Since Ω_0 is a Jordan domain, $u_n = \operatorname{Re} f_n$, where f_n is a holomorphic function on $\overline{\Omega}_0$. Again by Runge's theorem there is a holomorphic polynomial P_n such that $\sup_{\overline{\Omega}_0} |f_n(z) - P_n(z)| < 1/n$, and taking real parts we are done.

Next we see that (2.11) implies that Γ is an Euclidean circle with center at a.

Suppose first $X = \mathbb{D}$. So $H \equiv 1$, $\Gamma_0 = \Gamma$ and $a_0 = a$. Then (2.11) shows that

$$\frac{\Gamma'(s)}{\Gamma(s) - a} \frac{ds}{2\pi i}$$

coincides with the harmonic measure of Ω at a. (Here s denotes arc length.) In particular, $-i \frac{\Gamma'(s)}{\Gamma(s)-a} > 0$, for every s. And if $\Gamma(s) = x_1(s) + ix_2(s)$ and

$a = a_1 + ia_2$ we have that the following differential equation

$$x_1'(x_1 - a_1) + x_2'(x_2 - a_2) = 0$$

is satisfied, and that shows Γ is an Euclidean circle centered at a.

Now consider the last case left, $X = \mathbb{U}$. Undoing the change of variables made below the statement of the lemma, it follows from (2.11) that

$$\int_\Gamma \frac{u(z)}{z-a}\frac{dz}{2\pi i} = u(a), \qquad \text{for every } u \in C(\overline{\Omega}), \text{ which is harmonic in } \Omega,$$

and the above argument shows that Γ is an Euclidean circle centered at a. □

2.2. Recall that we have reformulated our problem so that (1.2) is equivalent to (2.3), so we only deal with $X = \mathbb{C}$ and $X = \mathbb{D}$. In the first case we put $G = M(2)$, while in the second one $G = \mathrm{Aut}\,(\mathbb{D})$.

Then using the projection $\pi : G \to X$ given by $\pi(g) = g(0)$ we may identify X as the homogeneous space G/K, where $K = SO(2)$ is the subgroup of the rotations.

We will follow the notations of [**6**], so we just remind briefly the main ones, referring the reader to the above reference for more details.

We identify locally integrable functions with distributions on X by means of the measure $d\mu(z) = dm(z)$ for $X = \mathbb{C}$, and $d\mu(z) = dm(z)/(1-|z|^2)^2$ for $X = \mathbb{D}$, where m is the Lebesgue measure on $\mathbb{C}$. The lifting dg of $d\mu$ by π is a Haar measure on G, which is bi-invariant under the action of K. We use that measure dg to identify locally integrable functions with distributions on G.

We denote by dk the normalized Haar measure on K, which considered as a distribution will be denoted as δ_K.

If φ is a function on G, $\check{\varphi}(g) = \varphi(g^{-1})$, for $g \in G$. Similarly, is defined $\check{T}$, for $T \in \mathcal{D}(G)$. Recall that the space $\mathcal{E}_0'(G)$ of compactly supported bi-invariant (under K) distributions on G is a topological convolution algebra.

$\widetilde{T}$ denotes the lifting to G of the distribution (or function) T on X, while S_π denotes the projection to X of the distribution (function) S on G. The operator $\tilde{\cdot}$ is a bijection from the usual spaces of distributions (functions) on X onto the corresponding spaces of right-invariant (under K) distributions (resp., functions) on G, with inverse $\cdot_\pi$.

Let Δ_0 be either the Euclidean Laplacian Δ for $X = \mathbb{C}$, or the hyperbolic Laplacian $\Delta_h = (1-|z|^2)^2\Delta$ for $X = \mathbb{D}$.

2.3. Now we will sketch the general group approach introduced by Berenstein and Zalcman [**6**] (see also [**2**]) which we will use to prove our results.

Recall that we have rewritten (1.2) as (2.3), for any case. Now (2.3) means that the following convolution equation on G holds:

$$\tilde{f} * \check{\tilde{T}} = 0, \tag{2.12}$$

where $T = T_{\Gamma_0, a_0}$ is the compactly supported Radon measure on X given by

$$T\varphi = \int_{\Gamma_0} \frac{\varphi(z)}{z - a_0} H(z) \frac{dz}{2\pi i} - \operatorname{Ind}(\Gamma_0, a_0) \cdot \varphi(a_0) \qquad (\varphi \in C(X)). \tag{2.13}$$

Let $\mathcal{J}$ be the closed convolution ideal in $\mathcal{E}'_0(G)$ generated by all the distributions of the form $\check{\tilde{T}} * \tilde{S}$, for

$$S = \frac{\partial^n \delta_0}{\partial z^n}, \frac{\partial^n \delta_0}{\partial \bar{z}^n} \qquad (n = 0, 1, 2, \dots),$$

where δ_0 is the Dirac measure at the origin.

Recall that a *spherical function* on $X \cong G/K$ is a radial eigenfunction of Δ_0 such that $\phi(0) = 1$.

For every $\lambda \in \mathbb{C}$ there is only one spherical function ϕ_λ such that $\Delta_0 \phi_\lambda = -p(\lambda)\phi_\lambda$, where $p(\lambda) = \lambda^2$ for $X = \mathbb{C}$ and $p(\lambda) = 1 + \lambda^2$ for $X = \mathbb{D}$.

The *spherical Fourier transform of* $R \in \mathcal{E}'_0(G)$ is given by

$$(\mathcal{F}R)(\lambda) = (R * \tilde{\phi}_\lambda)(e) \qquad (\lambda \in \mathbb{C}).$$

A theorem of Paley–Wiener–Schwarz type shows that $\mathcal{F}$ is an algebra isomorphism between the convolution algebra $\mathcal{E}'_0(G)$ and the multiplication algebra $\mathbb{E}'$ of all even entire functions (of one complex variable) of exponential type which have polynomial growth on $\mathbb{R}$. So the spherical Fourier transform transports the topology of $\mathcal{E}'_0(G)$ to $\mathbb{E}'$ (see [**6**, pp. 606-608]). Then $I = \mathcal{F}(\mathcal{J})$ is a closed ideal of $\mathbb{E}'$. It is easy to check that every zero λ_0 of $p(\lambda)$ is also a common zero of the functions in I. In fact, ϕ_{λ_0} is a radial harmonic function on X, so it is constant and then so is $(\tilde{S} * \tilde{\phi}_{\lambda_0})_\pi$, for every $S \in \mathcal{E}(\mathbb{C})$. Thus

$$\mathcal{F}(\check{\tilde{T}} * \tilde{S})(\lambda) = T\left((\tilde{S} * \tilde{\phi}_\lambda)_\pi\right) \tag{2.14}$$

vanishes at $\lambda = \lambda_0$, since the constants obviously satisfy the Cauchy integral formula.

Using the Schwartz spectral synthesis theorem and a classical division theorem for entire functions in [**2**, §3] it is proved that condition (2.12) implies f is harmonic if and only if the zeros of $p(\lambda)$ are the only common zeroes of the functions in the closed ideal I of $\mathbb{E}'$, and their "common" multiplicities in I coincide with their multiplicity in $p(\lambda)$.

2.4. Finally we state a regularity result for a free boundary problem which is just an application of a general theorem [**18**, Thm. VI.3.3], and which will be a fundamental tool in the proofs of the "general" theorems.

Lemma 2.3. *Let $\Omega \subset\subset \mathbb{C}$ be a $C^{2,\varepsilon}$ Jordan domain. Let $\Gamma = \partial\Omega$. Let z^0 be a point on Γ and $\mathcal{U}$ an open neighborhood of z_0. Assume there are $C^{2,\varepsilon}$ real-valued functions, v and w, which are solutions to the following free boundary problem:*

$$\left\{\begin{array}{ll} \left.\begin{array}{l} \Delta v + cv - dw = 0 \\ \Delta w + cw + dv = 0 \end{array}\right\} & \text{in } \Omega \cap \mathcal{U} \\ \left.\begin{array}{l} v = w = 0 \\ \frac{\partial v}{\partial x_1} + \frac{\partial w}{\partial x_2} = g \\ \frac{\partial v}{\partial x_2} - \frac{\partial w}{\partial x_1} = h \end{array}\right\} & \text{on } \Gamma \cap \mathcal{U}, \end{array}\right. \tag{2.15}$$

where c, d, g and h are (real-valued) real-analytic functions on $\mathcal{U}$, and $g^2 + h^2 > 0$ on Γ.

Then Γ admits a real analytic parametrization in a neighborhood of z_0.

Proof. The proof is similar to the one given in [**2**, 4.11], so we will only sketch the main differences. We will follow the same notations used there, in particular, a semicolon and subscripts will denote partial derivatives.

Without loss of generality we may assume $z^0 = 0$. Moreover, the last two boundary conditions and the hypothesis on g and h imply that some first order derivative of v or w is different from zero at the origin. Hence, after a simple change of variables we also may assume that $v_{;2}(0) = 1$. The *zeroth hodograph transformation*, $y_1 = x_1$, $y_2 = v$ gives a local $C^{2,\varepsilon}$ change of variables at the origin. Then Γ, which is described by the equation $v(x_1, x_2) = 0$ in the x-coordinates, is given by $y_2 = 0$ in the y-coordinates, in a neighborhood of 0. Then $x_2 = \psi(y_1, y_2)$, where ψ is a function of class $C^{2,\varepsilon}$ in a neighborhood of the origin.

Since the higher order terms of our system in the new variables are essentially the same as in [**2**], the ellipticity with weights $s_1 = s_2 = 0$, $t_1 = t_2 = 2$, follows in the same way. We only recall for later use that, after linearization, the principal symbols associated with those weights are: $L'_{11}(y,\xi) = P(y,\xi)$, $L'_{12}(y,\xi) = 0$, $L'_{21}(y,\xi) = P(y,\xi)\, W_{;2}$, and $L'_{22}(y,\xi) = -P(y,\xi)\psi_{;2}$, where

$$P(y,\xi) = \frac{1}{\psi_{;2}}\left(\left(\xi_1 - \frac{\psi_{;1}}{\psi_{;2}}\xi_2\right)^2 + \left(\frac{\xi_2}{\psi_{;2}}\right)^2\right).$$

Here we are using the notations of [**18**].

The first boundary conditions of (2.15) in the new coordinates are: $W = 0$ and $G\,\psi_{;2} + \psi_{;1} - W_{;2} = 0$; where $G(y) = g(y_1, \psi(y))$, and they hold around the origin on $y_2 = 0$.

Thus we have $B_{11}(y,\xi) = 0$, $B_{12}(y,\xi) = 1$, $B_{21}(y,\xi) = i(\xi_1 + \xi_2\, G(y))$, and $B_{22}(y,\xi) = -i\xi_2$. Taking the weights $r_1 = -2$ and $r_2 = -1$, the "principal symbol" B'_{hj} of B_{hj} coincides with B_{hj}, for every $h, j = 1, 2$. And let us prove the coerciveness of the system with respect the given weights. This will be done by checking that the system of equations

$$\begin{aligned} &L'_{j1}(0,D)U + L'_{j2}(0,D)V = 0 \qquad && \text{in } \mathbb{R}^2_+,\ j = 1,2, \\ &B'_{j1}(0,D)U + B'_{j2}(0,D)V = 0 \qquad && \text{on } y_2 = 0,\ j = 1,2, \end{aligned}$$

admits no nontrivial bounded exponential solutions of the form:

$$U(y) = e^{i\xi y_1}\phi(y_2), \qquad V(y) = e^{i\xi y_1}\varphi(y_2), \qquad \text{with } \xi \in \mathbb{R}\setminus\{0\}.$$

In fact, if the above functions are solutions to that system then both φ and ϕ satisfy the differential equation

$$\frac{1+\psi_{;1}^2}{\psi_{;2}^2}\, g''(t) - 2i\frac{\psi_{;1}}{\psi_{;2}}\xi\, g'(t) - \xi^2 g(y) = 0,$$

with boundary conditions: $\varphi(0) = 0$ and $\varphi'(0) = i\xi\,\phi(0) + G(y_1, 0)\,\phi'(0)$.

The general bounded solution to that equation is $g(t) = c \cdot e^{at}$, c being an arbitrary constant and

$$a = \left(-\frac{|\xi|}{|\psi_{;2}|} + i\frac{\psi_{;1}}{\psi_{;2}}\xi\right)\frac{\psi_{;2}^2}{1+\psi_{;1}^2}.$$

Thus the boundary conditions on ϕ and φ imply that $\varphi \equiv 0$ and $\phi(0)\,(i\xi + G(y_1,0)\,a) = 0$. But since G is real-valued it is easy to see, using the definition of a, that $i\xi + G(y_1,0)\,a$ never vanishes. Hence $\phi \equiv 0$.

Finally, applying the regularity theorem [**18**, Thm. VI.3.3], we obtain that both ψ and W are real analytic in some neighborhood of the origin. In particular, v is also real analytic in some neighborhood of z^0. Thus the proof is complete. □

3. The planar case.

3.1. Before carrying out the proof of the planar theorems, we study different examples of admissible and singular points.

Proposition 3.1. *Let $\Omega \subset\subset \mathbb{C}$ be a piecewise C^1 Jordan domain, $\Gamma = \partial\Omega$ and let $a \in \mathbb{C}\setminus\overline{\Omega}$. Assume there is a straight line separating a and Ω. Then a is an admissible point for Γ.*

Proof. Note that by the Stokes theorem a point $a \in \mathbb{C} \setminus \Gamma$ is admissible for Γ if and only if

$$\int_\Omega \frac{dm(z)}{z-a} \neq 0.$$

Assume $a \in \mathbb{C} \setminus \overline{\Omega}$ is separated by a line from Ω. Without loss of generality we may assume the separating line is the imaginary axis and Re $a > 0$. Then it is clear that the above integral has negative real part, so a is admisssible. □

Corollary 3.2. *All the points not lying in the convex hull of a Jordan curve are admissible. In particular, all the exterior points of a convex Jordan curve are admissible.*

The following examples give ways to construct interior admissible and singular points.

Example 1. Let Γ be a piecewise C^1 Jordan curve which is invariant under a rotation of angle $2\pi/n$, for some integer $n \geq 2$, around a point a. Then a is singular for Γ.

In fact it is clearly enough to consider the case $a = 0$. Now let Γ_0 be the arc of Γ lying on the sector $0 \leq \operatorname{Arg} z \leq \frac{2\pi}{n}$. Then

$$\Gamma = \sum_{k=0}^{n-1} e^{\frac{2\pi k i}{n}} \Gamma_0,$$

and

$$\int_\Gamma \frac{\overline{z}}{z} \frac{dz}{2\pi i} = \sum_{k=0}^{n-1} \int_{e^{\frac{2\pi k i}{n}} \Gamma_0} \frac{\overline{z}}{z} \frac{dz}{2\pi i} = \left(\sum_{k=0}^{n-1} e^{\frac{2\pi k i}{n}} \right) \left(\int_{\Gamma_0} \frac{\overline{z}}{z} \frac{dz}{2\pi i} \right) = 0.$$

Example 2. Assume Γ is a piecewise C^1 curve which is the boundary of a bounded starlike region Ω with respect to a point $a \in \mathbb{C}$. Suppose the following condition holds:

There exists a straight line ℓ passing through a such that it divides Ω in two parts, Ω_1 and Ω_2, and the image of Ω_1 by the symmetry with respect to ℓ is strictly included in Ω_2.

Then a is an admissible point for Γ.

Without loss of generality, using the invariance of the problem we may assume that ℓ is the real axis, and Ω_1 is in the upper half-plane. Since Ω is starlike with respect to a, we can parametrize Γ as $\Gamma(\theta) = a + r(\theta)e^{i\theta}$,

$-\pi \le \theta \le \pi$. Then the hypotheses show that there exists a non-empty open set I in $[0,\pi]$ such that $r(\theta) < r(-\theta)$, for $\theta \in I$, and $r(\theta) = r(-\theta)$, for $\theta \in [0,\pi] \setminus I$. Now

$$\begin{aligned}\int_\Gamma \frac{\overline{z}-\overline{a}}{z-a}\frac{dz}{2\pi i} &= \int_{-\pi}^{\pi} e^{-i\theta}(r'(\theta)+ir(\theta))\,\frac{d\theta}{2\pi i}\\ &= \int_{-\pi}^{\pi}((r(\theta)\cos\theta - r'(\theta)\sin\theta) - i(r(\theta)\sin\theta + r'(\theta)\cos\theta))\,\frac{d\theta}{2\pi}.\end{aligned}$$

Thus

$$\begin{aligned}\operatorname{Im}\left(\int_\Gamma \frac{\overline{z}-\overline{a}}{z-a}\frac{dz}{2\pi i}\right) &= -\int_{-\pi}^{\pi}(r(\theta)\sin\theta + r'(\theta)\cos\theta)\frac{d\theta}{2\pi}\\ &= -\int_{-\pi}^{\pi} r(\theta)\sin\theta\frac{d\theta}{\pi},\end{aligned}$$

since

$$\int_{-\pi}^{\pi}(r(\theta)\sin\theta - r'(\theta)\cos\theta)\frac{d\theta}{2\pi} = 0.$$

Therefore

$$\begin{aligned}\operatorname{Im}\left(\int_\Gamma \frac{\overline{z}-\overline{a}}{z-a}\frac{dz}{2\pi i}\right) &= -\int_{-\pi}^{\pi} r(\theta)\sin\theta\frac{d\theta}{\pi}\\ &= \int_0^{\pi}(r(-\theta)-r(\theta))\sin\theta\,\frac{d\theta}{\pi} = \int_I (r(-\theta)-r(\theta))\sin\theta\,\frac{d\theta}{\pi} > 0.\end{aligned}$$

Remark. There are many curves Γ and points a under the conditions of the above example. For instance:

(i) Any boundary Γ of a regular polygonal region and any interior point a different from its barycenter.

(ii) Any ellipse Γ and any interior point different from its center.

(iii) Any boundary Γ of a curvilinear triangle satisfying that its three sides are equal arcs of Euclidean circles and the center of each one is the opposite corner (the so called *Reuleaux triangle*), and any interior point different from its barycenter.

Our next result will give examples of Jordan curves Γ without rotation invariance which, nevertheless, have singular interior points. It also somehow shows the difficulty to characterize the interior singular points for general Jordan curves.

Example 3. Let r be a piecewise C^1 function on $\mathbb{R}$ which is positive and 2π-periodic. Let Γ be the Jordan curve parametrized by $r(\theta)e^{i\theta}$, $0 \le \theta \le 2\pi$.

Now it is clear from the previous example that

$$\int_\Gamma \frac{\overline{z}}{z}\frac{dz}{2\pi i} = \int_0^{2\pi} r(\theta)e^{-i\theta}\frac{d\theta}{\pi}.$$

So the origin is singular for Γ if and only if the Fourier coefficient $\hat{r}(1)$ vanishes.

Moreover, we can construct Γ such that the origin is singular, but the curve does not have the rotation invariance of Example 1, for any n. In fact, that invariance means that the Fourier coefficients $\hat{r}(k)$ satisfy the equation $\hat{r}(k)(1-e^{2\pi ik/n})=0$, i.e. $\hat{r}(k)=0$ for any k which is not a multiple of n.

For instance, $r(\theta)=5+2\cos(2\theta)+2\cos(3\theta)$ is a positive function such that for any $n\geq 2$ there exists a k, not multiple of n, and $\hat{r}(k)\neq 0$. Since $\hat{r}(1)=0$ the corresponding curve Γ has the required properties.

Remark. Note that choosing r conveniently it is possible to construct examples of non real analytic curves Γ which the boundary of a Jordan $C^{2,\varepsilon}$ domain Ω and such that $0\in\Omega$ is an admissible point.

Example 4. If Γ is the circle $|z-c|=r$ and $a\in\mathbb{C}\setminus(\Gamma\cup\{c\})$, then a is admissible for Γ.

In fact, parametrizating the circle Γ in polar coordinates we obtain that

$$\int_\Gamma \frac{\overline{z}}{z-a}\frac{dz}{2\pi i} = \begin{cases} \overline{c}\neq a & \text{if } a \text{ is interior to } \Gamma \\ \frac{r^2}{c-a}\neq 0, & \text{otherwise.}\end{cases}$$

3.2. In this subsection we are going to prove Theorem 1.1 for $X=\mathbb{C}$.

Let $\Gamma=\partial D(c,r)$ and $a\in\mathbb{C}\setminus\Gamma$, $a\neq c$. As we have just seen in Example 4, a is admissible for Γ and, by Lemma 2.1, we must show that any $f\in C(\mathbb{C})$ satisfying (1.2), for any $\sigma\in M(2)$, is harmonic.

Using the invariance of the problem we may assume that $c=0$, and we denote $D=D(0,r)$. Then consider the Radon measure $T=T_{\Gamma,a}$ given by (2.13), and the associated closed ideal I in $\mathbb{E}'$. By **2.2** we only have to prove that 0 is the only common zero of the functions in I, and its common multiplicity equals 2.

Recall that the spherical functions on $\mathbb{C}\cong M(2)/SO(2)$ are given by

$$\text{(3.1)}\quad \phi_\lambda(z)=\int_0^{2\pi} e^{i\lambda(x\cos\theta+y\sin\theta)}\frac{d\theta}{2\pi}=J_0(\lambda|z|)\qquad (z=x+iy\in\mathbb{C}),$$

where as usual J_0 is the Bessel function of order 0 (see [**16**, p. 404]).

Now we compute the generators of I.

By the change of variables $t = \frac{\pi}{2} - \theta$ from (3.1) we get

$$\phi_\lambda(z) = \int_0^{2\pi} e^{\frac{\lambda}{2}(ze^{it} - \bar{z}e^{-it})} \frac{dt}{2\pi}.$$

Next recall the generating function of the Bessel functions J_n of integer order:

$$e^{\frac{1}{2}z(\zeta+\zeta^{-1})} = \sum_{n=-\infty}^{\infty} c_n(z)\zeta^n \qquad (\zeta \in \mathbb{C} \setminus \{0\}), \tag{3.2}$$

where $c_n(z) = J_n(z)$, if $n \geq 0$, and $c_n(z) = (-1)^n J_{-n}(z)$, if $n < 0$ (see [**20**, p.100-1, (5.2.10-11)]). Using that, we have

$$(\widetilde{S} * \widetilde{\phi}_\lambda)_\pi(z) = \begin{cases} \left(-\frac{\lambda^2 z}{2}\right)^n G_{n,z}(\lambda) \text{ if } S = \frac{\partial^n \delta_0}{\partial \bar{z}^n} \\ \left(-\frac{\lambda^2 \bar{z}}{2}\right)^n G_{n,z}(\lambda) \text{ if } S = \frac{\partial^n \delta_0}{\partial z^n}, \end{cases}$$

where $G_{n,z}$ is the even entire function given by:

$$G_{n,z}(\lambda) = \frac{J_n(\lambda|z|)}{(\lambda|z|)^n}.$$

Thus using (2.14) we get:

$$\mathcal{F}\left(\check{\widetilde{T}} * \left(\widetilde{\frac{\partial^n \delta_0}{\partial \bar{z}^n}}\right)\right)(\lambda) = \left(-\frac{\lambda^2}{2}\right)^n (\alpha_n G_{n,r}(\lambda) - \alpha_n G_{n,a}(\lambda))$$

$$\mathcal{F}\left(\check{\widetilde{T}} * \left(\widetilde{\frac{\partial^n \delta_0}{\partial z^n}}\right)\right)(\lambda) = \left(-\frac{\lambda^2}{2}\right)^n (\beta_n G_{n,r}(\lambda) - \bar{\alpha}_n G_{n,a}(\lambda))$$

where

$$\alpha_n = \int_{|z|=r} \frac{z^n}{z-a} \frac{dz}{2\pi i} = \operatorname{Ind}(\partial D(0,r), a)a^n, \qquad \text{and}$$

$$\beta_n = \int_{|z|=r} \frac{\bar{z}^n}{z-a} \frac{dz}{2\pi i}.$$

Evaluating β_n by straightforward calculations we obtain that

$$\beta_0 = \left\{ \begin{matrix} 1 \text{ if } |a| < r \\ 0 \text{ if } |a| > r \end{matrix} \right\} \qquad \text{and, for } n \geq 1, \qquad \beta_n = \begin{cases} 0 & \text{if } |a| < r \\ -\left(\frac{r^2}{a}\right)^n & \text{if } |a| > r. \end{cases}$$

Thus we have to consider the even entire functions

$$F_{n,z}(\lambda) = (\lambda|z|)^n J_n(\lambda|z|)$$

and distinguish two cases:

Case 1: $|a| < r$. Then $\alpha_0 = \beta_0 = 1$, $\alpha_n = a^n$ and $\beta_n = 0$, for $n \geq 1$. Since $a \neq 0$, by hypothesis, it follows that the closed ideal I is generated by the functions $F_{0,r} - F_{0,a}$, $F_{n,r}$ and $F_{n,a}$, $n \geq 1$.

Case 2: $|a| > r$. Then $\alpha_n = \beta_n = 0$, for $n \geq 0$, and $\beta_n \neq 0$, for $n \geq 1$. Therefore the functions $F_{n,r}$, $n \geq 1$, generate I.

Since in any case the functions $F_{n,r}$, $n \geq 1$, are between the considered generators of I, to finish the proof we only have to check that the functions $F_n(z) = z^n J_n(z)$, $n \geq 1$, have no common zeroes except 0, and its common multiplicity equals 2.

Taking into account (3.2) we deduce that if $z_0 \neq 0$ is a common zero of all the F_n, $n \geq 1$, then $c_n(z_0) = 0$, for $n \neq 0$, and so we get a contradiction.

Since J_1 has a simple zero at the origin, F_1 has a double zero at that point, so the common multiplicity equals 2.

3.3. In order to prove Theorem 1.4, observe, following Berenstein and Zalcman [**6**, §9], that there is a common zero $\lambda \in \mathbb{C} \setminus \{0\}$ of the functions in I if and only if there exists $\alpha \in \mathbb{C} \setminus \{0\}$ such that the Euclidean Fourier transform $\widehat{T}$ of T, given by

$$\widehat{T}(\xi) = \langle T, z \mapsto e^{-iz\cdot\xi} \rangle,$$

vanishes on the complex circle $M_\alpha = \{ z \in \mathbb{C}^2 : z_1^2 + z_2^2 = \alpha \}$.

Indeed that equivalence comes from the following relationship between the spherical Fourier transform on $\mathbb{C} \cong M(2)/SO(2)$ and the Euclidean Fourier transform:

$$\mathcal{F}(\check{\widetilde{T}} * \widetilde{S})(\xi) = \int_0^{2\pi} \check{T}(-e^{i\theta}\xi) \cdot \widehat{S}(e^{i\theta}\xi)\, d\theta.$$

Now it is not difficult to adapt the argument in [**8**] to show that, under the hypothesis of (i), for every $\alpha \in \mathbb{C} \setminus \{0\}$ there is a curve $z_\alpha(t)$, $t > 0$, in M_α such that $|\widehat{T}(z_\alpha(t))|$ grows exponentially to $+\infty$, as $t \to +\infty$, and that gives the first part of Theorem 1.4.

3.4. In order to prove the second part of Theorem 1.4, assume that (ii) holds but there is some $f \in C(\mathbb{C})$ satisfying (1.2), for every $\sigma \in M(2)$, which is not harmonic. Then the above argument shows there is some $\alpha \neq 0$ such that $\widehat{T}$ vanishes on M_α.

As was remarked by Brown, Schreiber and Taylor (see [**21**, Thm. 1]) using Euclidean Fourier transforms, that fact means that there exists a solution $S \in \mathcal{E}'(\mathbb{C})$ to the partial differential equation

$$\Delta S + \alpha S = -T. \tag{3.3}$$

Since $\Delta + \alpha$ is an elliptic operator and T is supported on $\Gamma \cup \{a\}$, S coincides with a real analytic function on $\mathbb{C} \setminus (\Gamma \cup \{a\})$. Moreover, the fact that S is compactly supported gives by analytic continuation that S vanishes on $\mathbb{C} \setminus (\overline{\Omega} \cup \{a\})$.

Let $\mathcal{N}_\alpha$ be the fundamental solution of $\Delta + \alpha$ given by

$$\mathcal{N}_\alpha(z) = \frac{1}{4} N_0(\sqrt{\alpha}|z|),$$

where $\sqrt{\alpha}$ is either square root of α and N_0 is the Neumann function of order 0.

Then $S = -(\mathcal{N}_\alpha * T)$, because $S \in \mathcal{E}'(\mathbb{C})$. Hence S is given by the locally integrable function

$$u(z) = \int_\Gamma \frac{\mathcal{N}_\alpha(z-\zeta)}{\zeta - a} \frac{d\zeta}{2\pi i} - \operatorname{Ind}(\Gamma, a)\, \mathcal{N}_\alpha(z-a). \tag{3.4}$$

The right hand side of (3.4) is continuous as a function of z on $\mathbb{C} \setminus \{a\}$. Indeed, that can be proved essentially as in [**2**], taking into account that $N_0(z) = A(|z|)\log|z| + B(|z|)$, where A and B are entire functions. In particular, u is a continuous function on a neiborhood of Γ which vanishes on $\mathbb{C} \setminus \overline{\Omega}$, so $u = 0$ on Γ. Furthermore, (3.3) means that u satisfies the equation $\Delta u + \alpha u = -T$, which gives that $\Delta u + \alpha u = 0$ in $\Omega \setminus \{a\}$, so u is real analytic on $\Omega \setminus \{a\}$. We will obtain more boundary conditions on u using the second Green formula. First observe that, by the regularity theorem [**14**, Thm. 6.19], u is of class $C^{2,\varepsilon}$ on $\overline{\Omega}$ since so is Ω by hypothesis. Now let $\varphi \in \mathcal{D}(\mathbb{C} \setminus \{a\})$. Then

$$\begin{aligned}
-\int_\Gamma \frac{\varphi(z)}{z-a} \frac{dz}{2\pi i} &= -\langle T, \varphi\rangle = \langle \Delta u + \alpha u, \varphi\rangle \\
&= \int_{\Omega_\varepsilon} u(z)(\Delta\varphi(z) + \alpha\varphi(z))\, dm(z) \\
&= \int_{\Omega_\varepsilon} (\Delta u(z) + \alpha u(z))\varphi(z)\, dm(z) \\
&\quad + \int_\Gamma \left(u\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial u}{\partial n}\right) ds - \int_{\partial D(a,\varepsilon)} \left(u\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial u}{\partial n}\right) ds \\
&= -\int_\Gamma \varphi\frac{\partial u}{\partial n}\, ds,
\end{aligned}$$

where n is the outward normal to $\Omega_\varepsilon = \Omega \setminus \overline{D(a,\varepsilon)}$, and $\varepsilon > 0$ is small enough. Therefore, if $\Gamma(s)$ is the arc length parametrization of Γ, the above identity gives that

$$\frac{\partial u}{\partial n}(\Gamma(s)) = \frac{-i \cdot \Gamma'(s)}{2\pi(\Gamma(s) - a)}. \tag{3.5}$$

Now let $u = v + iw$, $a = a_1 + ia_2$ and $\Gamma(s) = x_1(s) + ix_2(s)$. Then $n = x_2' - ix_1'$ and therefore, since v and w vanish on Γ, (3.5) is equivalent to

$$\begin{aligned}
\frac{\partial v}{\partial x_1} &= \frac{x_2'(x_1 - a_1) - x_1'(x_2 - a_2)}{(x_1 - a_1)^2 + (x_2 - a_2)^2}\frac{x_2'}{2\pi};\\
\frac{\partial v}{\partial x_2} &= \frac{x_2'(x_1 - a_1) - x_1'(x_2 - a_2)}{(x_1 - a_1)^2 + (x_2 - a_2)^2}\frac{(-x_1')}{2\pi}\\
\frac{\partial w}{\partial x_1} &= -\frac{x_1'(x_1 - a_1) + x_2'(x_2 - a_2)}{(x_1 - a_1)^2 + (x_2 - a_2)^2}\frac{x_2'}{2\pi};\\
\frac{\partial w}{\partial x_2} &= -\frac{x_1'(x_1 - a_1) + x_2'(x_2 - a_2)}{(x_1 - a_1)^2 + (x_2 - a_2)^2}\frac{(-x_1')}{2\pi}.
\end{aligned}$$

Since $(x_1'(s))^2 + (x_2'(s))^2 = 1$, we easily deduce that

$$\begin{aligned}
\frac{\partial v}{\partial x_1} + \frac{\partial w}{\partial x_2} &= \frac{1}{2\pi}\frac{x_1 - a_1}{(x_1 - a_1)^2 + (x_2 - a_2)^2} && \text{on } \Gamma.\\
\frac{\partial v}{\partial x_2} - \frac{\partial w}{\partial x_1} &= \frac{1}{2\pi}\frac{x_2 - a_2}{(x_1 - a_1)^2 + (x_2 - a_2)^2} && \text{on } \Gamma.
\end{aligned}$$

Hence v and w are solutions to the following boundary value problem:

$$\begin{cases}
\Delta v + \alpha_1 v - \alpha_2 w = 0 & \text{in } \Omega \setminus \{a\}.\\
\Delta w + \alpha_2 v + \alpha_1 w = 0 & \text{in } \Omega \setminus \{a\}.\\
v = w = 0 & \text{on } \Gamma\\
\frac{\partial v}{\partial x_1} + \frac{\partial w}{\partial x_2} = g & \text{on } \Gamma\\
\frac{\partial v}{\partial x_2} - \frac{\partial w}{\partial x_1} = h & \text{on } \Gamma,
\end{cases} \tag{3.6}$$

where

$$g(z) = \frac{1}{2\pi}\operatorname{Re}\left(\frac{1}{z-a}\right) \quad \text{and} \quad h(z) = \frac{1}{2\pi}\operatorname{Im}\left(\frac{1}{z-a}\right).$$

Finally, applying Lemma 2.3 to (3.6) we conclude that Γ is real-analytic which is a contradiction. Hence part (ii) of Theorem 1.4 is proved.

3.5. The proofs of the remaining parts of Theorem 1.4 are just a consequence of the methods used by Williams [**21**, **22**] in order to deal with the Pompeiu problem.

First observe that we may assume, by regularization, that our function f is C^∞. Then (1.2) can be written by Stokes' theorem as follows:

$$\int_\Omega \frac{\frac{\partial f}{\partial \bar z} \circ \sigma}{z-a}\, dm(z) = 0, \qquad \text{for every } \sigma \in M(2). \tag{3.7}$$

(Here it is essential that the derivative of an holomorphic Euclidean motion is constant.)

Thus it is clear that (3.7) can be written as the convolution equation

$$\widetilde{\left(\frac{\partial f}{\partial \bar z}\right)} * \left(\frac{\chi_\Omega(z)}{z-a}\right)^{\tilde{}} = 0. \tag{3.8}$$

Using the above arguments (see **2.2**, **2.3** and **3.3**) it is not difficult to see that (3.8) implies f is entire if and only if the Euclidean Fourier transform F of the function $\chi_\Omega(z)/(z-a)$ does not vanish at the origin and also it is not identically zero on any M_α, $\alpha \neq 0$.

Since a is an admissible point for Γ, $F(0) \neq 0$. Hence, if there exists some non entire function $f \in C^\infty(\mathbb{C})$ satisfying (1.2), there is $\alpha \neq 0$ such that $F \equiv 0$ on M_α. Then it follows that the equation

$$\Delta S + \alpha S = -\frac{1}{z-a} \tag{3.9}$$

has a solution $S \in \mathcal{E}'(\mathbb{C})$.

Now the proof of [**21**, Thm. 3] under the hypothesis of (iii) works. Just observe that the problem is localized near Γ and the main requirement on the data of the partial differential equation (3.9) is that it does not vanish on Γ. Moreover the Cauchy-Kowalewsky and Holmgrem's uniqueness theorems are also true for non-characteristic systems of elliptic equations. Thus the third part of Theorem 1.4 is proved. Finally, the hypothesis of (iv) and the second Green formula gives that α is real, and it is not difficult to adapt the arguments in [**22**] (who used the deep regularity methods of Caffarelli [**9**]) to obtain the desired conclusion, which completes the proof of Theorem 1.4.

Remark. Note that in part (ii) of Theorem 1.4 we could have considered instead of (1.2) the equivalent Pompeiu-type condition (3.7). Proceeding similarly with the proof of **3.4** we would have obtained a different kind of boundary value problem. Then in order to obtain the real-analyticity of its solutions using the method of the proof of Lemma 2.3, we would have to use a first hodograph transformation. But the lowest regularity required for Ω to apply [**18**, Thm. VI.3.3] would then have been $C^{4,\varepsilon}$.

4. The unit disk and upper half-plane cases.

4.1. In this subsection we will prove Theorem 1.1 when X is either $\mathbb{D}$ or $\mathbb{U}$. Let $\Gamma \subset\subset X$ be the circle $\partial D(c,r)$. As we saw in Section 2 condition (1.2) is equivalent to (2.3), so we are dealing with the circle $\Gamma_0 = \partial D(c_0, r_0)$.

For computational reasons it is convenient to rewrite (2.3) as an integral over an Euclidean circle centered at the origin. Let $\Delta_{\mathbb{D}}(b,\rho)$ be the closed pseudohyperbolic disk which coincides with the closed Euclidean disk $D = D(c_0, r_0)$ (see (2.5)). Then it is clear that in (2.3) we can replace σ by $\sigma \circ \tau_b$, where τ_b is given by (2.8).

Making the change of variables $z = \tau_b(w)$, we obtain that

$$\int_{\Gamma_0} \frac{(f\circ\sigma\circ\tau_b^{-1})(z)}{z-a_0} H(z)\,\frac{dz}{2\pi i} = \frac{1}{2\pi i}\frac{1-|b|^2}{1-a_0\bar{b}}\int_{\Gamma_1}\frac{(f\circ\sigma)(w)}{w-d}H(\tau_b(w))\frac{dw}{1+\bar{b}w},$$

where $d = \tau_b^{-1}(a_0)$, and $\Gamma_1 = \tau_b^{-1}(\Gamma_0) = \partial\Delta_{\mathbb{D}}(0,\rho) = \partial D(0,\rho)$.

Thus we observe that condition (2.3) can be written as the convolution equation (2.12) in the group $G = \operatorname{Aut}(\mathbb{D})$, where now T is defined by

$$T\varphi = \frac{1}{2\pi i}\frac{1-|b|^2}{1-a_0\bar{b}}\int_{\Gamma_1}\frac{\varphi(z)}{z-d}K(z)\,dw - \operatorname{Ind}(\Gamma_1,d)\cdot\varphi(d) \qquad (\varphi\in C(\mathbb{D})),$$

and

$$K(z) = \frac{H(\tau_b(z))}{1+\bar{b}z} = \frac{H(b)}{1+\overline{w}z}$$

with $w = b$ if $H \equiv 1$, and $w = \frac{\bar{b}-1}{1-b}$ otherwise.

4.2. Recall that the spherical functions on $\mathbb{D} \cong \operatorname{Aut}(\mathbb{D})/SO(2)$ are given by

$$\phi_\lambda(z) = F\left(\frac{1+i\lambda}{2},\frac{1-i\lambda}{2};1;-\frac{|z|^2}{1-|z|^2}\right) \qquad (z\in\mathbb{D}),$$

$F(\alpha,\beta;\gamma;z)$ being the classical hypergeometric function.

As we discuss in Section 2 we have to show that $\pm i$ are the *only* common zeroes of the functions in I and its common multiplicity equals 1. So let us compute the generators of the ideal I in the present case. First observe that

$$(\tilde{S}*\tilde{\phi}_\lambda)_\pi(z) = \begin{cases} \left(\frac{-z}{1-|z|^2}\right)^n F_{n,z}(\lambda) \text{ if } S = \frac{\partial^n\delta_0}{\partial\bar{z}^n} \\ \left(\frac{-\bar{z}}{1-|z|^2}\right)^n F_{n,z}(\lambda) \text{ if } S = \frac{\partial^n\delta_0}{\partial z^n}, \end{cases}$$

(see [**2**, 3.7]) where

$$F_{n,z}(\lambda) = \frac{\partial^n}{\partial\zeta^n}F\left(\frac{1+i\lambda}{2},\frac{1-i\lambda}{2};1;\zeta\right)\Big|_{\zeta=-|z|^2/(1-|z|^2)}.$$

Then using (2.14) we get that

$$\mathcal{F}\left(\check{\check{T}} * \left(\widetilde{\frac{\partial^n \delta_0}{\partial \bar{z}^n}}\right)\right)(\lambda) = \alpha_n F_{n,\rho}(\lambda) - \beta_n F_{n,d}(\lambda)$$
$$\mathcal{F}\left(\check{\check{T}} * \left(\widetilde{\frac{\partial^n \delta_0}{\partial z^n}}\right)\right)(\lambda) = \gamma_n F_{n,\rho}(\lambda) - \overline{\beta}_n F_{n,d}(\lambda),$$

where

$$\alpha_n = \frac{1-|b|^2}{1-\bar{b}a_0}\left(\int_{|z|=\rho} \frac{(-z)^n}{z-d} K(z)\,\frac{dz}{2\pi i}\right)\frac{1}{(1-\rho^2)^n}$$
$$\beta_n = \mathrm{Ind}(\partial D(0,\rho), d)\left(\frac{-d}{1-|d|^2}\right)^n$$
$$\gamma_n = \frac{1-|b|^2}{1-\bar{b}a_0}\left(\int_{|z|=\rho} \frac{(-\bar{z})^n}{z-d} K(z)\,\frac{dz}{2\pi i}\right)\frac{1}{(1-\rho^2)^n}.$$

Now it is clear that

$$\alpha_n = \frac{1-|b|^2}{1-\bar{b}a_0} K(d)\left(\frac{-d}{1-\rho^2}\right)^n \mathrm{Ind}(\partial D(0,\rho), d) = \left(\frac{-d}{1-\rho^2}\right)^n \mathrm{Ind}(\partial D(0,\rho), d).$$

On the other hand, by a straightforward (and tedious) calculation one obtains that

$$\gamma_n = \begin{cases} \left(\frac{\rho^2 \overline{w}}{1-\rho^2}\right)^n & \text{if } d = 0 \\ \frac{1}{d^n}\left\{1 + \mathrm{Ind}(\partial D(0,\rho), d) - (-\overline{w}d)^n\right\}\left(\frac{-\rho^2}{1-\rho^2}\right)^n & \text{if } d \neq 0. \end{cases}$$

(Here we have to note that $d = 0$ means $b = a_0$.)

Now we have to distinguish three cases:

Case 1: $|d| > \rho$. Then $\alpha_n = \beta_n = 0$, for $n \geq 0$, $\gamma_0 = 0$ and $\gamma_n = 1$, for $n \geq 1$. Therefore the functions $F_{n,\rho}$, $n \geq 1$, generate the closed ideal I.

Case 2: $d = 0$. Then $\alpha_0 = \beta_0 = \gamma_0 = 1$, $\alpha_n = \beta_n = 0$, for $n \geq 1$. Moreover, for $n \geq 1$, $\gamma_n = 0$ if and only if $w = 0$, which only happens when $X = \mathbb{D}$, $\Gamma = \partial D(0, r)$ and $a = 0$. But this case is excluded in the hypothesis. Thus $\gamma_n \neq 0$, for $n \geq 1$. Therefore the functions $F_{0,\rho} - 1$ and $F_{n,\rho}$, $n \geq 1$, are generators of I.

Case 3: $|d| < \rho$. Then

$$\begin{vmatrix} \alpha_n & -\beta_n \\ \gamma_n & -\overline{\beta}_n \end{vmatrix} = \left(\frac{-1}{1-|d|^2}\right)^n \frac{1}{(1-\rho^2)^n}\left((2 - (-\overline{w}d)^n)(-\rho^2)^n - (-|d|^2)^n\right) \neq 0,$$

since $|w| \leq 1$ and $|d| < \rho < 1$. Therefore the functions $F_{n,\rho}$ and $F_{n,d}$, $n \geq 1$, generate I.

In any case the functions $F_{n,\rho}$, $n \geq 1$, are between the considered generators of I. And the argument of [**2**, 3.8] based in the hypergeometric equation shows that those functions only have $\pm i$ as common zeroes with common multiplicity equal to 1. So the proof of Theorem 1.1 is complete.

Remark. If $\Gamma = \partial D(c, r) \subset\subset \mathbb{U}$ and $a = c$ then (1.2) is just (1.3) so following carefully the above proof we obtain the Corollary 1.2.

4.1. The general case for $\mathbb{D}$ and $\mathbb{U}$.

Recall that in Section 2 we have rewritten the problems in $\mathbb{D}$ and $\mathbb{U}$ in a unified way.

Now we are going to rewrite (2.3) in a more convenient way for our purposes. Roughly speaking, we change the role of a by the origin. By making the change of variables $z = \tau_{a_0}(w)$ we obtain

$$\int_{\Gamma_0} \frac{(f \circ \sigma \circ \tau_{-a_0})(z)}{z - a_0} H(z) \frac{dz}{2\pi i} = \int_{\Gamma_1} \frac{(f \circ \sigma)(w)}{w(1 + \overline{a_0} w)} H(\tau_{a_0}(w)) \frac{dw}{2\pi i},$$

where $\Gamma_1 = \tau_{-a_0}(\Gamma_0)$. Note that $0 \notin \Gamma_1$ and, when $H(z) = \frac{1-a_0}{1-z}$,

$$\frac{H(\tau_{a_0}(w))}{1 + \overline{a_0} w} = \frac{1}{1 + \frac{\overline{a_0} - 1}{1 - a_0} w}.$$

Thus (2.3) is equivalent to

$$\int_{\Gamma_1} \frac{(f \circ \sigma)(z)}{z(1 + \bar{b} z)} \frac{dz}{2\pi i} = \operatorname{Ind}(\Gamma_1, 0) f(\sigma(0)) \qquad \text{for every } \sigma \in \operatorname{Aut}(\mathbb{D}),$$

where b is either a_0 ($X = \mathbb{D}$) or $(\overline{a}_0 - 1)/(1 - a_0)$ ($X = \mathbb{U}$). Note that in any case $|b| \leq 1$.

Consider the compactly supported Radon measure $T = T_{\Gamma_1}$ given by

$$T\varphi = \int_{\Gamma_1} \frac{\varphi(z)}{z(1 + \bar{b} z)} \frac{dz}{2\pi i} - \operatorname{Ind}(\Gamma_1, 0)\varphi(0) \qquad (\varphi \in \mathbb{C}(\mathbb{D})).$$

Then (2.3) can be rewritten as the convolution equation (2.12) in the group $\operatorname{Aut}(\mathbb{D})$.

Let $\mathcal{J}$ be the closed ideal in $\mathcal{E}'(\operatorname{Aut}(\mathbb{D}))$ generated by the distributions $\check{\tilde{T}} * \tilde{S}$, $S \in \mathcal{E}'(\mathbb{D})$, and let I be the image under the spherical Fourier transform of $\mathcal{J}$.

Now assume there is a non harmonic function f on $\mathbb{D}$ satisfying the above convolution equation. Then the argument given in Section 2 shows that there are two posibilities:

(a) $\pm i$ are common zeros of the functions in I with common multiplicity bigger than 1.

(b) There exists a common zero $\lambda_0 \neq \pm i$ of the functions in I.

Our next goal will be to prove that (a) cannot happen and furthermore, that the λ_0 of (b) is different from $i(1+2k)$, for any $k \in \mathbb{Z}^+$, i.e. it is a non-simple point (see [**16**, p. 46, Prop. 4.8]). That will be done by showing that the function $h = \mathcal{F}(\check{\tilde{T}} * \tilde{\delta}_0) \in I$ does not vanish at $\lambda = i(1+2k)$, for any $k \in \mathbb{Z}^+$, and has a zero of multiplicity equal to 1 at $\lambda = i$.

By Stokes' theorem we have that

$$T\varphi = 2i \int_{\Omega_1} \frac{\frac{\partial \varphi}{\partial \bar{z}}(z)}{z(1+\bar{b}z)}\, dm(z) \qquad (\varphi \in \mathcal{E}(\mathbb{D})),$$

where Ω_1 is the interior of Γ_1. So $h(\lambda) = -(i/2)(1+\lambda^2)\, h_0(\lambda)$, where

$$h_0(\lambda) = \int_{\Omega_1} \frac{F_0\left(\lambda, -\frac{|z|^2}{1-|z|^2}\right)}{(1+\bar{b}z)(1-|z|^2)^2}\, dm(z),$$

with $F_0(\lambda, x) = F((3+i\lambda)/2, (3-i\lambda)/2; 2; x)$. And we have to show that $h_0(i(1+2k)) \neq 0$, for every $k \in \mathbb{Z}^+$.

Since $(1+\bar{b}z)(1-|z|^2)^2$ has positive real part for $z \in \mathbb{D}$, the above fact will follow as soon as we prove that $F_0(i(1+2k), x) > 0$, for every $x < 0$ and $k \in \mathbb{Z}^+$. Indeed, $F_0(i, x) = 1/(1-x) > 0$, for $x < 0$, and

$$F_0(i(1+2k), x) = \sum_{n=0}^{k-1} \binom{k-1}{n} \binom{k+n+1}{n} \frac{(-x)^n}{n+1} > 0,$$

for every $x < 0$ and $k \geq 1$.

The fact that the common zero λ_0 of I is a non-simple point implies that the Helgason Fourier Transform of T vanishes at $(\pm\lambda_0, \zeta)$, for every $\zeta \in \partial\mathbb{D}$. Then using a division theorem due to Helgason [**17**, Thm. 8.5] one obtains that there exists $S \in \mathcal{E}'(\mathbb{D})$ such that $\Delta_h S - \alpha S = T$, where $\alpha = -(1+\lambda_0^2)$ *(see [**2**, 4.5, 4.6]). Now consider the fundamental solution N_α of $\Delta_h - \alpha$ in* $\mathbb{D}$ given by

$$N_\alpha(z) = -\frac{1}{2\pi} Q_\nu(\cosh(2r)) = -\frac{1}{2\pi} Q_\nu\left(\frac{1+|z|^2}{1-|z|^2}\right),$$

where $\alpha = \nu(\nu+1)$ and Q_ν is the Legendre function of degree ν of the second kind.

Then we can write S as $(\widetilde{T} * \widetilde{N}_\alpha)_\pi$. And it follows that S coincides with the locally integrable function

$$u(z) = \int_{\Gamma_1} \frac{G_\alpha(z,w)}{w(1+\bar{b}w)} \frac{dw}{2\pi i} - \operatorname{Ind}(\Gamma_1, 0)\, N_\alpha(z), \tag{4.1}$$

where

$$G_\alpha(z,w) = N_\alpha\left(\frac{z-w}{1-\overline{w}z}\right)$$

is the Green function of $\Delta_h - \alpha$. Now note that the right hand side of (4.1) defines a continuous function on $\mathbb{D}\setminus\{0\}$. An argument like the one given in the planar case shows that

$$\Delta_h u - \alpha u = 0 \qquad \text{in } \Omega_1. \tag{4.2}$$

$$u = 0 \ \text{ on } \Gamma_1 \text{ and } \frac{\partial u}{\partial n}(\Gamma_1(s)) = \frac{-i\,\Gamma_1'(s)}{2\pi\,\Gamma_1(s)(1+\bar{b}\Gamma_1(s))} \tag{4.3}$$

where s denotes arc length. Moreover, a general regularity theorem [**14**, Thm. 6.19] shows that (4.2) u is of class $C^{2,\varepsilon}$ on $\overline{\Omega_1}$, since so is Ω_1. From (4.3) we deduce that if $z = x_1 + ix_2$ and $u = v + iw$, then

$$\frac{\partial v}{\partial x_1} + \frac{\partial w}{\partial x_2} = g, \tag{4.4}$$

$$\frac{\partial v}{\partial x_2} - \frac{\partial w}{\partial x_1} = h, \tag{4.5}$$

where

$$g(z) = \frac{1}{2\pi}\operatorname{Re}\left(\frac{1}{z(1+\bar{b}z)}\right) \quad \text{and} \quad h(z) = \frac{1}{2\pi}\operatorname{Im}\left(\frac{1}{z(1+\bar{b}z)}\right).$$

Note that (4.2) is equivalent to

$$\Delta_h v - \alpha_1 v + \alpha_2 w = 0 \qquad \text{in } \Omega_1. \tag{4.6}$$

$$\Delta_h w - \alpha_2 v - \alpha_1 w = 0 \qquad \text{in } \Omega_1. \tag{4.7}$$

Finally, (4.6), (4.7), (4.5) and (4.5) together with $v = w = 0$ on Γ_1 mean that the real valued $C^{2,\varepsilon}$ functions v and w satisfy (2.15) with

$$c(z) = \frac{-\alpha_1}{(1-|z|^2)^2} \quad \text{and} \quad d(z) = \frac{-\alpha_2}{(1-|z|^2)^2}.$$

Since $g^2(z) + h^2(z) > 0$ on Γ_1, by lemma 2.3, we deduce that Γ_1 is real-analytic, so Γ is also real-analytic, and the proof of Theorem 1.3 is complete. □

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References

[1] C.A. Berenstein, *An inverse spectral theorem and its relation to the Pompeiu problem*, J. Analyse Math., **37** (1980), 128–144.

[2] C.A. Berenstein and D. Pascuas, *Morera and mean-value type theorems in the hyperbolic disk*, Israel J. Math., **4** (1993), 61–106.

[3] C.A. Berenstein and M. Shahshahani, *Harmonic analysis and the Pompeiu problem*, Amer. J. Math., **105** (1983), 1217–1229.

[4] C. A. Berenstein and P. C. Yang, *An inverse Neumann problem*, J. Reine Angew. Math., **382** (1987), 1–21.

[5] C.A. Berenstein and L. Zalcman, *Pompeiu's problem on spaces of constant curvature*, J. Analyse Math., **30** (1976), 113–130.

[6] C.A. Berenstein and L. Zalcman, *Pompeiu's problem on symmetric spaces*, Comment. Math. Helvetici, **55** (1980), 593–621.

[7] L. Brown, J-P. Kahane, *A note on the Pompeiu problem for convex domains*, Math. Ann., **259** (1982), 107–110.

[8] L.Brown, B. Schreiber and B. A. Taylor, *Spectral synthesis and the Pompeiu poblem*, Ann. Inst. Fourier, **23** (1973), 125–154.

[9] L. A. Caffarelli, *The regularity of free boundaries in higher dimensions*, Acta Math., **139** (1977), 155–184.

[10] P. Ebenfelt, *Some results on the Pompeiu problem*, preprint (1992).

[11] J.B. Garnett, Bounded analytic functions, Pure and Applied Mathematics 96, Academic Press, New York, 1981.

[12] N. Garofalo and F. Segala, *New results on the Pompeiu problem*, Trans. Amer. Math. Soc., **325** (1991), 273-286.

[13] N. Garofalo and F. Segala, *Asymptotic expansions for a class of Fourier integrals and applications to the Pompeiu problem*, J. Analyse Math., **56** (1991), 273–286.

[14] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Grundlehren der mathematischen Wissenschaften 224, Springer-Verlag, Berlin, 1977.

[15] S. Helgason, Differential Geometry, Lie Groups and Symmetric Spaces, Pure and Applied Mathematics 80, Academic Press, San Diego, 1978.

[16] S. Helgason, Groups and Geometric Analysis, Pure and Applied Mathematics 113, Academic Press, Orlando, Florida, 1984.

[17] S. Helgason, *The surjectivity of invariant differential operators I*, Ann. Math., **98** (1973), 451–479.

[18] D. Kinderlehrer and G. Stampacchia, An Introduction to Variational Inequalities and Their Applications, Pure and Applied Mathematics 88, Academic Press, New York, 1980.

[19] N. S. Landkof, Foundations of Modern Potential Theory, Die Grundlehren der mathematischen Wissenschaften, Springer-Verlag, Berlin, 1972.

[20] N. N. Lebedev, Special Functions and their Applications, Dover, New York, 1972.

[21] S. A. Williams, *A partial solution of the Pompeiu problem*, Math. Ann., **223** (1976), 183–190.

[22] S. A. Williams, *Analyticity of the boundary for Lipschitz domains without the Pompeiu property*, Indiana Math. J., **30** (1981), 357–369.

[23] L. Zalcman, *Analyticity and the Pompeiu problem*, Arch. Rat. Mech. Anal., **47** (1972), 237–254.

[24] L. Zalcman, *Offbeat integral geometry*, Amer. Math. Month., **87** (1980), 161–175.

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