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ON SPECTRA OF SIMPLE RANDOM WALKS ON ONE-RELATOR GROUPS

PIERRE-ALAIN CHERIX AND ALAIN VALETTE
WITH AN APPENDIX BY PAUL JOLISSAINT

For a one relator group $\Gamma = \langle X : r \rangle$, we study the spectra of the transition operators h_X and h_S associated with the simple random walks on the directed Cayley graph and ordinary Cayley graph of Γ respectively. We show that, generically (in the sense of Gromov), the spectral radius of h_X is $(\#X)^{-1/2}$ (which implies that the semi-group generated by X is free). We give upper bounds on the spectral radii of h_X and h_S. Finally, for Γ the fundamental group of a closed Riemann surface of genus $g \geq 2$ in its standard presentation, we show that the spectrum of h_S is an interval $[-r, r]$, with $r \leq g^{-1}(2g-1)^{1/2}$. Techniques are operator-theoretic.

1. Introduction.

Let Γ be a finitely generated group. Fix a finite, not necessarily symmetric generating subset X, and let $S = X \cup X^{-1}$ be the symmetrization of X. With X and S are classically associated the usual Cayley graph $G(\Gamma, S)$, but also the Cayley digraph (or directed graph) $G(\Gamma, X)$, where the set of vertices is Γ and, for any $\gamma \in \Gamma$ and $s \in X$, an oriented edge is drawn from γ to γs. We denote by $\#E$ the number of elements in the set E.

We consider the normalized adjacency operators, or transition operators, h_X and h_S; these are operators of norm at most 1 on $l^2(\Gamma)$, defined by:

$$(h_X\xi)(x) = \frac{1}{\#X}\sum_{s\in X}\xi(xs)$$
$$(h_S\xi)(x) = \frac{1}{\#S}\sum_{s\in S}\xi(xs) \qquad (\xi \in l^2(\Gamma), x \in \Gamma).$$

Consider the nearest neighbour simple random walk on $G(\Gamma, X)$ obtained by assigning probability $1/(\#X)$ to each neighbour of a given vertex $\gamma \in \Gamma$ (where a neighbour of γ is the extremity of an oriented edge with origin γ); then, for any $x, y \in \Gamma$, the probability $p^{(n)}(x, y)$ of a transition in n steps from x to y is given by $\langle h_X^n \delta_x | \delta_y \rangle$ (where $(\delta_x)_{x\in\Gamma}$ is the canonical basis of $l^2(\Gamma)$); the analogous probabilistic interpretation of h_S is classical. We denote by $\mathrm{Sp}(T)$ and $r(T)$ the spectrum and spectral radius of a bounded operator T on a Hilbert space. That the spectra of h_X and h_S capture important information about the pairs (Γ, X) or (Γ, S) follows from the following results of Day and Kesten (see [**Day**], [**Ke1**], [**Ke2**]).

Theorem 1.1.

(a) *The following are equivalent:*

(i) $r(h_X) = 1$;

(ii) $1 \in \mathrm{Sp}(h_X)$;

(iii) Γ *is amenable.*

(b) *Assume* $\#X \geq 2$; *then* $\frac{\sqrt{2(\#X)-1}}{\#X} \leq r(h_S)$, *with equality if and only if* Γ *is isomorphic to the free group* $\mathbb{F}(X)$ *on* X; *in this case*

$$\mathrm{Sp}(h_S) = \left[-\frac{\sqrt{2(\#X)-1}}{\#X}, \frac{\sqrt{2(\#X)-1}}{\#X}\right].$$

In passing, we recall that, in the symmetric case, $r(h_S)$ is the inverse of the radius of convergence of the Green kernel

$$G(x, y; z) = \sum_{n=0}^{\infty} p^{(n)}(x, y) z^n \qquad (x, y \in \Gamma).$$

The qualitative study of the spectra of h_X and h_S was pursued in [**HRV1**] and [**HRV2**], where the following result was proved, with the exception of assertions concerning a symmetric X, that were obtained by Cartwright [**Car**] and Kesten [**Ke1**] respectively.

Theorem 1.2.

(a) *Let* $\mathbb{T}$ *be the group of complex numbers of modulus 1; fix* $z \in \mathbb{T}$. *If there exists a character* $\chi : \Gamma \to \mathbb{T}$ *such that* $\chi(x) = z$ *for any* $x \in X$, *then* $\mathrm{Sp}(h_X)$ *is invariant under multiplication by* z. *The converse is true if either* Γ *is amenable or* X *is symmetric (i.e.* $X = X^{-1}$*).*

(b) *Assume* $\#X \geq 2$. *Set* $\sigma(X) = \limsup_{k\to\infty} \| h_X^k \|_2^{1/k}$, *where* h_X *is now viewed as the normalized characteristic function of* X *and* h_X^k *denotes the* k^{th} *convolution power of* h_X. *Then*

$$\frac{1}{\sqrt{\#X}} \leq \sigma(X) \leq r(h_X)$$

with $\frac{1}{\sqrt{\#X}} = \sigma(X)$ *if and only if* X *generates a free semi-group, and* $\sigma(X) = r(h_X)$ *if either* X *is symmetric or* Γ *is hyperbolic in the sense of Gromov (but not in general).*

(c) *Let Γ be either the free group $\mathbb{F}(X)$, with $\#X \geq 2$, or the surface group $\Gamma_g \cong \langle a_1, b_1, \cdots, a_g, b_g : \prod_{i=1}^{g}[a_i, b_i]\rangle$ with $X = a_1, b_1, \cdots, a_g, b_g$ and $g \geq 2$; then* $\mathrm{Sp}(h_X) = \left\{ z \in \mathbb{C} : |z| \leq \frac{1}{\sqrt{\#X}} \right\}$.

In the case of h_S, quantitative results on the spectrum were obtained mainly for virtually abelian groups (using Fourier analysis, as in [**KeS**]) or, at the other extreme, for virtually free groups or groups for which the Cayley graph is tree-like (using methods from combinatorics on trees, see e.g. [**CS1**], [**CS2**], [**IoP**], [**KuS**], [**Mlo**]). In the present paper, we deal with one-relator groups, i.e. groups of the form $\Gamma = \langle X : r\rangle$ where r, the relator, is a cyclically reduced word in $\mathbb{F}(X)$. This class of groups contains the fundamental groups of all compact surfaces (even non-orientable ones), and one-relator groups share a number of interesting properties with surface groups (e.g., it follows from famous results of Lyndon [**Lyn**] and Stallings [**Sta**] that a torsion-free one-relator group which is not free must have cohomological dimension 2). To avoid degeneracies, we shall always assume $\#X \geq 2$ and $|r| \geq 3$, i.e. the word length of r in $\mathbb{F}(X)$ is at least 3.

Here is a summary of our results:

(a) We propose the statistical result that "most" presentations $\Gamma = \langle X : r\rangle$ give $r(h_X) = \frac{1}{\sqrt{\#X}}$ (which implies in particular that the semi-group generated by X in Γ is free). More precisely, we prove that the ratio

$$\frac{\#\left\{\text{presentation } r \text{ with } r(h_X) = (\#X)^{-1/2} \text{ and } |r| = N\right\}}{\#\{\text{presentation } r \text{ with } |r| = N\}}$$

tends (exponentially fast) to 1 when N tends to $+\infty$. This is exactly the sense of genericity introduced by Gromov ([**Gro**], 0.2(A)), and studied further by Champetier [**Ch2**].

(b) Let H_r (resp. H_l) be the subgroup of $\mathbb{F}(X)$ generated by all quotients xy^{-1}, with $x, y \in X$ (resp. $x^{-1}y$, with $x, y \in X$). Suppose that r is not in the union $H_r \cup H_l$; then $\| h_X \| = \frac{2\sqrt{\#X - 1}}{\#X}$ and $\max\{r(h_X), r(h_S)\} \leq \frac{2\sqrt{\#X-1}}{\#X}$.

(c) If r is in the exceptional set $H_r \cup H_l$, then $r(h_X) = \frac{1}{\sqrt{\#X}}$ and $\mathrm{Sp}(h_X)$ is a union of concentric circles centered at 0; this is proved using Jolissaint's result from the Appendix.

(d) Without restriction on r but with $\#X \geq 4$, we have

$$\max\{r(h_X), r(h_S)\} \leq \frac{2\sqrt{\#X-2}+1}{\#X} < 1.$$

(e) For the surface group Γ_g with $g \geq 2$, Peter Sarnak asked for the exact value of $r(h_S)$ in terms of the genus g. We prove that $\mathrm{Sp}(h_S)$ is an interval $[-r, r]$ with the non-trivial estimate $r \leq \dfrac{\sqrt{2g-1}}{g}$.

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2. Rotational symmetries of spectra.

As is well-known, existence of homomorphisms is the easiest thing to check in the case of a finitely presented group. We examplify this in the case of a one-relator group $\Gamma = \langle X : r\rangle$; denote by Σ the sum of all exponents in r and fix $z \in \mathbb{T}$; then:

- For z a primitive d-th root of 1, there exists a character $\chi : \Gamma \to \mathbb{T}$ such that $\chi(x) = z$ for any $x \in X$ if and only if $\Sigma \equiv 0 (\mathrm{mod}\ d)$;
- for z not a root of 1, there exists a character $\chi : \Gamma \to \mathbb{T}$ such that $\chi(x) = z$ for any $x \in X$ if and only if $\Sigma = 0$.

From this and Theorem 1.2 above, we immediately deduce:

Proposition 2.1.

(a) *If* $\Sigma \equiv 0 (\mathrm{mod}\, d)$, *then* $\mathrm{Sp}(h_X)$ *is invariant under multiplication by* $\exp(2\pi i/d)$*;*

(b) *If* $\Sigma = 0$, *then* $\mathrm{Sp}(h_X)$ *is a union of concentric circles, centered at* 0*;*

(c) $\mathrm{Sp}(h_S)$ *is symmetric with respect to* 0 *if and only if* Σ *is even.*

3. Free semi-groups and small cancellation.

Definition 3.1. A word $w \in \mathbb{F}(X)$ is **positive** if it involves only generators with positive exponents. Any non-empty reduced word $r \in \mathbb{F}(X)$ can be written in a unique way as a product without cancellation, either $r = w_1 w_2^{-1} w_3 \cdots w_n^{\pm 1}$ or $r = w_1^{-1} w_2 w_3^{-1} \cdots w_n^{\pm 1}$, where the w_i's are positive words. We say that r **alternates enough** if $n \geq 4$, i.e. there are at least 3 changes of signs in the exponents of r.

Lemma 3.2. *Each of the following statements implies the next one:*

(i) $r(h_X) = \dfrac{1}{\sqrt{\#X}}$;

(ii) *X generates a free semi-group;*

(iii) *the relator r alternates enough.*

Proof. i) $\Rightarrow$ ii) follows immediately from Theorem 1.2. To show ii) $\Rightarrow$ iii), we assume that r does not alternate enough and prove that X does not generate a free semi-group. There are 3 cases to consider.

(a) r has no change of signs in its exponents, i.e. r or r^{-1} is a positive word; then we have a positive word that represents the identity in Γ;

(b) r has exactly one change of sign, say $r = w_1 w_2^{-1}$, with w_1, w_2 positive, distinct words; then w_1 and w_2 represent the same element in the semi-group generated by X in Γ;

(c) r has exactly two changes of sign, i.e. r or r^{-1} is of the form $w_1 w_2^{-1} w_3$, with w_1, w_2, w_3 positive words; then by cyclically permuting we get $w_3 w_1 w_2^{-1}$, i.e. we are back to the preceding case.

□

Example 3.3. We give an example showing that the converse implication ii) $\Rightarrow$ i) does not hold in general. It seems that this example was known to Y. Guivarc'h (private communication). Consider the one-relator group $\Gamma = \langle y, z : yzy^{-1}z^{-1}yz^{-1}\rangle$. We claim that, for $X = \{y, z\}$, we have $r(h_X) = 1$ and X generates a free semi-group. To see it, set $x = zy^{-1}$; in the generators x, y, the group Γ has the famous presentation

$$\Gamma = \langle x, y : yxy^{-1}x^{-2}\rangle$$

(Γ is the first Baumslag-Solitar group). Γ is solvable, hence amenable, thus $r(h_X) = 1$. Let H be the subgroup of Γ generated by x. The relation

$$(*) \qquad yxy^{-1} = x^2$$

exhibits Γ as an HNN-extension of H with respect to the monomorphism $\Theta : H \to H$ such that $x^k \to x^{2k}$. Therefore, Γ acts on a tree T, whose construction we now recall (see [**Ser**], I.1.4, I.5.1). The homogeneous space Γ/H will be both the set of vertices and the set of edges of T: We define the extremity of the edge γH as the vertex γH, and the origin of γH as the vertex $\gamma y^{-1}H$; it follows from relation (*) that this is well-defined. The resulting tree T is the homogeneous tree of degree 3 with, at each vertex, one incoming edge and two outgoing edges. We call descendant of order n of the vertex H any of the 2^n vertices at distance n from H that can be reached from H by a positively oriented path. To prove that y and z generate a free semi-group in Γ, it suffices to prove the following

Claim. Any descendant of order n of H can be written as wH, where w is a positive word of length n in y and z.

Note that this writing is necessarily unique, since there are 2^n positive words of length n in y and z. We prove the claim by induction over n, the case $n = 0$ being obvious. So, let γH be a descendant of order $n+1$ of H; then $\gamma y^{-1}H$ is a descendant of order n of H, so by the induction assumption we have $\gamma y^{-1}H = wH$ for some positive word w of length n in y and z. Thus $w = \gamma y^{-1}x^k$ for some $k \in \mathbb{Z}$. If k is even, we have $w = \gamma x^{k/2}y^{-1}$, i.e. $\gamma H = wyH$; if k is odd, we have $w = \gamma x^{(k+1)/2}y^{-1}x^{-1}$, i.e. $\gamma H = wzH$. Both wy and wz are positive words of length $n+1$ in y and z.

Example 3.4. In Lemma 3.2, the converse implication iii) $\Rightarrow$ ii) does not hold either. To see it, let $n \geq 1$ be an integer, and consider the group $\Gamma = \langle a, b : a(ab^{-1})^{n+1}\rangle$. The relator r alternates enough. Set $r' = (ab^{-1})^n a^2 b^{-1}$, a cyclic permutation of r; then $r^{-1}ar'a^{-1} = bab^{-1}a^{-1}$, so that X does not generate a free semi-group, since $ab = ba$. (Actually, this also shows that Γ is a quotient of $\mathbb{Z}^2$; since the vector $(n+2, -n-1)$ is primitive in $\mathbb{Z}^2$, one checks easily that Γ is isomorphic to $\mathbb{Z}$.)

This example typically displays absence of small cancellation, whose definition we recall now (see e.g. [**LyS**] for an extensive study).

Definition 3.5. Let $\Gamma = \langle X : r\rangle$ be a one-relator group. Denote by R the set of words obtained by cyclic permutations of r and r^{-1}. A **piece** is a prefix u which is common to two distinct elements of R (by prefix we mean any not empty initial part of a word; a word is a prefix of itself). Fix $\lambda \in]0,1[$. We say that r satisfies **the small cancellation condition** $C'(\lambda)$ if, for any piece u, one has:

$$|u| < \lambda |r|.$$

Definition 3.6. A one-relator group $\Gamma = \langle X : r\rangle$ satisfies a **Dehn's algorithm** if, for any reduced word $w \in \mathbb{F}(X)$ that represents 1 in Γ, there exists a prefix u of some word in R such that u is a subword of w and $|u| > \frac{1}{2}|r|$.

It is known that groups satisfying the small cancellation property $C'(\lambda)$, with $\lambda \leq \frac{1}{6}$, also satisfy a Dehn's algorithm (see [**LyS**], Theorem 4.4 of Chapter V; [**Str**], Theorem 25). On the other hand, by a result of Gromov, groups with a Dehn's algorithm are hyperbolic ([**Gro**], Theorem 2.3.D; see also [**Str**], Theorem 36 for a direct proof that $C'(1/6)$-groups are hyperbolic).

After these standard definitions, here is another one of our own.

Let $\Gamma = \langle X : r\rangle$ be a one relator-group presentation. For any $r' \in R$, express r' as a reduced product in $\mathbb{F}(X)$ either $r' = w_1 w_2^{-1} w_3 \cdots w_n^{\pm 1}$ or $r' = w_1^{-1} w_2 w_3^{-1} \cdots w_n^{\pm 1}$, with the w_i's positive words.

Definition 3.7. The presentation $\Gamma = \langle X : r \rangle$ is **balanced** if one has $|w_i| \le \frac{|r|}{4}$ for $i = 1, \cdots, n$.

Clearly, a balanced presentation alternates enough.

We are now in position to present a class of one-relator presentations for which the three conditions of Lemma 3.2 are equivalent.

Lemma 3.8. *Suppose that the presentation $\Gamma = \langle X : r \rangle$ is balanced and satisfies a Dehn's algorithm (this latter assumption being verified if the presentation satisfies condition $C'(1/6)$). Then $r(h_X) = \frac{1}{\sqrt{\#X}}$.*

Proof. We first show that X generates a free semi-group in Γ. Let N be the normal subgroup generated by r in $\mathbb{F}(X)$ (N is the set of consequences of the relation r). Fix $w \in N$, a reduced word. Thanks to the Dehn's algorithm, we find a subword u of w which is also a prefix of some $r' \in R$, with $|u| > \frac{|r|}{2}$. Because the presentation is balanced, we see that u, and a fortiori w, must contain at least 2 changes of sign in their exponents. Now, let v_1, v_2 be distinct positive words. Since $v_1 v_2^{-1}$ has exactly one change of sign in its exponents, we see that $v_1 v_2^{-1}$ does not belong to N, i.e. v_1 is distinct from v_2 in Γ. This shows that the semi- group generated by X in Γ is free [1]. Since Γ is hyperbolic, Theorem 1.2 applies to give $r(h_X) = \frac{1}{\sqrt{\#X}}$. □

Remarks.

(1) It is stated in Theorem 3 of [**New**] (for a proof, see [**LyS**], Theorem 5.5 of Chapter IV) that a one-relator group with torsion satisfies a Dehn's algorithm, and hence is hyperbolic.

(2) Lemma 3.8 and its proof naturally raise the question: Which one-relator groups are hyperbolic? It is conjectured that a one-relator group is hyperbolic if and only if every non-identity element has a cyclic centralizer (Conjecture 2 in [**Juh**]).

The following definition is basically due to Gromov ([**Gro**], 0.2(A)) and was made precise by Champetier ([**Ch2**]; this paper also contains many impressive results on "generic" properties).

Definition 3.9. Let $\#X \ge 2$ be fixed. For any integer $N \ge 1$, denote by $C(N)$ the number of cyclically reduced words of length N in $\mathbb{F}(X)$. Let (P) be a property of one-relator presentations. We say that (P) is **asymptotically almost sure** if the ratio

$$\frac{\#\{\text{ presentation } \langle X : r \rangle \text{ with property } (P) \text{ and } |r| = N\}}{C(N)}$$

[1] On p. 100 of [**HRV2**], it was stated without proof that, in the surface group Γ_g ($g \ge 2$) with the standard presentation, X generates a free semi-group. This provides a proof of this statement.

tends to 1 for N tending to $+\infty$.

Lemma 3.10. *Fix $\lambda \in]0,1[$. Condition $C'(\lambda)$ is asymptotically almost sure.*

Proof. See [**Ch2**], Lemma 4.4. Note that the proof reveals in this case that, for $N \to \infty$, the convergence of the above ratio to 1 is exponentially fast. □

Lemma 3.11. *A one-relator presentation is asymptotically almost surely balanced.*

Proof. Set $\#X = k$ for simplicity. First, notice that $C(N)$ is not smaller than the number of reduced words of length N in $\mathbb{F}(X)$ whose last letter is not the inverse of the first one, i.e.

$$C(N) \geq 2k(2k-1)^{N-2}(2k-2). \tag{1}$$

Now, we estimate the number B(N) of "bad" presentations, i.e those presentations $\langle X : r \rangle$ such that there exists $r' \in R$ beginning with a positive subword of length larger than $N/4$. Since there are at most $2N$ elements in R, we have

$$B(N) \leq 2N \sum_{l=[N/4]+1}^{N} C(N,l)$$

where $C(N,l)$ is the number of cyclically reduced words of length N beginning with a positive subword of length exactly l. Thus we certainly have:

$$B(N) \leq 2N \sum_{l=[N/4]+1}^{N} k^l(2k-1)^{N-l}. \tag{2}$$

Dividing (2) by (1), we estimate the proportion of non-balanced presentations:

$$\begin{aligned}\frac{B(N)}{C(N)} &\leq \frac{N(2k-1)^2}{2k(k-1)} \sum_{l=[N/4]+1}^{N} k^l(2k-1)^{-l} \\ &= \frac{N(2k-1)^2}{2k(k-1)} \frac{k^{[n/4]+1}(2k-1)^{-[n/4]-1} - k^{N+1}(2k-1)^{-N-1}}{1-k(2k-1)^{-1}}.\end{aligned}$$

Since $k \geq 2$, this ratio tends exponentially fast to 0 for $N \to \infty$. □

From this, we deduce:

Theorem 3.12. *Let $\#X \geq 2$ be fixed. A presentation $\Gamma = \langle X : r \rangle$ has asymptotically almost surely $r(h_X) = \frac{1}{\sqrt{\#X}}$.*

Proof. We combine Lemma 3.10 (with $\lambda = 1/6$) and Lemma 3.11, and use the fact that the conjunction of two asymptotically almost sure properties is asymptotically almost sure. Thus, asymptotically almost surely, a one-relator presentation is balanced and satisfies $C'(1/6)$, so also satisfies $r(h_X) = \frac{1}{\sqrt{\#X}}$, by Lemma 3.8. □

Remark. Fix an integer $k \geq 1$. Let $\Gamma_n = \langle X_n : r_n \rangle$ be a sequence of one-relator groups on k generators, with $|r_n|$ tending to infinity for $n \to \infty$. Set $S_n = X_n \cup X_n^{-1}$. It was proved by Grigorchuk [**Gri**] (and recently reproved by Champetier [**Ch1**]) that, if all Γ_n's satisfy the small cancellation condition $C'(\lambda)$, with $\lambda < 1/6$, then

$$\lim_{n\to\infty} r(h_{S_n}) = \frac{\sqrt{2(\#X) - 1}}{\#X}.$$

This corresponds to the intuitive idea that, as $|r_n|$ becomes larger, the Cayley graph of Γ_n looks more and more like a tree.

4. Estimates on norms and spectral radii.

First, we recall that, for any group Γ, the **right regular representation** is the representation ρ of Γ on $l^2(\Gamma)$ defined by:

$$(\rho(g)\xi)(h) = \xi(hg) \qquad (\xi \in l^2(\Gamma), g, h \in \Gamma).$$

If X is a finite generating subset of Γ and $S = X \cup X^{-1}$, our transition operators h_X, h_S may be expressed simply in terms of ρ as:

$$h_X = \frac{1}{\#X} \sum_{s \in X} \rho(s);$$
$$h_S = \frac{1}{\#S} \sum_{s \in S} \rho(s).$$

We shall need the following result of Akemann-Ostrand [**AkO**] (see also [**Woe**]).

Lemma 4.1. *Let $x_1, x_2, \cdots, x_n$ be elements of Γ that generate a free subgroup on n generators. Then*

$$\left\| \sum_{i=1}^{n} \rho(x_i) \right\| = 2\sqrt{n-1};$$

$$\left\|1+\sum_{i=1}^{n}\rho(x_i)\right\| = 2\sqrt{n}.$$

With this we may estimate the norm of h_X.

Proposition 4.2. *Let $\Gamma = \langle X : r\rangle$ be a one-relator group, with $\#X \geq 4$, $|r| > 2$ and r cyclically reduced. Then*

$$\max\{r(h_X), r(h_S)\} \leq \| h_X \| \leq \frac{2\sqrt{\#X-2}+1}{\#X} < 1.$$

Proof. The inequality $r(h_X) \leq \| h_X \|$ holds for any bounded operator. Now, since $|r| > 2$ and r is cyclically reduced, the intersection $X \cap X^{-1}$ is empty, so

$$r(h_S) = \| h_S \| = \left\| \frac{h_X + h_X^*}{2} \right\| \leq \| h_X \|,$$

where the first equality holds for any bounded self adjoint operator. This proves the first inequality in the statement. To prove the second, set $X = \{x_1, \cdots, x_k\}$; without loss of generality, we may assume that x_k appears in the relator r. Then

$$\| h_X \| \leq \frac{1}{k}\left(\left\|\sum_{i=1}^{k-1}\rho(x_i)\right\| + 1\right).$$

Now, by Magnus'Freiheitssatz (see [**LyS**], Proposition 5.1 of Chapter II), the subgroup of Γ generated by $x_1, \cdots, x_{k-1}$ is free on $k-1$ generators, so Lemma 4.1 applies. Finally, the expression is $1/k(2\sqrt{k-2}+1)$ less than 1 provided $k \geq 4$. □

Remark. It was observed in Proposition 4 (iv) of [**HRV2**] that, for $\#X = 2$, one always has $\| h_X \| = 1$.

Example 4.3. For $\#X = 3$, Proposition 4.2 just gives the obvious bound $\| h_X \| \leq 1$. The following example shows that we cannot expect better in this case. Indeed, consider the group $\Gamma = \langle a, b, c : [ac^{-1}, bc^{-1}]\rangle$. Then, factoring out $\rho(c)$ on the right, we get:

$$\| h_X \| = \frac{1}{3} \| \rho(ac^{-1}) + \rho(bc^{-1}) + 1 \| .$$

Now, ac^{-1} and bc^{-1} commute and actually they generate a subgroup H isomorphic to $\mathbb{Z}^2$. Under Fourier transform $C_r^*(H)$ is isometrically isomorphic to $C(\mathbb{T}^2)$, the C*-algebra of continous functions on the 2-torus

$\mathbb{T}^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = |z_2| = 1\}$, the norm on $C(\mathbb{T}^2)$ being the sup-norm. Thus

$$\| \rho(ac^{-1}) + \rho(bc^{-1}) + 1 \| = \sup_{(z_1,z_2)\in\mathbb{T}^2} |z_1 + z_2 + 1| = 3.$$

We are now going to improve the bound of Proposition 4.2, and drop the assumption $\#X \geq 4$. For that, we shall have to exclude some presentations (like the one in Example 4.3). First we fix some notations. Let $\Gamma = \langle X : r \rangle$ be a one-relator group; fix $y \in X$. We define a new generating subset X_y as

$$X_y = \{xy^{-1} : x \in X, x \neq y\} \cup \{y\};$$

note that $\#X_y = \#X$. Now, let r_y be the word r written in the alphabet X_y; more precisely, if we set $x' = xy^{-1}$ for $x \in X$, $x \neq y$, the word r_y is obtained from r by the change of variables (Nielsen transformation)

$$T_y : \begin{cases} x \to x'y & (x \neq y) \\ y \to y \end{cases}.$$

Then we define r'_y as the word obtained from r_y by cyclically reducing.

Lemma 4.4. *If r is a cyclically reduced word in $\mathbb{F}(X)$, then y is the only element that may disappear in r when T_y is applied. More precisely, if we denote by $(\alpha_1, \cdots, \alpha_l)$ the ordered set of elements in $X \cup X^{-1} - \{y, y^{-1}\}$ appearing in r (i.e. $r = y^{\nu_1}\alpha_1 y^{\nu_2}\alpha_2 y^{\nu_3} \cdots y^{\nu_l}\alpha_l y^{\nu_{l+1}}$ where $\nu_i \in \mathbb{Z}$), then the ordered set in r_y and hence in r'_y is $(\alpha'_1, \cdots, \alpha'_l)$.*

Proof. We view T_y as an isomorphism from $\mathbb{F}(X)$ to $\mathbb{F}(X_y)$. Note that T_y^{-1} is defined on the generators of $\mathbb{F}(X_y)$ by $T_y^{-1}(x') = xy^{-1}$ for $x' \in X_y - \{y\}$ and $T_y^{-1}(y) = y$. Then for $r = y^{\nu_1}\alpha_1 y^{\nu_2}\alpha_2 y^{\nu_3} \cdots y^{\nu_l}\alpha_l y^{\nu_{l+1}}$:

$$\begin{aligned} r_y = T_y(r) &= T_y(y^{\nu_1}\alpha_1 y^{\nu_2}\alpha_2 y^{\nu_3} \cdots y^{\nu_l}\alpha_l y^{\nu_{l+1}}) \\ &= T_y(y^{\nu_1})T_y(\alpha_1)T_y(y^{\nu_2})T_y(\alpha_1) \cdots T_y(y^{\nu_l})T_y(\alpha_l)T_y(y^{\nu_{l+1}}) \\ &= y^{\mu_1}\alpha'_1 y^{\mu_2}\alpha'_2 y^{\mu_3} \cdots y^{\mu_l}\alpha'_l y^{\mu_{l+1}}. \end{aligned}$$

Suppose that α'_i and α'_{i+1} cancel out in r_y. Then applying T_y^{-1}, we see that r cannot contain α_i and α_{i+1}, a contradiction. So the ordered set of r_y is exactly $(\alpha'_1, \cdots, \alpha'_l)$.

As r is cyclically reduced, by the same argument, we conclude that we cannot cancel α'_1 and α'_l by cyclic permutation of r_y. So $(\alpha'_1, \cdots, \alpha'_l)$ is also the ordered set of r'_y. That concludes the proof of 4.4. □

Recall from the introduction that we defined a subgroup H_r (resp. H_l) of $\mathbb{F}(X)$ as the subgroup generated by all right quotients xy^{-1}, with $x, y \in X$

(resp. all left quotients $x^{-1}y$, with $x, y \in X$); in passing, notice that H_r and H_l are free on $(\#X) - 1$ generators.

Lemma 4.5. *For r cyclically reduced in $\mathbb{F}(X)$, the following are equivalent:*
(i) $r \in H_r \cup H_l$;
(ii) *For any $y \in X$, the letter y does not appear in r'_y ;*
(iii) *There exists $y \in X$ such that y does not appear in r'_y.*

Proof. (i) $\Rightarrow$ (ii) Suppose that r is in H_r; write $r = \prod_{i=1}^{n} a_i b_i^{-1}$ with a_i, $b_i \in X$.

We can have three kinds of factors $a_i b_i^{-1}$:
(1) a, $b \in X - \{y\}$: then $T_y(ab^{-1}) = a'yy^{-1}(b')^{-1} = a'(b')^{-1}$
(2) $T_y(ay^{-1}) = a'yy^{-1} = a'$
(3) $T_y(ya^{-1}) = y(a'y)^{-1} = (a')^{-1}$.

Thus r_y does not contain y and so does r'_y.

If r is in H_l then the element $s = yry^{-1}$ belongs to H_l. So y does not appear in s_y, s_y is cyclically reduced (because r is cyclically reduced and Lemma 4.4) and $s_y = r'_y$.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i) We assume that y does not appear in r'_y and analyse r in several steps. Write $r = a_1 \cdots a_n$, a word on $X \cup X^{-1}$, and look at the ordered subset $(a_{i_1}, \cdots, a_{i_l})$ of all letters in r from $X \cup X^{-1} - \{y, y^{-1}\}$. By Lemma 4.4, the corresponding ordered set in r'_y is $(a'_{i_1}, \cdots, a'_{i_l})$. We will systematically use the following argument: Suppose that, after applying T_y to each letter of r, we find a word r_y in which y appears and there is no obvious cancellation to remove it: Then it is really impossible to remove y, because it would be necessary first to remove some a'_{i_j}, contradicting Lemma 4.4.

First step: r has no subword of the form ab or $a^{-1}b^{-1}$, with a, $b \in X - \{y\}$. Indeed, if this would be the case, by applying T_y, we would get either $a'yb'$ or $(a')^{-1}y^{-1}(b')^{-1}$ and y would appear in r'_y.

2nd Step: r does not contains y^2 or y^{-2}. To see this, we suppose by contradiction that r contains such a subword, and show that r'_y contains y or y^{-1}. There are three cases to consider.

(1) r contains $a^{\pm 1}y^n b^{\pm 1}$ with $n \geq 2$, $(a \neq y \neq b)$. If the exponents of a and b are positive, then $ay^n b$ becomes $a'y^{n+1}b'$ after T_y. If the exponents of a and b are different, $ay^n b^{-1}$ becomes $a'yy^n y^{-1}(b')^{-1} = a'y^n(b')^{-1}$ and $a^{-1}y^n b$ becomes $(a')^{-1}y^n b'$. Finally if the exponents of a and b are negative, $a^{-1}y^n b^{-1}$ becomes $(a')^{-1}y^{n-1}(b')^{-1}$. The same can be done for n negative with $|n| \geq 2$.

(2) r begins with $y^n b^{\pm 1}$ $(b \in X - \{y\})$. First assume $n \geq 2$. Since r is cyclically reduced, r ends with some letter $a \in X \cup X^{-1} - \{y^{-1}\}$. Then

$r_y = T_y(r)$ cannot end with y^{-1}, so there will be no cancellation when cyclically reducing r_y to get r'_y. On other hand, if r begins with $y^n b$, then r_y begin with $y^n b'$, and if r begins with $y^n b^{-1}$, then r_y begin with $y^{n-1}(b')^{-1}$. Since $n \geq 2$, y appears in r'_y. The same can be done for $n \leq -2$.

(3) Similar arguments hold if r ends with $b^{\pm 1} y^n$ ($b \in X - \{y\}$, $|n| \geq 2$).

Note that cases 2 and 3 also show that r cannot begin with yb or $y^{-1}b^{-1}$, neither end with by or $b^{-1}y^{-1}$ ($b \in X - \{y\}$).

3rd step: All exponents in r are equal to ± 1, and exponents alternate in sign, i.e. $r = a_1^{-1} a_2 a_3^{-1} \cdots a_n^{\pm 1}$ or $r = a_1 a_2^{-1} a_3 \cdots a_n^{\pm 1}$. Indeed, we have to show that r contains no subword of the form ab or $a^{-1}b^{-1}$, for $a, b \in X$. We already know that this holds if either $a, b \in X - \{y\}$ (first step) or $a = b = y$ (second step). It remains to show that r contains no subword of the form ay or yb ($a, b \in X - \{y\}$), or an inverse of these. The remark at the end of the second step already shows that r cannot begin or end with such a subword. If $ayb^{\pm 1}$ appears then we see that after applying T_y, $ayb^{\pm 1}$ becomes either $a'yyb'y$ or $a'yyy^{-1}(b')^{-1} = a'y(b')^{-1}$. Similar arguments holds for yb, $a^{-1}y^{-1}$, $y^{-1}b^{-1}$.

Final step: To see that r is in $H_r \cup H_l$, we have to see that the exponent of a_n is the opposite of the exponent of a_1. By contradiction suppose that a_1 and a_n have the same exponent $+1$ (resp. -1). Then $r = a_1 a_2^{-1} a_3 \cdots a_n$ becomes, after applying T_y, $a'_1 (a')_2^{-1} a'_3 \cdots a'_n y$ so y appear in r'_y (because $a_1 \neq y^{-1}$). The argument for $r = a_1^{-1} a_2 a_3^{-1} \cdots a_n^{-1}$ is similar. That ends the proof. □

Remark. The group $\Gamma = \langle X : r \rangle$ clearly also admits the presentation $\Gamma = \langle X_y : r'_y \rangle$. If $r \in H_r \cup H_l$, Lemma 4.5 reveals that Γ is the free product of $\mathbb{Z} = \langle y \rangle$ with the one-relator group $\Gamma_y = \langle X_y - y : r'_y \rangle$. Now Shenitzer [**She**] has characterized those presentations $\Gamma = \langle X : r \rangle$ such that Γ is isomorphic to the free product of $\mathbb{Z}$ with another group; the criterion is that at least one generator of X must disappear from r by applying Nielsen transformations. Our Lemma 4.5 however does not seem to be a consequence of the result in [**She**] because we only consider very special Nielsen transformations, namely the T_y's.

Theorem 4.6. *Let $\Gamma = \langle X : r \rangle$ be a one-relator group, with $\#X \geq 2$, $|r| > 2$, r cyclically reduced and $r \notin H_r \cup H_l$. Then:*

$$\max\{r(h_X), r(h_S)\} \leq \| h_X \| = \frac{2\sqrt{\#X - 1}}{\#X}.$$

Proof. The inequality is proved as in Proposition 4.2. Fix $y \in X$; then

$$\| h_X \| = \frac{1}{\#X} \left\| 1 + \sum_{x \in X - \{y\}} \rho(xy^{-1}) \right\| = \frac{1}{\#X} \left\| 1 + \sum_{x' \in X_y - \{y\}} \rho(x') \right\| .$$

Since r is not in $H_r \cup H_l$, it follows from Lemma 4.5 that y appears in r'_y; again by Magnus'Freiheitssatz, $X_y - \{y\}$ freely generates a free group on $(\#X) - 1$ generators; Lemma 4.1 then applies to give the result.

Note that Example 4.3 above of a presentation with $r \in H_r, \#X = 3$ and $\| h_X \| = 1$ shows that the assumption on r in Theorem 4.6 cannot be dropped. We now discuss somewhat the "exceptional" presentations in $H_r \cup H_l$. We choose to work with H_r (the analogous results for H_l following by interchanging left and right). □

Proposition 4.7. *Let* $\Gamma = \langle X : r \rangle$ *be a one-relator presentation, with* $r \in H_r$. *Then* X *generates a free semi-group in* Γ *and* $r(h_X) = \frac{1}{\sqrt{\#X}}$. *Moreover* $\mathrm{Sp}(h_X)$ *is a union of concentric circles centered at* 0.

Proof. Fix $y \in X$; as mentioned in the remark following Lemma 4.5, Γ is the free product of $\mathbb{Z} = \langle y \rangle$ with the one-relator group $\Gamma_y = \langle X_y - y : r_y \rangle$. We first prove that X generates a free semi-group. So, let w_1, w_2 be two distinct positive words in $\mathbb{F}(X)$; using the change of variables T_y together with the normal form for elements in a free product, we see that w_1 and w_2 define distinct elements of Γ. It follows from Theorem 1.2 that $\sigma(X) = \frac{1}{\sqrt{\#X}} \leq r(h_X)$. To prove the converse inequality $r(h_X) \leq \sigma(X)$, we appeal to Jolissaint's result from the Appendix: there exists a constant $C > 0$ such that, for any integer $k \geq 0$:

$$\| h_X^k \| \leq C(1+k)^3 \, \| h_X^k \|_2 \, .$$

The desired inequality follows then straight from the definition of $\sigma(X)$. The final assertion follows from Proposition 2.1 by noticing that the sum of all exponents in r is 0. □

Remarks.

(1) In the case of h_S, it would of course be desirable to find an upper bound on $r(h_S)$ that depends on the relator r (for example on the length of r); but we did not succeed in achieving that. Note that such a lower bound for $r(h_S)$ was recently obtained by Paschke [**Pas**]: one has

$$r(h_S) \geq \min_{s>0} \left\{ \cosh(s) + (\#X - 1)\, Q \left(\frac{\cosh(|r|s) + 1}{\mathrm{sh}(s)\, \mathrm{sh}(|r|s)} \right) \right\}$$

where $Q(t) = \dfrac{\sqrt{t^2+1}-1}{t}$.

(2) Proposition 4.2 and Theorem 4.6 above show that one-relator group are, in a certain sense, uniformly non-amenable: if we fix the number of generators, then the spectral radius of h_S is uniformly bounded away from 1. Since Proposition 4.2 and Theorem 4.6 provide upper bounds on $r(h_S)$, it is possible to deduce from them lower bounds on quantities that are known to depend on the width of the spectral gap, i.e. the quantity $\epsilon = 1 - r(h_S)$. One such quantity is the *Kazhdan constant* of the right regular representation ρ with respect to S, defined as

$$\kappa(\rho, S) = \inf_{\xi \in l^2(\Gamma), \|\xi\|=1} \max_{s \in S} \|\rho(s)\xi - \xi\| .$$

It is proved in Proposition I(6) of **[HRV1]** that $\kappa(\rho, \Gamma) \geq \sqrt{2\epsilon}$. Another such quantity is the isoperimetric constant of the Cayley graph $G(\Gamma, S)$, defined as

$$\iota(G(\Gamma, S)) = \inf \left\{ \frac{\#\partial A}{\#A} \ : \ A \text{ finite subset of } \Gamma \right\};$$

here ∂A is the boundary of A, i.e the set of edges of $G(\Gamma, S)$ with one extremity in A and the other in $\Gamma - A$. It is well-known that

$$\iota(G(\Gamma, S)) \geq |S|\epsilon$$

(see e.g. Theorem 3.3 of **[Moh]** for a slightly better inequality).

5. Some computations of spectra.

We recall that the reduced C^*-algebra of the group Γ, denoted by $C_r^*(\Gamma)$, is the C^*-algebra generated by $\rho(\Gamma)$. If Γ is torsion free, a tantalizing conjecture of Kaplansky and Kadison states that $C_r^*(\Gamma)$ has no idempotent except 0 and 1 (see **[Val]** for a survey). We explore here some consequences of this conjecture for one-relator groups.

Proposition 5.1. *Let $\Gamma = \langle X : r \rangle$ be a torsion-free one-relator group satisfying the Kaplansky-Kadison conjecture. Denote by Σ the sum of all exponents in r. Then:*

(i) *If $\Sigma = 0$, then $\mathrm{Sp}(h_X)$ is either a disk or an annulus centered at 0;*

(ii) *If S is even, then $\mathrm{Sp}(h_S)$ is an interval symmetric with respect to 0.*

Proof. Any element in $C_r^*(\Gamma)$ has a connected spectrum (otherwise, by holomorphic functional calculus, we construct non- trivial idempotents in $C_r^*(\Gamma)$).

If $\Sigma = 0$, Proposition 2.1 says that $\mathrm{Sp}(h_X)$ is a union of concentric circles; by connectedness, it is either a disk or an annulus. Similarly, by connectedness $\mathrm{Sp}(h_S)$ must be an interval, symmetric with respect to 0 if Σ is even. □

Corollary 5.2. *Let Γ_g be the surface group $\langle a_1, b_1, \cdots, a_g, b_g : \prod_{i=1}^n [a_i, b_i]\rangle$ with $X = \{a_1, b_1, \cdots, a_g, b_g\}$ and $g \geq 2$; then $\mathrm{Sp}(h_S) = [-r, r]$ with $r \leq \frac{\sqrt{2g-1}}{g}$.*

Proof. The fact that Γ_g satisfies the Kaplansky-Kadison conjecture was proved by Kasparov [**Kas**]. The result then follows by combining Proposition 5.1 with Theorem 4.6.

Of course, in the case of Γ_g, it is somewhat frustrating that we are able to compute explicitly the spectrum of h_X (see Theorem 1.2), but not the spectrum of its real part h_S. □

Example 4.3 revisited.

We consider again the group $\Gamma = \langle a, b, c : [ac^{-1}, bc^{-1}]\rangle$ with $X = \{a, b, c\}$. We claim that $\mathrm{Sp}(h_X)$ is the disk $\{z \in \mathbb{C} : |z| \leq \frac{1}{\sqrt{3}}\}$. That the spectral radius is $\frac{1}{\sqrt{3}}$ follows from Proposition 4.7. Now, Γ is the free product $\mathbb{Z}^2 * \mathbb{Z}$, so Γ satisfies the Kaplansky-Kadison conjecture as a corollary of a result of Rosenberg ([**Ros**], Proposition 2.10). By Proposition 5.1, $\mathrm{Sp}(h_X)$ is either a disk or an annulus centered at 0. To prove that it is a disk, we just have to prove that h_X is not invertible. But since $h_X = \frac{1}{3}(\rho(ac^{-1}) + \rho(bc^{-1}) + 1)\rho(c)$, it is enough to show that $\rho(ac^{-1}) + \rho(bc^{-1}) + 1$ is not invertible. Since ac^{-1} and bc^{-1} generate a subgroup isomorphic to $\mathbb{Z}^2$, this follows from the fact that the function $(z_1, z_2) \to z_1 + z_2 + 1$ is not invertible on the 2-torus.

Appendix : An upper bound for the norms of powers of normalised adjacency operators.

By Paul Jolissaint

The aim of the present note is to prove a result needed in Proposition 3 of the previous article.

Let Γ be a group, let X be a finite subset of Γ containing 1, and let G be the free product $\Gamma * \langle y\rangle$ of Γ with an infinite cyclic group generated by y.

Set also $G_1 = \Gamma$, $G_2 = \langle y\rangle$, $X^* = X - \{1\}$ and $G_j^* = G_j - \{1\}$ for $j = 1, 2$.

Recall that any element ω of G can be uniquely written as a product $\omega = \omega_1 ... \omega_n$ with $\omega_j \in G_{i_j}^*$, and $i_j \neq i_{j+1}$ for $j \leq n-1$. The integer n is called the length of ω, and we denote by Λ_n the set of words of length n.

The main result of this appendix is :

Proposition 5.3. *With the notation above, set*

$$h = \frac{1}{|X|} \sum_{x \in X} \rho(xy) .$$

Then there exists a positive constant C such that for every positive integer k, one has :

$$\| h^k \| \leq C(1+k)^3 \, \| h^k \|_2 ,$$

where $\| \;\; \|$ in the left hand side is the operator norm and h^k is the k^{th} convolution power of h.

Let $\chi = \frac{1}{|X|} \sum_{x \in X} \delta_{xy}$, so that $h = \rho(\chi)$.

It turns out that it will be more convenient to prove the inequality in Proposition 5.3 for $\lambda(\chi)$ instead of h, where λ denotes the left regular representation of G. (As λ and ρ are equivalent representations, this will prove Proposition 5.3, as well.)

If k is a positive integer, it is easy to check by induction on k that the function χ^k $(= \chi * \ldots * \chi$, k times) is supported in $\bigcup_{1 \leq j \leq 2k} \Lambda'_j$, where Λ'_l is the set of reduced words $\omega = \omega_1 \ldots \omega_l$ such that

(a) either $\omega_j \in X^*$ or $\omega_j = y^{\mu_j}$ with $\mu_j > 0$;

(b) $\omega_l = y^{\mu_l}$;

(c) $\sum \mu_j \leq l$.

Hence Proposition 5.3 follows immediately from the sligthly more general:

Proposition 5.4. *There exists a positive constant C such that for any finitely supported function φ on G whose support lies in $\bigcup_{1 \leq j \leq k} \Lambda'_j$ for some k, then*

$$\| \lambda(\varphi) \| \leq C(1+k)^3 \, \| \varphi \|_2 .$$

The proof of Proposition 5.4 is similar to that of Theorem 2.2.2 of [**Jol**] and is based on an idea due to U. Haagerup in the case of the free group [**Haa**].

Using the same arguments as in the proof of Proposition 1.2.6 in [**Jol**], Proposition 5.4 is a consequence of the following result:

Proposition 5.5. *There exists a positive constant c such that for non-negative integers k, l and m satisfying: $|k-l| \leq m \leq k+l$, and for functions φ and ψ on G supported in Λ'_k and Λ_l respectively, one has*

$$\| (\varphi * \psi) \chi_{\Lambda_m} \|_2 \leq c(1+k) \, \| \varphi \|_2 \| \psi \|_2 .$$

Let us recall the following result which is a special case of Lemma 2.2.1 of [**Jol**]:

Lemma 5.6. *Let k, l, m and q be non-negative integers such that $m = k + l - q$.*

If $\omega \in \Lambda_m$, let $\omega = \omega_1 ... \omega_m$ be its reduced form.

Set also: $E_{k,l}(\omega) = \{(u,v) \in \Lambda_k \times \Lambda_l \mid u\,v = \omega\}$.

Then

(1) *If $q = 2p$ is even, set $u_\omega = \omega_1 ... \omega_{k-p}$ and $v_\omega = \omega_{k-p+1} ... \omega_m$; then $E_{k,l}(\omega)$ is the set of pairs $(u,v) \in \Lambda_k \times \Lambda_l$ such that there exists $a \in \Lambda_p$ with*

$$u = u_\omega\, a \text{ and } v = a^{-1} v_\omega .$$

(2) *If $q = 2p + 1$ is odd, set $u_\omega = \omega_1 ... \omega_{k-p-1}$ and $v_\omega = \omega_{k-p+1} ... \omega_m$; then $E_{k,l}(\omega)$ is the set of pairs $(u,v) \in \Lambda_k \times \Lambda_l$ such that there exists $a \in \Lambda_p$ and $b_1, b_2 \in \Lambda_1$ satisfying:*

$$b_1\, b_2 = \omega_{k-p}\,,\quad u = u_\omega\, b_1\, a \text{ and } v = a^{-1}\, b_2\, v_\omega .$$

Proof of Proposition 5.5.

Set $m = k + l - q$; then q is an integer such that $0 \leq q \leq \min(k,l)$. We divide the proof into two cases:

Case 1. Assume that q is even. Using the first part of the above lemma, one shows more generally that if φ is supported in Λ_k then:

$$\| (\varphi * \psi)\chi_{\Lambda_m} \|_2 \leq \| \varphi \|_2 \| \psi \|_2 \,.$$

The proof is exactly the same as that of Lemma 1.3 in [**Haa**].

Case 2. Assume that $q = 2\,p + 1$ is odd.

For every $\omega \in \Lambda_m$, set $E'_{k,l}(\omega) = \{(u,v) \in \Lambda'_k \times \Lambda_l \mid u\,v = \omega\}$. Let $\omega \in \Lambda_m$ be such that $E'_{k,l}(\omega) \neq \emptyset$ and let us write ω as $\omega = \omega_1 ... \omega_{k-p-1}\omega_{k-p}\omega_{k-p+1} ... \omega_m$ in its reduced form. Set $u_\omega = \omega_1 ... \omega_{k-p-1}$ and $v_\omega = \omega_{k-p+1} ... \omega_m$ as in the second part of the lemma. If $(u,v) \in E'_{k,l}(\omega)$, then $u = u_1 ... u_{k-1} y^{\mu_k}$ and $v = v_1 ... v_l$ for some u_i, v_i and $\mu_k > 0$.

Moreover there exists $a \in \Lambda_p$ and $b_1, b_2 \in \Lambda_1 = G_1^* \cup G_2^*$ such that $u = u_\omega\, b_1\, a$, $v = a^{-1}\, b_2\, v_\omega$ and $b_1\, b_2 = \omega_{k-p}$.

Then two cases may occur:

(i) $\omega_{k-p} \in G_1^*$: Thus $b_1, b_2 \in G_1^*$ and, since $u \in \Lambda'_k$, we must have $b_1 \in X^*$, $a \in \Lambda'_p$ and a begins with y^ν for some $\nu > 0$. (We will write: $a \in \Lambda'_p$, $a_1 = y^{\nu_1}$.)

(ii) $\omega_{k-p} \in G_2^*$: Then b_1, $b_2 \in G_2^*$ and $a \in \Lambda_p'$ begins with some $x \in X^*$.

One has:

$$\begin{aligned}
(1) \qquad \| (\varphi * \psi)\chi_{\Lambda_m} \|_2^2 &= \sum_{\omega \in \Lambda_m} |(\varphi * \psi)(\omega)|^2 \\
&= \sum_{\omega \in \Lambda_m} \left| \sum_{(u,v) \in E'_{k,l}(\omega)} \varphi(u)\psi(v) \right|^2 \\
&= \sum_{\substack{\omega \in \Lambda_m \\ \omega_{k-p} \in G_1^*}} \left| \sum_{(u,v) \in E'_{k,l}(\omega)} \varphi(u)\psi(v) \right|^2 \\
&\quad + \sum_{\substack{\omega \in \Lambda_m \\ \omega_{k-p} \in G_2^*}} \left| \sum_{(u,v) \in E'_{k,l}(\omega)} \varphi(u)\psi(v) \right|^2 .
\end{aligned}$$

Denote by Σ_1 (resp. Σ_2) the first (resp. the second) sum in the right handside of (1).
Let us estimate Σ_1 first:

$$\begin{aligned}
\Sigma_1 = &\sum_{\substack{\omega \in \Lambda_m \\ \omega_{k-p} \in G_1^*}} \left| \sum_{\substack{a \in \Lambda_p' \\ a_1 = y^\nu}} \sum_{x \in X^*} \varphi(u_\omega\, x\, a)\psi(a^{-1}(x^{-1}\omega_{k-p})v_\omega) \right|^2 \\
&\leq \sum_{\substack{u \in \Lambda_{k-p-1} \\ u_{k-p-1} \in G_2^*}} \sum_{\substack{v \in \Lambda_{l-p-1} \\ v_1 \in G_2^*}} \sum_{\omega_{k-p} \in G_1^*} \left| \sum_{\substack{a \in \Lambda_p' \\ a_1 = y^\nu}} \sum_{x \in X^*} \varphi(uxa)\psi(a^{-1}(x^{-1}\omega_{k-p})v) \right|^2 \\
&\leq \sum_{\substack{u \dots \\ v \dots}} \sum_{\omega \in G_1^*} \left(\sum_{\substack{a \in \Lambda_{p+1}' \\ a_1 \in X^*}} |\varphi(u\, a)|\, |\psi(a^{-1}\omega v)| \right)^2 \\
&\leq \sum_{\substack{u \dots \\ v \dots \\ \omega \dots}} \left(\sum_{a \dots} |\varphi(u\, a)|^2 \right) \left(\sum_{a \dots} |\psi(a^{-1}\omega v)| \right)^2 \\
&= \| \varphi \|_2^2 \sum_{\substack{v \in \Lambda_{l-p} \\ v_1 \in G_1^*}} \sum_{\substack{a \in \Lambda_{p+1}' \\ a_1 \in X^*}} |\psi(a^{-1}v)|^2 .
\end{aligned}$$

But

$$\sum_{\substack{v\in\Lambda_{l-p}\\ v_1\in G_1^*}} \sum_{\substack{a\in\Lambda'_{p+1}\\ a_1\in X^*}} |\psi(a^{-1}v)|^2 \leq (|X|-1)\, \|\psi\|_2^2 \,.$$

Let us finally estimate Σ_2:
We are going to use the fact that if f is a finitely supported function on $G_2(=\mathbb{Z})$, then $\|\lambda(f)\| \leq \frac{\pi}{\sqrt{3}} \|f\|_{2,1}$, where $\|f\|_{2,1}^2 = \sum_{x\in\mathbb{Z}} |f(x)|^2(1+|x|)^2$. (See [**Jol**], Example 1.2.3.)
Then:

$$\Sigma_2 = \sum_{\substack{\omega\in\Lambda_m\\ \omega_{k-p}\in G_2^*}} \left| \sum_{\substack{a\in\Lambda'_p\\ a_1\in X^*}} \lambda_{G_2}(\varphi_{u_\omega,a})\psi_{v_\omega,a}(\omega_{k-p}) \right|^2 ,$$

where we set, for $u\in\Lambda_{k-p-1}$, $a\in\Lambda'_p$ and $v\in\Lambda_{l-p-1}$: $\varphi_{u,a}(x)=\varphi(uxa)$ and $\psi_{v,a}(x)=\psi(a^{-1}xv)$ for $x\in G_2$.
Hence

$$\begin{aligned}
\Sigma_2 &\leq \sum_{\substack{u\in\Lambda_{k-p-1}\\ u_{k-p-1}\in X^*}} \sum_{\substack{v\in\Lambda_{l-p-1}\\ v_1\in G_1^*}} \left\| \sum_{\substack{a\in\Lambda'_p\\ a_1\in X^*}} \lambda_{G_2}(\varphi_{u,a})\psi_{v,a} \right\|_2^2 \\
&\leq \sum_{\substack{u\ldots\\ v\ldots}} \left(\sum_{a\ldots} \|\lambda_{G_2}(\varphi_{u,a})\|^2 \right) \left(\sum_{a\ldots} \|\psi_{v,a}\|_2^2 \right) \\
&\leq \frac{\pi^2}{3} \left(\sum_{\substack{u\ldots\\ a\ldots}} \|\varphi_{u,a}\|_{2,1}^2 \right) \left(\sum_{\substack{v\ldots\\ a\ldots}} \|\psi_{v,a}\|_2^2 \right) .
\end{aligned}$$

But

$$\sum_{\substack{v\ldots\\ a\ldots}} \|\psi_{v,a}\|^2 = \|\psi\|_2^2 \qquad \text{and}$$

$$\begin{aligned}
\sum_{\substack{u\ldots\\ a\ldots}} \|\varphi_{u,a}\|_{2,1}^2 &= \sum_{u\ldots}\sum_{a\ldots} \sum_{x\in\{y^\nu;1\leq\nu\leq k\}} |\varphi(uxa)|^2(1+|x|)^2 \\
&\leq (1+k)^2\, \|\varphi\|_2^2 \,.
\end{aligned}$$

□

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RUE EMILE-ARGAND 11
CH-2007 NEUCHATEL
SWITZERLAND

Added on proof:

1) It has been pointed out to us by P. de la Harpe that, by combining Kesten's results [**Ke1**] with the Freiheitssatz, the upper bound on $r(h_S) = \|h_S\|$ in Proposition 4.2 can be improved, to the effect that

$$r(h_S) \leq \frac{\sqrt{2(\#X) - 3} + 1}{\#X}.$$

In particular, for $\#X \to \infty$, one sees that $r(h_S)$ behaves like $\sqrt{\frac{2}{\#X}}$.

2) The first author has recently extended the genericity result, Theorem 3.12, to all finitely presented groups; see P.-A. Cherix, *Generic result for the existence of a free semi-group*, Sèminaire de théorie spectrale et géométrie, **13** (1994-95), Institut Fourier, Grenoble.

3) The Kaplansky-Kadison conjecture has now been proved for the class of torsion-free one-relator groups (see C. Béguin, H. Bettaieb & A. Valette, *The Baum-Connes conjecture for torsion-free one-relator groups*, preprint Neuchâtel, 1996). So this extra assumption can be dropped from Proposition 5.1.