PACIFIC JOURNAL OF MATHEMATICS
Vol. 175, No. 2, 1996

BANACH ALGEBRAS WITH UNITARY NORMS

MOGENS L. HANSEN AND RICHARD V. KADISON

Dedicated to Edward G. Effros, for his pioneering contributions, on the occasion of his sixtieth birthday.

The metric nature of the unitary group of a (unital) C*-algebra is studied in a Banach-algebra framework.

1. Introduction and preliminaries.

The group $\mathfrak{A}_u$ of unitary elements in a (unital) C*-algebra $\mathfrak{A}$ is one of the critical structural components of $\mathfrak{A}$. From [**K**], each U in $\mathfrak{A}_u$ is an extreme point of the unit ball $(\mathfrak{A})_1$ of $\mathfrak{A}$; in case $\mathfrak{A}$ is abelian, $\mathfrak{A}_u$ is precisely the set of extreme points of $\mathfrak{A}$. (More generally, when $\mathfrak{A}$ has a separating family of tracial states, $\mathfrak{A}_u$ is precisely the set of extreme points of $(\mathfrak{A})_1$.) Proceeding from this, Phelps [**P**] shows that the Krein–Milman property holds for $\mathfrak{A}_u$ and $(\mathfrak{A})_1$ when $\mathfrak{A}$ is abelian—namely, $(\mathfrak{A})_1$ is the norm closure of the convex hull $\mathrm{co}(\mathfrak{A}_u)$ of $\mathfrak{A}_u$. In [**RD**], Dye and Russo remove the commutativity restriction—the Phelps result is valid for every unital C*-algebra. (This has become known as the "Russo–Dye theorem.") Gardner [**G**] gives a short and much simplified proof of the Russo–Dye theorem. A significant strengthening of the Russo–Dye theorem [**KP**] (based on a device in [**G**]) states that each A in $\mathfrak{A}$ with $\|A\| < 1$ is the (arithmetic) mean of a finite number of elements of $\mathfrak{A}_u$ —in finer detail, of n elements of $\mathfrak{A}_u$ when $\|A\| < 1 - \frac{2}{n}$ with $n = 3, 4, \ldots$. Haagerup [**H**] establishes a conjecture of Olsen and Pedersen [**OP**] by showing that this same is valid even when $\|A\| = 1 - \frac{2}{n}$ (a deep result).

Are these approximation properties of $\mathfrak{A}_u$ in $(\mathfrak{A})_1$ characteristic of C*-algebras? Is a Banach algebra $\mathfrak{A}$ with a subgroup $\mathfrak{G}$ of the group $\mathfrak{A}_{\mathrm{inv}}$ of invertible elements in $(\mathfrak{A})_1$ whose (norm-) closed, convex hull is $(\mathfrak{A})_1$ (isometrically, isomorphic to) a C*-algebra? We shall see that the answer to these questions is in the negative. In Section 4, we note that the Wiener algebra (functions with absolutely convergent Fourier series—equivalently, the group algebra $l_1(\mathbb{Z})$ of the additive group $\mathbb{Z}$ of integers) has the Russo–Dye (R–D) approximation property and is not isomorphic to a C*-algebra (even algebraically).

Definition 1. If $\mathfrak{A}$ is a Banach algebra and $\mathfrak{G}$ is a subgroup of $\mathfrak{A}_{\text{inv}} \cap (\mathfrak{A})_1$ such that $(\mathfrak{A})_1$ is the norm closure of $\text{co}(\mathfrak{G})$, we say that the norm on $\mathfrak{A}$ is *unitary* (*with group* $\mathfrak{G}$).

On the other hand, when our Banach algebra $\mathfrak{A}$ is an algebra of bounded operators on a Hilbert space $\mathcal{H}$ equipped with the norm it acquires by assigning to each operator in $\mathfrak{A}$ its bound, if $(\mathfrak{A}, \mathfrak{G})$ satisfies the R–D approximation property, then $\mathfrak{A}$ is a C*-algebra. (See Theorem 4.) Effros, Ruan, and Choi ([**CE**], [**ER**], and [**R**]) have taught us the importance of 'matricial norming' and 'operator spaces' in studying operator algebras. With that background, Blecher, Ruan, and Sinclair have produced a striking characterization [**BRS**] of those Banach algebras that are isometrically isomorphic to an algebra of operators on a Hilbert space (equipped with the operator-bound norm). Combining [**BRS**] with our proposition, we single out those Banach algebras that are isometrically isomorphic to a C*-algebra in terms of the norm and potential unitary group. The group $\mathfrak{G}$ may map onto a proper subgroup of the unitary group of the C*-algebra under the isomorphism; a C*-algebra $\mathfrak{A}$ and a proper (norm-) dense subgroup $\mathfrak{G}$ of $\mathfrak{A}_{\text{u}}$ will illustrate this situation. In Proposition 3, we note that, with $\mathfrak{A}$ a C*-algebra, $\mathfrak{A}_{\text{u}}$ is maximal in the set of norm-bounded subgroups of $\mathfrak{A}_{\text{inv}}$. When $\mathfrak{G}$ is a maximal bounded subgroup of $\mathfrak{A}_{\text{inv}}$ and the elements of $\mathfrak{G}$ have norm 1, then an isometric isomorphism of $\mathfrak{A}$ with a C*-algebra carries $\mathfrak{G}$ onto the unitary group of that C*-algebra (Theorem 8).

Definition 2. If the norm on a Banach $\mathfrak{A}$ is unitary for the group $\mathfrak{G}$ and $\mathfrak{G}$ is a maximal bounded subgroup of $\mathfrak{A}_{\text{inv}}$, the norm on $\mathfrak{A}$ is said to be *maximal unitary.*

In Theorem 6, we show that each finite-dimensional Banach algebra with a maximal unitary norm is isometrically isomorphic to a C*-algebra. Section 3, contains a study of Banach-algebra norms on $C(X)$, the algebra of all complex-valued continuous functions on a compact Hausdorff space X under pointwise operations. Of course, the supremum norm on $C(X)$ is unitary (with that norm, $C(X)$ is a C*-algebra). We find other unitary norms. When X is a (finite) set of n points (so that $C(X)$ is $\mathbb{C}^n$, with $\mathbb{C}$ the complex numbers), we construct a family of distinct unitary norms for groups $\mathfrak{G}$ containing $\mathbb{T}_1$, the group of constant functions on X of modulus 1, such that $\mathfrak{G}/\mathbb{T}_1$ is finite. In this case, we construct a Banach-algebra norm on $C(X)$ that is not unitary. On the other hand, if a Banach algebra algebraically isomorphic with $C(X)$ has a maximal unitary norm, the isomorphism is an isometry when $C(X)$ is equipped with its supremum norm (X an arbitrary compact Hausdorff space—Theorem 19).

The purely C*- and von Neumann algebra properties of the maximal bounded subgroups of $\mathfrak{A}_{\text{inv}}$ form a topic of considerable independent interest. We examine that topic in another article.

2. Inverse Russo–Dye theorems.

In this section, we study some of the partial converses to the Russo–Dye theorem. We note, in the proposition that follows, an important structural feature of the unitary group of a C*-algebra.

Proposition 3. *Let $\mathfrak{A}$ be a C^*-algebra (with identity I) and $\mathcal{U}$ its group of unitary elements. Then $\mathcal{U}$ is a maximal bounded subgroup of the group $\mathfrak{A}_{\text{inv}}$ of invertible elements in $\mathfrak{A}$.*

Proof. Suppose that $\mathfrak{G}$ is a norm-bounded subgroup of $\mathfrak{A}_{\text{inv}}$ and that $\mathcal{U} \subseteq \mathfrak{G}$. If $T \in \mathfrak{G}$ and UH is the polar decomposition of T, then $U \in \mathcal{U}$ because T is invertible. Thus $H = U^*UH = U^*T \in \mathfrak{G}$. Let λ be an element of $\text{sp}(H)$. Then $\lambda^n \in \text{sp}(H^n)$, so $|\lambda|^n \leq \|H^n\| < \infty$ for all integers n (since $H^n \in \mathfrak{G}$). It follows that $|\lambda| = 1$, and since H is positive, $\lambda = 1$. Hence $\text{sp}(H) = \{1\}$, and $H = I$. Therefore $T = UH = U \in \mathcal{U}$, and it follows that $\mathfrak{G} = \mathcal{U}$, so $\mathcal{U}$ is maximal. □

It follows from Proposition 3 and the Russo–Dye theorem that if $(\mathfrak{A}, \| \ \|)$ is a C*-algebra, then $\| \ \|$ is maximal unitary (relative to the group $\mathfrak{A}_{\text{u}}$ of unitary elements of $\mathfrak{A}$).

Theorem 4. *Let $\mathcal{H}$ be a Hilbert space and $\mathfrak{A}$ be a unital Banach subalgebra of $\mathcal{B}(\mathcal{H})$ (with the norm $\| \ \|$ inherited from $\mathcal{B}(\mathcal{H})$). If $\| \ \|$ is unitary, then $\mathfrak{A}$ is a C^*-algebra (when equipped with the involution inherited from $\mathcal{B}(\mathcal{H})$).*

Proof. Since $\| \ \|$ is unitary, we can find a subgroup $\mathfrak{G}$ of $\mathfrak{A}_{\text{inv}}$ containing the scalars of modulus 1 such that each U in $\mathfrak{G}$ has norm 1, and such that each A in the unit ball of $(\mathfrak{A}, \| \ \|)$ is the $\| \ \|$-limit of convex combinations of elements of $\mathfrak{G}$. We show first that any element of $\mathfrak{G}$ is a unitary operator in $\mathcal{B}(\mathcal{H})$. Let U be an element of $\mathfrak{G}$. Then $\|U\| = 1$, $U^{-1} \in \mathfrak{G}$, and $\|U^{-1}\| = 1$. For any x in $\mathcal{H}$, $\|x\| = \|U^{-1}Ux\| \leq \|U^{-1}\|\|Ux\| = \|Ux\| \leq \|U\|\|x\| = \|x\|$, so $\|Ux\| = \|x\|$. Thus U is an invertible isometry and, therefore, a unitary operator.

If $U \in \mathfrak{G}$, then $U^* = U^{-1} \in \mathfrak{G}$. Thus the linear space generated by $\mathfrak{G}$ is a self-adjoint subalgebra of $\mathfrak{A}$ as is its norm closure (by norm continuity of the adjoint operation). By choice of $\mathfrak{G}$, this norm closure is $\mathfrak{A}$. Hence $\mathfrak{A}$ is a C*-algebra. □

Remark 5. In the preceding proof, we have not used the full force of the conditions imposed on $\mathfrak{G}$. We have proved that the norm-closed linear subspace of $\mathcal{B}(\mathcal{H})$ generated by a group of (invertible) norm 1 operators is a C*-algebra.

From Proposition 3, the unitary group of a C*-algebra is maximal bounded. With this added assumption on our group, we obtain the desired inverse Russo–Dye theorem in the case of a finite dimensional Banach algebra.

Theorem 6. *Let* $(\mathfrak{A}, \|\ \|)$ *be a (unital) finite dimensional Banach algebra. If* $\|\ \|$ *is maximal unitary, then* $(\mathfrak{A}, \|\ \|)$ *is (isometrically isomorphic to) a C*-algebra.*

Proof. Let $\mathfrak{G}$ be a maximal bounded subgroup of $\mathfrak{A}_{\text{inv}}$ relative to which $\|\ \|$ is unitary. Let $\mathfrak{A}$ act on itself by left multiplication; that is, let $\phi_1(A)B$ be AB for all A, B in $\mathfrak{A}$. Then ϕ_1 is an isometric representation of $\mathfrak{A}$.

As $\mathfrak{A}$ is finite dimensional, we can find (an inner product and) a norm $\|\ \|_{\mathcal{H}}$ such that $(\mathfrak{A}, \|\ \|_{\mathcal{H}})$ is a Hilbert space. The two norms $\|\ \|$ and $\|\ \|_{\mathcal{H}}$ are equivalent because $\mathfrak{A}$ is finite dimensional (see [**KR**, Proposition 1.2.16]). Thus, the identity mapping $\iota : (\mathfrak{A}, \|\ \|) \to (\mathfrak{A}, \|\ \|_{\mathcal{H}})$ is bicontinuous. For A in $\mathfrak{A}$, let $\phi_2(A)$ be $\iota \circ \phi_1(A) \circ \iota^{-1}$. Then ϕ_2 is a faithful, bicontinuous representation of $\mathfrak{A}$ on $(\mathfrak{A}, \|\ \|_{\mathcal{H}})$. Therefore, the restriction of ϕ_2 to $\mathfrak{G}$ is a bounded representation of the group $\mathfrak{G}$ on the finite dimensional Hilbert space $(\mathfrak{A}, \|\ \|_{\mathcal{H}})$. In this case, $\mathfrak{G}$ is compact and, therefore, amenable. Thus [**D**, Théorème 6] applies, and there is an invertible operator T in $\mathcal{B}((\mathfrak{A}, \|\ \|_{\mathcal{H}}))$ such that $U \mapsto T\phi_2(U)T^{-1}$ is a unitary representation of $\mathfrak{G}$ on $(\mathfrak{A}, \|\ \|_{\mathcal{H}})$. (This follows, as well, from the Peter–Weyl theory.) For A in $\mathfrak{A}$, let $\psi(A)$ be $T\phi_2(A)T^{-1}$. Then ψ is a faithful, bicontinuous representation of $\mathfrak{A}$ on $(\mathfrak{A}, \|\ \|_{\mathcal{H}})$ such that $\psi(U)$ is unitary whenever $U \in \mathfrak{G}$.

By choice of $\mathfrak{G}$, every A in $\mathfrak{A}$ is the norm limit of linear combinations of elements of $\mathfrak{G}$. Hence $\psi(\mathfrak{A})$ is the norm closed linear subspace of $\mathcal{B}((\mathfrak{A}, \|\ \|_{\mathcal{H}}))$ generated by the group $\psi(\mathfrak{G})$ of unitary (hence norm 1) operators. By Remark 5, $\psi(\mathfrak{A})$ is a C*-algebra.

Let $\mathcal{U}$ denote the (full) group of unitary operators in $\psi(\mathfrak{A})$. Since ψ is bicontinuous, $\psi^{-1}(\mathcal{U})$ is a bounded group of (invertible) elements of $(\mathfrak{A}, \|\ \|)$, and $\psi^{-1}(\mathcal{U}) \supseteq \mathfrak{G}$. Since $\mathfrak{G}$ is assumed to be maximal, $\psi^{-1}(\mathcal{U}) = \mathfrak{G}$, whence $\mathcal{U} = \psi(\mathfrak{G})$.

Let A be an element of $\mathfrak{A}$, and suppose that $\|\psi(A)\| \leq 1$. By the Russo–Dye theorem, $\psi(A)$ is a norm limit of convex combinations of elements of $\mathcal{U}$ $(= \psi(\mathfrak{G}))$. Hence A is a norm limit of convex combinations of elements of $\mathfrak{G}$. All elements of $\mathfrak{G}$ have norm 1, so $\|A\| \leq 1$. On the other hand, if $A \in \mathfrak{A}$, and $\|A\| \leq 1$, then A is a norm limit of convex combinations of elements of $\mathfrak{G}$. Thus $\psi(A)$ is a norm limit of convex combinations of elements of $\psi(\mathfrak{G})$. All elements of $\psi(\mathfrak{G})$ have norm 1, so $\|\psi(A)\| \leq 1$. With A non-zero, then, $\|\psi(\|A\|^{-1}A)\| \leq 1$ and $\|\psi(A)\| \leq \|A\|$. At the same time, $\|\psi(\|\psi(A)\|^{-1}A)\| \leq 1$, whence $\|\psi(A)\|^{-1}\|A\| \leq 1$ and $\|A\| \leq \|\psi(A)\|$. It follows that ψ is an isometric isomorphism of $\mathfrak{A}$ with the C*-algebra $\psi(\mathfrak{A})$. □

Lemma 7. *Let $(\mathfrak{A}, \|\ \|)$ be a (unital) Banach algebra, and ϕ be a representation of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$ such that $\phi(I) = I$ and $\|\phi\| \leq 1$. If $(\mathfrak{A}, \|\ \|, \mathfrak{G})$ satisfies the K–P condition, then $\phi(\mathfrak{A})$ is a self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$. If $(\mathfrak{A}, \|\ \|, \mathfrak{G})$ satisfies the R–D condition (that is, $\|\ \|$ is unitary), then the norm closure $\phi(\mathfrak{A})^=$ of $\phi(\mathfrak{A})$ is self-adjoint.*

Proof. Let $\mathfrak{G}$ be the group relative to which the K–P condition (or the R–D condition) is satisfied. With A in $\mathfrak{G}$, $\|\phi(A)\| \leq 1$ and $\|\phi(A)^{-1}\| \leq 1$. By polar decomposition, $\phi(A) = UH$ and $\phi(A^{-1}) = H^{-1}U^*$ where U is unitary and H is positive in $\mathcal{B}(\mathcal{H})$. Now, $\|H\| = \|U^*\phi(A)\| \leq \|\phi(A)\| \leq 1$, and $\|H^{-1}\| \leq \|\phi(A^{-1})U\| \leq \|\phi(A^{-1})\| \leq 1$. Since H is positive, it follows that $H = I$, whence $\phi(A)$ $(= U)$ is a unitary operator in $\mathcal{B}(\mathcal{H})$ and $\phi(A^{-1}) = \phi(A)^*$. Thus $\phi(\mathfrak{G})$ is a self-adjoint subset of $\mathcal{B}(\mathcal{H})$ as is its linear span and the norm closure $\mathfrak{B}$ of that linear span.

If $(\mathfrak{A}, \mathfrak{G})$ satisfies the K–P condition, then each element of $\mathfrak{A}$ has a multiple in co $\mathfrak{G}$, whence $\phi(\mathfrak{A})$ is the linear span of $\phi(\mathfrak{G})$, and $\phi(\mathfrak{A})$ is self-adjoint.

Suppose that $(\mathfrak{A}, \mathfrak{G})$ satisfies the R–D condition. Then each T in $\mathfrak{A}$ has a positive multiple that is in the norm closure of the convex combinations of elements of $\mathfrak{G}$. By continuity of ϕ, $\phi(T)$ is in the norm closure $\mathfrak{B}$ of the linear span of $\phi(\mathfrak{G})$. Thus $\phi(\mathfrak{A}) \subseteq \mathfrak{B}$ and $\phi(\mathfrak{A})^= \subseteq \mathfrak{B}$. Of course, the linear span of $\phi(\mathfrak{G})$ is contained in $\phi(\mathfrak{A})$. Thus $\mathfrak{B} \subseteq \phi(\mathfrak{A})^=$. Hence $\phi(\mathfrak{A})^= = \mathfrak{B}$, and $\phi(\mathfrak{A})^=$ is self-adjoint. □

In **[CE]**, Choi and Effros characterize self-adjoint linear spaces of operators acting on Hilbert spaces (*operator systems*) up to complete order isomorphisms in terms of *matrix orderings.* In **[ER]** and **[R]**, Effros and Ruan characterize (not necessarily self-adjoint) linear spaces of operators acting on Hilbert spaces (*operator spaces*) up to complete isometry in terms of their concept of *matricial norming.*

With V a vector space over $\mathbb{C}$, the set $M_n(V)$ of $n \times n$ matrices with entries from V is, again, a vector space over $\mathbb{C}$ (with entrywise operations). With A in $M_n(\mathbb{C})$ $(= M_n)$ and v_n in $M_n(V)$, we denote by Av_n and v_nA the elements of $M_n(V)$ formed by left and right multiplication of v_n by A (in the obvious sense of matrix multiplication). With these actions of M_n on $M_n(V)$, $M_n(V)$ becomes an M_n-bimodule. Let $v_n \oplus v_m$ denote the element of $M_{n+m}(V)$ whose principal diagonal blocks (from top to bottom) are v_n and v_m and whose off-diagonal blocks have 0 (in V) at each entry.

Effros and Ruan say that V is *L^∞-matricially normed* when each $M_n(V)$

is equipped with a norm $\| \|_n$ such that the family of norms satisfies two conditions.

(1) $\|v_n \oplus v_m\|_{n+m} = \max\{\|v_n\|_n, \|v_m\|_m\}$

(2) $\|Av_nB\| \leq \|A\|\|v_n\|_n\|B\|$

when $v_n \in M_n(V)$, $v_m \in M_m(V)$, and A, $B \in M_n$, where $\|A\|$ and $\|B\|$ are the norms the matrices A and B acquire as bounded operators on $\mathbb{C}^n$ *equipped with its l_2-norm* (that is, *its Hilbert-space norm*). In [**R**], Ruan shows that an L^∞-matricially normed space V is completely isometric with a subspace $\mathfrak{A}$ of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Of course, each such subspace $\mathfrak{A}$ is L^∞-matricially normed by the norms it acquires from the subspaces $M_n(\mathfrak{A})$ of $\mathcal{B}(\mathcal{H}^n)$ where $\mathcal{H}^n$ is the n-fold direct sum of $\mathcal{H}$ with itself (each matrix of operators from $\mathfrak{A}$ acting in the usual fashion on the elements of $\mathcal{H}^n$ as column vectors).

We make some observations in connection with the preceding discussion. Condition (1) causes the norms on an L^∞-matricially normed space V to behave like the supremum norm (bound) of operators on a normed space. Condition (2) imparts to the normed space on which the operators act its Hilbert-space structure (the *quadratic character* of its norm) by virtue of the norm we have chosen on M_n (by choosing the Hilbert-space norm on $\mathbb{C}^n$). That we are dealing with matricial norming and *complete* isometry is of the essence in this discussion for *each* normed space is *isometric* with an operator space. To see this, note that with V a normed space and $(V^\#)_1$ the unit ball of its dual space, the mapping that assigns to each v in V the function $\hat{v}$ on $(V^\#)_1$ defined by $\hat{v}(\rho) = \rho(v)$, for each ρ in $(V^\#)_1$ is an isometric linear mapping of V into $C((V^\#)_1)$, from the Hahn–Banach theorem, where $C((V^\#)_1)$ is equipped with its supremum norm and $(V^\#)_1$, provided with its weak* topology, is a compact Hausdorff space. Of course, $C((V^\#)_1)$ is a (commutative) C*-algebra, and as such, has an isometric * representation on a Hilbert space $\mathcal{H}$. Composing this representation and the mapping $v \mapsto \hat{v}$, V is now isometric with a linear subspace of $\mathcal{B}(\mathcal{H})$.

When our L^∞-matricially normed space $\mathfrak{A}$ is an algebra with unit I such that $\|I\| = 1$ and $\|A_nB_n\|_n \leq \|A_n\|_n\|B_n\|_n$ for all A_n and B_n in $M_n(\mathfrak{A})$ and each positive integer n, Blecher, Ruan, and Sinclair [**BRS**] tell us that $\mathfrak{A}$ is completely isometric and isomorphic with an algebra of operators on a Hilbert space. On the other hand, when $\mathfrak{A}$ is a normed algebra isometric with an algebra of operators $\mathfrak{B}$ on a Hilbert space $\mathcal{H}$, $\mathfrak{A}$ acquires the L^∞-matricial norming of $\mathfrak{B}$. The assumption that $\mathfrak{A}$ is isomorphic and (*simply*, rather than, *completely*) isometric with an operator algebra $\mathfrak{B}$ acting on $\mathcal{H}$, is a more serious restriction on $\mathfrak{A}$ than the assumption of simple isometry is in the case of a normed space. In general, a unital Banach algebra can fail to be isomorphic and isometric with an algebra of operators on a Hilbert space. We can use our unitary-norm techniques and results to see that the Wiener algebra is not isomorphic and isometric with an algebra of operators on a Hilbert space. Suppose it were. We note (Lemma 20) that the norm on the Wiener algebra is unitary. From Theorem 4, then, the algebra of operators on a Hilbert space to which it is (as assumed) isometrically isomorphic is a C*-algebra. This C*-algebra is commutative and, therefore, (isometric and isomorphic with) a $C(X)$ for some compact Hausdorff space X. However, as we note with the aid of Lemma 21, the Wiener algebra is not even algebraically isomorphic to a $C(X)$.

Combining the Blecher–Ruan–Sinclair characterization with our Theorem 4, we have a Banach-space characterization of those Banach algebras that are C*-algebras up to complete isometry. The characterization of the unitary group of a C*-algebra as a maximal bounded subgroup of elements in the unit ball of an L^∞-matricially normed Banach algebra follows, now, from Proposition 3.

Theorem 8. *A unital algebra $\mathfrak{A}$ that is an L^∞-matricially normed, Banach algebra with a unitary norm is completely isometric and isomorphic to a C*-algebra. If $\mathfrak{G}$ is the group for the unitary norm, that norm is maximal if and only if $\mathfrak{G}$ maps* onto *the unitary group of the C*-algebra.*

3. Unitary Norms on $C(X)$.

In this section, we discuss the possibility of introducing unitary norms on $C(X)$, when X is a compact Hausdorff space. We study the properties of such norms. Let $\| \ \|$ denote the supremum norm on $C(X)$, let $C(X)_{\mathrm{inv}}$ be the invertible elements in $C(X)$, and $C(X)_{\mathrm{u}}$ be the unitary functions in $C(X)$. Our approach is to examine the properties of a subgroup $\mathfrak{G}$ of $C(X)_{\mathrm{inv}}$ for which the $\| \ \|$-closure $\mathcal{S}$ of $\mathrm{co}\,\mathfrak{G}$ is the closed unit ball relative to a normed-algebra norm on $C(X)$. We develop some general results and apply them to the case where X has a finite number of points (so that $C(X) = \mathbb{C}^n$, where $|X| = n$).

Lemma 9. *If $(C(X), \| \ \|')$ is a Banach algebra, $\mathfrak{G}$ is a subgroup of $C(X)_{\mathrm{inv}}$ such that the $\| \ \|'$-closure $\mathcal{S}$ of $\mathrm{co}\,\mathfrak{G}$ is the closed unit ball of $(C(X), \| \ \|')$, then $\mathfrak{G}$ is $\| \ \|$-bounded and separates the points of X and $\mathfrak{G} \subseteq C(X)_{\mathrm{u}}$.*

Proof. Since $\| \ \|'$ is a Banach-algebra norm on $C(X)$, $|f(x)| \le \|f\|'$ for each x in X (as $f(x) \in \mathrm{sp}(f)$). Thus $\|f\| \le \|f\|'$ for each f in $C(X)$. With u in $\mathfrak{G}$, $u^n \in \mathfrak{G}$ for each integer n. Thus $|u(x)|^n = |u^n(x)| \le \|u^n\| \le \|u^n\|' \le 1$ for each integer n; and $|u(x)| = 1$ for all x in X. It follows that $\mathfrak{G} \subseteq C(X)_{\mathrm{u}}$ and that $\mathfrak{G}$ is $\| \ \|$-bounded.

Since X is completely regular, $C(X)$ separates the points of X as does every subset that spans a dense linear manifold in $C(X)$. As $\operatorname{co}\mathfrak{G}$ is $\|\ \|'$-dense in $\mathcal{S}$, the unit ball of $(C(X), \|\ \|')$, we have that $\mathfrak{G}$ separates the points of X. □

Lemma 10. *Let* $(\mathfrak{A}, \|\ \|)$ *be a complex Banach algebra and* $\mathfrak{G}$ *be a subgroup of* $\mathfrak{A}_{\text{inv}}$. *If the* $\|\ \|$*-closure* $\mathcal{S}$ *of* $\operatorname{co}\mathfrak{G}$ *is the (closed) unit ball for some norm* $\|\ \|'$ *on* $\mathfrak{A}$, *then* $(\mathfrak{A}, \|\ \|')$ *is a normed algebra.*

Proof. We show that if A and B are in $\mathcal{S}$, then $AB \in \mathcal{S}$. From this, we conclude that if $\|A\|' \leq 1$ and $\|B\|' \leq 1$, then $\|AB\|' \leq 1$. With T and S arbitrary (non-zero) elements of $\mathfrak{A}$, $T\|T\|'^{-1}$ and $S\|S\|'^{-1}$ are in $\mathcal{S}$. Thus $\|TS\|'\|T\|'^{-1}\|S\|'^{-1} \leq 1$ and $\|TS\|' \leq \|T\|'\|S\|'$.

To prove that $AB \in \mathcal{S}$ when A and B are in $\mathcal{S}$, assume that $\{A_n\}$ and $\{B_n\}$ are sequences in $\operatorname{co}\mathfrak{G}$ such that $\|A_n - A\| \to 0$ and $\|B_n - B\| \to 0$. Then $A_n = \sum_{j=1}^{r_n} a_{j,n}U_{j,n}$ and $B_n = \sum_{k=1}^{s_n} b_{k,n}V_{k,n}$, where $a_{j,n}$ and $b_{k,n}$ are positive real numbers, $\sum_j a_{j,n} = \sum_k b_{k,n} = 1$, and $U_{j,n}$ and $V_{k,n}$ are in $\mathfrak{G}$. We have that $A_nB_n = \sum_{j=1}^{r_n}\sum_{k=1}^{s_n} a_{j,n}b_{k,n}U_{j,n}V_{k,n}$. Now, $U_{j,n}V_{k,n} \in \mathfrak{G}$ and $\sum_{j,k} a_{j,n}b_{k,n} = \left(\sum_j a_{j,n}\right)\left(\sum_k b_{k,n}\right) = 1$. Of course, $a_{j,n}b_{k,n} > 0$ and $\|A_nB_n - AB\| \to 0$. Thus A_nB_n is in the convex hull of $\mathfrak{G}$ and $AB \in \mathcal{S}$. □

Recall, that a subset Y of a linear space is said to be *balanced* if $ay \in Y$ whenever $y \in Y$, $a \in \mathbb{C}$, and $|a| \leq 1$.

Lemma 11. *If* $(\mathfrak{A}, \|\ \|)$ *is a complex Banach algebra and* $\mathfrak{G}$ *is a bounded subgroup of* $\mathfrak{A}_{\text{inv}}$ *such that* $\theta U \in \mathfrak{G}$ *whenever* $\theta \in \mathbb{T}_1$ *and* $U \in \mathfrak{G}$, *then* $\operatorname{co}\mathfrak{G}$ *and its norm closure* $\mathcal{S}$ *are balanced sets. For each* T *in* $\mathfrak{A}$, *define* $\|T\|'$ *to be* $\inf\{t \in \mathbb{R}^+ : T \in t\mathcal{S}\}$. *If each element of* $\mathfrak{A}$ *has a positive multiple in* $\mathcal{S}$, *then* $\|\ \|'$ *is a norm on* $\mathfrak{A}$ *for which* $\mathcal{S}$ *is the closed unit ball.*

Proof. Since both I and $-I$ are in $\mathfrak{G}$, 0 is in $\operatorname{co}\mathfrak{G}$. Thus, for each A in $\operatorname{co}\mathfrak{G}$, $tA \in \operatorname{co}\mathfrak{G}$ when $0 \leq t \leq 1$. With U_j in $\mathfrak{G}$ and $\sum_{j=1}^n a_jU_j$ a convex combination of $U_1, \ldots, U_n$, the product $a\sum_{j=1}^n a_jU_j \in \operatorname{co}\mathfrak{G}$, since $\theta U_j \in \mathfrak{G}$, where $0 < |a| \leq 1$ and $\theta = a|a|^{-1}$. It follows that $\operatorname{co}\mathfrak{G}$ is balanced. As multiplication by a scalar is a continuous mapping from $\mathfrak{A}$ into $\mathfrak{A}$, $\mathcal{S}$ is also balanced.

As noted, $0 \in \mathcal{S}$. By assumption, each A in $\mathfrak{A}$ has a positive multiple t_0A in $\mathcal{S}$. Thus $tA \in \mathcal{S}$ when $0 \leq t \leq t_0$, since $\mathcal{S}$ is convex. Hence 0 is an internal point of $\mathcal{S}$. It follows from [**KR**, Proposition 1.1.5], that $\|\ \|'$ is a semi-norm.

If $\|T\|' = 0$, then $t^{-1}T \in \mathcal{S}$ for arbitrarily small positive t. Hence $\|T\| = 0$, since $\mathcal{S}$ is $\|\ \|$-bounded, and $T = 0$. Thus $\|\ \|'$ is a norm on $\mathfrak{A}$.

If $T \in \mathcal{S}$, then $\|T\|' \leq 1$ by the definition of $\| \ \|'$. If $\|S\|' \leq 1$, then $t^{-1}S \in \mathcal{S}$ for t near $\|S\|'$ and hence $t_n^{-1}S \in \mathcal{S}$ for some monotone sequence $\{t_n\}$ decreasing to 1. As $\|t_n^{-1}S - S\| \to 0$ and $\mathcal{S}$ is $\| \ \|$-closed, we have that $S \in \mathcal{S}$. Thus $\mathcal{S} = \{T \in \mathfrak{A} : \|T\|' \leq 1\}$. □

Lemma 12. *If $\mathfrak{G}$ is a $\| \ \|$-bounded subgroup of $C(X)_{\text{inv}}$, $\mathfrak{G}$ contains the scalars of modulus 1, and $\mathfrak{G}$ separates the points of X, then the set $\mathbb{R}^+ \operatorname{co} \mathfrak{G}$ of positive multiples of $\operatorname{co} \mathfrak{G}$ is a $\| \ \|$-dense subalgebra of $C(X)$. Let $\mathcal{S}$ be the $\| \ \|$-closure of $\operatorname{co} \mathfrak{G}$. If $\mathbb{R}^+\mathcal{S}$ is $\| \ \|$-closed, then there is a norm $\| \ \|'$ on $C(X)$ such that $(C(X), \| \ \|')$ is a normed algebra and $\mathcal{S}$ is its unit ball.*

Proof. Since $\mathfrak{G}$ contains the scalars of modulus 1, $\mathbb{R}^+ \operatorname{co} \mathfrak{G}$ is the (complex) algebra generated by $\mathfrak{G}$. As $\mathfrak{G}$ is a $\| \ \|$-bounded subgroup of $C(X)_{\text{inv}}$, each element of $\mathfrak{G}$ is a unitary function on X by Lemma 9. Thus, if $u \in \mathfrak{G}$, then $\bar{u} = u^{-1} \in \mathfrak{G}$. It follows that $\mathbb{R}^+ \operatorname{co} \mathfrak{G}$ is a subalgebra of $C(X)$ stable under complex conjugation, containing the constant functions, and separating the points of X. From the Stone–Weierstrass theorem (compare [**KR**, Theorem 3.4.14]), $\mathbb{R}^+ \operatorname{co} \mathfrak{G}$ is $\| \ \|$-dense in $C(X)$.

If $\mathbb{R}^+\mathcal{S}$ is $\| \ \|$-closed, then $\mathbb{R}^+\mathcal{S} = C(X)$. Hence each element of $C(X)$ has a positive multiple in $\mathcal{S}$. From Lemma 11, there is a norm $\| \ \|'$ on $C(X)$ with $\mathcal{S}$ as its closed unit ball.

From Lemma 10, $(C(X), \| \ \|')$ is a normed algebra. □

Example 13. With $\mathcal{K}$ a $\| \ \|$-closed convex subset of $C(X)$, $\mathbb{R}^+\mathcal{K}$ may fail to be closed. To see this, let X be the unit interval $[0, 1]$, and $\mathcal{K}$ the set of functions f in $C([0, 1])$ such that $|f| \leq \iota$, where ι denotes the identity transform on $[0, 1]$ (that is, $\iota(t) = t$ for all t in $[0, 1]$). Of course, $\mathcal{K}$ is $\| \ \|$-closed and convex. In addition, $\mathcal{K}$ is stable under complex conjugation and separates the points of $[0, 1]$.

We suppose, now, that $\mathbb{R}^+\mathcal{K}$ is $\| \ \|$-closed and draw a contradiction from this assumption. Since $af \in \mathcal{K}$ when $f \in \mathcal{K}$ and $|a| \leq 1$, $\mathbb{R}^+\mathcal{K}$ is a $\| \ \|$-closed, complex subalgebra of $C([0, 1])$ stable under complex conjugation and separating the points of $[0, 1]$. The algebra obtained from $\mathbb{R}^+\mathcal{K}$ by adjoining the one-dimensional space of constant functions has the same properties, and of course, contains the constants. From the Stone–Weierstrass theorem, this algebra is $C([0, 1])$. Thus each g in $C([0, 1])$ has the form $a + rf$ with a a constant, $r > 0$, and f in $\mathcal{K}$. If $g(0) = 0$, then $a = 0$, since $f(0) = 0$. Hence $g = rf$ in this case.

Let $g(t)$ be $(t - t^2)^{1/2}$ for t in $[0, 1]$. (The graph of g is the upper semi-circle with center at $\frac{1}{2}$ on the real axis and radius $\frac{1}{2}$.) Then $g = rf \leq r\iota$ for some positive r and some f in $\mathcal{K}$. But $g \not\leq r\iota$ for all r in $\mathbb{R}^+$. Hence $\mathbb{R}^+\mathcal{K}$ is not $\| \ \|$-closed.

Theorem 14. *Let X be a finite set of points and $\mathfrak{G}$ be a $\| \, \|$-bounded subgroup of $C(X)_{\text{inv}}$ containing the scalars of modulus 1. The $\| \, \|$-closure $\mathcal{S}$ of $\operatorname{co}\mathfrak{G}$ is the closed unit ball for a Banach-algebra norm $\| \, \|'$ on $C(X)$ if and only if $\mathfrak{G}$ separates the points of X. If $\mathfrak{G}$ separates the points of X, then $\| \, \|$ and $\| \, \|'$ coincide if and only if $\mathfrak{G}$ is $\| \, \|$-dense in the group $C(X)_{\text{u}}$ of unitary functions on X.*

Proof. With X a finite set, $C(X)$ $(= \mathbb{C}^n$, where $n = |X|)$ is finite dimensional, all norms on $C(X)$ are equivalent to $\| \, \|$, $(C(X), \| \, \|')$ is a Banach algebra for each normed algebra norm $\| \, \|'$ on $C(X)$, and each linear subspace of $C(X)$ is $\| \, \|$-closed. If $\mathcal{S}$ is a unit ball, then $\mathfrak{G}$ separates the points of X from Lemma 9. Since the algebra $\mathbb{R}^+\mathcal{S}$ is $\| \, \|$-closed, we have that if $\mathfrak{G}$ separates points, then Lemma 12 applies and $\mathcal{S}$ is the unit ball for a Banach-algebra norm $\| \, \|'$ on $C(X)$.

If $\mathfrak{G}$ is $\| \, \|$-dense in $C(X)_{\text{u}}$, then without further assumption, $\mathcal{S}$ is the $\| \, \|$-closure of $\operatorname{co}(C(X)_{\text{u}})$. From the Russo–Dye theorem, the $\| \, \|$-closure of $\operatorname{co}(C(X)_{\text{u}})$ is the unit ball of $(C(X), \| \, \|)$, so that $\mathcal{S}$ is this unit ball.

Again, with X a finite set, if $\mathfrak{G}$ is not dense in $C(X)_{\text{u}}$, its $\| \, \|$-closure $\mathfrak{G}^=$ is a proper compact subgroup of $C(X)_{\text{u}}$. Since $\mathcal{S}$ is the $\| \, \|$-closure of $\operatorname{co}(\mathfrak{G}^=)$, we have that $\mathfrak{G}^=$ contains the extreme points of $\mathcal{S}$, from [**KR**, Theorem 1.4.5]. As the unitary functions in $C(X)$ are the extreme points of the unit ball (compare [**KR**, Theorem 7.3.1]), $\mathcal{S}$ is not this unit ball, whence $\| \, \|'$ and $\| \, \|$ are different. □

Remark 15. With X finite and $\mathfrak{G}$ a subgroup of $C(X)_{\text{u}}$ separating the points of X and containing the scalars of modulus 1, there is a Banach-algebra norm $\| \, \|'$ on $C(X)$ with $\mathcal{S}$ its unit ball. From the proof of Theorem 14, each extreme point of $\mathcal{S}$ lies in the norm closure $\mathfrak{G}^=$ of $\mathfrak{G}$. On the other hand, $\|f\| \le \|f\|' \le 1$ when $f \in \mathcal{S}$, so that $\mathcal{S}$ is contained in the unit ball of $(C(X), \| \, \|)$. Thus each unitary in $\mathcal{S}$ is an extreme point (of the unit ball of $(C(X), \| \, \|)$ and *a fortiori*) of $\mathcal{S}$. Hence each element of $\mathfrak{G}^=$ is an extreme point of $\mathcal{S}$ and the set of extreme points of $\mathcal{S}$ is precisely $\mathfrak{G}^=$.

Example 16. We construct some subgroups $\mathfrak{G}$ of $C(X)_{\text{u}}$ giving rise to unitary norms on $C(X)$ distinct from one another and from the supremum norm $\| \, \|$. The group $\mathfrak{G}$ will be $\| \, \|$-closed and $\mathfrak{G}/\mathbb{T}_1$ will be finite in each case.

We assume, again, that X is finite, say $X = \{1, \dots, n\}$. We denote by $\mathbb{Z}_m$ the group of mth roots of unity. Let $m_2, \dots, m_n$ be integers not less than 2 and define

$$\mathfrak{G}(m_2, \dots, m_n) = \left\{ u \in C(X)_{\text{u}} : \overline{u(1)}u(j) \in \mathbb{Z}_{m_j}, \forall j \in \{2, \dots, n\} \right\}.$$

Then $\mathfrak{G}(m_2, \dots, m_n)$ is a closed subgroup of $C(X)_{\text{u}}$ containing the scalars of modulus 1, $\mathbb{T}_1$, and $\mathfrak{G}(m_2, \dots, m_n)/\mathbb{T}_1$ is $\mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_n}$ (to see this, note that the mapping $u \mapsto \left(\overline{u(1)}u(2), \ldots, \overline{u(1)}u(n)\right)$ has $\mathbb{T}_1$ as kernel and is a homomorphism of $\mathfrak{G}(m_2, \ldots, m_n)$ onto $\mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_n}$). With j and k distinct elements of $\{1, \ldots, n\}$, assume that $k \neq 1$, and let $u(h)$ be 1 when $h \neq k$ and $u(k)$ be an element of $\mathbb{Z}_{m_k}$ different from 1. Then $u \in \mathfrak{G}(m_2, \ldots, m_n)$ and u separates j and k. Thus $\mathfrak{G}(m_2, \ldots, m_n)$ separates the points of X and gives rise to a unitary norm with unit ball $\mathrm{co}(\mathfrak{G}(m_2, \ldots, m_n))^=$ whose set of extreme points is $\mathfrak{G}(m_2, \ldots, m_n)$ by Remark 15. Of course, $\mathfrak{G}(m_2, \ldots, m_n)$ and $\mathfrak{G}(m_2', \ldots, m_n')$ are different groups unless $m_2 = m_2', \ldots, m_n = m_n'$. In case they are different groups, the extreme point set of the unit ball for the unitary norm of one group is different from the corresponding set of the other. Thus the two unitary norms are distinct and different from $\| \;\|$.

Are all (normalized) Banach-algebra norms on $C(X)$, with X finite, unitary norms? We shall answer this in the negative in Example 18.

Lemma 17. *Let X be a finite set, $\| \;\|'$ a norm on $C(X)$ $(= \mathbb{C}^n$, when $n = |X|)$, $\mathcal{S}$ the closed unit ball of $(C(X), \| \;\|')$, and $\mathcal{E}$ the set of extreme points of $\mathcal{S}$. Then $\| \;\|'$ is a (normalized) Banach-algebra norm on $C(X)$ if and only if the constant function 1 is in $\mathcal{S}$ (and $\mathcal{E}$) and $fg \in \mathcal{S}$ when f and g are in $\mathcal{E}$.*

Proof. Since $C(X)$ is finite dimensional, $\| \;\|'$ and $\| \;\|$ are equivalent, $\| \;\|'$ is a Banach-space norm on $C(X)$, and multiplication in $C(X)$ is jointly continuous relative to $\| \;\|'$. In the broad sense, $\| \;\|'$ is a Banach-algebra norm on $C(X)$ without further discussion. What is at issue is the question of when $\| \;\|'$ is *normalized.* We have assumed that $\|1\|' \leq 1$; if we have the multiplication inequality for $\| \;\|'$ $(\|fg\|' \leq \|f\|'\|g\|')$, then $\|1\|' = \|1 \cdot 1\|' \leq \|1\|'^2$, whence $\|1\|' \geq 1$, and $\|1\|' = 1$.

If $\| \;\|'$ is a normalized Banach-algebra norm, then $1 \in \mathcal{S}$, and, for f and g in $\mathcal{E}$, $\|fg\|' \leq \|f\|'\|g\|' = 1$, so $fg \in \mathcal{S}$.

Suppose $1 \in \mathcal{S}$ and $fg \in \mathcal{S}$ when $f, g \in \mathcal{E}$. If $\sum_{j=1}^n a_j f_j$ $(= f)$ and $\sum_{k=1}^m b_k g_k$ $(= g)$ are convex combinations of elements f_j and g_k of $\mathcal{E}$, then

$$fg = \left(\sum_{j=1}^n a_j f_j\right)\left(\sum_{k=1}^m b_k g_k\right) = \sum_{j=1}^n \sum_{k=1}^m a_j b_k f_j g_k.$$

By assumption, $f_j g_k \in \mathcal{S}$. Moreover, $a_j b_k \geq 0$ and $\sum_{j=1}^n \sum_{k=1}^m a_j b_k = 1$. Thus fg is a convex combination of elements of $\mathcal{S}$, so $fg \in \mathcal{S}$.

Suppose $\|f - f_n\|' \to 0$ and $\|g - g_n\|' \to 0$ as $n \to \infty$, with f_n and g_n in $\mathrm{co}\,\mathcal{E}$. Then $f_n g_n \in \mathcal{S}$, from the preceding argument, and $\|fg - f_n g_n\|' \to 0$ as $n \to \infty$. Thus $fg \in \mathcal{S}$. From the Krein–Milman theorem ($\mathcal{S}$ is compact and convex), $\mathcal{S}$ is the $\| \;\|'$-closure of $\mathrm{co}\,\mathcal{E}$. Thus $fg \in \mathcal{S}$ when f and g

are in $\mathcal{S}$; that is, $\|fg\|' \le 1$ when $\|f\|' \le 1$ and $\|g\|' \le 1$. It follows that $\|fg\|' \le \|f\|'\|g\|'$, and $\|\ \|'$ is a (normalized) Banach-algebra norm. □

Lemma 17 indicates a technique for constructing (normalized) Banach-algebra norms on $C(X)$. We start with a compact set $\mathcal{F}$ stable under multiplication by scalars of modulus 1, included in which are the potential extreme points of the closed unit ball. In addition, $\mathcal{F}$ should contain 1, have the property that a product of two of its elements lies in its closed convex hull $\mathcal{S}$, and $\mathcal{S}$ should contain some $\|\ \|$-ball of positive radius with center 0.

Example 18. Using the technique just described and Example 16, we construct a Banach-algebra norm on $C(X)$ that is not unitary. Let $\mathfrak{G}$ be the group $\mathfrak{G}(3,\ldots,3)$ of Example 16, where X is $\{1,\ldots,n\}$. Let a be a constant in $(0,1)$, define $\mathcal{F}$ to be

$$\left\{u \in \mathfrak{G} : \overline{u(1)}u(2) = 1\right\} \cup \left\{au : u \in \mathfrak{G}, \overline{u(1)}u(2) \ne 1\right\},$$

and let $\mathcal{S}$ be the $\|\ \|$-closed convex hull of $\mathcal{F}$.

We note, first, that $\mathcal{F}$ is $\|\ \|$-closed. If $f_m \in \mathcal{F}$ and $\|f_m - f\| \to 0$ as $m \to \infty$, then either $\overline{f_m(1)}f_m(2) \ne 1$ and $\left(\overline{f_m(1)}f_m\right)^3 = a^6$ or $\overline{f_m(1)}f_m(2) = 1$ and $\left(\overline{f_m(1)}f_m\right)^3 = 1$. Since $\overline{f_m(1)}f_m(2) \to \overline{f(1)}f(2)$ as $m \to \infty$, either $\overline{f_m(1)}f_m(2) = 1$ for all large m, so that $\overline{f(1)}f(2) = 1$ and $\left(\overline{f(1)}f\right)^3 = 1$, or $\overline{f_m(1)}f_m(2) = a^2\theta$ for all large m, where $1 \ne \theta \in \mathbb{Z}_3$, and $\left(\overline{f_m(1)}f_m\right)^3 = a^6$ for all large m, so that $\overline{f(1)}f(2) = a^2\theta$ and $\left(\overline{f(1)}f\right)^3 = a^6$. In either case, $f \in \mathcal{F}$ and $\mathcal{F}$ is closed.

By construction, $\mathcal{F}$ contains the scalars of modulus 1. With f and g in $\mathcal{F}$, f is u or au and g is v or av, with u and v in $\mathfrak{G}$. Thus fg is a^2uv, auv, or uv. If fg is uv, then f is u and g is v, so that u and v are in $\mathcal{F}$, $\overline{u(1)}u(2) = 1$, $\overline{v(1)}v(2) = 1$, and $\overline{(uv)(1)}(uv)(2) = 1$. Thus $fg = uv \in \mathcal{F}$, in this case. If $fg = auv$, then either $f = au$ and $g = v$, or $f = u$ and $g = av$. In the first case, $\overline{u(1)}u(2) \ne 1$ and $\overline{v(1)}v(2) = 1$. Thus $\overline{(uv)(1)}(uv)(2) \ne 1$, and $fg = auv \in \mathcal{F}$, in this case. Similarly, for the second case. If $fg = a^2uv$, then $fg \in \mathcal{S}$ since either uv or auv is in $\mathcal{F}$ and $0 \in \mathcal{S}$. In any event, $fg \in \mathcal{S}$.

From Example 16, $\operatorname{co}(\mathfrak{G})^=$ is the unit ball for a (Banach-algebra) norm on $C(X)$. Since all norms on $C(X)$ are equivalent, $\operatorname{co}(\mathfrak{G})^=$ contains a $\|\ \|$-ball of positive radius with center 0, as does $a\operatorname{co}(\mathfrak{G})^=$ $(= \operatorname{co}(a\mathfrak{G})^=)$. Now, $a\mathfrak{G} \subseteq \mathcal{S}$ since $0 \in \mathcal{S}$ and for each u in $\mathfrak{G}$, either u or au belongs to $\mathcal{F}$. Thus $\operatorname{co}(a\mathfrak{G})^= \subseteq \mathcal{S}$, and $\mathcal{S}$ contains a $\|\ \|$-ball of positive radius with center 0.

It follows (from Lemma 12) that $\mathcal{S}$ is the closed unit ball of a Banach-algebra norm $\|\ \|'$ on $C(X)$. Each u in $\mathfrak{G}$ such that $\overline{u(1)}u(2) = 1$ is an extreme point of $\mathcal{S}$. Moreover, all extreme points of $\mathcal{S}$ are in $\mathcal{F}$. Now, $\mathfrak{G}\cap\mathcal{F}$ consists of unitary elements u in $C(X)$ such that $u(1) = u(2)$. Thus $\text{co}(\mathfrak{G}\cap\mathcal{F})^{=}$ does not separate 1 and 2 in X, whence $\text{co}(\mathfrak{G}\cap\mathcal{F})^{=} \neq \mathcal{S}$. It follows that $\mathfrak{G}\cap\mathcal{F}$ does not contain all the extreme points of $\mathcal{S}$ and that there are extreme points in $\mathcal{F}$ of the form av with v in $\mathfrak{G}$. Since $(av)^{-1} = a^{-1}v^* \notin \mathcal{F}$, we conclude that the set of extreme points of $\mathcal{S}$ does not form a group (and includes functions that are not unitary). Thus $\|\ \|'$ is a (normalized) Banach-algebra norm on $C(X)$ that is not unitary.

With some further effort, we can show that $\mathcal{F}$ is precisely the set of extreme points of $\mathcal{S}$. To see this, note that if $u \in \mathfrak{G}\cap\mathcal{F}$, then $uf \in \mathcal{F}$ when $f \in \mathcal{F}$. If $M_u g$ is ug for g in $C(X)$, then with u in $\mathfrak{G}\cap\mathcal{F}$, M_u is a linear isomorphism of $C(X)$ onto itself that carries $\mathcal{S}$ into $\mathcal{S}$. The same is true for the inverse $M_{\bar{u}}$ of M_u. Hence M_u and $M_{\bar{u}}$ have bound not exceeding 1 relative to $\|\ \|'$. Thus $\|M_u f\|' \le \|f\|'$, and

$$\|f\|' = \|M_{\bar{u}}M_u f\|' \le \|M_u f\|'.$$

It follows that $\|M_u f\|' = \|f\|'$ and that M_u is a $\|\ \|'$-isometric linear isomorphism of $C(X)$ onto $C(X)$. Thus M_u maps the set of extreme points of $\mathcal{S}$ onto itself. With av an extreme point of $\mathcal{S}$ and $\overline{v(1)}v(2)$ the element θ of $\mathbb{Z}_3$, where $\theta \neq 1$, by choosing u appropriately in $\mathfrak{G}\cap\mathcal{F}$, we can arrange that uv is any previously assigned element w of $\mathfrak{G}$ such that $\overline{w(1)}w(2) = \theta$. Since $M_u(av) = auv = aw$, aw is an extreme point of $\mathcal{S}$ for each w in $\mathfrak{G}$ such that $\overline{w(1)}w(2) = \theta$. From our earlier argument, we know that there is an extreme point of the form av, where $\overline{v(1)}v(2)$ is one of the elements θ of the group $\mathbb{Z}_3$ different from 1. What we need, to complete the argument that the set of extreme points of $\mathcal{S}$ is precisely $\mathcal{F}$, is that au is an extreme point of $\mathcal{S}$ for some u in $\mathfrak{G}$ such that $\overline{u(1)}u(2) = \theta^2$. We prove this.

Note that the mapping $f \mapsto \bar{f}$, complex conjugation, is a real-linear isomorphism of $C(X)$ onto itself that carries $\mathfrak{G}$ onto $\mathfrak{G}$. To see this, observe that if $u \in C(X)_{\text{u}}$ and $\overline{u(1)}u(j) \in \mathbb{Z}_3$ for j in $\{2, \ldots, n\}$, then $\overline{\bar{u}(1)}\bar{u}(j) = \overline{\overline{u(1)}u(j)} \in \mathbb{Z}_3$ since $\overline{\overline{u(1)}u(j)}$ is the inverse of $\overline{u(1)}u(j)$ (in $\mathbb{Z}_3$). In addition, complex conjugations carries $\mathcal{F}$ onto $\mathcal{F}$. To see this, note that, with f in $\mathcal{F}$, either $f = u \in \mathfrak{G}$ and $\overline{u(1)}u(2) = 1$ or $f = av$ with v in $\mathfrak{G}$ and $\overline{v(1)}v(2)$ is one of the two elements θ and θ^2 of $\mathbb{Z}_3$ different from 1. In the first case, $\overline{\bar{u}(1)}\bar{u}(2) = \overline{\overline{u(1)}u(2)} = 1$ and $\bar{u} \in \mathfrak{G}$, so $\bar{f} \in \mathcal{F}$; in the second case, $\bar{f} = a\bar{v}$ and $\overline{\bar{v}(1)}\bar{v}(2) = \overline{\overline{v(1)}v(2)}$. Since $\bar{\theta} = \theta^2$, $\overline{\theta^2} = \theta$, and $\overline{v(1)}v(2)$ is one of θ or θ^2, we have that $\bar{f} \in \mathcal{F}$.

It follows, now, that complex conjugation maps $\mathcal{S}$ onto $\mathcal{S}$ and the set of extreme points of $\mathcal{S}$ onto itself. If v in $\mathfrak{G}$ is such that $\overline{v(1)}v(2)$ is one of θ or θ^2, then $\overline{\bar{v}(1)}\bar{v}(2)$ is the other. In one case, av $(\in \mathcal{F})$ is an extreme point of

$\mathcal{S}$ from our earlier argument and, hence, $\overline{a\bar{v}} = a\bar{v}$ is an extreme point of $\mathcal{S}$ as well; in the other case $a\bar{v}$ is an extreme point of $\mathcal{S}$ and, hence, $\overline{a\bar{v}} = av$ is an extreme point. It follows that the set of extreme points of $\mathcal{S}$ is precisely $\mathcal{F}$.

Theorem 19. *Suppose $(\mathfrak{A}, \|\ \|')$ is a Banach algebra and $\mathfrak{G}$ is a $\|\ \|'$-bounded subgroup of $\mathfrak{A}_{\text{inv}}$ such that the $\|\ \|'$-closure of $\text{co}\,\mathfrak{G}$ is the closed unit ball of $(\mathfrak{A}, \|\ \|')$. Suppose, moreover, that $\mathfrak{G}$ is a maximal bounded subgroup of $\mathfrak{A}_{\text{inv}}$ and that ϕ is an algebraic isomorphism of $\mathfrak{A}$ onto $C(X)$ for some compact Hausdorff space X. Then ϕ is an isometry of $(\mathfrak{A}, \|\ \|')$ onto $(C(X), \|\ \|)$ carrying $\mathfrak{G}$ onto $C(X)_{\text{u}}$.*

Proof. By means of the mapping ϕ, we may identify $\mathfrak{A}$ with $C(X)$ and regard $\|\ \|'$ as a Banach-algebra norm on $C(X)$. With this identification, $\|\ \|'$ and $\|\ \|$ are equivalent and the same sets are bounded relative to $\|\ \|'$ and $\|\ \|$. Thus $\mathfrak{G}$ is $\|\ \|$-bounded and is a subgroup of $C(X)_{\text{u}}$ by Lemma 9. Moreover, $C(X)_{\text{u}}$ is $\|\ \|'$-bounded. Since $\mathfrak{G}$ is maximal, $\mathfrak{G} = C(X)_{\text{u}}$. From the Russo–Dye theorem (Phelps **[P]** in this case), $\text{co}(C(X)_{\text{u}})^{=}$ is the unit ball of $(C(X), \|\ \|)$. By assumption, $\text{co}(\mathfrak{G})^{=}$ is the unit ball of $(C(X), \|\ \|')$. Thus the unit balls coincide as do $\|\ \|$ and $\|\ \|'$. □

4. The Wiener algebra.

Suppose $(\mathfrak{A}, \|\ \|)$ is a complex, commutative, semi-simple Banach algebra with unit I and $\|\ \|$ is unitary relative to the subgroup $\mathfrak{G}$ of $\mathfrak{A}_{\text{inv}}$. Let X be the space of non-zero, multiplicative linear functionals on $\mathfrak{A}$. Then X is compact in the weak* topology [**KR**, Proposition 3.2.20]. With A in $\mathfrak{A}$, let $\hat{A}(\rho)$ be $\rho(A)$ for each ρ in X. Since each ρ in X has norm 1, the mapping $A \mapsto \hat{A}$ from $\mathfrak{A}$ to $C(X)$ is a homomorphism of norm 1 taking I to the constant function 1. As $\mathfrak{A}$ is semi-simple, this mapping is an isomorphism. Moreover, $\widehat{\mathfrak{G}}$ is a bounded group of invertible elements in $C(X)$. Hence $\widehat{\mathfrak{G}}$ consists of unitary functions on X.

Let $\|\hat{A}\|'$ be $\|A\|$ for each A in $\mathfrak{A}$ and $\|f\|$ be the supremum norm of f for each f in $C(X)$. By assumption, if $\|A\|' \le 1$ and a positive ϵ is given, then there is a convex combination $\sum_{j=1}^{n} a_j\hat{U}_j$ of elements $\hat{U}_j$ in $\widehat{\mathfrak{G}}$ such that $\left\|\hat{A} - \sum_{j=1}^{n} a_j\hat{U}_j\right\|' < \epsilon$. On $C(X)$ the supremum norm is the smallest possible Banach-algebra norm. Hence $\left\|\hat{A} - \sum_{j=1}^{n} a_j\hat{U}_j\right\| < \epsilon$, so $\left\|\hat{A}^* - \sum_{j=1}^{n} a_j\hat{U}_j^{-1}\right\| < \epsilon$, where $\hat{A}^*$ is the complex conjugate $\bar{\hat{A}}$ of $\hat{A}$. It follows that $\hat{A}^*$ is in the $\|\ \|$-closure of $\widehat{\mathfrak{A}}$. By passing to a suitable positive multiple of $\hat{T}$, for an arbitrary T in $\mathfrak{A}$, we see that $\hat{T}^*$ is in the $\|\ \|$-closure of $\widehat{\mathfrak{A}}$. Suppose $\hat{A}_n \in \widehat{\mathfrak{A}}$ and $\left\|f - \hat{A}_n\right\| \to 0$. Then $\left\|\bar{f} - \hat{A}_n^*\right\| \to 0$, and as just noted, $\hat{A}_n^*$ is in the $\|\ \|$-closure of $\widehat{\mathfrak{A}}$. Thus $\bar{f}$ is in the $\|\ \|$-closure of $\widehat{\mathfrak{A}}$ when f is. It follows that this $\|\ \|$-closure is stable under complex conjugation, contains the constants, and separates the points of X (since $\widehat{\mathfrak{A}}$ does). From the Stone–Weierstrass theorem, this norm closure is $C(X)$. We also have that the complex algebra generated by $\widehat{\mathfrak{G}}$ is stable under complex conjugation and $\|\ \|$-dense in $C(X)$. Under the assumption that $(\mathfrak{A}, \|\ \|, \mathfrak{G})$ satisfies the K–P condition, this last algebra is $\widehat{\mathfrak{A}}$.

Can a *proper* $\|\ \|$-dense subalgebra of $C(X)$, stable under complex conjugation and containing the constant functions, admit a unitary, Banach-algebra norm? We consider the additive group $\mathbb{Z}$ of integers, the Banach algebra $(l_1(\mathbb{Z}), \|\ \|_1)$ under convolution multiplication, and the Hilbert space $l_2(\mathbb{Z})$ $(= \mathcal{H})$. We show that $\|\ \|_1$ is unitary with group generated by the multiples by scalars of modulus 1 of the elements of $l_1(\mathbb{Z})$ corresponding to the functions u_n that are 1 at some integer n and 0 at all other integers. We show that this algebra (the *Wiener algebra*) is not a C*-algebra.

Let the elements of $l_1(\mathbb{Z})$ act by (left) multiplication on $l_2(\mathbb{Z})$. This mapping is a * isomorphism of $l_1(\mathbb{Z})$ with a self-adjoint subalgebra $\mathcal{A}_0$ of $\mathcal{B}(\mathcal{H})$, where $f^*(n) = \overline{f(-n)}$ for all n in $\mathbb{Z}$ and $f \mapsto f^*$ is the involution on $l_1(\mathbb{Z})$. (See [**KR**, Exercises 3.5.33-3.5.35].) The norm-closure $\mathcal{A}$ of $\mathcal{A}_0$ is a commutative C*-algebra isomorphic to $C(\mathbb{T}_1)$, where $\mathbb{T}_1$ is the unit circle in $\mathbb{C}$, the dual group to $\mathbb{Z}$. The mapping that assigns to the image (in $C(\mathbb{T}_1)$) of an element of $\mathcal{A}_0$ the function in $l_1(\mathbb{Z})$ corresponding to that element is the Fourier transform (assigning to the function in $C(\mathbb{T}_1)$ its Fourier series—the function in $l_1(\mathbb{Z})$).

Lemma 20. *Let $\mathfrak{G}$ be $\{\theta u_n : \theta \in \mathbb{C}, |\theta| = 1, n \in \mathbb{Z}\}$. Then $\mathfrak{G}$ is a (multiplicative) group of norm 1 elements of $l_1(\mathbb{Z})$ and $(l_1(\mathbb{Z}), \mathfrak{G})$ satisfies the R–D condition.*

Proof. Note that $u_n * u_{-n} = u_0$ and that u_0 is the (multiplicative) identity in $l_1(\mathbb{Z})$ (corresponding to I in $\mathcal{A}_0$ and the constant function 1 in $C(\mathbb{T}_1)$). Thus each u_n is an invertible element in $l_1(\mathbb{Z})$ and $\|u_n\|_1 = 1$. Suppose $f \in l_1(\mathbb{Z})$ and $1 = \sum_{j=-\infty}^{\infty} |f(j)| = \|f\|_1$. For large m, $\sum_{|j|>m} |f(j)| < \epsilon/2$, where ϵ is a preassigned positive number. At the same time,

$$\left\| f - \sum_{j=-n}^{n} f(j)u_j \right\|_1 \to 0 \qquad n \to \infty.$$

Thus for a large enough m, $(a =) \sum_{|j|>m} |f(j)| < \epsilon/2$ and $\left\|f - \sum_{j=-m}^{m} f(j)u_j\right\|_1 < \epsilon/2$. If $f(j) \neq 0$, let θ_j be $f(j)|f(j)|^{-1}$. Then

$(v_j =)\ \theta_j u_j \in \mathfrak{G}$ and

$$\left\| f - \sum_{j=-m}^{m} |f(j)| v_j \right\|_1 < \epsilon/2.$$

In addition, $1 - \sum_{j=-m}^{m} |f(j)| = \sum_{|j|>m} |f(j)| = a < \epsilon/2$. Hence

$$\left\| f - \left(\sum_{j=-m}^{m} |f(j)| v_j + a u_0 \right) \right\|_1 < \epsilon$$

and $\sum_{j=-m}^{m} |f(j)| v_j + a u_0 \in \operatorname{co} \mathfrak{G}$.

If $0 < \|g\|_1 \leq 1$, then $\|tg\|_1 = 1$, where $t = \|g\|^{-1} \geq 1$. We have just noted that tg is in the norm closure $\operatorname{co}(\mathfrak{G})^=$ of $\operatorname{co} \mathfrak{G}$. Moreover, $0 = \frac{1}{2}(u_0 + (-u_0)) \in \operatorname{co} \mathfrak{G}$. Since $\|g\|_1 \leq 1$, we have that $g = \|g\|_1 tg = \|g\|_1 tg + (1 - \|g\|_1)0 \in \operatorname{co}(\mathfrak{G})^=$. Thus $(l_1(\mathbb{Z}), \mathfrak{G})$ satisfies R–D condition. □

We note, next, that $l_1(\mathbb{Z})$ is not (even algebraically) isomorphic to a C*-algebra.

Lemma 21. *If $\mathcal{A}$ is a norm-dense subalgebra of $C(X)$ for some compact Hausdorff space X, and $\mathcal{A}$ is* algebraically *isomorphic to $C(Y)$ for some compact Hausdorff space Y, then $\mathcal{A} = C(X)$.*

Proof. Let ϕ be an (algebraic) isomorphism of $\mathcal{A}$ onto $C(Y)$. For each x in X, let $\rho_x(f)$ be $f(x)$, where $f \in C(X)$. Let $\sigma_x(g)$ be $\rho_x(\phi^{-1}(g))$ for each g in $C(Y)$. Then σ_x is a non-zero, multiplicative, linear functional on $C(Y)$ and corresponds to a (unique) point $\zeta(x)$ in Y such that $\sigma_x(g) = g(\zeta(x))$ for each g in $C(Y)$ [**KR**, Corollary 3.4.2]. Thus

$$\phi^{-1}(g)(x) = \rho_x(\phi^{-1}(g)) = \sigma_x(g) = (g \circ \zeta)(x) \qquad x \in X.$$

It follows that $\phi^{-1}(g) = g \circ \zeta$ for each g in $C(Y)$. Since ζ transforms a continuous function g on Y to a continuous function $\phi^{-1}(g)$ on X, the mapping ζ is continuous.

If x and x' are distinct points of X, then $f(x) \neq f(x')$ for some f in $\mathcal{A}$ (otherwise $h(x) = h(x')$ for each h in the norm closure, $C(X)$, of $\mathcal{A}$). We have that

$$\phi(f)(\zeta(x)) = f(x) \neq f(x') = \phi(f)(\zeta(x')),$$

and $\zeta(x) \neq \zeta(x')$. Thus ζ is a one-to-one, continuous mapping of X into Y. It follows that $\zeta(X)$ is a compact (hence, closed) subset of Y and $Y \setminus \zeta(X)$ is an open subset of Y. Let g be a function in $C(Y)$ that vanishes on $\zeta(X)$. Then

$$\phi^{-1}(g)(x) = g(\zeta(x)) = 0 \qquad x \in X.$$

Thus $\phi^{-1}(g) = 0$ and $g = \phi(\phi^{-1}(g)) = 0$. It follows that $Y \backslash \zeta(X)$ is null, from complete regularity of Y. Thus ζ is a continuous, one-to-one mapping of X onto Y. Since X and Y are compact Hausdorff spaces, ζ is a homeomorphism of X onto Y.

With f in $C(X)$, let $g(\zeta(x))$ be $f(x)$ for each x in X. Then $g = f \circ \zeta^{-1}$, and $g \in C(Y)$. Moreover,

$$\phi^{-1}(g)(x) = g(\zeta(x)) = f(x) \qquad x \in X.$$

Thus $f = \phi^{-1}(g) \in \mathcal{A}$, and $\mathcal{A} = C(X)$. □

To apply the preceding lemma to the case of $l_1(\mathbb{Z})$, we use the fact that $l_1(\mathbb{Z})$ is isomorphic to the dense subalgebra $\mathcal{A}$ of $C(\mathbb{T}_1)$ and note that not all continuous functions on $\mathbb{T}_1$ have absolutely convergent Fourier series. Thus $\mathcal{A} \neq C(\mathbb{T}_1)$. If $\mathcal{A}$ were isomorphic to a C*-algebra, it would be a commutative C*-algebra, hence isomorphic to some $C(Y)$. From Lemma 21, then, $\mathcal{A}$ would be $C(\mathbb{T}_1)$.

From these considerations, the Wiener algebra provides us with an example of a semi-simple Banach algebra (isomorphic to a dense subalgebra of $C(\mathbb{T}_1)$) with a unitary norm that is not (even isomorphic to) a C*-algebra. It seems likely to us that the group $\mathfrak{G}$ we have used to establish that the L_1-norm is unitary on $l_1(\mathbb{Z})$ is a maximal bounded subgroup of the invertible elements in $l_1(\mathbb{Z})$, but we have not proved that. If this is so, then $l_1(\mathbb{Z})$ provides an example of a semi-simple Banach algebra with a maximal unitary norm that is not a C*-algebra.

References

[BRS] D.P. Blecher, Z.-J. Ruan and A.M. Sinclair, *A Characterization of Operator Algebras*, J. Funct. Anal., **89** (1990), 188-201.

[CE] M.-D. Choi and E.G. Effros, *Injectivity and Operator Spaces*, J. Funct. Anal., **24** (1977), 156-209.

[D] J. Dixmier, *Les moyennes invariantes dans les semi-groupes et leur applications*, Acta Sci. Math. Szeged, **12** (1950), 213-227.

[ER] E. Effros and Z.-J. Ruan, *On matricially normed spaces*, Pacific J. Math., **132** (1988), 243-264.

[G] L.T. Gardner, *An Elementary proof of the Russo–Dye Theorem*, Proc. Amer. Math. Soc., **90** (1984), 171.

[H] U. Haagerup, *On Convex Combinations of Unitary Operators in C*-Algebras*, Mappings of Operator Algebras, Proceedings of the Japan–U. S. Joint Seminar, University of Pennsylvania, 1988, 1-13 (eds. H. Araki and R. V. Kadison), Birkhäuser, Boston, 1991, Progress in Mathematics, **84**.

[K] R.V. Kadison, *Isometries of operator algebras*, Ann. Math., **54** (1951), 325-338.

[KP] R.V. Kadison and G.K. Pedersen, *Means and Convex Combinations of Unitary Operators*, Math. Scand., **57** (1987), 249-266.

[KR] R.V. Kadison and J.R. Ringrose, *Fundamentals of the Theory of Operator Algebras*, Academic Press, Orlando, Florida, 1983, 1986, Pure and Applied Mathematics, **100**, volumes III and IV containing solutions to all the exercises, Birkhäuser, Boston, 1991, 1992.

[OP] C.L. Olsen and G.K. Pedersen, *Convex Combinations of Unitary Operators in von Neumann Algebras*, J. Funct. Anal., **66** (1986), 365-380.

[P] R.R. Phelps, *Extreme Points in Function Algebras*, Notices Amer. Math. Soc., **11** (1964), 538.

[Ru] Z.-J. Ruan, *Subspaces of C^*-algebras*, J. Funct. Anal., **76** (1988), 217-230.

[RD] B. Russo and H.A. Dye, *A Note on Unitary Operators in C^*-Algebras*, Duke Math. J., **33** (1966), 413-416.

Received May 12, 1994.

UNIVERSITY OF PENNSYLVANIA
PHILADELPHIA, PA 19104-6395
E-mail address: liming@math.upenn.edu