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RIGIDITY OF ISOTROPIC MAPS

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We consider a rigidity question for isotropic harmonic maps from a compact Riemann surface to a complex projective space. In the case of the projective plane, we prove that ridigity holds if the degree is small in relation to the genus. For a projective space of any dimension we obtain coarser results about rigidity and rigidity up to finitely many choices.

Introduction.

Let $f, g : X \to \mathbb{P}^r$ denote two isotropic harmonic maps from a compact Riemann surface to complex projective space. In this article we study whether from the isometry of f and g one may conclude their unitary equivalence.

Using Calabi's rigidity theorem, this question may be reduced to one in the algebraic category, involving certain curves of osculating spaces to a holomorphic curve. We obtain some rigidity results mostly by analyzing the quadrics containing those curves.

After recalling some definitions and basic facts, we show in §1 that our unitary question may be reduced to a projective one. Then in §2 we record some rigidity statements that follow easily from the use of projective invariants. In §3 and §4 we consider plane curves; we prove in (3.8) and (4.13) that ridigidy holds, roughly speaking, if the degree is small compared to the genus, providing a partial answer to a question posed by Quo-Shin Chi [**C**].

Motivated by the method of proof of Theorem (3.8), in §4 we begin a study of the ideal of associated curves and of the curves $f^{\langle k \rangle}(X)$ introduced in §1. This is related to some aspects of Brill-Noether theory that we plan to pursue in another article.

§1.

(1.1) We consider harmonic maps $X \to \mathbb{P}^r$ from a compact Riemann surface to complex projective space. One way of constructing such harmonic maps is the following: start with a holomorphic non-degenerate $f : X \to \mathbb{P}^r$ and, using the Fubini-Study metric of $\mathbb{P}^r$, construct a Frenet frame $f = f_0, f_1, \ldots, f_r$. The maps $f_i : X \to \mathbb{P}^r$ are harmonic, and, for the purpose of this paper, harmonic maps obtained by this process will be called <u>isotropic</u> maps. We refer to [**EW**] for definitions and details on this construction.

One of the main problems treated in [**EW**] is to classify isotropic maps among all harmonic maps and, in particular, to give conditions that guarantee that a given harmonic map is isotropic. Here we take a different route and, following Chi [**C**], we consider the rigidity question

(1.2) If two isotropic maps $F : X \to \mathbb{P}^r$ and $F' : X \to \mathbb{P}^r$ are isometric, does it follow that F and F' are unitarily equivalent ?

(1.3) In order to phrase (1.2) in more convenient terms, we introduce some more notation and discuss a global way of defining the Frenet curves f_k (see [**EW**]).

Let $\mathcal{F}_k$ $(k = 1, 2, \dots, r)$ denote the flag variety

$$\mathcal{F}_k = \{(A, B)/A \subset B\} \subset \mathrm{Grass}(k-1, \mathbb{P}^r) \times \mathrm{Grass}(k, \mathbb{P}^r).$$

For $(A, B) \in \mathcal{F}_k$ let A' and B' denote the corresponding k and $k+1$ dimensional vector subspaces of $\mathbb{C}^{r+1}$ and $L = B' \cap A'^{\perp}$ the (one dimensional) orthogonal complement of A' in B', with respect to the standard hermitian inner product on $\mathbb{C}^{r+1}$. Then we have a well defined differentiable map

$$\pi_k : \mathcal{F}_k \to \mathbb{P}^r$$

sending (A, B) to L.

(1.4) On the other hand, if $f : X \to \mathbb{P}^r$ is holomorphic, let $f^{(k)} : X \to \mathrm{Grass}(k, \mathbb{P}^r)$ $(k = 0, 1, \dots, r-1)$ denote the k-th associated map [**ACGH**], sending a point $x \in X$ to the osculating k-plane to f at x. Our f induces a holomorphic map $(f^{(k-1)}, f^{(k)}) : X \to \mathcal{F}_k$ and the k-th member f_k of the Frenet frame is obtained as the composition

$$\begin{array}{ccc} X & \xrightarrow{(f^{(k-1)}, f^{(k)})} & \mathcal{F}_k \\ & & \Big\downarrow{\scriptstyle \pi_k} \\ & & \mathbb{P}^r \end{array}$$

The harmonicity of f_k follows from the fact that π_k is a Riemannian submersion and $(f^{(k-1)}, f^{(k)})$ is horizontal (see [**EW**]). It also follows that

(1.5) f_k and $(f^{(k-1)}, f^{(k)})$ induce the same metric on X.

(Here $\mathcal{F}_k$ is given the metric induced by the product metrics of the Grassmanians, which in turn inherit a metric from their Plucker embeddings.)

(1.6) Denote ϕ_k the projective embedding of $\mathcal{F}_k$ obtained by composing the Segre with the Plucker embeddings

$$\begin{aligned} \mathcal{F}_k &\subset \mathrm{Grass}(k-1, \mathbb{P}^r) \times \mathrm{Grass}(k, \mathbb{P}^r) \\ &\subset \mathbb{P}(\wedge^k \mathbb{C}^{r+1}) \times \mathbb{P}(\wedge^{k+1} \mathbb{C}^{r+1}) \subset \mathbb{P}(\wedge^k \mathbb{C}^{r+1} \otimes \wedge^{k+1} \mathbb{C}^{r+1}) = \mathbb{P}_k \end{aligned}$$

and for $f : X \to \mathbb{P}^r$ holomorphic let us define

$$f^{\langle k \rangle} = \phi_k \circ (f^{(k-1)}, f^{(k)}) : X \to \mathbb{P}_k$$

so we may rephrase (1.5) as

(1.7) f_k and $f^{\langle k \rangle}$ induce the same metric on X.

(1.8) Now suppose that F and F' are as in (1.2), so that there are holomorphic f and f' such that $F = f_h$, $F' = f'_k$ and such that f_h and f'_k are isometric. By (1.7), $f^{\langle h \rangle}$ and $f'^{\langle k \rangle}$ are isometric, and by Calabi's rigidity theorem, they are unitarily equivalent. Hence, the basic question (1.2) is equivalent to

(1.9) Suppose that $f : X \to \mathbb{P}^r$ and $f' : X \to \mathbb{P}^r$ are holomorphic and such that $f^{\langle h \rangle}$ is unitarily equivalent to $f'^{\langle k \rangle}$ for some h and k. Does it follow that f is unitarily equivalent to f' ? This method of reduction, using the lifting map $f^{\langle k \rangle}$ plus Calabi rigidity, is borrowed from [**C**]. Now we have a question in the holomorphic, or algebraic, category and we will approach it by translating into the language of linear series.

(1.10) Remark. If $f : X \to \mathbb{P}^r$ is holomorphic non-degenerate and $0 \le h \le r$ then f_h is isometric to $f^{(r-1)}_{r-h}$. This follows from (1.5) and the relation $f^{(r-1)(h)} = f^{(r-1-h)}$ (see [**ACGH**]).

The next Proposition will allow us to relate projective and unitary equivalence.

(1.11) Proposition. *Let V denote a finite dimensional complex vector space with hermitian inner product $\langle \, , \rangle$. Fix $0 \le k < \dim V$ and consider $V' = \wedge^k V \otimes \wedge^{k+1} V \subset V'' = V^{\otimes 2k+1}$ with their naturally induced hermitian inner products $\langle \, , \rangle'$ and $\langle \, , \rangle''$. Denote the groups of projective (i.e. modulo scalars) linear automorphisms $G = \mathrm{Aut}(V)$, $H = \mathrm{Aut}(V, \langle \, , \rangle)$, with similar meaning for G' and H'. We consider G as a subgroup of G' in the natural way. Then $G \cap H' = H$.*

Proof. For $g : V \to V$ linear, denote g^* the adjoint of g with respect to $\langle , \rangle$. Also, denote g' and g'' the induced endomorphisms of V' and V'' respectively. It is easy to check that $(g^*)'' = (g'')^*$. Since $g' = g''|_{V'}$ and $\langle , \rangle' = \langle , \rangle''|_{V'}$, it follows that $(g^*)' = (g')^*$ also. Now suppose that $f \in G \cap H'$, that is, $f = g'$ with $g \in G$ and $f^* = f^{-1}$. Then, $(g^*)' = (g')^* = (g')^{-1} = (g^{-1})'$, which easily implies $g^* = g^{-1}$, as wanted.

(1.12) Corollary. *In the situation of (1.9), if the isometry between $f^{\langle h \rangle}$ and $f'^{\langle k \rangle}$ is induced by a projective automorphism σ of $\mathbb{P}^r$ then σ is unitary.*

(1.13) Corollary. *Also in the situation of (1.9), and assuming that $f^{\langle h \rangle}(X)$ does not have projective automorphisms (this happens in particular if X does* *not have automorphisms) then f and f' are unitarily equivalent if and only if they are projectively equivalent.*

§2.

(2.1) In order to fix notation we recall some definitions from [**H**] or [**GH**]. If X is a compact Riemann surface, a linear series on X is a pair (L, V) where L is a line bundle on X and $V \subset H^0(X, L)$ is a linear subspace of the space of global sections of L. We denote $e : V_X \to L$ the induced bundle map from the trivial vector bundle on X with fiber V, obtained by composing with the natural bundle map $H^0(X, L)_X \to L$. We assume that e is surjective (the linear series does not have base points).

If $f : X \to \mathbb{P}^r$ is holomorphic then f induces a linear series on X by taking $L = f^*\mathcal{O}(1)$ ($\mathcal{O}(1)$ is the line bundle on $\mathbb{P}^r$ defined by a hyperplane) and as V the image of the pull-back map $H^0(\mathbb{P}^r, \mathcal{O}(1)) \to H^0(X, L)$.

Conversely, from the linear series $V \subset H^0(X, L)$ we may reconstruct f up to a projective equivalence since we may construct a holomorphic map $f : X \to \mathbb{P}(V^*)$ by sending $x \in X$ to the hyperplane in V consisting of sections vanishing at x.

Two linear series (L, V) and (L', V') are said to be isomorphic (written $(L, V) \cong (L', V')$) if there exists an isomorphism $\phi : L \to L'$ such that $H^0(\phi)(V) = V'$. Equivalently, the induced maps to projective space are projectively equivalent.

A similar construction applies for maps into Grassmanians: if V is a finite dimensional vector space, morphisms $f : X \to \text{Grass}(k, V^*)$ correspond to surjective bundle maps $e : V_X \to E$ where E is a vector bundle on X of rank k. Notice that e determines and is determined by the vector space map $H^0(e) : V \to H^0(X, E)$.

(2.2) Consider a linear series $V \subset H^0(X, L)$ on X corresponding to $f : X \to \mathbb{P}(V^*)$, with $r + 1 = \dim(V)$. Let $P^k(L)$ denote the bundle of jets of order k of sections of L ([**G1**, **G2**, **P**]) and

$$t_k : V_X \to P^k(L)$$

the composition $V_X \to H^0(X, L)_X \to P^k(L)$ of the inclusion with the natural (truncated Taylor expansion) maps. For $0 \leq k \leq r - 1$, denote by P^k the image of t_k; P^k is a locally free subsheaf of $P^k(L)$ of the same rank $k + 1$, and the cokernel of t_k is supported on the hyperosculation points of f (see (2.6), [**P**], [**ACGH**]). The k-th associated map $f^{(k)} : X \to \text{Grass}(k + 1, V^*)$ corresponds to $t_k : V_X \to P^k$ and the composition with the Plucker embedding of the Grassmanian corresponds to

$$\wedge^{k+1}(t_k) : \wedge^{k+1} V_X \to \wedge^{k+1} P^k$$

and hence the map $f^{\langle k\rangle} : X \to \mathbb{P}_k$ of §1 is given by the linear series $(L_{V,k}, V^{\langle k\rangle})$ where $V^{\langle k\rangle}$ is the image of the composition

$$(2.3)\quad \wedge^k V\otimes\wedge^{k+1}V \to H^0(\wedge^k P^{k-1})\otimes H^0(\wedge^{k+1}P^k) \to H^0(\wedge^k P^{k-1}\otimes\wedge^{k+1}P^k)$$

and $L_{V,k} = \wedge^k P^{k-1}\otimes\wedge^{k+1}P^k$.

Now suppose that $f : X \to \mathbb{P}^r$ and $f' : X \to \mathbb{P}^r$ are as in (1.9). Let $V \subset H^0(X, L)$ and $V' \subset H^0(X, L')$ be the corresponding linear series. It follows from (2.3) that if $f^{\langle h\rangle}$ is projectively equivalent to $f'^{\langle k\rangle}$ then we have an isomorphism (2.1) of the corresponding linear series

$$(2.4)\qquad \left(L_{V,h}, V^{\langle h\rangle}\right) \cong \left(L'_{V',k}, V'^{\langle k\rangle}\right).$$

From the projective –rather than unitary– viewpoint, question (1.9) may be formulated as

(2.5) If (2.4) is satisfied, to what extend does it follow that

$$(L, V) \cong (L', V')?$$

(2.6) The line bundle $L_{V,k}$ of (2.3) may be expressed in terms of ramification indices: one has an exact sequence

$$0 \to P^k \to P^k(L) \to \mathrm{Coker}(t_k) \to 0$$

and by [**ACGH**], page 39, $\mathrm{Coker}(t_k)$ is the structure sheaf of the divisor

$$R_k = \sum_{x\in X}\sum_{0\le j\le k} \alpha_j(x).x$$

where $\alpha_j(x)$ is the j-th ramification index of (L, V) at x. We obtain

$$\wedge^{k+1}P^k(L) = \wedge^{k+1}P^k \otimes \mathcal{O}_X(R_k).$$

(2.7) From the standard exact sequences

$$0 \to \Omega^{\otimes k}\otimes L \to P^k(L) \to P^{k-1}(L) \to 0$$

where Ω denotes the sheaf of 1-forms on X, we obtain

$$\wedge^{k+1}P^k(L) = \wedge^k P^{k-1}(L)\otimes\Omega^{\otimes k}\otimes L.$$

Multipliying these equalities we find $\wedge^k P^{k-1}(L) = \Omega^{\otimes\binom{k}{2}}\otimes L^{\otimes k}$ and hence

$$L_{V,k} = \wedge^k P^{k-1}\otimes\wedge^{k+1}P^k = \Omega^{\otimes k^2}\otimes L^{\otimes 2k+1}\otimes\mathcal{O}_X(-R_{k-1}-R_k).$$

(2.8) If, for instance, f does not have hyperosculation points of order k (i.e. $R_k = 0$) then $L_{V,k} = L_k$ does not depend on V and $f^{\langle k\rangle} : X \to \mathbb{P}_k$ corresponds to the linear series $(L_k, V^{\langle k\rangle})$ where $V^{\langle k\rangle}$ is the image of

$$\wedge^k V \otimes \wedge^{k+1} V \to H^0\left(\Omega^{\otimes\binom{k}{2}} \otimes L^{\otimes k}\right) \otimes H^0\left(\Omega^{\otimes\binom{k+1}{2}} \otimes L^{\otimes k+1}\right)$$
$$\to H^0\left(\Omega^{\otimes k^2} \otimes L^{\otimes 2k+1}\right).$$

(2.9) Proposition. *Let X be a Riemann surface of genus $g \geq 0$. Consider holomorphic maps $f : X \to \mathbb{P}^r$ and $f' : X \to \mathbb{P}^r$. Let d denote the degree of f and $r_k = \deg(R_k)$, with similar primed notation for f'. Suppose that $f^{\langle h\rangle}$ is projectively equivalent to $f'^{\langle k\rangle}$. Then*

$$2(g-1)h^2 + d(2h+1) - r_{h-1} - r_h = 2(g-1)k^2 + d'(2k+1) - r'_{k-1} - r'_k.$$

Proof. If f (resp. f') corresponds to (L, V) (resp. (L', V')) then $f^{\langle h\rangle}$ projectively equivalent to $f'^{\langle k\rangle}$ implies, by (2.4) and (2.7), that

$$\Omega^{\otimes h^2} \otimes L^{\otimes 2h+1} \otimes \mathcal{O}_X(-R_{h-1} - R_h) \cong \Omega^{\otimes k^2} \otimes L'^{\otimes 2k+1} \otimes \mathcal{O}_X(-R'_{k-1} - R'_k).$$

Taking degree we obtain the Proposition.

(2.10) For the rest of this §, we restrict our attention to holomorphic maps $f : X \to \mathbb{P}^r$ without special hyperosculation points, that is, we assume $R_{r-1} = 0$.

(2.11) Corollary. *Using the notation in (2.9), suppose that $d = d'$ and $h \neq k$. Then f_h is not isometric to f'_k, except that f_h may be isometric to f_{d-h} when $g = 0$.*

Proof. when $d = d'$ the equality in (2.9) may be written as $(g-1)(h-k)(h+k) = -d(h-k)$.

(2.12) Remark. Regarding the case $g = 0$, fix an hermitian inner product on $H^0(\mathbb{P}^1, \mathcal{O}(1))$ and consider the d-tuple Veronese embedding $f : \mathbb{P}^1 \to \mathbb{P}^d$ defined by an orthonormal basis of $H^0(\mathbb{P}^1, \mathcal{O}(d)) = \mathrm{Symm}^d(H^0(\mathbb{P}^1, \mathcal{O}(1)))$. Then f_h and f_{d-h} are isometric but not unitarily equivalent, since they have different Kahler angle (see [**BJRW**], Theorem (5.2)).

(2.13) Proposition. *If $f : X \to \mathbb{P}^r$ and $f' : X \to \mathbb{P}^r$ are such that $f^{\langle h\rangle}$ is projectively equivalent to $f'^{\langle h\rangle}$ then $L^{\otimes 2h+1} \cong L'^{\otimes 2h+1}$.*

Proof. Follows from (2.4) and (2.7).

(2.14) For a linear series (L, V) on a curve X of genus $g \geq 0$, let us denote $\deg(L) = d$, $\dim(V) = r+1$. The series (L, V) is said to be complete if $V = H^0(X, L)$. We remark that if $d \geq 2g-1$ and $r \geq d-g$ then it follows from Riemann-Roch that (L, V) is complete and $r = d - g$. Hence, a non-degenerate map $f : X \to \mathbb{P}^{d-g}$ with $d \geq 2g - 1$ determines and is determined (up to projective equivalence) by a line bundle L on X of degree d.

(2.15) Proposition. *Suppose $f : X \to \mathbb{P}^r$ is a non-degenerate holomorphic map of degree $d \geq 2g-1$, with $r = d - g$, and fix $0 \leq h \leq r$. Then, up to unitary equivalence, there exist at most $(2h+1)^{2g}$ non-degenerate holomorphic maps $f' : X \to \mathbb{P}^r$ such that f'_h is isometric to f_h.*

Proof. By (2.14), f (resp. f') corresponds to a complete linear series with line bundle L (resp. L'). Suppose that f'_h is isometric to f_h and that f' is not isometric to f. We claim that L is not isomorphic to L': otherwise f and f' would be projectively equivalent and hence, by (1.12), unitarily equivalent, contrary to our assumption. On the other hand, according to (2.13), we have $L^{\otimes 2h+1} = L'^{\otimes 2h+1}$ and hence the possible choices for L' correspond to the $(2h+1)^{2g}$ points of $(2h+1)$-torsion in the group $\mathrm{Pic}^0(X)$ of isomorphism classes of line bundles of degree zero on X.

(2.16) Remark. using the notation of (2.15), there are at most D^{2g} maps f' such that f_h is isometric to f'_h and f_k is isometric to f'_k, where D is the greatest common divisor of $2h+1$ and $2k+1$. The argument is the same as in (2.15).

(2.17) Proposition. *Suppose $f : X \to \mathbb{P}^2$ maps X birationally onto a curve $Y = f(X)$ of degree d with $\delta \leq d - 3$ nodes as only singularities. Then f_1 is rigid (among maps as in (2.10)).*

More precisely, if $f' : X \to \mathbb{P}^2$ is such that f'_1 is isometric to f_1 and f' does not have cusps (i.e. $R'_1 = 0$) then f' is unitarily equivalent to f.

Proof. Let f (resp. f') be given by the linear series (L, V) (resp. (L', V')). It follows from (2.7) that $L^{\otimes 3} \cong L'^{\otimes 3}$ and hence $d = d'$. It is known (see [**ACGH**], page 56) that (L, V) is the unique linear series on X with $\deg(L) = d$ and $\dim(V) \geq 3$. It follows that f and f' are projectively equivalent and hence, by (1.12), they are unitarily equivalent.

(2.18) Remark. A similar proposition holds for maps $f : X \to \mathbb{P}^r$ such that the corresponding linear series is unique with the given degree and dimension. See [**CL**] for examples of this situation.

(2.19) Remark. A stronger result than (2.17) will be proved in (3.8) using a different method.

§3.

(3.1) In this section we specialize to the case of plane curves. We consider a holomorphic map $f : X \to \mathbb{P}^2$ from a compact Riemann surface X of genus $g \geq 0$. We assume that f maps X birationally onto a curve $Y = f(X)$ of degree d.

(3.2) As in §2, f_1 is isometric to $f^{\langle 1 \rangle} : X \to \mathcal{F}_1 \subset \mathbb{P}^8$. Identifying $\mathbb{P}^2$ with its dual, we may think of $\mathcal{F} = \mathcal{F}_1$ as

$$\mathcal{F} = \{(x, y)/x_0y_0 + x_1y_1 + x_2y_2 = 0\} \subset \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8.$$

(3.3) As in (1.10), let us notice that if $f^{(1)}$ denotes the first associated (or dual) curve of f then f_1 is isometric to $f_1^{(1)}$. In fact, if τ is the unitary automorphism of $\mathbb{P}^8$ defined by $\tau(x \otimes y) = (y \otimes x)$ then the known biduality $f^{(1)(1)} = f$ means that $\tau f^{\langle 1 \rangle} = f^{(1)\langle 1 \rangle}$.

(3.4) We will say that f_1 is <u>rigid</u> if it is true that given a map $f' : X \to \mathbb{P}^2$ as in (3.1), with degree d', such that f_1 and f'_1 are isometric, it follows that f' is unitarily equivalent to f or to $f^{(1)}$.

(3.5) Since $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ is cut out by quadrics and $\mathcal{F}$ is a hyperplane section of $\mathbb{P}^2 \times \mathbb{P}^2$, $\mathcal{F}$ is equal to the intersection of the quadrics containing it. For a variety Z in projective space, let us denote $I_2(Z)$ the space of quadrics containing Z.

Clearly we have $I_2(\mathcal{F}) \subset I_2(f^{\langle 1 \rangle}(X))$.

(3.6) Proposition. *If $I_2(\mathcal{F}) = I_2(f^{\langle 1 \rangle}(X))$ then f_1 is rigid.*

Proof. If f_1 is isometric to f'_1 then, as in §2, there exists a unitary linear isomorphism $\sigma : \mathbb{P}^8 \to \mathbb{P}^8$ such that $\sigma(f^{\langle 1 \rangle}(X)) = f'^{\langle 1 \rangle}(X)$. Then σ transforms the quadrics containing $f^{\langle 1 \rangle}(X)$ into the quadrics containing $f'^{\langle 1 \rangle}(X)$. Using our hypothesis it follows that $\sigma(I_2(\mathcal{F})) = I_2(\mathcal{F})$. Since $\mathcal{F}$ is an intersection of quadrics, we obtain that $\sigma(\mathcal{F}) = \mathcal{F}$. But $\mathrm{Aut}(\mathcal{F})$ is the direct product of $\mathrm{Aut}(\mathbb{P}^2)$ (acting by $g(x, y) = (g(x), g(y))$) and the subgroup generated by τ (as in (3.3)), and hence it follows that either σ or $\tau\sigma$ is induced from $\mathbb{P}^2$. It follows from (1.11) that the automorphism of $\mathbb{P}^2$ is in fact unitary. This proves the Proposition.

(3.7) Remark. A proposition similar to (3.6) holds for maps $f : X \to \mathbb{P}^r$ due to the fact that $\mathcal{F}_k$ (as in §2) is also an intersection of quadrics. In fact, $\mathrm{Grass}(k-1, \mathbb{P}^r) \times \mathrm{Grass}(k, \mathbb{P}^r) \subset \mathbb{P}_k$ (as in (1.6)) is the intersection of the Grassmann and the Segre quadrics, and $\mathcal{F}_k$ is a linear section of the product of the Grassmannians (see [**D**], page 184).

(3.8) Theorem. *Let $f : X \to \mathbb{P}^2$ be a holomorphic map as in* (3.1). *Denote $r_1 = \deg(R_1)$ the total number of cusps of f, as in* (2.6). *If $3d < 2g - 2 - r_1$ then f_1 is rigid.*

Proof. We will prove that $I_2(\mathcal{F}) = I_2(f^{\langle 1\rangle}(X))$. The result then follows from (3.6). Suppose that there exists a quadric $Q \subset \mathbb{P}^8$ containing $f^{\langle 1\rangle}(X)$ and not containing $\mathcal{F}$. Denote $Z = (f(X) \times \mathbb{P}^2) \cap Q \cap \mathcal{F} \subset \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$. We claim that all components of Z have dimension one. Let h and k denote the pull-backs to $\mathcal{F}$ of the hyperplane classes in $\mathbb{P}^2$, so that h and k form a basis of $\mathrm{Pic}(\mathcal{F})$. The claim follows by observing that no component of the divisor $Q \cap \mathcal{F}$, with class $2h + 2k$, could contain the irreducible divisor $(f(X) \times \mathbb{P}^2) \cap \mathcal{F}$ with class dh, because $d > 2$.

Since $f^{\langle 1\rangle}(X) \subset Z$, we must have $\deg(f^{\langle 1\rangle}(X)) \leq \deg(Z)$. Computing in $\mathrm{Pic}(\mathbb{P}^2 \times \mathbb{P}^2)$, $\deg(Z) = dh.(2h + 2k).(h + k).(h + k) = 6d$. On the other hand, we obtain from (2.7) $\deg(f^{\langle 1\rangle}(X)) = \deg(\Omega \otimes L^{\otimes 3} \otimes \mathcal{O}_X(-R_1)) = 2g - 2 + 3d - r_1$, so that the existence of Q implies $2g - 2 - r_1 \leq 3d$. This contradiction proves the Theorem.

(3.9) Corollary. *Suppose for instance that the only singularities of $f(X)$ are δ nodes and $\kappa = r_1$ ordinary cusps. If $2\delta + 3\kappa < d(d-6)$ then f_1 is rigid.*

Proof. Follows from (3.8) and the formula $g = \binom{d-1}{2} - \delta - \kappa$.

§4.

(4.1) In view of (3.7) and (3.8), it seems interesting to determine the equations of the projective curves $f^{(k)}(X)$ and $f^{\langle k\rangle}(X)$ for a given $f : X \to \mathbb{P}^r$.

We remark that the space of hyperplanes containing $f^{(k)}(X)$ is the kernel of the higher Gauss map $\wedge^{k+1}V \to H^0(\wedge^{k+1}P^k)$, with notation as in (2.2). When this map is onto, the ideal of $f^{(k)}(X)$ is generated by quadrics since the degree of $\wedge^{k+1}P^k$ is large (see [**G**]); also, these quadrics may be represented as two by two minors of a certain matrix of linear forms [**EKS**]. We leave a more detailed study of this situation for a future paper.

Now we start by looking at the hyperplanes containing $f^{\langle 1\rangle}(X)$ for a given $f : X \to \mathbb{P}^2$.

(4.2) Proposition. *Let $f : X \to \mathbb{P}^2$ be a holomorphic immersion given by the linear series (L, V) of dimension 3 and degree d, and consider $f^{\langle 1\rangle} : X \to \mathbb{P}^8$ as in (3.2). Denote by E the hyperplane $E = \{(x, y)/x_0y_0 + x_1y_1 + x_2y_2 = 0\} \subset \mathbb{P}^8$. Then the following holds*

(a) *if $g = g(X) > 0$ then $Y = f^{\langle 1\rangle}(X) \subset E$ is non-degenerate,*

(b) *if $X = \mathbb{P}^1$ then $Y \subset E$ is degenerate if and only if f is given (up to projective transformations) by $f(t) = (1, t^n, t^m)$ for some $m, n \in \mathbb{N}$.*

Proof. Let the plane curve $f(X) \subset P^2$ be given by the equation $F = 0$, so that

$$Y = \{(x, y)/F(x) = 0, y = \Delta F\} \subset \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$$

where we set $\Delta F = (F_0, F_1, F_2)$ and the subindices indicate partial derivatives. In these terms, a hyperplane $H \subset \mathbb{P}^8$ with equation $\sum a_{ij}\ x_i \otimes y_j$ containing Y is the same as a relation

$$\sum L_i(x) F_i(x) = 0 \mod F \tag{4.3}$$

where the L_i are linear in x. When $(L_0, L_1, L_2) = (x_0, x_1, x_2)$, H is E, and the idea is that when $H \neq E$ then (4.3) implies the existence of a non-zero regular vector field on X. In fact, an (L_0, L_1, L_2) as in (4.3) is the same as an element in the kernel of $\Delta F \circ \beta$ in the diagram

$$\begin{tikzcd} H^0(P, \mathcal{O}_P(1))^3 \arrow[r, "\Delta F"] \arrow[d, "\beta"'] & H^0(P, \mathcal{O}_P(d)) \arrow[d] \\ H^0(X, L)^3 \arrow[r, "\Delta F"] & H^0(X, L^{\otimes d}) \end{tikzcd} \tag{4.4}$$

where the vertical maps are pull-back of sections. Consider the diagram

$$\begin{tikzcd} & & 0 & \\ 0 \arrow[r] & TX(D) \arrow[r] & f^*TP \arrow[r] \arrow[u] & f^*\mathcal{O}_P(d) = L^{\otimes d} \\ & & L^3 \arrow[u] \arrow[ur, "\varphi"'] & \\ & & \mathcal{O}_X \arrow[u] & \\ & & 0 \arrow[u] & \end{tikzcd} \tag{4.5}$$

where $\varphi = (F_0, F_1, F_2)$, the column is the pull-back to X of the Euler sequence on $P = \mathbb{P}^2$ and the row is the normal bundle sequence of f (D is the divisor of zeroes of the differential df and the normal bundle N_f is realized as a subbundle of $f^*\mathcal{O}_P(d)$ via φ). Taking global sections in (4.5) we see that if $H^0(X, TX(D)) = 0$ then $\ker(\Delta F)$ is one-dimensional, and hence $\dim(\ker(\Delta F \circ \beta)) = 1$, as wanted. The condition $H^0(X, TX(D)) = 0$ is satisfied if $\deg(D) < 2g - 2$; in particular, it is satisfied if f is an immersion ($D = 0$) and $g \geq 2$.

It remains to analyze the case $g = 1$. Suppose that X is an elliptic curve, represented as the complex plane modulo a lattice. We may assume that $f : X \to \mathbb{P}^2$ is given by $f(z) = (1, f_1(z), f_2(z))$ where f_1 and f_2 are elliptic functions with $0 < \operatorname{ord}(f_1) < \operatorname{ord}(f_2)$; here ord denotes order of pole at the origin. As before, denote by $F = 0$ the equation of $f(X)$, so that we have $F(1, f_1(z), f_2(z)) = 0$ for all z. Differentiating this, we obtain the second of the equalities below. The other two are the Euler relation and (4.3).

$$
\begin{aligned}
F_0(1,f_1,f_2) + F_1(1,f_1,f_2)f_1 + F_2(1,f_1,f_2)f_2 &= 0\\
F_1(1,f_1,f_2)f_1' + F_2(1,f_1,f_2)f_2' &= 0\\
F_0(1,f_1,f_2)L_0(1,f_1,f_2) \qquad\qquad\qquad\qquad & \\
+F_1(1,f_1,f_2)L_1(1,f_1,f_2) + F_2(1,f_1,f_2)L_2(1,f_1,f_2) &= 0.
\end{aligned}
$$

It follows that $(L_0, L_1, L_2) = a(1, f_1, f_2) + b(0, f_1', f_2')$ for some constants a and b. We get a contradiction by looking at orders of pole at the origin.

For (b) consider the exact diagram of sheaves on X

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_X(1)^2 & \longrightarrow & TX & \longrightarrow & 0\\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{Jac}(f)} & & \downarrow{\scriptstyle df} & & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_X(d)^3 & \longrightarrow & f^*(TP) & \longrightarrow & 0
\end{array}
$$

Using coordinates x_0 and x_1 on $X = \mathbb{P}^1$, a syzygy as in (4.3) may be represented by an element $(L_0, L_1) \in H^0(\mathcal{O}_X(1)^2)$ such that

$$
\text{(4.6)} \qquad \mathrm{Jac}(f).(L_0, L_1) \in V^3 \subset H^0(X, L)^3.
$$

Considering the differential operator $\delta : H^0(X, \mathcal{O}_X(d)) \to H^0(X, \mathcal{O}_X(d))$ defined by $\delta = L_0\frac{\partial}{\partial x_0} + L_1\frac{\partial}{\partial x_1}$, we see that (4.6) is equivalent to the condition $\delta(V) \subset V$. Now, $\mathrm{Aut}(\mathbb{P}^1)$ acts on the space of vector fields on $\mathbb{P}^1$ with two orbits, so we may assume after a change of variables that $\delta = x_1\frac{\partial}{\partial x_0}$ (a vector field with a double zero) or that $\delta = x_0\frac{\partial}{\partial x_0}$ (a vector field with two simple zeroes). Denote $e_i = x_0^i x_1^{d-i}$ the standard basis of $H^0(X, \mathcal{O}_X(d))$. In the first case, we have $\delta(e_i) = i.e_{i-1}$ and we see that δ is nilpotent and its only three-dimensional invariant subspace is the one generated by e_0, e_1, e_2. In the second case we have $\delta(e_i) = i.e_i$, so that the e_i are eigenvectors with different eigenvalues, and the three-dimensional δ-invariant subspaces are the ones generated by three of the e_i's. From this, (b) easily follows in both cases.

Now we consider the quadrics containing $Y = f^{\langle 1\rangle}(X)$ for a given $f : X \to \mathbb{P}^2$ with linear system (V, L). We maintain the notation introduced above. It is easy to see that such a quadric is the same as a relation

$$
\sum Q_{ij}(x)F_i(x)F_j(x) = 0 \mod F
$$

where the Q_{ij} are homogeneous polynomials of degree two. In other words, such a quadric corresponds to an element

$$q \in S^2(H^0(P, \mathcal{O}_P(1))^3) \text{ such that } S^2(\Delta F)(q) = 0 \tag{4.7}$$

using notation as in (4.4); S^2 denotes second symmetric power.

Let us recall (see [**H**], Exercise (5.16)) that for each exact sequence $0 \to A \to B \to C \to 0$ of locally free sheaves, with A of rank one, one has a natural exact sequence

$$0 \to A \otimes B \to S^2(B) \to S^2(C) \to 0. \tag{4.8}$$

Applying (4.8) twice to (4.5) (assuming $D = 0$) we obtain the exact diagram (4.9)

$$\begin{tikzcd}
& 0 & 0 & & \\
0 \arrow[r] & f^*TP \otimes TX \arrow[u] \arrow[r] & S^2(f^*TP) \arrow[u] \arrow[r] & S^2(N_f) \arrow[r] & 0 \\
& \ker S^2(\varphi) \arrow[u] \arrow[r] & S^2(L^3) \arrow[u] \arrow[ur, "S^2(\varphi)"'] & & \\
& L^3 \arrow[u] \arrow[r] & L^3 \arrow[u] & & \\
& 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

Taking cohomology in (4.9) we find that

$$H^0\left(\ker S^2(\varphi)\right)/H^0\left(L^3\right) \cong \ker\left(H^0\left(f^*TP \otimes TX\right) \to H^1\left(L^3\right)\right). \tag{4.10}$$

On the other hand, considering the natural maps

$$S^2\left(V^3\right) \to S^2\left(H^0(L)^3\right) \to H^0\left(S^2\left(L^3\right)\right)$$

it follows that the space of quadrics containing Y modulo those containing $\mathbb{P}^2 \times \mathbb{P}^2$ is a subspace of $H^0\left(\ker S^2(\varphi)\right)$. The quadrics that are multiples of the Euler relation are represented by elements in $H^0(L^3)$ and, therefore, the space of quadrics containing Y modulo those containing $\mathcal{F}$ may be identified with a subspace of $\ker(H^0(f^*TP \otimes TX) \to H^1(L^3))$.

Combining this with Proposition (3.6) we obtain

(4.11) Proposition. *If the space in* (4.10) *is zero (for instance, if* $H^0(f^*TP \otimes TX) = 0$*) then* f_1 *is rigid.*

Now we assume $g \geq 2$ and turn our attention to $H^0(B)$, where we set $B = f^*TP \otimes TX$. Let us remark first that the Euler characteristic of B is $3(d-(2g-2))$, so $H^0(B) = 0$ is possible only if $d \leq 2g-2$. Tensoring the Euler sequence $0 \to \mathcal{O}_X \to V^* \otimes L \to f^*TP \to 0$ by TX and taking cohomology, we obtain an exact sequence

$$0 \to V^* \otimes H^0(L \otimes TX) \to H^0(B) \to H^1(TX) \to V^* \otimes H^1(L \otimes TX)$$

where the last map is Serre-dual to the multiplication map

$$\mu : V \otimes H^0(2K - L) \to H^0(2K)$$

(here K denotes a canonical divisor and, as it is customary, we use additive notation).

Hence,

(4.12) For $d < 2g-2$, $H^0(B) = 0$ if and only if μ is onto.

By the H^0 Lemma ([**G**], (4.e.1)), μ is onto if $h^1(2K-2L) \leq 1$. Since this condition is satisfied if $d \leq g-1$, we obtain

(4.13) Proposition. *If* $g(X) \geq 2$ *and* $f : X \to \mathbb{P}^2$ *is an immersion of degree* $d \leq g-1$ *then* f_1 *is rigid.*

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