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M-HYPERBOLIC REAL SUBSETS OF COMPLEX SPACES

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The aim of this paper is to make a first attempt to study real analytic subsets of complex manifolds (or more generally of complex analytic spaces) from the viewpoint of the theory of metric spaces.

1. Introduction.

Our starting point was inspired by the definition of the so-called Kobayashi pseudodistance on complex manifolds We recall briefly that such a pseudodistance is defined on any complex analytic space M using only the space of all holomorphic maps sending the open unit disk Δ in $\mathbb{C}$ in the space M. Moreover the complex space M is said to be "hyperbolic" if such a pseudodistance actually is a real distance, namely it assigns non vanishing values to pair of distinct points of M. In our situation, we introduce a similar pseudodistance $d_{V,M}$ on any subset of V of a complex analytic space M using the space of all holomorphic maps from Δ to M sending the open interval $I =]-1, 1[$ in V, and we introduce the concept of M-hyperbolicity (cf. Section 2).

We are primarily interested in the case when M is a smooth complex manifold and V is a (closed) real analytic smooth submanifold of M, but the definitions work in this more general context as well.

Any holomorphic map between complex manifolds is distance decreasing when the manifolds are endowed with the Kobayashi distances. Our pseudodistances also fulfill this fundamental property. A unexpected phenomenon is that there are some classes of non holomorphic mappings which enjoy this property. A description of such mappings is given in the Section 3 of the paper. As an application, some hyperbolicity criteria are given, and some Liouville type theorems are proved.

We also extend the construction of the Kobayashi-Royden pseudometric when V is a smooth real analytic submanifold of a complex manifold M (Section 4) and we establish some results on the behaviour of a complex Lie group G acting holomorphically on M and leaving V invariant (Section 5). Moreover we define and study the "geodesics" for such a metric Some examples are given (Section 6).

2. Main definitions.

Let us fix some notations. We denote by I the open real interval $]-1,1[$, and by D the open unit disk in $\mathbb{C}$. The Poincaré hyperbolic distance on D will be denoted by ρ.

We denote by $D(R)$, $0 < R \leq +\infty$, the set of complex number z such that $|z| < R$, and also put $I(R) = D(R) \cap \mathbb{R}$.

Let M be a complex analytic (reduced) complex space and let V be a subset of M. By an *M-analytic arc* in V, or simply an analytic arc in V, we mean a holomorphic map $f : D \to M$ such that $f(I) \subset V$. Given two points p and q in V, an *analytic chain* γ in V joining p and q is given by the following data:

(i) points $a_0, \dots, a_k$ in I;

(ii) M-analytic arcs $f_1, \dots, f_k$ in V such that $f_1(a_0) = p$, $f_k(a_k) = q$ and $f_j(a_j) = f_{j+1}(a_j)$ for $j = 1, \dots, k-1$.

The length of the analytic chain γ is by definition the number

$$\rho(\gamma) = \sum_{j=0}^{k-1} \rho(a_j, a_{j+1}).$$

We denote by $C_{p,q}(V, M)$ the set af all the M-analytic chains in V joining p and q.

Using the analytic arcs so defined we introduce a pseudodistance on V by the formula

$$d_{V,M}(p,q) = \inf\{\rho(\gamma) \mid \gamma \in C_{p,q}(V, M)\},$$

where by definition the second member in the definition is $+\infty$ if the set $C_{p,q}(V, M)$ is empty.

Clearly the function $d_{V,M}(p, q)$ so defined is a pseudodistance that vanishes when $p = q$, it is symmetric in p and q, and satisfies the triangle inequality.

We say that V is *hyperbolic* with respect to M, or simply *M-hyperbolic* if $d_{V,M}(p, q) > 0$ whenever $p \neq q$.

On the other hand we say that V is *M-hyperbolically flat*, or simply *M-flat*, if the pseudodistance $d_{V,M}$ vanishes identically.

In this paper we are interest in the case when V is a real analytic subset (even a real analytic submanifold) of M. Nevertheless the definition makes sense with no additional structure on V.

We begin by noting some elementary properties:

(i) If $V = M$, then $d_{V,M}$ is the usual Kobayashi pseudodistance on M;

(ii) If $M = D$ and $V = I$, then the Schwarz Lemma implies that the pseudodistance $d_{V,M}$ is the restriction to I of the Poincaré distance on D;

(iii) If (V_1, M_1) and (V_2, M_2) are pairs of complex spaces as above and $f : M_1 \to M_2$ is a holomorphic map sending V_1 in V_2, then for every p and q in V_1

$$d_{M_2,V_2}(f(p), f(q)) \leq d_{M_1,V_1}(p, q);$$

(iv) If $\delta : V \times V \to [0, +\infty]$ is a pseudodistance such that

$$\delta(f(t), f(t)) \leq \rho(t, s)$$

for all M-analytic arcs f in V then $\delta \leq d_{V,M}$.

(v) If $M = \mathbb{C}$ and $V = \mathbb{R}$ then $d_{V,M}$ vanishes identically, that is, $\mathbb{R}$ is $\mathbb{C}$-flat; indeed, given $y \in \mathbb{R}$, let f be the analytic arc $z \mapsto nyz$, $n \in \mathbb{N}$; then $f(0) = 0$, $f(1/n) = y$ and hence

$$d_{V,M}(0, y) \leq \rho(0, 1/n).$$

Taking the limit for $n \to +\infty$ we obtain $d_{V,M}(0, y) = 0$.

3. Hyperbolicity and "good" mappings.

We say that an arbitrary map $F : M_1 \to M_2$ between complex spaces is *good*, if, for every holomorphic map $f : D(R) \to M_1$, there exists a holomorphic map $\tilde{f} : D(R) \to M_2$ such that $\tilde{f}(t) = F(f(t))$ for every $t \in I(R)$.

The proofs of the following two Propositions are straightforward.

Proposition 3.1. *Let M_1 and M_2 be complex spaces, V_1 and V_2 be subsets of M_1 and M_2 respectively, and let $F : M_1 \to M_2$ be a good map satisfying $F(V_1) \subset V_2$. Then, for every pair of points p and q in V_1,*

$$d_{V_2,M_2}(F(p), F(q)) \leq d_{V_1,M_1}(p, q).$$

Proposition 3.2. *Let M_1, M_2, V_1, V_2 and F as in the previous Proposition.*

(i) *If V_2 is M_2-hyperbolic and $F_{|V_1}$ is injective, then V_1 is M_1-hyperbolic.*

(ii) *If V_1 is M_1-flat and $F(V_1) = V_2$, then V_2 is M_2-flat.*

Every holomorphic map is clearly good. However there also are not holomorphic good maps:

Proposition 3.3. *The map $F : \mathbb{C}^n \to \mathbb{C}^{2n}$ defined by*

$$z = (z_1, \ldots, z_n) \mapsto F(z) = (z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n)$$

is good.

Proof. Let $f : D(R) \to \mathbb{C}^n$ be a holomorphic map. Define $f^* : D(R) \to \mathbb{C}^n$ by the formula

$$f^*(z) = f(\bar{z})), \quad z \in D(R).$$

Clearly f^* is holomorphic and the map $\tilde{f} : D(R) \to \mathbb{C}^{2n}$ given by

$$\tilde{f}(z) = (f_1(z), f_1^*(z), \dots, f_n(z), f_n^*(z)),$$

where $f_\imath$ and f_i^* are the i-th component respectively of f and f^*, satisfies $\tilde{f}(t) = F(f(t))$ for every $t \in I(R)$. $\square$

Since compositions of good maps are good we immediatly obtain

Proposition 3.4. *Let $H : \mathbb{C}^{2n} \to M$ be a holomorphic (or simply a good) map. Then the maps $F, G : \mathbb{C}^n \to M$ defined by*

$$F(z_1, \dots, z_n) = H(z_1, \bar{z}_1, \dots, z_n, \bar{z}_n),$$
$$G(x_1 + iy_1, \dots, x_n + iy_n) = H(x_1, y_1, \dots, x_n, y_n),$$

are good.

For the projective space we have:

Proposition 3.5. *The map $F : \mathbb{CP}^n \to \mathbb{CP}^\nu$, $\nu = (n+1)^2 - 1$, defined by*

$$w_{ij} = z_i \bar{z}_j, \quad i, j = 0, \dots, n \tag{3.1}$$

is good.

Proof. The assertion follows from the Propositions 3.3 and 3.4, and the fact that for every k there is a one to one correspondence between holomorphic maps $f : D \to \mathbb{CP}^k$ and holomorphic maps $g = (g_0, \dots, g_k) : D(R) \to \mathbb{C}^{k+1}$ satisfying $g_i \neq 0$ for some $i = 0, \dots, k$. $\square$

In order to find hyperbolic spaces the following (almost trivial) remark is useful.

Proposition 3.6. *Let M be a complex space and let V be a subset of M. If N is a closed complex subspace of M containing V, then*

$$d_{V,M} = d_{V,N}.$$

In particular, if N is hyperbolic (as complex space), then V is M-hyperbolic.

Proof. It suffices to show that if $f : D \to M$ is a holomorphic arc in V, then $f(D) \subset N$, that is $f^{-1}(N) = D$. But this is obvious, since $f^{-1}(N)$ is a closed complex subspace of D containing I, and any such a subspace must coincide with D. $\square$

We now give some example of flat spaces.

Proposition 3.7. *Any interval of the real line is flat.*

Proof. It suffices to prove the assertion for the interval $J = [0,1]$. Indeed, the map $z \mapsto \exp(-z^2)$ shows that $d_{J,\mathbb{C}}(1,t) = 0$ for every $t \in]0,1]$. Analogously, the map $z \mapsto 1 - \exp(-z^2)$ yields $d_{J,\mathbb{C}}(t,0) = 0$ for every $t \in [0,1[$. Finally, one has $d_{J,\mathbb{C}}(1,0) \leq d_{J,\mathbb{C}}(1, 1/2) + d_{J,\mathbb{C}}(1/2, 0) = 0$. □

As consequence of this Proposition we obtain the following Liouville type Theorem.

Theorem 3.1. *Let V be a subset of a complex space M. If V is M-hyperbolic then every holomorphic map $f : \mathbb{C} \to M$ sending some non-trivial real interval $J \subset \mathbb{R}$ in V is a constant map.*

Proof. Since V is M-hyperbolic and J is $\mathbb{C}$-flat the map f must be constant on J and hence it is constant on all $\mathbb{C}$. □

Other examples of flat space are given in the following three Propositions.

Proposition 3.8. *Any connected subset of a non-singular real conic in $\mathbb{C} = \mathbb{R}^2$ is flat.*

Proof. Since real affine self map of $\mathbb{C}$ are good, any conic is isometric either to the unit circle $x^2 + y^2 = 1$, or to the equilateral hyperbola $xy = 1$, or to the parabola $y = x^2$. Any connected subset of such a conic is the image of an interval of some real line in $\mathbb{C}$ under the maps $z \mapsto \cos(z) + i\sin(z)$, $z \mapsto \exp(z) + i\exp(-z)$, and $z \mapsto z + iz^2$ respectively. □

Proposition 3.9. *The boundary S of the unit ball in $\mathbb{C}^n$ (with respect to the standard euclidean norm) is flat.*

Proof. Given two arbitrary distinct points p and q in S, the complex line L joining p and q intersect S along a circumference, that is, a conic in L, and hence, by the previous Proposition, one has

$$d_{S,\mathbb{C}^n}(p,q) \leq d_{S\cap L,L}(p,q) = 0,$$

and the assertion follows. □

Proposition 3.10. *Every real ellipsoid in $\mathbb{C}^n$ is flat.*

Proof. Indeed the unit ball of $\mathbb{C}^n$ can be mapped onto any real ellipsoid under a suitable real linear map of $\mathbb{C}^n$, and any such map is good. □

And now here are some examples of hyperbolic sets. The following Proposition is immediate consequence of Propositions 3.4 and 3.6.

Proposition 3.11. *Let $V \subset \mathbb{C}^n = \mathbb{R}^{2n}$ be a subset defined by k equations*

$$f_i(x_1, y_1, \dots, x_n, y_n) = 0, \quad i = 1, \dots, k, \tag{3.2}$$

where x_i and y_i are the standard real coordinates in $\mathbb{C}^n$, and $f_1, \dots, f_k$ are real analytic functions defined by real power series converging over all $\mathbb{R}^{2n}$. Let $V_{\mathbb{C}}$ be the subset of $\mathbb{C}^{2n}$ defined by the same set of equations 3.2, where now x_i and y_i represent the complex coordinates of $\mathbb{C}^{2n}$. Under these hypotheses, if $V_{\mathbb{C}}$ is hyperbolic (as complex space) then V is $\mathbb{C}^n$-hyperbolic.

Example. Let $z = x + iy$ be the standard coordinate in $\mathbb{C}$. Let $V \subset \mathbb{C}$ the graph of the real function $y = \log(1 + x^2)$. Then V is $\mathbb{C}$-hyperbolic. Indeed according to the previous Proposition it suffices to prove that the complex curve

$$V_{\mathbb{C}} = \{(z, w) \in \mathbb{C}^2 \mid \exp(w) = 1 + z^2\}$$

is hyperbolic. Clearly $V_{\mathbb{C}}$ is regular everywhere, that is it is a closed Riemann surface in $\mathbb{C}^2$. Let denote by $g : V_{\mathbb{C}} \to \mathbb{C}$ the restriction to $V_{\mathbb{C}}$ of the projection map $(z, w) \mapsto z$. The map g is a non constant holomorphic map on $V_{\mathbb{C}}$. Since the exponential function never vanishes, then the map g necessarily omits the values i and $-i$, the zeroes of the function $1 + z^2$. The little Picard Theorem therefore implies that the universal covering of $V_{\mathbb{C}}$ can not be the complex plane, and hence $V_{\mathbb{C}}$ is covered by the unit disc D, that is, $V_{\mathbb{C}}$ must be hyperbolic, as asserted.

The following assertion gives a criterion for $\mathbb{CP}^1$-hyperbolicity.

Proposition 3.12. *Let $V \subset \mathbb{C} \subset \mathbb{CP}^1$ be a subset defined by an equation*

$$f(x, y) = 0, \quad z = x + iy \in \mathbb{C}, \tag{3.3}$$

where $f(x, y)$ is a polynomial in the variables x and y of degree d. Let $\bar{V}$ be the (topological) closure of V in $\mathbb{CP}^1$. Let $V_{\mathbb{C}}$ be the complex curve in $\mathbb{C}^2$ of equation 3.3, where now x and y are considered as complex coordinates in $\mathbb{C}^2$, and finally let $\bar{V}_{\mathbb{C}}$ be the closure of $V_{\mathbb{C}}$ in $\mathbb{CP}^2$. If $\bar{V}_{\mathbb{C}}$ is hyperbolic then $\bar{V}$ (and hence V also) is $\mathbb{CP}^1$-hyperbolic.

Proof. Let z_0 and z_1 be homogeneous coordinates in $\mathbb{CP}^1$, that is, $z = x + iy = z_1/z_0$.

Let $g \in \mathbb{C}[X_0, X_1, X_2]$ be the homogeneous polynomial defined by the equation

$$g(X_0, X_1, X_2) = X_0^d f\left(\frac{1}{2X_0}(X_1 + X_2), \frac{1}{2iX_0}(X_1 - X_2)\right).$$

Choosing X_0, X_1, X_2 and X_3 as homogeneous coordinates in $\mathbb{CP}^3$, let W be the quasiprojective algebraic subset of $\mathbb{CP}^3$ defined by

$$\begin{cases} g(X_0, X_1, X_2) = 0 \\ X_0 X_3 - X_1 X_2 = 0 \\ X_0 \neq 0 \end{cases} ,$$

and let $\bar{W}$ be the closure in $\mathbb{CP}^3$ of W.

Consider now the map $F : \mathbb{CP}^1 \to \mathbb{CP}^3$ defined by

$$\begin{cases} X_0 = z_0 \bar{z}_0 \\ X_1 = \bar{z}_0 z_1 \\ X_2 = z_0 \bar{z}_1 \\ X_3 = z_1 \bar{z}_1 \end{cases}$$

Such a map is injective, and Proposition 3.5 says that the map F so defined is a good map. By construction one clearly has $F(V) \subset W$ and hence $F(\bar{V}) \subset \bar{W}$. By Proposition 3.2, in order to check the $\mathbb{CP}^1$-hyperbolicity of $\bar{V}$, it suffices to prove that the curve $\bar{W}$ is hyperbolic.

It is easy to show that W and $V_{\mathbb{C}}$ are isomorphic (as affine algebraic varieties), and therefore $\bar{W}$ and $\bar{V}_{\mathbb{C}}$ are birationally equivalent as projective algebraic curves. Since hyperbolicity is preserved under birational isomorphisms between (compact algebraic) curves, it follows from our hypotheses that $\bar{W}$ is hyperbolic, as asserted. □

For algebraic varieties of higher dimension hyperbolicity is no longer a birational invariant. So the previous argument does not apply to higher dimensional projective spaces. Nevertheless the following Liouville type Theorem for meromorphic mappings holds:

Proposition 3.13. *Let $V \subset \mathbb{C}^n$ be a subset defined by k equations as in the Proposition* 3.11. *Let $V_{\mathbb{C}}$ be defined as in Proposition* 3.11, *and let $\bar{V}_{\mathbb{C}}$ be the closure in $\mathbb{CP}^{2n}$ of $V_{\mathbb{C}}$. Let $f_1, \dots, f_n : \mathbb{C} \to \mathbb{C}$ be meromorphic functions. Assume that*

(i) *there exists a non-degenerate interval $J \subset \mathbb{R}$ such that every f_i has no poles on J and $(f_1(t), \dots, f_n(t)) \in V$ for every $t \in J$;*

(ii) *the complex space $\bar{V}_{\mathbb{C}}$ is hyperbolic.*

Then every f_i is a constant function.

Proof. Let P be the set of all the poles of the functions f_i. The set P is discrete and closed in $\mathbb{C}$. It is easy to check that, for every $i = 1, \dots, n$, the function $f_i^*(z) = f(\bar{z}))$ is an entire meromorphic function, and the mapping $F : \mathbb{C} \setminus P \to \mathbb{C}^{2n}$ defined by

$$F(z) = \Big(\frac{1}{2}(f_1(z) + f_1^*(z)), \frac{1}{2i}(f_1(z) - f_1^*(z)), \dots ,\\ \frac{1}{2}(f_n(z) + f_n^*(z)), \frac{1}{2i}(f_n(z) - f_n^*(z))\Big)$$

is a holomorphic map sending the real interval J in $V_{\mathbb{C}}$. Since $V_{\mathbb{C}}$ is closed in $\mathbb{CP}^{2n}$, it follows that $F(\mathbb{C} \setminus P) \subset V_{\mathbb{C}}$. Moreover, by hypothesis, $\bar{V}_{\mathbb{C}}$ is a compact hyperbolic complex space. Thus the map F extends throughout all $\mathbb{C}$ (cf. Corollary 3.2. of Chapter VI of [**4**]). Again by the hyperbolicity of $V_{\mathbb{C}}$, the map F must be constant, and this yields our assertion. □

The following Proposition follows immediatly from [**8**, Theorem 3].

Proposition 3.14. *Let M be a complex manifold and let V be a subset of M. Assume that there exists a bounded plurisubharmonic function $u : M \to \mathbb{R}$ of class C^2. If u is strictly plurisubharmonic at every point of V, then V is M-hyperbolic.*

4. Real analytic submanifolds.

In this section we assume that M is a (connected) complex manifold and $V \subset M$ is a (connected) closed real analytic submanifold of M.

Proposition 4.1. *Let $p_0 \in V \subset M$ be a point and let (U, x) be a local real coordinate system on V around p_0. Then there exists a neighbourhood $U' \subset U$ of p_0 and a positive finite constant C such that for every p and q in V one has*

$$d_{V,M}(p, q) \leq C \, \|x(p) - x(q)\| .$$

In particular the function $d_{V,M}$ is continuous in $V \times V$.

Proof. Let m be the real dimension of V. Thus the map x is a real analytic diffeomorphism of U onto $x(U) \subset \mathbb{R}^m$. Put $x_0 = x(p_0)$. Since the map $x^{-1} : x(U) \to V$ is real analytic, there exists a neighbourhood $U' \subset U$ of p_0 and a small ball $B \subset \mathbb{C}^m$ centered at x_0 and a holomorphic map $F : B \to M$ such that $x(U') \subset\subset B$, $F(B \cap \mathbb{R}^m) \subset U \subset V$, and $F(x(p)) = p$ for every $p \in U'$. It follows that if p and q are arbitrarily chosen points of U' then

$$d_{V,M}(p, q) = d_{V,M}(F(x(p)), F(x(q))) \leq d_{B\cap\mathbb{R}^m,B}(x(p), x(q)). \tag{4.1}$$

Since $x(U') \subset\subset B$ it easy to prove, using images under complex affine mappings of the unit disc D, that there exists a constant C such that for every pair of points y' and y'' in $x(U')$ one has

$$d_{B\cap\mathbb{R}^m,B}(y',y'') \leq C\,\|y'-y''\|\,. \tag{4.2}$$

Combining 4.1 and 4.2 our assertion follows. □

The following assertion is an immediate consequence of this proposition.

Proposition 4.2. *If V is M-hyperbolic then the distance $d_{V,M}$ induces the topology of V.*

Proof. As $d_{V,M}$ is continuous we only have to prove that for every $p_0 \in V$ the open balls $B(r) = \{p \in V \mid d_{V,M}(p,q) < r\}$ form a fundamental system of neighbourhoods of p_0.

Let U be an arbitrary neighbourhood of p_0. We need to prove that there exists a ball $B(\varepsilon)$ contained in U for some $\varepsilon > 0$. Pick a connected neigbourhood U' of x_0 contained in U with compact boundary $S = \partial U'$. Every analytic chain in V connecting p_0 and an arbitrary point q in $V \setminus U'$ must intersect the boundary S of U' and therefore one has

$$\inf_{p\in V\setminus U} d_{V,M}(p_0,p) \geq \inf_{p\in V\setminus U'} d_{V,M}(p_0,p) \geq \inf_{p\in S} d_{V,M}(p_0,p) = \varepsilon > 0,$$

where the last inequality follows from the M-hyperbolicity of V, the continuity of $d_{V,M}$ and the compactess of S. But this implies that $B(\varepsilon) \subset U' \subset U$, as asserted. □

We now introduce a pseudometric on $V \subset M$ which generalizes the contruction of the Kobayashi-Royden pseudometric on complex manifolds, and then we will prove that its integrated form is the pseudodistance $d_{V,M}$.

Let us fix some notation. For every $p \in V$ we identify the real tangent space of M at p with the holomorphic tangent space of M at p, so that the (real) tangent space T_pV of V at p will be identified with a subspace of the holomorphic tangent space $T_p^{\mathbb{C}}M$ of M at p. For later use we denote by $\mathbb{C}T_pV$ the smallest complex vector subpace of $T_p^{\mathbb{C}}M$ containing T_pV.

If $f : D \to M$ is a holomorphic map sending I in V, for every $t \in I \subset D$ we then denote by $f'(t)$ either the image of the (real) tangent vector $\partial/\partial t$ under the differential of $f_{|I}$ at t, or the image of the holomorphic tangent vector $\partial/\partial z$ under the (holomorphic) differential of f at t.

With this notation, for every $p \in V$ and every $\xi \in T_pV$ we define $[F_{V,M}](p,\xi)$ as the infimum of the positive real numbers $a > 0$ for which there exists an M-analytic arc f in V such that $f(0) = p$ and $f'(0) = a^{-1}\xi$.

It is easy to check that all properties $(i), \ldots, (v)$ of the Section 2 stated for the pseudodistance $d_{V,M}$, with the necessary modifications hold for the pseudometric $[F_{V,M}]$. Moreover one sees that this pseudometric decreases under differentiable good mappings, and that the analogous estimate to that in Proposition 4.1 can also be given for this pseudodistance.

Up to now very little can be said about the regularity of $[F_{V,M}]$. Denoting the (real) tangent bundle of V by TV with its usual topological structure, the best result we can prove is the following:

Proposition 4.3. *The pseudometric* $[F_{V,M}] : TV \to [0, +\infty[$ *is a Borel function.*

Proof. Denote $[F_{V,M}]$ simply by F. We will prove our assertion finding a decreasing sequence of lower semicontinuous pseudometrics $F_n : TV \to [0, +\infty[$ such that for every $p \in V$ and $\xi \in T_pV$ one has

$$F(p, \xi) = \inf_n F_n(p, \xi). \tag{4.3}$$

Fix a complete hermitian metric h on M and denote by d its associated distance. For every $n \in \mathbb{N}$ let denote by $\mathcal{A}_n$ the class of all analytic arcs f in V satisfying $d(f(z), f(w)) \leq n \|z - w\|$ for every z and w in D. Let F_n be the pseudometric defined as the pseudometric F but using analytic arcs in $\mathcal{A}_n$ instead of all analytic arcs in V. As consequence of the Ascoli Theorem, by the completeness of the metric h and the closure of M, it follows that if f_ν is an arbitrary sequence of analytic arcs in $\mathcal{A}_n$ such that the sequence $f_\nu(0)$ converges to some point $p \in V$, then a subsequence of f_ν converges uniformly on all compact subsets of D to an analytic arc $f \in \mathcal{A}_n$ such that $f(0) = p$. Moreover the derivatives at 0 of such a subsequence converge to $f'(0)$. It is then an easy matter to derive the lower semicontinuity of the pseudometric F_n from this fact.

Let now $p \in V$ and $\xi \in T_pV$ be given. Let f be an analytic arc in V such that $f(0) = p$ and $f'(0) = a^{-1}\xi$. For every $\varepsilon > 0$ small put $f_\varepsilon(z) = f((1 - \varepsilon)z)$, $z \in D$. Then $f_\varepsilon \to f$ uniformely on compact subsets of D, and each f_ε belongs to $\mathcal{A}_n$, for some $n = n(\varepsilon)$. All this clearly implies the formula 4.3. The proof is so completed. □

If $\gamma : [0, 1] \to V$ is an absolutely continuous curve, the length of γ (with respect to the pseudometric $[F_{V,M}]$) is the number

$$\int_0^1 [F_{V,M}](\gamma(s), \dot{\gamma}(s)) ds.$$

The integrated form $\tilde{d}_{V,M}(p, q)$ of the pseudometric $[F_{V,M}]$ is the infimum of the lengths of the absolutely continuous curves $\gamma : [0, 1] \to V$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

Proposition 4.4. *The pseudodistance $d_{V,M}$ and the integrated form of the pseudometric $[F_{V,M}]$ coincide.*

Proof. It is a direct consequence of the Theorem 2.1 of [**9**]. $\square$

5. Group actions.

In this section M will stand for a complex manifold, V for a closed real analytic submanifold of M, and G for a complex Lie group of holomorphic transformation of M. We denote by $G(V)$ the subgroup of G of the transformations which leave the submanifold V invariant. Being V closed in M, then $G(V)$ is a closed subgroup of G, and therefore is a (real) Lie group. We also denote by $\mathbf{g}$ and $\mathbf{g}(V)$ the Lie algebras respectively of G and of $G(V)$, and by J the complex structure of $\mathbf{g}$.

Theorem 5.1. *If $G(V)$ acts transitively on V, then V is M-flat.*

Proof. Let $p \in V$. Then there is a neighbourhood U of p in V such that every $q \in V$ belongs to a real one parameter subgroup $t \mapsto \exp(tX)$, for some $X \in \mathbf{g}(V)$, which extends holomorphically to a entire holomorphic map by $\mathbb{C} \ni z \mapsto f(z) = \exp(zX)$. Clearly $f(\mathbb{R}) \subset V$, and therefore $d_{V,M}(p,q) = 0$. The triangle inequality then implies that $d_{V,M}$ vanishes everywhere, that is V is M-flat. $\square$

Theorem 5.2. *If $G(V)$ acts effectively on V and V is M-hyperbolic then $G(V)$ is discrete.*

Proof. It suffices to prove that $\mathbf{g} = 0$. Pick $X \in \mathbf{g}$. Consider the real one-parameter subgroups

$$t \mapsto \exp(tX), \quad t \mapsto \exp(tJX).$$

We have $[X, JX] = 0$ and consequentely these two one-parameter subgroups generate a complex one-parameter subgroup H of G. Thus, taking $\mathbb{C}$, the universal covering of H, we obtain a holomorphic action $\mathbb{C} \times M \to M$ which extends the real action on V given by $(t,p) \mapsto \exp(tX)p$. Then, from the Theorem 3.1 it follows that $\exp(tX)p = p$ for every $t \in \mathbb{R}$, $p \in V$ and this implies $X = 0$, because $G(V)$ by hypothesis acts effectively on V. $\square$

Corollary 5.1. *If V is M-hyperbolic and $\dim_{\mathbb{R}} G(V) > 0$, then G acts trivially on V. In particular, if there is a point $p_0 \in V$ such that $\mathbb{C}T_{p_0}V = T_{p_0}M$, then G acts trivially on M.*

Corollary 5.2. *Let M be compact, V be M-hyperbolic and suppose that there is a point p_0 such that $\mathbb{C}T_{p_0}V = T_{p_0}M$. Denote with* Aut$(M)$ *the group* *of all the holomorphic automorphisms of M. Then the set*

$$\{\sigma \in \operatorname{Aut}(M) \mid \sigma(V) \subset V\}$$

is a discrete subgroup of $\operatorname{Aut}(M)$.

Proof. Indeed $\operatorname{Aut}(M)$ is a complex Lie group wich acts on M effectively. □

Example. Let $M = \mathbb{C}^2$; then $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ acts on $\mathbb{C}^2$ by

$$(z, w) \mapsto (\lambda z, w + (\lambda^2 - 1)z^2).$$

Let

$$V = \{(t, t^2) \mid t \in \mathbb{R}\}, \quad V' = \{(t + it, 2it^2) \mid t \in \mathbb{R}\}.$$

Then $G(V) = G(V') = \mathbb{R}^* = \mathbb{R}\setminus\{0\}$ acts effectively on V and V' respectively. Observe that V and V' in this example are flat.

6. Geodesics.

Let M be complex space and V be a subset of M. We say that an analytic arc $f : \Delta \to M$ such that $f(I) \subset V$ is a *M-geodesic* if it is a local isometry with respect to the distances $d_{I,\Delta}$ and $d_{V,M}$, that is, for every $t_0 \in I$ there exists a open interval $J \subset I$ containing t_0 such that

$$d_{V,M}(f(t), f(s)) = d_{I,\Delta}(t, s)$$

for every t and s in J. With abuse of language we also call *M-geodesic* in V a one dimensional real submanifold of M contained in V which is the image of the interval I under a M-geodesic $f : \Delta \to M$ in V.

Remark. If M is a hyperbolic Riemann surface and $V = M$ then the distance $d_{V,M}$ is the distance associated to a Hermitian metric h_M, and a M-geodesic in V is a holomorphic map $f : \Delta \to M$ such that $f_{|I}$ is a geodesic with respect to the metric h_M.

The following Proposition on geodesics on Riemann surfaces is useful for finding geodesics.

Proposition 6.1. *Let M be an hyperbolic irreducible complex curve, that is an irreducible complex space of (complex) dimension 1, and let M_r be the set of regular points of M. Let $\varphi : M \to M$ be an antiholomorphic map and let X be the set of the fixed points of φ. Then each connected component of X contained in M_r is (the image of) a geodesic of M.*

Proof. Let X_0 be a connected component of X contained in M_r and let $x_0 \in X_0$. Let $\pi : \tilde{M} \to M$ be the normalization of M and let $\tilde{x}_0 \in \tilde{M}$ be the unique point such that $\pi(\tilde{x}_0) = x_0$. Let $f : \Delta \to \tilde{M}$ be a universal covering of $\tilde{M}$ such that $f(0) = \tilde{x}_0$ and let $\sigma : \Delta \to \Delta$ be the unique continuous map such that $\sigma(0) = 0$ and $\pi \circ f \circ \sigma = \varphi \circ \pi \circ f$. Then X_0 is the image under $\pi \circ f$ of the set Z of the fixed point set of σ. But σ is an antiholomorphic automorphism of Δ such that $\sigma(0) = 0$ and hence there exists $\theta \in \mathbb{R}$ such that

$$\sigma(z) = e^{i\theta}\bar{z}.$$

Thus the set Z is the intersection of Δ and a straight (real) line through the origin, and therefore it is a geodesic in Δ (for the Poincaré metric of Δ). Since both the covering map f and the restriction of π to $\pi^{-1}(M_r)$ are (local) isometries for the Kobayashi distance, the set X_0 also is a geodesic in M, as asserted. $\square$

Example. Let $X \subset \mathbb{C}^2$ be the image of the periodic map $f : \mathbb{R} \to \mathbb{C}^2$ defined by

$$f(t) = \left(e^{it}, \frac{e^{-it}}{(e^{it}-2)(2e^{it}-1)}\right).$$

Then X is a $\mathbb{C}^2$ geodesic. Indeed let $M = \mathbb{C} \setminus \{0, 1/2, 2\}$ and let $F : M \to \mathbb{C}^2$ be the map defined by

$$F(z) = \left(z, \frac{1}{z(z-2)(2z-1)}\right).$$

Then F is a holomorphic embedding of M into $\mathbb{C}^2$ and X is the image under F of $S \subset M$, the unit circle in $\mathbb{C}$. Hence it suffices to prove that S is a geodesic in M (for the Kobayashi metric). But this follows immediatly from the previous proposition, observing that S is the fixed point set of the antiholomorphic automorphism $\varphi : M \to M$ defined by

$$\varphi(z) = 1/\bar{z}.$$

Proposition 6.2. *Let $V \subset \mathbb{C}^n = \mathbb{R}^{2n}$ be a subset defined by k real equations as in Proposition* 3.11. *Assume furthermore that V is a real smooth submanifold of (real) dimension one. If V is $\mathbb{C}^n$-hyperbolic then each connected component of V is a $\mathbb{C}^n$-geodesic.*

Proof. Let $F : \mathbb{C}^n \to \mathbb{C}^{2n}$ be defined by

$$z = (x_1 + iy_1, \dots, x_n + iy_n) \mapsto (x_1, y_1, \dots, x_n, y_n).$$

Let $V_{\mathbb{C}} \subset \mathbb{C}^{2n}$ be defined as in the Proposition 3.11. Let $L : \mathbb{C}^{2n} \to \mathbb{C}^n$ be the holomorphic map defined by

$$(x_1, y_1, \dots, x_n, y_n) \mapsto (x_1 + iy_1, \dots, x_n + iy_n).$$

Obviously $L(F(z)) = z$ for every $z \in \mathbb{C}^n$. Thus, given $z, w \in V$, one has

$$\begin{aligned} (6.1) \qquad d_{V,\mathbb{C}^n}(z,w) &\geq d_{F(V),\mathbb{C}^{2n}}(F(z), F(w)) \\ &\geq d_{L(F(V)),\mathbb{C}^n}(L(F(z)), L(F(w))) = d_{V,\mathbb{C}^n}(z,w). \end{aligned}$$

It follows that the map $L : F(V) \to V$ is an isometry with respect to the distances $d_{F(V),\mathbb{C}^{2n}}$ and $d_{V,\mathbb{C}^n}$, and hence in order to prove our assertion it suffices to prove that each connected component of $F(V)$ is a $\mathbb{C}^{2n}$-geodesic.

Let $F(V_0)$ be a connected component of $F(V)$, where V_0 is a connected component of V, and let W be the smallest complex analytic subspace of $\mathbb{C}^{2n}$ containing $F(V_0)$. Since W is closed in $\mathbb{C}^{2n}$ then, by Proposition 3.6, one has

$$d_{F(V_0),\mathbb{C}^{2n}}(F(z), F(w)) = d_{F(V_0),W}(F(z), F(w)).$$

Since V is $\mathbb{C}^n$-hyperbolic, by 6.1 it follows that W is not flat for the Kobayashi metric, and hence, since W is a complex one dimensional curve, it is hyperbolic.

Let $\varphi : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ the map defined by

$$(x_1, y_1, \dots, x_n, y_n) \mapsto (\bar{x}_1, \bar{y}_1, \dots, \bar{x}_n, \bar{y}_n).$$

Since $V_{\mathbb{C}}$ is defined by real equations, the space W is invariant under φ. Clearly the restriction of the map φ to W is an antiholomorphic automorphism of W. We end the proof observing that $F(V_0)$ is a connected component of the fixed point set in W of the map φ and hence the Proposition 6.1 applies. □

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