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DIMENSIONS OF NILPOTENT ALGEBRAS OVER FIELDS OF PRIME CHARACTERISTIC

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In this short paper we consider the conjecture that for a finite dimensional commutative nilpotent algebra M over a perfect field of prime characteristic p, $\dim M \geq p \dim M^{(p)}$ where M^p is the subalgebra of M generated by $x^p, x \in M$. We prove that for any finite dimensional nilpotent algebra M (not necessarily commutative) over any field of prime characteristic p, $\dim M \geq p \dim M^{(p)}$ for $\dim M^{(p)} \leq 2$.

1. Introduction.

It is generally accepted that the structure of nilpotent algebras is not well understood. In this paper we discuss some questions which could lead to significant further development of this structure theory.

Let M be an algebra over a field K. The algebra M is said to be nilpotent if $M^n = 0$ for some $n \geq 1$. (Recall that for $j \geq 1$, M^j is the subalgebra of M generated by all monomials of degree j in the elements of M.) If n is the least such integer, then n is called the nilpotency index of M. If $M^j \neq 0$, then it follows from the nilpotency of M that $M^j \supset M^{j+1}$. Let $d_i = \dim_K(M^i/M^{i+1})$, where $\dim_K V$ or $\dim V$ denotes of course the dimension of V as a vector space over K. Then clearly $\dim M = \sum_{i=1}^{n-1} d_i$.

In [**1**] N. Eggert conjectures that for a finite commutative nilpotent algebra over a field K of characteristic p,

$$\dim_K M \geq p \ \dim_K M^{(p)} \tag{1.1}$$

where $M^{(p)}$ is the subalgebra of M generated by x^p $x \in M$. In [**1**] Eggert proves (1.1) when $\dim M^{(p)} \leq 2$. His proof however does not appear to be easy to generalise. In view of this we provide a simpler proof in Section 2. Since these proofs do not require the nilpotent algebra to be finite, it is believed that all that is required in (1.1) is that M be finite dimensional.

Now (1.1) obviously holds when $\dim M^{(p)} = 0$. If $\dim M^{(p)} = 1$ then it follows from the remarks above, that $\dim M \geq \sum_{i=1}^{p} d_i \geq p$, and so (1.1) holds in this case. In the next section we prove that for any finite dimensional nilpotent algebra M over any field of characteristic p, $\dim M \geq p \ \dim M^{(p)}$ if $\dim M^{(p)} = 2$. This yields a second proof of Eggert's theorem. Our proof however does not require the field K to perfect, not indeed the algebra M to be commutative, and in this sense will be a slight extension.

2. The Two Dimensional Case.

Some Notation. Let V be a vectorspace over a field K of characteristic p. If $X \subseteq V$ by $\langle X\rangle$ we mean the linear span (over K) of X. If X is a singleton say $\{x\}$ we write Kx for $\langle X\rangle$. Let $N^{(p)}$ denote the *subspace* of M generated by x^p, $x \in M$. Of course if M is commutative, then $M^{(p)} = N^{(p)}$.

We now prove the following theorem, of which our main theorem will be just a simple corollary.

Theorem 1. *Let M be any nilpotent algebra over the field K of prime characteristic, and let $d_k = 1$ for some $1 \le k \le p-1$, then if $N^{(p)} = \langle z^p, z \in M\rangle$, then $N^{(p)} = \langle y^{pi}, i \ge 1\rangle$ for some $y \in M$.*

Proof. If $M^p = 0$, then $N^{(p)} = 0$ and there is nothing to prove. Hence we may assume that $M^p \neq 0$. Since $d_k = 1$, there exists $m = m_1 m_2 \ldots m_k$ for which $\langle m + M^{k+1}\rangle = M^k/M^{k+1}$. Write $m = y\sigma$ where $y = m_1 \in M$ and $\sigma = m_2 \ldots m_k \in M^{k-1}$ (if $k = 1$, regard σ as an empty product). Now

$$M^k = Ky\sigma + M^{k+1}, \tag{2.1}$$

and so using the algebra properties of M, $yM^k \subseteq Ky^2\sigma + M^{k+2}$. Consequently since $M^{k+1} = M^k M$ we have that

$$M^{k+1} = Ky\sigma M + M^{k+2} \subseteq yM^k + M^{k+2} \subseteq Ky^2\sigma + M^{k+2}. \tag{2.2}$$

Assuming $d_k = 1$, choose now $1 \le i \le k$ maximal with $M^k = Ky^i\sigma + M^{k+1}$, with $\sigma = a_1 a_2 \ldots a_{k-i}$ a product of $k - i$ elements of M. From (2.1) it is clear that such an i exists. We now show that $i = k$. If not, then σ is not an empty product and by applying (2.2) to this formula we get that $M^{k+1} = Ky^{i+1}\sigma + M^{k+2}$. But $M^{k+1} \neq M^{k+2}$, since otherwise $M^{k+1} = 0$ and $k+1 \le p$ would imply that $M^p = 0$ - a contradiction. Hence $y^{i+1}\delta \notin M^{k+2}$, and therefore $y^{i+1}a_1 \ldots a_{k-i-1} \notin M^{k+1}$. But then $y^{i+1}a_1 \ldots a_{k-i-1} \in M^k \setminus M^{k+1}$, and so since $d_k = 1$, $M^k = Ky^{i+1}a_1 a_2 \ldots a_{k-i-1} + M^{k+1}$, contradicting the maximality of i. Thus $i = k$ and $M^k = Ky^k + M^{k+1}$.

Now equation (2.2) and induction imply that $M^{k+j} = Ky^{k+j} + M^{k+j+1}$ for all $j \ge 0$ and hence $M^{k+j} = \langle y^{k+j}, y^{k+j+1}, \ldots\rangle$. Since $k \le p-1$, we have in particular that

$$M^{p-1} = \langle y^{p-1}, y^p, y^{p+1}, \ldots\rangle. \tag{2.3}$$

Now let $Y = \langle y, y^2, \ldots\rangle$. If $p = 2$ then by (2.3), $M = Y$. Hence M is commutative and $N^{(p)} = \langle y^p, y^{2p}, \ldots\rangle$ as required. We may assume therefore that $p \ge 3$. Let $A = \mathrm{Annih}_M(y^{p-2})$. The map $m \to my^{p-2}$ (of additive groups) sends M into M^{p-1}, has kernal A and by (2.3) maps Y onto M. It is easy to verify that this forces $Y + A = M$.

We next claim that any product of the elements of Y and A which involves p factors, with at least one from A must equal 0. For consider the first A term in such a product $y_1 \dots y_j a m_1 \dots m_t$ where $j + t = p - 1$, $y_i \in Y$ and $a_i \in A$. Notice that $y_2 \dots y_j a m_1 \dots m_y \in M^{p-1}$. Sine Y clearly centralises M^{p-1}, we may slide the factor y_1 above to the far right . Thus $x = y_2 \dots y_j a m_1 \dots m_t y_1$, and continuing in this manner we get that $x = a m_1 \dots m_t y_1 \dots y_j \in a M^{p-1} = 0$. But then $x = 0$ proving our claim.

Finally, if z is an arbitrary element of M then since $M = Y + A$, we may write $z = b + a$, $(b \in Y,\ a \in A.)$ Then $z^p = b^p +$ a sum of terms involving p factors of Y^s and of A^s with at least one A factor occuring. Thus, by the above this last sum is zero and so $z^p = b^p$ and so is a $K-$linear sum of the $y^{pi'}s, i \geq 1$. Thus $N^{(p)} \subseteq \langle y^{pi}, i \geq 1 \rangle$ completing the proof of the theorem. □

We now have our main theorem as a corollary of this:

Theorem 2. *Let M be a finite dimensional nilpotent algebra over a fields K of characteristic p, and let $M^{(p)}$ be the subalgebra of M generated by the p^{th} powers of elements of M. If* $\dim M^{(p)} = 2$ *then* $\dim M \geq 2p$.

Proof. Assume the theorem is false and let M be a counter example of least dimension. Then $\dim M < 2p$ and so $M^{2p} = 0$. In particular we have that $m^p n^p = 0$ for every $m, n \in M$ and so $M^{(p)}$ will just be the linear span of elements of the form $x^p, x \in M$, i.e $M^{(p)} = N^{(p)}$. Since $\dim M < 2p$ and $\dim M^p \geq 2$, it follows that there exists k, $1 \leq k \leq p - 1$ for which $d_k = 1$. But then, by Theorem 1, $M^{(p)} = N^{(p)} = \langle y^{pi}, i \geq 1 \rangle$. But since $M^{2p} = 0$, we must have that $M^{(p)} = K y^p$, and so $\dim M^{(p)} \leq 1$ a contradiction. This completes the proof of the theorem. □

We remark that the nonnecessity of the assunptions of commutativity and perfectness of the field in out previous theorem suggests perhaps (and believed to be the case by the author) and (2.1) may hold for finite dimensional noncommutative nilpotent algebras and perhaps also over fields of prime characteristic.

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