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BOUNDARY BEHAVIOR OF THE BERGMAN CURVATURE IN STRICTLY PSEUDOCONVEX POLYHEDRAL DOMAINS

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In this article, we present an explicit description of the boundary behavior of the holomorphic curvature of the Bergman metric of bounded strictly pseudoconvex polyhedral domains with piecewise C^2 smooth boundaries. Such domains arise as an intersection of domains with strongly pseudoconvex domains with C^2 smooth boundaries, creating normal singularities in the boundary. Our results in particular yield an optimal generalization of the well-known theorem of Klembeck, in terms of the boundary regularity. As an application, we demonstrate generalization of several theorems which were previously known only for the cases of eveywhere C^∞ (essentially) smooth boundaries.

1. Introduction.

Let D be a bounded domain in $\mathbb{C}^n$. Consider the space

$$\mathcal{H}^2(D) := \left\{ f : D \to \mathbb{C} \,|\, f \text{ is holomorphic}, \int_D |f|^2 d\mu < \infty \right\}$$

where $d\mu$ is the standard volume form of $\mathbb{C}^n$. This space is usually called the *Bergman space.* Equipped with the standard L^2 norm, it is a separable Hilbert space. Therefore, we choose an orthonormal basis $\{\varphi_j\}_{j=0}^{\infty}$ for the Bergman space. Then the *Bergman kernel function* $K : D \times D \to \mathbb{C}$ can be obtained by

$$K(z, \bar{\zeta}) := \sum_{j=1}^{\infty} \varphi_j(z)\overline{\varphi_j(\zeta)},$$

where $z, \zeta \in D$. This function gives rise to the well-known *Bergman metric* of D as follows:

$$\sum_{\alpha,\beta=1}^{n} g_{\alpha\bar{\beta}} dz^\alpha \otimes d\overline{z^\beta} := \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \log K(z, \bar{z})}{\partial z^\alpha \partial \overline{z^\beta}} dz^\alpha \otimes d\overline{z^\beta}.$$

One of the important features of this metric is that it is one of the invariant Kähler metrics, in the sense that the biholomorphic mappings are isometries with respect to the Bergman metric.

The goal of this article is to present an effective analysis of the boundary behavior of the holomorphic curvature (or, more traditionally, holomorphic sectional curvature) of the Bergman metric near the boundary of a *strictly pseudoconvex polyhedral domain* that is obtained by taking intersections of the bounded domains with C^2 smooth strictly pseudoconvex boundaries, where the intersections are allowed to produce the normal-crossing singularities.

Definition 1. A bounded domain Ω in $\mathbb{C}^n$ is called a (*pseudoconvex*) *polyhedral domain*, if there are C^2 smooth real valued functions $\rho_1, \ldots, \rho_k : \mathbb{C}^n \to \mathbb{R}$ such that

(1) $\Omega = \{z \in \mathbb{C}^n \mid \rho_1(z) < 0, \ldots, \rho_k(z) < 0\}$,

(2) the gradient vectors $\nabla\rho_{i_1}(q), \ldots, \nabla\rho_{i_\ell}(q)$ are linearly independent over $\mathbb{C}$ whenever $\rho_{i_1}(q) = \ldots = \rho_{i_\ell}(q) = 0$, for all appropriate indices $\{i_1, \ldots, i_\ell\} \subset \{1, \ldots, k\}$,

(3) $\partial\Omega$ is pseudoconvex at every smooth boundary point.

If, in addition to (3), $\partial\Omega$ is strongly pseudoconvex at every smooth boundary point, then the domain Ω is called a *strictly pseudoconvex polyhedral domain.*

As indicated in [**Kim**], even in the case for the simplest singular boundary points, there is no uniformity in the boundary behavior of the Bergman curvature in the sense that the boundary behavior is heavily depending upon the tangency of the orbit as well as the target boundary point. The first and simplest case of our results is the case when the target boundary point is smooth.

Theorem 1. *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$, and let p be a boundary point of D at which the boundary ∂D is C^2 smooth and strongly pseudoconvex. Then for any sequence of points p_j in D and any holomorphic sections Π_j at p_j the limit of the holomorphic curvatures of the Bergman metric at p_j in the direction Π_j is $-4/(n+1)$.*

Main contribution of this result is that the regularity required at the smooth strictly pseudoconvex boundary point is only C^2. This differentiates our results from the results of [**Kle**, **GK1-2**] and others that use Fefferman's asymptotic expansion formula of the Bergman kernel function. We would like to point out also that work of Diederich ([**Di**] and others) on the Bergman kernel and metric deals with C^2 smooth boundary case effectively. Thus, in principle, Theorem 1 above may also follow along the ideas and the estimates of [**Di**] after some considerable amount of modification and computation. However, deriving the above conclusion from such work seems at least long

and tedious if not formidable. Since our method here is simpler and more flexible and general as far as the analysis of the curvature of the Bergman metric is concerned, it should be of separate interest concerning the above theorem.

The methods presented in this article do not depend upon either the asymptotic expansion formula or the pseudo-local estimate ([**FK**]). Rather, we use the L^2 estimate of the $\bar{\partial}$ operator by Hörmander [**Hör1**, **2**] with appropriate weights to localize the minimum integrals which give rise to a formula ([**Ber1**, **2**, **Fuks**] and others) for the holomorphic curvature of the Bergman metric in a bounded domain. Then we use the scaling technique which converts the problem of limiting curvature near the boundary to the interior stability problem of the Bergman curvature, which was initiated in an earlier work of the first author [**Kim**]. Then a simple modification of a theorem of Ramadanov [**Ram**] on the interior stability of the Bergman kernel function, observed earlier in [**Kim**], yields the result. We would like to emphasize at this point that our method does not require any regularity of the boundary in its application. Thus it is flexible enough to handle the following cases where the target boundary point is in fact singular.

Indeed, the main thrust of the methods we present in this article is aimed toward the study of curvature behavior of the Bergman metric near the boundary point where the boundary is not smooth.

Theorem 2. *Let Ω be a bounded strictly pseudoconvex polyhedral domain in $\mathbb{C}^2$ and let $p \in \partial\Omega$ be a boundary point at which the boundary $\partial\Omega$ is singular. Let $\{p_j\}_{J=1}^{\infty} \subset \Omega$ be a sequence of "radial type" that converges to p in $\mathbb{C}^2$ to the boundary of Ω. Then the holomorphic sectional curvature tensor of the Bergman metric of Ω at p_j converges to the holomorphic sectional curvature tensor of the Bergman metric of the bi-disk in $\mathbb{C}^2$ as $j \to \infty$.*

In the complex dimension higher than two, a similar but somewhat weaker conclusion can be obtained, and they are also presented in the end of Section 4. As for the tangential sequences of reference points, we also have the following:

Theorem 3. *Let Ω be a strictly pseudoconvex polyhedral domain in $\mathbb{C}^n$ and let p be its singular boundary point. Let $\{p_j\}_{j=1}^{\infty} \subset \Omega$ be a sequence of "q-tangential type" approaching p, then the holomorphic sectional curvature tensor of the Bergman metric of Ω along p_j converges to the constant $-4/(n+1)$.*

The precise concepts of the *radial type* and the *q-tangential type* are introduced in Section 4. Roughly speaking, a sequence in a strictly pseudoconvex polyhedral domain is called *q-tangential* if the sequence is tangential to the

smooth portion of the boundary at least to order 2. The radial type sequences are the ones that are either not tangential at all to the boundary or tangential "only to the singular locus" of the boundary. There is one remaining intermediate type of sequences to consider which we call "mixed type." Along such sequences our method yields the conclusion that the limit of the holomorphic curvature converges to that of the Bergman metric at a precisely determined point in the intersection of the ball and the half spaces. In conclusion, in terms of subsequences, Theorems 1, 2 and 3 provide an effective sequential analysis of the boundary behavior of the holomorphic curvature of the Bergman metric in complex dimension 2.

The final section of this article enlists several applications of our curvature analysis of the Bergman metric. Among many applications possible, we introduce only perhaps most straightforward cases. However, we would like to point out that Pagano's recent work ([**Pa**]) demonstrates that there are more problems that may be solved using our main results.

2. Localization of Curvature and Minimum Integrals.

2.1. Minimum Integrals and a Formula of Bergman-Fuks. Let $\zeta \in D$, and let $\xi = (\xi_1, \ldots, \xi_n) \in T_\zeta D = \mathbb{C}^n$ be a nonzero vector. Then consider the minimum integrals

$$I_0(\zeta) = \inf\left\{\int_D |f|^2 d\mu \,|\, f \in \mathcal{H}^2(D), f(\zeta) = 1\right\}$$

$$I_1(\zeta) = \inf\left\{\int_D |f|^2 d\mu \,|\, f \in \mathcal{H}^2(D), f(\zeta) = 0, \sum_{j=1}^{n} \xi_j \frac{\partial f}{\partial z_j}(\zeta) = 1\right\}$$

$$I_2(\zeta) = \inf\left\{\int_D |f|^2 d\mu \,|\, f \in \mathcal{H}^2(D), f(\zeta) = \frac{\partial f}{\partial z_1}(\zeta) = \ldots = \frac{\partial f}{\partial z_n}(\zeta) = 0, \right.$$
$$\left. \sum_{j,k=1}^{n} \xi_j \xi_k \frac{\partial^2 f}{\partial z_j \partial z_k}(\zeta) = 1\right\}.$$

Then the following formula for the holomorphic curvature was proven in 1930's by Bergman ([**Ber1**, **2**]) and Fuks ([**Fuks**]):

Theorem (Bergman-Fuks). *Let ζ, ξ, D be as above. Then the holomorphic curvature $R = R_\zeta(\xi)$ at ζ in the direction ξ satisfies*

$$R = 2 - \frac{(I_1(\zeta))^2}{I_0(\zeta) I_2(\zeta)}.$$

The holomorphic curvature R above is of course given by

$$R = \frac{R_{\bar{h}jk\bar{\ell}}\bar{\xi}^h\xi^j\xi^k\bar{\xi}^\ell}{(g_{j\bar{k}}\xi^j\bar{\xi}^k)^2}$$

where

$$R_{\bar{h}jk\bar{\ell}} = -\frac{\partial^2 g_{j\bar{h}}}{\partial z^k \partial z^{\bar{\ell}}} + g^{\nu\bar{\mu}}\frac{\partial g_{j\bar{\mu}}}{\partial z^k}\frac{\partial g_{\nu\bar{h}}}{\partial z^{\bar{\ell}}}.$$

In the above, the summation convention was used. Moreover, $g^{\bar{\mu}\nu}$ represents the inverse matrix of $g_{\alpha\bar{\beta}}$.

Remark. Perhaps the simplest proof of the Bergman-Fuks formula is by representing the curvature formula above as well as the corresponding minimum integrals by the special basis for $\mathcal{H}^2(D)$ with derivative control which can be found in detail in p. 146–148 of [**GW**]. (For the use of special basis preceding [**GW**], see for instance [**Ber1**, **2**], [**Kob**].)

2.2. Localization of the Minimum integrals. In the light of the formula of Bergman and Fuks above, in order to localize the holomorphic curvature of the Bergman metric of a given domain, it is enough to localize the minimum integrals I_i, $i = 1, 2$. In this section, we will denote the minimum integrals by I_i^D instead of I_i to emphasize the dependence of the minimum integrals upon the domain D.

The main result of this section is

Theorem 4. *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$. Suppose that $p \in \partial D$ is a local peak point. Then for any neighborhood U of p in $\mathbb{C}^n$,*

$$\lim_{\zeta \to p} \frac{I_i^{D\cap U}(\zeta)}{I_i^D(\zeta)} = 1, \qquad i = 0, 1, 2. \tag{1}$$

Moreover, the convergence is uniform on the choices of the unit vectors $\xi \in \mathbb{C}^n$ in the definition of the minimum integrals.

In the statement above, a boundary point $p \in \partial D$ is called a *local peak point* of D if there are a neighborhood V of p in $\mathbb{C}^n$ and a function h that is continuous on $\overline{D} \cap V$ and holomorphic on $D \cap V$ satisfying: $|h(z)| < 1$ for all $z \in \overline{D} \cap V \setminus \{p\}$ and $h(p) = 1$.

Proof of Theorem 4. For this proof and throughout the paper, we will use the following notation:

$$|z| = |(z_1, \ldots, z_n)| = \sqrt{|z_1|^2 + \ldots + |z_n|^2}$$

$$\|f\|_D = \left(\int_D |f|^2 d\mu\right)^{\frac{1}{2}}.$$

Let h, V be as above. We may assume, without loss of generality, that $U \subset V$. Choose another open neighborhood U_0 of p in $\mathbb{C}^n$ such that $U_0 \subset\subset U$ and $h \neq 0$ on U_0. Then there is a constant $a \in (0,1)$ such that $|h| \leq a$ on $\overline{(U \setminus U_0) \cap D}$. Choose a cut-off function $\chi \in C_o^\infty(U)$ satisfying: $\chi = 1$ on U_0 and $0 \leq \chi \leq 1$ on U.

Given any function $f \in \mathcal{H}^2(U \cap D)$, for each integer $k \geq 1$, set $\alpha = \overline{\partial}(\chi f h^k)$. Then α is a smooth closed (0,1)-form on D with $supp\, \alpha \subset (U \setminus U_0) \cap D$. For any fixed $\zeta \in U_0$, we write $\varphi(z) = (2n+4)\log|z-\zeta|$. Clearly φ is plurisubharmonic on $\mathbb{C}^n$. Applying Theorem 4.42 in [**Hör**], one gets a solution u to the equation $\overline{\partial} u = \alpha$ on D such that

$$\int_D |u(z)|^2 e^{-\varphi(z)} (1+|z|^2)^{-2}\, d\mu \leq \int_D |\alpha(z)|^2 e^{-\varphi(z)}\, d\mu.$$

Here, as above, $d\mu$ denotes the standard Lebesgue measure on $\mathbb{C}^n$. The inequality above then becomes

$$\int_D \frac{|u|^2}{|z-\zeta|^{2n+4}(1+|z|^2)^2}\, d\mu \leq \int_{D\cap(U\setminus U_0)} \frac{|\alpha|^2}{|z-\zeta|^{2n+4}}\, d\mu. \tag{2}$$

Since the right hand side of (2) is bounded, so is the left hand side. This implies in particular that

$$\frac{\partial^{|A|+|B|} u}{\partial z^A \partial \bar{z}^B}(\zeta) = 0, \qquad \text{for all } A,B \text{ with } |A|+|B| \leq 2. \tag{3}$$

Moreover, since D is bounded, there are positive constants c_1, c_2 independent of k such that

$$\int_D \frac{|u|^2}{|z-\zeta|^{2n+4}(1+|z|^2)^2}\, d\mu \geq c_1 \int_D |u|^2\, d\mu,$$

and

$$\begin{aligned}
\int_{D\cap(U\setminus U_0)} \frac{|\alpha|^2}{|z-\zeta|^{2n+4}}\, d\mu &= \int_{D\cap(U\setminus U_0)} \frac{|\overline{\partial}\chi|^2 |f|^2 |h|^{2k}}{|z-\zeta|^{2n+4}}\, d\mu \\
&\leq c_2 \int_{D\cap(U\setminus U_0)} |f|^2 |h|^{2k}\, d\mu \\
&\leq c_2 a^{2k} \int_{D\cap(U\setminus U_0)} |f|^2\, d\mu \qquad \text{by the choice of } h,\ U_0 \\
&\leq c_2 a^{2k} \|f\|^2_{D\cap U}.
\end{aligned}$$

Therefore it follows from (2) that

$$\|u\|_D \leq c a^k \|f\|_{D\cap U}. \tag{4}$$

Here, $c = c_2/c_1$.

Now set $F_k = \chi f h^k - u$ for any $k \geq 1$. Then $F_k \in \mathcal{H}^2(D)$. Moreover, (4) implies that

$$\begin{aligned} \|F_k\|_D &\leq \|\chi f h^k\|_D + \|u\|_D \\ &\leq \|f\|_{D\cap U} + ca^k \|f\|_{D\cap U} \\ &= (1 + ca^k)\, \|f\|_{D\cap U}. \end{aligned} \tag{5}$$

Let $f \in \mathcal{H}^2(D \cap U)$ be the minimizing function for $I_2^{D\cap U}(\zeta)$. That is, $f(\zeta) = \frac{\partial f}{\partial z_i}(\zeta) = 0, i = 1, \ldots, n$, $\sum_{j,k=1}^n \xi_j \xi_k (\partial^2 f/\partial z_j \partial z_k)(\zeta) = 1$, and $\|f\|^2 = I_2^{D\cap U}(\zeta)$. For any $z \in D$, set $g(z) = F_k(z)/(h(\zeta))^k$. Then $g \in A^2(D)$. Moreover, we have

$$g(\zeta) = \frac{\partial g}{\partial z_i}(\zeta) = \frac{\partial f}{\partial z_i}(\zeta) = 0,\ 1 \leq i \leq n, \text{ and } \sum_{j,k=1}^{n} \xi_j \xi_k \frac{\partial^2 g}{\partial z_j \partial z_k}(\zeta) = 1.$$

Hence, by the minimality of $I_2^D(\zeta)$, we obtain

$$\begin{aligned} I_2^D(\zeta) \leq \|g\|_D^2 &= \frac{1}{|h(\zeta)|^{2k}} \|F_k\|_D^2 \\ &\leq \frac{1}{|h(\zeta)|^{2k}} \left(1 + ca^k\right)^2 \|f\|_{D\cap U}^2 \\ &= \frac{1}{|h(\zeta)|^{2k}} \left(1 + ca^k\right)^2 I_2^{D\cap U}(\zeta). \end{aligned}$$

This implies that

$$\frac{I_2^D(\zeta)}{I_2^{D\cap U}(\zeta)} \leq \frac{\left(1 + ca^k\right)^2}{|h(\zeta)|^{2k}}. \tag{6}$$

Letting $\zeta \to p$ in the above inequality, we get

$$\limsup_{\zeta \to p} \frac{I_2^D(\zeta)}{I_2^{D\cap U}(\zeta)} \leq \left(1 + ca^k\right)^2.$$

Then let $k \to \infty$. Since $0 \leq a \leq 1$ and since c is independent of k, we get

$$\limsup_{\zeta \to p} \frac{I_2^D(\zeta)}{I_2^{D\cap U}(\zeta)} \leq 1.$$

On the other hand, from the definition of the minimum integrals we have $I_2^D \geq I_2^{D\cap U}$. Therefore, we conclude that

$$\lim_{\zeta \to p} \frac{I_2^D(\zeta)}{I_2^{D\cap U}(\zeta)} = 1.$$

Furthermore, this convergence is uniform on the choices of the unit vector ξ in the definition of I_2^D because the right hand side of (6) is independent of choices of ξ. This proves (1) for I_2. The proof of (1) for I_0 and I_1 is almost identical. So we do not include further details to avoid repeating similar arguments. This completes the proof of Theorem 4. □

An immediate consequence of Theorem of Bergman-Fuks and Theorem 4 above is the following localization of the holomorphic Bergman curvature:

Corollary. *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$. If $p \in \partial D$ is a local peak point of D then, for any neighborhood U of p in $\mathbb{C}^n$, we have*

$$\lim_{\zeta_j \to p} \frac{2 - R^{D\cap U}_{\zeta_j}(\xi_j)}{2 - R^{D}_{\zeta_j}(\xi_j)} = 1, \text{ for any sequence } \{\xi_j\}_j \subset \mathbb{C}^n \setminus 0,$$

where $R^D_\zeta(\xi)$ denotes the holomorphic curvature of the Bergman metric of the domain D at ζ in the direction ξ with respect to the standard coordinate system of $\mathbb{C}^n$.

Remark. An effective localization for the strictly pseudoconvex boundary points was obtained for the Bergman kernel (on the diagonal) and the Bergman metric earlier by Diederich and others. (Cf. [**Di**, **DFH**, **McN**, **Ohs**, **Yu**].)

3. Curvature behavior at C^2 strongly pseudoconvex points.

To make the exposition as clear as possible, we would like to present first the proof of Theorem 1 in the special case when p_j approaches $p \in \partial D$ *nontangentially* to ∂D. Then in Section 3.2 below, we will present the arguments which reduce the general case to the nontangential case.

3.1. Nontangential behavior. We will use the following standard notations:

$$z = (z_1, \dots, z_n) = (z_1, z'), \ z' = (z_2, \dots, z_n).$$

Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ which possesses a boundary point p admitting an open neighborhood U such that the boundary $\partial D \cap U$ is C^2 smooth and strongly pseudoconvex. Applying a global quadratic change of complex coordinates of $\mathbb{C}^n$ at p, and shrinking U to a smaller neighborhood, we may assume that $p = 0$ and that $E = D \cap U$ is defined by

$$E = \{z \in U \mid 2 \operatorname{Re} z_1 < -|z'|^2 + o(|z'|^2, |z_1|), \ |z| \le r\}.$$

Let $\{p_j\}_j$ be a sequence in D converging to p, and let $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$ fixed. Then, as soon as we prove that

$$\lim_{j\to\infty} R^E_{p_j}(\xi_j) = \frac{-4}{n+1}, \text{ for any sequence } \{\xi_j\}_j \subset \mathbb{C}^n \setminus 0,$$

the corollary to Theorem 4 in the preceding section will immediately imply that

$$\lim_{j\to\infty} R^D_{p_j}(\xi) = \frac{-4}{n+1}.$$

Therefore, to prove Theorem 1, we concentrate on the limiting behavior of the holomorphic curvature of the Bergman metric of E.

Assume momentarily that p_j converges to $p = 0 \in \partial D$ nontangentially, meaning that there exists an acute cone V with the vertex at p such that, for a neighborhood W of p, all of the points p_j belong to $\in V \cap W \subset D$. Replacing $r > 0$ in the definition of E above by a smaller positive value, we may assume without loss of generality that there exists a constant $c > 0$ such that

$$E \subset G := \{(z_1, z') \in \mathbb{C}^n \mid 2 \text{ Re } z_1 < -c|z'|^2\}.$$

Then we consider the sequence of complex linear maps of $\mathbb{C}^n$ defined by

$$L_j(z_1, z') := (\lambda_j z_1, \sqrt{\lambda_j} z')$$

where $\lambda_j = |p_j - p|^{-1} = |p_j|^{-1}$. Then $L_j(G) = G$ for all j. Therefore, $L_j(E) \subset G$ for every j. Observe that the sequence of convex domains $L_j(E)$ converges in the sense of the local Hausdorff set-convergence to the Siegel upper half-space

$$H := \{(z_1, z') \in \mathbb{C}^n \mid 2 \text{ Re } z_1 < -|z'|^2\}.$$

Furthermore, observe that, because $\{p_j\}$ converges to p nontangentially to the boundary, the sequence $\{L_j(p_j)\}$ converges to a single point on the Re z_1 axis bounded away from the boundary of H as well as the boundaries of the domains $L_j(E)$ for sufficiently large values of j.

Now consider the linear fractional mapping

$$\Phi(z_1, z') = \left(\frac{1+z_1}{1-z_1}, \frac{z'}{1-z_1}\right).$$

Φ maps biholomorphically the domains H, G and $L_j(E)$ onto the bounded domains defined by

$$\Phi(H) = \{(z_1, z') \in \mathbb{C}^n \mid |z_1|^2 + c|z'|^2 < 1\}$$
$$\Phi(G) = \{z \in \mathbb{C}^n \mid |z| < 1\}$$
$$\Phi(L_j(E)) \subset \Phi(G), \ \forall j$$

respectively. Moreover, $\Phi(L_j(E))$ converges to $\Phi(H)$ in the sense of *global* Hausdorff set-convergence. Notice that $\Phi(H)$ is convex and contains the origin inside, and that for every $\epsilon > 0$, there exists j_0 such that

$$(1-\epsilon)\Phi(H) \subset \Phi(L_j(E)) \subset (1+\epsilon)\Phi(H), \ \forall j > j_0. \tag{7}$$

Furthermore, since each $L_j : E \to L_j(E)$ is a biholomorphism, we have

$$R^E_{p_j}(\xi) = R^{\Phi\circ L_j(E)}_{\Phi\circ L_j(p_j)}(d(\Phi \circ L_j)(\xi)) \tag{8}$$

for every $j > j_0$. Notice that the convergence of the right hand side of (8) concerns only the interior stability of the Bergman holomorphic curvature of the domain under the perturbation of the boundary. Therefore, the identity (8) in fact *converts the boundary behavior problem of the Bergman holomorphic curvatures to the interior stability problem.*

Now we will show that the right hand side of (7) converges to $-4/(n+1)$. This will follow from the following mild modification of Theorem 2 of [**Ram**], and from the fact that the holomorphic curvature of the Bergman metric of $\Phi(H)$ is identically equal to $-4/(n+1)$ at every point in every direction.

Proposition. *Let $\{D_j\}_{j=1}^{\infty}$ be a sequence of bounded domains in $\mathbb{C}^n$ that converges to a convex bounded domain $D \subset \mathbb{C}^n$ in such a way that there exists a common interior point q of D and D_j for all j and such that for every $\epsilon > 0$ there exists j_0 satisfying*

$$(1-\epsilon)(D-q) \subset D_j - q \subset (1+\epsilon)(D-q),$$

where $D-q$ denotes the affine translation by $-q$ of the set D in $\mathbb{C}^n$. Then for each $z \in D$ that admits a compact neighborhood F which is uniformly bounded away from ∂D, the Bergman kernel function $K_{D_j}(z,\bar{z})$ converges uniformly on F to $K_D(z,\bar{z})$ on all derivative levels.

We do not include any detailed proof of the proposition above, since it follows from the arguments that are almost identical to the proof in [**Ram**], which proves that $K_{D_j}(z,\bar{\zeta})$ converges in L^2 norm to $K_D(z,\bar{\zeta})$ on $F \times F$. The conclusion of the proposition above then follows immediately by Cauchy estimates on $D \times \bar{D}$, where $\bar{D}$ denotes the domain D with the conjugate complex structure.

3.2. General Case. Now consider a sequence of points p_j in D converging to $p = 0 \in \partial D$ in an arbitrary manner. As in the preceding section, we will begin our proof with

$$E = \{(z_1, z') \in \mathbb{C}^n \mid 2 \operatorname{Re} z_1 + |z'|^2 + \epsilon(z) < 0, |z| < r\}$$

where $\epsilon(z) = o(|z'|^2, |z_1|)$.

Let $\eta > 0$ be given. Without loss of generality, we may assume that all p_j are in the η-neighborhood W of the origin. We may choose η so small that $W \subset\subset U$. Then the statement and the proof of Lemma 2.2 of [**Pin**] also yield the following.

Lemma. *There exists a positive value for η such that for each $\zeta \in \partial E \cap W$, there exists a complex linear coordinate change A^ζ, depending continuously on ζ, that satisfies*

(1) $A^\zeta(\zeta) = 0$, $A^0 =$ *identity;*

(2) A^ζ *maps the normal vector to ∂E at ζ to the normal vector to the boundary of $A^\zeta(E)$ at the origin,*

(3) *the new domain $A^\zeta(E \cap W)$ has a defining function near the origin as follows:*

$$2\,\mathrm{Re}\, z_1 + \sum_{j,k=1}^{n} a_{jk}(\zeta) z_j z_k + \sum_{j,k=1}^{n} b_{j\bar{k}}(\zeta) z_j \bar{z}_k + \psi(\zeta, z) < 0 \tag{9}$$

where:

(4) *the functions $a_{jk}, b_{j\bar{k}}$ and ψ depend continuously on ζ, and*

(5) $\psi = o(|z'|^2, |z_1|)$.

The proof of this lemma follows directly from part of the proof of Lemma 2.2 of [**Pin**], and so we again omit the details here.

Now we deal with the curvature behavior for general p_j converging to $p = 0 \in \partial D$. The key idea is that there is a certain uniformity to our methods in a neighborhood of a strongly pseudoconvex point. Note that without loss of generality we may replace E by $E \cap W$, keeping the notation E. Let $\zeta_j \in \partial E$ be such that $|p_j - \zeta_j| = \mathrm{dist}\,(p_j, \partial E)$ and such that the line joining p_j and ζ_j is perpendicular to ∂E. Then, by the Lemma, the defining function of E can be written as in (3)–(5) in Lemma above, with ζ_j replacing ζ. Since the linear change of coordinates does not destroy the strong convexity of $\partial E \cap U$, there exists $R > 0$ such that

$$A^\zeta(E) \subset B := \{(z_1, z') \in C^n \,|\, |z_1 + R|^2 + |z'|^2 < R^2\}$$

for any $\zeta \in \partial E \cap U$. Let

$$\hat{B} := \{(z_1, z') \in \mathbb{C}^n \,|\, 2\ \mathrm{Re}\ z_1 < -|z'|^2/R\}$$

then clearly $B \subset \hat{B}$. Now let $\lambda_j = |p_j - \zeta_j|^{-1}$ and apply the linear map

$$L_j(z_1, z') = \left(\lambda_j z_1, \sqrt{\lambda_j} z'\right)$$

to $A^{\zeta_j}(E)$ and $\hat{B}$. First of all $L_j(\hat{B}) = \hat{B}$ for every j. Then, the properties (1)–(5) of Lemma imply that the local Hausdorff set-limit of the sequence $L_j(A^{\zeta_j}(E))$ is

$$H = \{(z_1, z') \in \mathbb{C}^n \mid 2 \text{ Re } z_1 < -|z'|^2\}.$$

Since $L_j(A^{\zeta_j}(p_j))$ defines a sequence in the common interior of H and $L_j(A^{\zeta_j}(p_j))$ bounded away from their boundaries, the rest of the proof is identical with that for the nontangential case in the preceding section. Therefore, the proof of Theorem 1 is now complete.

4. Curvature behavior at the singular boundary points.

As will be seen in the subsequent sections, the boundary behavior of the Bergman curvature in a polyhedral domain near a singular boundary point of the holomorphic curvature of the Bergman metric is in general very sensitive to the order of tangency of the orbit of the reference points to the smooth faces of the boundary of the domain. So, we will begin this portion of exposition by defining the classes of the orbits of the reference points in complex dimension two in which the conclusions are the strongest and most explicit. At the end of this chapter, we will discuss how certain methods and cases can be generalized to higher dimensions.

4.1. Types of orbits of the reference points. Let $\Omega \subset \mathbb{C}^2$ be a polyhedral domain as in Definition 1 of Section 1 defined by

$$\Omega = \{z \in \mathbb{C}^2 \mid \rho_1(z) < 0, \dots, \rho_k(z) < 0\}.$$

Call the strongly pseudoconvex hypersurfaces $\Sigma_j = \{\rho_j(z) = 0\}$ $(j = 1, \dots, k)$ the *faces* of the boundary surface $\partial\Omega$.

Let p be a singular boundary point of Ω. It turns out that the boundary behavior of the sectional curvature tensor of the Bergman metric is *dependent* upon the tangency of the orbit of the reference points to the smooth portion of the boundary as the reference points approach the boundary point p. Therefore, as in what follows, the asymptotic behavior of the holomorphic curvature tensor of the Bergman metric of Ω can only be made sense in terms of subsequential limits along subsequences of the sequence of the reference points. Now, the orbits can be classified into three mutually exclusive classes as follows:

Let us denote by $S_{\partial\Omega}$ the set of boundary points at which the boundary $\partial\Omega$ is singular, and by $R_{\partial\Omega}$ the set of smooth boundary points.

Let $p \in S_{\partial\Omega}$. According to the definition of the polyhedral domains (Definition 1), the singularity set $S_{\partial\Omega}$ is locally an intersection of exactly two distinct faces of $\partial\Omega$ in $\mathbb{C}^2$. We may assume without loss of generality that

the two faces are Σ_1 and Σ_2. Further, assume that p is the origin of $\mathbb{C}^2$. Then at the origin, apply a complex linear change of the coordinates of $\mathbb{C}^2$ such that the gradient vectors to Σ_1 and Σ_2 are parallel to the Im z and Im w axes, respectively.

Let p_j denote a sequence of points in Ω converging to p. We now define the orbit-types. Denote by

$$\lambda_j = \text{dist}\,(p_j, \Sigma_1)$$
$$\mu_j = \text{dist}\,(p_j, \Sigma_2)$$

for each j. Then

Definition 2. The sequence p_j above is said to be *of radial type* if there exists a positive constant C independent of j such that

$$\frac{1}{C} \leq \frac{\lambda_j}{\mu_j} \leq C, \text{ for all } j.$$

The sequence p_j is said to be *of q-tangential type* if either

$$\lim_{j\to\infty} \mu_j^{-1}\sqrt{\lambda_j} = 0$$

or

$$\lim_{j\to\infty} \lambda_j^{-1}\sqrt{\mu_j} = 0.$$

The sequence p_j is said to be *of mixed type* if it is of neither radial type nor q-tangential type.

4.2. Curvature behavior along a sequence of radial type. Let Ω be a bounded strictly pseudoconvex polyhedral domain in $\mathbb{C}^2$, let $p = (0,0) \in S_{\partial\Omega} \subset \partial\Omega$, and let p_j denote a sequence of radial type in Ω approaching p.

We now fix a vector $\xi = (\xi_1, \xi_2) \in \mathbb{C}^2$ and compute

$$\lim_{j\to\infty} R_{p_j}(\xi)$$

in what follows.

Angle factor at p. Recall from the paragraph preceding Definition 2 that we assumed p is the origin of $\mathbb{C}^2$. Then at the new origin p, apply a complex linear change, say A, of the coordinates of $\mathbb{C}^2$ such that the gradient vectors to Σ_1 and Σ_2 are parallel to the Im z and Im w axes, respectively. This A will play a role in the computation of the limit of the holomorphic curvature.

Controlled scaling along a radial orbit. Now choose, for each j, $s_j \in \Sigma_1$ and $t_j \in \Sigma_2$ such that

$$\begin{aligned} \operatorname{dist}(s_j, p_j) &= \operatorname{dist}(p_j, \Sigma_1) \\ \operatorname{dist}(t_j, p_j) &= \operatorname{dist}(p_j, \Sigma_2) \end{aligned}$$

respectively. Due to the smoothness of the "faces" and due to the definition of the polyhedral domains, the following two unit vectors in $\mathbb{C}^2$ given by

$$\begin{aligned} \nu_{1j} &= \frac{p_j - s_j}{\|p_j - s_j\|} \\ \nu_{2j} &= \frac{p_j - t_j}{\|p_j - t_j\|} \end{aligned}$$

are well-defined and linearly independent over $\mathbb{C}$.

For each j, consider the complex affine transform $B_j : \mathbb{C}^2 \to \mathbb{C}^2$ which maps p_j to the origin and, ν_{1j} and ν_{2j} to the unit vectors in Im z and Im w directions, respectively. More specifically, we define B_j for each j by

$$\begin{aligned} B_j(p_j) &= (0,0), \\ B_j(\nu_{1j}) &= \left(\sqrt{-1}, 0\right), \\ B_j(\nu_{2j}) &= \left(0, \sqrt{-1}\right). \end{aligned}$$

Notice that $\lim_{j\to\infty} B_j$ is the identity mapping of $\mathbb{C}^2$.

Choose a small open ball $U = U(p, \epsilon)$ with radius ϵ centered at p. For each j, change coordinates by passing to A and B_j in order. Then, in U the domain Ω is represented by the inequalities

$$\begin{aligned} \operatorname{Im}\left(z - s_j^{(1)}\right) &> Q_1(w, \bar{w}) + o\left(|z - s_j^{(1)}| + |w|^2\right) \\ \operatorname{Im}\left(w - t_j^{(2)}\right) &> Q_2(z, \bar{z}) + o\left(|w - t_j^{(2)}| + |z|^2\right) \end{aligned}$$

where Q_1 and Q_2 denote the real-valued quadratic polynomials, and where $s_j = (s_j^{(1)}, s_j^{(2)})$ and $t_j = (t_j^{(1)}, t_j^{(2)})$ in $\mathbb{C}^2$. Now, apply to $U \cap \Omega$ the mapping $L_j : \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$L_j(z, w) = \left(\lambda_j^{-1} z, \mu_j^{-1} w\right)$$

and consider the sequence of the domains $L_j(U \cap \Omega)$ To be more specific, consider the domains $L_j \circ B_j \circ A(V \cap \Omega)$ for some small open ball V in $\mathbb{C}^2$ centered at p. Then, one easily observes the following:

(a) The sequence of domains $L_j \circ B_j \circ A(V \cap \Omega)$ converges in the local Hausdorff sense to the domain defined by the inequalities

$$\operatorname{Im}(z + c_1) > 0, \text{ and } \operatorname{Im}(w + c_2) > 0$$

for some positive constants c_1, c_2.

(b) For every $r > 0$, there exist positive constants j_0 such that

$$L_j \circ B_j \circ A(V \cap \Omega) \subset E \equiv \{(z,w) \in \mathbb{C}^2 \mid \operatorname{Im} z > -c_1 - r,\ \operatorname{Im} w > -c_2 - r\}.$$

for all $j > j_0$.

(c) $L_j \circ B_j \circ A(p_j) = (0,0)$ for every j.

Bergman curvature along a radial orbit. Continuing from the above, denote by $\hat{\Omega}$ the domain in $\mathbb{C}^2$ defined by

$$\operatorname{Im} z > -c_1, \text{ and } \operatorname{Im} w > -c_2.$$

There exists a linear fractional transformation Φ of $\mathbb{C}^2$ which embeds $\hat{\Omega}$ into $\mathbb{C}^2$ biholomorphically so that

$$\Phi(\hat{\Omega}) = \Delta^2 \equiv \{(z,w) \in \mathbb{C}^2 \mid |z| < 1, |w| < 1\}$$

and

$$\Phi(0,0) = (0,0).$$

Now, let E be the domain in (a) above. By choosing a sufficiently small value for $r > 0$, we obtain that $\Phi(E)$ is a bounded domain in $\mathbb{C}^2$ biholomorphic to E. The effect of such simultaneous bounded realization is the following:

(i) $(\Phi \circ L_j \circ B_j \circ A)(V \cap \Omega) \subset \Phi(E)$ for all $j \geq j_0$.

(ii) For every $\epsilon > 0$, there exists j_1 such that $(1-\epsilon)\Delta^2 \subset (\Phi \circ L_j \circ B_j \circ A)(V \cap \Omega) \subset (1+\epsilon)\Delta^2$ for all $j > j_1$.

Now, we apply the interior stability of the Bergman metric (Proposition in Section 3.1) to get the curvature results. First choose a subsequence of p_j so that

$$\lim_{j\to\infty} \frac{\lambda_j}{\mu_j} = m$$

and let

$$L = \lim_{j\to\infty} \lambda_j L_j = \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}.$$

Then for every $\xi \in \mathbb{C}^2 \setminus \{0\}$, we obtain

$$\begin{aligned} \lim_{j\to\infty} R^{V\cap\Omega}_{p_j} &= \lim_{j\to\infty} R^{(\Phi\circ L_j \circ B_j \circ A)(V\cap\Omega)}_{(0,0)} \left((\Phi_* \circ L_j \circ (B_j)_* \circ A)(\xi)\right) \\ &= \lim_{j\to\infty} R^{\Delta^2}_{(0,0)} \left((\Phi_* \circ L \circ A)(\xi)\right). \end{aligned}$$

Notice that Φ, L and A can be explicitly computed whenever the strictly pseudoconvex polyhedral domain and the radial type sequence of the reference points p_j are given explicitly.

Combining the discussion above with the sharp localization of the holomorphic curvature of the Bergman metric in Theorem 4 and its corollary in Section 2 of this article, we now obtain the following refinement of Theorem 2:

Theorem 2′. *Let Ω be a strictly pseudoconvex polyhedral domain in $\mathbb{C}^2$ with a singular boundary point p, through which two C^2 smooth strongly pseudoconvex faces Σ_1 and Σ_2 pass. Let p_j be a sequence of radial type converging to p from inside Ω such that $m = \lim_{j\to\infty} \operatorname{dist}(p_j, \Sigma_1)/\operatorname{dist}(p_j, \Sigma_2)$. Then there exists a nonsingular 2×2 matrix S depending only on the number m and the normal vectors to the faces Σ_1 and Σ_2 such that the boundary behavior of the holomorphic curvature of the Bergman metric is described by*

$$\lim_{j\to\infty} R^{\Omega}_{p_j}(\xi) = R^{\Delta^2}_{(0,0)}(S\xi), \ \forall \xi \in \mathbb{C}^2$$

where the curvature tensor is represented with respect to the standard Euclidean coordinate system of $\mathbb{C}^2$. Moreover, S can be computed explicitly depending upon the domain Ω and the sequence p_j.

Remark. The reasons why the above conclusion at a singular point is limited to the holomorphic curvatures are the following two: (1) We do not have a sharp localization of the full sectional curvature tensor of the Bergman metric and, (2) The holomorphic sectional curvature of the limit domain is not in general constant. However, *for a convex polyhedral domain* in $\mathbb{C}^2$, the localization argument is unnecessary in the discussion above and hence one obtains much stronger a conclusion on the asymptotic boundary behavior of the full sectional curvature of the Bergman metric. Such a result can be easily obtained by a line by line imitation of the above.

4.3. Curvature behavior along a sequence of q-tangential type. Now, staying in the complex dimension two mainly for the sake of simplicity, we consider the sequence of the reference points when it is of q-tangential type. The main differences of this case in comparison to the preceding radial case are (1) that the scaling limit yields the ball which is totally different from the bidisk in its Bergman geometry, and (2) that the "angle factor" does not play a role.

In detail, let p be a singular boundary point of Ω and assume that the faces Σ_1 and Σ_2 pass through p. Let $p_j \in \Omega$ form a q-tangential orbit of the reference points, and denote by

$$\lambda_j = \operatorname{dist}(p_j, \Sigma_1)$$
$$\mu_j = \operatorname{dist}(p_j, \Sigma_2)$$

for each j. Taking a subsequence, we assume further that the orbit $\{p_j\}$ is q-tangential to Σ_1. More precisely speaking,

$$\lim_{j\to\infty} \mu_j^{-1}\sqrt{\lambda_j} = 0.$$

Controlled scaling along a q-tangential orbit. As in the paragraph preceding Definition 2 in Section 4.1, we use the setting that the normal vectors to the faces Σ_1 and Σ_2 are parallel to the Im z and Im w axes, respectively.

Denote by $p_j = (p_j^{(1)}, p_j^{(2)})$ for each j. Restrict ourselves for a moment to a closed ball $B(p,r)$ in $\mathbb{C}^2$ centered at p with radius r. For sufficiently small a positive number $r > 0$, the surfaces Σ_1 and Σ_2 can be represented by the equalities

$$\begin{aligned}
\text{Im}\left(z - p_j^{(1)}\right) + \lambda_j &= Q_1\left(w - p_j^{(2)}\right) + o\left(|z - p_j^{(1)}| + |w - p_j^{(2)}|^2\right) \\
\text{Im}\left(w - p_j^{(2)}\right) + \mu_j &= Q_2\left(z - p_j^{(1)}\right) + o\left(|w - p_j^{(2)}| + |z - p_j^{(1)}|^2\right)
\end{aligned}$$

where Q_1 and Q_2 are real-valued strictly subharmonic quadratic polynomials. Under the same restriction, the set Ω is obviously represented by the inequalities

$$\begin{aligned}
\text{Im}\left(z - p_j^{(1)}\right) + \lambda_j &> Q_1\left(w - p_j^{(2)}\right) + o\left(|z - p_j^{(1)}| + |w - p_j^{(2)}|^2\right) \\
\text{Im}\left(w - p_j^{(2)}\right) + \mu_j &> Q_2\left(z - p_j^{(1)}\right) + o\left(|w - p_j^{(2)}| + |z - p_j^{(1)}|^2\right).
\end{aligned}$$

Now consider the scaling by the sequence of scaling mappings $L_j : \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$L_j(z,w) = \left(\frac{1}{\lambda_j}(z - p_j^{(1)}), \frac{1}{\sqrt{\lambda_j}}(w - p_j^{(2)})\right).$$

First notice that in the sense of local Hausdorff set convergence we have

$$\lim_{j\to\infty} L_j(\Omega) = \lim_{j\to\infty} L_j(B(p,r) \cap \Omega),$$

since $\lambda_j \to 0$ and $p_j \to p$ as $j \to \infty$. Now, the set $L_j(B(p,r) \cap \Omega)$ for each j is represented by

$$\begin{aligned}
\text{Im}\,(\lambda_j\zeta) + \lambda_j &> Q_1\left(\sqrt{\lambda_j}\xi\right) + o\left(\lambda_j\left(|\zeta| + |\xi|^2\right)\right) \\
\text{Im}\left(\sqrt{\lambda_j}\xi\right) + \mu_j &> Q_2(\lambda_j\zeta) + o\left(\sqrt{\lambda_j}|\xi| + \lambda_j^2|\zeta|^2\right)
\end{aligned}$$

in (ζ, ξ) coordinates of $\mathbb{C}^2$. To compute the set convergence, we normalize the first inequality by dividing by λ_j and the second by $\sqrt{\lambda_j}$. It follows immediately that the second inequality yields the entire $\mathbb{C}^2$ as its local Hausdorff limit, whereas the first inequality converges to

$$\operatorname{Im} \zeta + 1 > Q_1(\xi).$$

Since $Q_1(\xi)$ is a real-valued strictly subharmonic quadratic polynomial, the above expression yields that the limit set

$$\lim_{j\to\infty} L_j(\Omega) = \lim_{j\to\infty} L_j(B(p,r) \cap \Omega)$$

is indeed biholomorphic to the unit ball in $\mathbb{C}^2$.

Bergman curvature along a q-tangential orbit. Now we are ready to complete the proof of Theorem 3. Let Ω be a strictly pseudoconvex polyhedral domain (recall that it has a piecewise C^2 smooth, but not entirely smooth boundary), and let $p_j \in \Omega$ form a sequence of q-tangential type approaching the singular boundary point p of Ω. For each j, choose a real two dimensional holomorphic section Π_j in $T_{p_j}\Omega$. Then combining (1) the arguments of localization (Chapter 2 of this article), (2) the conversion of limiting boundary behavior problem of Bergman curvature to the interior stability problem (as shown in Chapter 3 and the preceding sections of Chapter 4), and (3) the fact that the scaled limit is the unit ball in the present case, one can easily repeat the methods of 3.2 to obtain the conclusion that every subsequence of the sequence $R^{\Omega}_{p_j}(\Pi_j)$ contains a subsequence converging to $-4/(n+1)$. This yields the conclusion of Theorem 3.

4.4. Holomorphic Bergman curvature along the orbits of mixed type. As above, let p be a singular boundary point of Ω through which two faces Σ_1 and Σ_2 pass. This time, let $\{p_j\}_{j=1}^{\infty} \subset \Omega$ be a sequence of reference points of *mixed type*. Let us use the same notations as above:

$$\lambda_j = \operatorname{dist}(p_j, \Sigma_1)$$
$$\mu_j = \operatorname{dist}(p_j, \Sigma_2).$$

Then there are essentially only two typical types of orbits of mixed type as follows:

Case 1: $\lim_{j\to\infty} \frac{\lambda_j}{\mu_j} = 0$ and $\lim_{j\to\infty} \frac{\sqrt{\lambda_j}}{\mu_j} = m > 0$

Case 2: $\lim_{j\to\infty} \frac{\lambda_j}{\mu_j} = 0$ and $\lim_{j\to\infty} \frac{\sqrt{\lambda_j}}{\mu_j} = \infty$.

In both cases, we scale the domain Ω in a small neighborhood of p using the scaling mappings $L_j : \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$L_j(z,w) = (\lambda_j^{-1}(z - p_j^{(1)}), \mu_j^{-1}(w - p_j^{(2)}))$$

for each j, where $p_j = (p_j^{(1)}, p_j^{(2)})$. Then the scaled limit in this case will be given by the two inequalities

$$\begin{aligned} \operatorname{Im} \zeta + 1 &> \lim_{j\to\infty} \frac{\mu_j^2}{\lambda_j} Q_1(\xi) \\ \operatorname{Im} \xi + 1 &> 0. \end{aligned}$$

Therefore, one gets a Siegel domain as the limit for both cases. The second case yields in fact the bidisk as the limit and hence the curvature behavior is well understood. However, notice that the holomorphic curvature of the Bergman metric of the limiting domains in these cases are not constant. Therefore, to complete the procedure of the limit curvature computation, we consider the behavior of the holomorphic sections under the scaling. The holomorphic sections are mapped into another in both cases by the sequence of linear mappings represented by the matrix

$$\begin{pmatrix} \lambda_j/\mu_j & 0 \\ 0 & 1 \end{pmatrix}$$

and the reference points are mapped to the origin for every j.

According to the methods presented in this article, the limiting behavior of the holomorphic Bergman curvature is completely understood in the second case. But the limiting boundary behavior of the holomorphic Bergman curvature along the first case of the mixed orbit is not explicitly computed due to two reasons: First, the Bergman metric of the limiting Siegel domain is not well understood, and secondly, passing to the simultaneous bounded realization of $L_j(\Omega)$, the limiting Siegel domain is not convex. However, this is not unexpected, since we are dealing with the domains with necessarily a complicated curvature behavior to begin with. Also, it is expected that the curvature behavior of the Bergman metric of an arbitrary domain cannot be always a combination of relatively simple domains such as symmetric domains.

4.5. Remarks on curvature behavior in higher dimensions. A careful examination of the methods presented above will show that the asymptotic boundary behavior of the holomorphic curvature of the Bergman metric in complex dimensions higher than two can be easily understood by a direct generalization, if the sequence of the reference points in consideration falls

on to one of the following three cases: radial type, q-tangential type to a particular face, or special cases of the mixed type that yields the polydisk as the limit domain of the scaling. Beyond such cases, it is not known to us at the time of this writing if any better analysis of the behavior of the Bergman curvature in higher dimensions is possible. The main difficulty is that, in complex dimension higher than two, there are many more possibilities for the typical subsequential types of the orbits of the reference points resulting in that the limit domain becomes a general Siegel domain. Since the Bergman curvature of the general Siegel domains are not known to us explicitly, one can only draw a general conclusion: *Along any sequence of the reference points in a strictly pseudoconvex polyhedral domain in $\mathbb{C}^n$ for any n, any subsequential limit of the holomorphic sectional curvature tensor of the Bergman metric is the holomorphic curvature tensor of the Bergman metric of a certain Siegel domain at a certain point.* Even though the Siegel domain and the point in the preceding statement can always be computed explicitly, lack of explicit understanding of the Bergman curvature of the Siegel domains leaves us only the result in the below. This is a direct consequence of our methods presented above, which useful for many purposes. See [Pa] for more application, for instance.

Proposition. *For any bounded (weakly) pseudoconvex polyhedral domain D in $\mathbb{C}^n$, there exists a sequence of points in D such that the holomorphic sectional curvature tensor of the Bergman metric of D along the sequence converges to the holomorphic sectional curvature tensor of either the unit ball in $\mathbb{C}^n$ or the polydisk (up to an angle factor for the polydisk) in $\mathbb{C}^n$.*

The proof follows immediately by considering the strictly convex boundary points and radial type orbit accumulating to it. We do not include any details in order to avoid repeating similar arguments.

5. Applications and Concluding Remarks.

Theorem 1 yields, for instance, the following improvement from the theorem on complex analyticity of Kähler isometries by Kobayashi and Nomizu [**KN1**], and also by Greene and Krantz [**GK1**]:

Proposition. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with a C^2 strongly pseudoconvex boundary point. Let Ω be equipped with the Bergman metric, and let M be a Kähler manifold. Then any smooth isometry between Ω and M is either holomorphic or conjugate holomorphic.*

Corollary. *Any smooth isometry from a bounded pseudoconvex domain with C^2 boundary in $\mathbb{C}^n$ onto a Kähler manifold is either holomorphic or*

conjugate holomorphic.

The proofs follow from Theorem 1 and a line by line imitation of the proof of Theorem 1.17 in [**GK1**, **2**]. The improvement we present here is again in the fact that we require only the local C^2 regularity for the boundary, which is most natural, at the strongly pseudoconvex point in consideration.

Theorem 1 also provides a verification of the validity of the geometric arguments (formerly known only for C^∞ strongly pseudoconvex cases) proving the full version of Rosay's generalization ([**Ros**]) of Wong's Theorem ([**Wo**]) which is:

Theorem (Wong (1977), Rosay (1979)). *Let Ω be a bounded domain in $\mathbb{C}^n$. If there exist a sequence of points q_j of Ω bounded away from the boundary $\partial\Omega$ of the domain Ω and a sequence of holomorphic automorphisms f_j of Ω such that the sequence $\{f_j(q_j)\}_j$ accumulates at a boundary point p of Ω at which the boundary $\partial\Omega$ is C^2 strongly pseudoconvex, then the domain Ω is biholomorphic to the unit ball in $\mathbb{C}^n$.*

The proof is as follows: Since the orbit $\{f_j(q_j)\}$ accumulates at $p \in \partial\Omega$ at which $\partial\Omega$ is C^2 strongly pseudoconvex, due to the existence of a local peaking function at p, the sequence $\{f_j(z)\}$ will also accumulate at the same p for every $z \in \Omega$ as $j \to \infty$. Therefore, Theorem 1 implies that the holomorphic curvature of the Bergman metric of this domain is identically equal to $-4/(n+1)$ at every point. Moreover, due to the existence of such special orbit of a point by a sequence of automorphisms accumulating at p, the domain is necessarily simply connected, as shown in detail in Lemma on p. 256 of [**Wo**]. A similar argument with the estimate of the Bergman distance near the C^2 smooth strongly pseudoconvex boundary points by Diederich ([**Di**]) also yields that the domain equipped with the Bergman metric is complete Kähler. Finally, Lu's theorem ([**Lu**]) then concludes that the bounded domain in consideration is indeed biholomorphic to the unit ball.

References

[Ber1] S. Bergman, *Über die Kernfunktion eine Bereiches und ihr Verhalten am Rande*, J. Reine Angew. Math., **169** (1933), 1-42.

[Ber2] ———, *The kernel function and conformal mapping*, Am. Math. Soc., Providence, R.I., 1970.

[Di] K. Diederich, *Das Rendverhalten der Bergmanschen Kernfunktion und Metrik in streng pseudokonvexen Gebieten*, Math. Ann., **187** (1970), 9-36.

[DFH] K. Diederich, J.E. Fornaess and G. Herbort, *Boundary behavior of the Bergman metric*, Proc. Symp. Pure Math., **41** (1984), 59-67.

[Fe] C. Fefferman, *The Bergman kernel and biholomorphic mappings of pseudoconvex domains*, Invent. Math., **26** (1974), 1-65.

[FK] G.B. Folland and J.J. Kohn, *The Neumann problem for the Cauchy Riemann complex*, Princeton U. Press, 1972.

[Fuks] B.A. Fuks, *Über geodätische Mannigfaltigkeiten einer bei pseudokonformen Abbildungen invarianten Riemannschen Geomtrie*, Mat. Sbornik, n.s., **44** (1937), 567-594.

[GK1] R.E. Greene and S.G. Krantz, *Deformation of complex structures, estimates for the $\bar{\partial}$-equation, and stability of the Bergman kernel*, Adv. Math., **43** (1982), 1-86.

[GK2] ———, *Invariants of Bergman geometry and results concerning the automorphism groups of domains in $\mathbb{C}^n$*, Proc. of the 1989 Conference in Cetraro (D. Struppa ed.), to appear.

[GW] R.E. Greene and H. Wu, *Function theory on manifolds which possess a pole*, Lect. Notes in Math. No. 699, Springer-Verlag, 1979.

[Hör1] L. Hörmander, *L^2 estimates and existence theorems for $\bar{\partial}$-operator*, Acta Math., **113** (1965), 89-152.

[Hör2] ———, *Introduction to complex analysis in several variables*, North-Holland, 1973.

[Kim] K.T. Kim, *Asymptotic behavior of the curvature of the Bergman metric of the thin domains*, Pacific J. Math., **155** (1992), 99-110.

[Kle] P. Klembeck, *Kähler metrics of negative curvature, the Bergman metric near the boundary and the Kobayashi metric on smooth bounded strictly pseudoconvex sets*, Indiana U. Math. J., **27** (1978), 275-282.

[Kob] S. Kobayashi, *The geometry of bounded domains*, Trans. A.M.S., **92** (1959), 267-290.

[KN1] S. Kobayashi and K. Nomizu, *On automorphisms of Kählerian structures*, Nagoya J. Math., **11** (1957), 115-124.

[KN2] ———, *Foundations of Differential Geometry* (Volume II), Interscience, New York, 1969.

[Kr] S.G. Krantz, *Function theory of several complex variables* (2/e), Wadsworth and BrooksCole Advanced Books and Software, Pacific Grove, California, 1992.

[Lu] Q.K. Lu, *On Kähler manifolds with constant curvature*, Chinese Math. (English), **9** (1966), 283-298.

[McN] J.D. McNeal, *Lower bounds on the Bergman metric near a point of finite type*, Annals of Math., **136** (1992), 339-360.

[Ohs] T. Ohsawa, *Boundary behavior of the Bergman kernel function on pseudoconvex domains*, Publ. Res. Inst. Math. Sci., **20** (1984), 897-902.

[Pa] A. Pagano, *Ph. D. Dissertation*, Brown University, Providence, R.I., 1994.

[Pin] S. Pinchuk, *Holomorphic inequivalences of some classes of domains in $\mathbb{C}^n$*, Mat. USSR Sbornik, **39** (1981), 61-86.

[Ram] I.P. Ramadanov, *Sur une propriété de la fonction de Bergman*, C.R. Acad. Bulgare des Sci., **20** (1967), 759-762.

[Ros] J.P. Rosay, *Sur une caracterisation de la boule parmi les domaines de $\mathbb{C}^n$ par son groupe d'automorphismes*, Ann. Inst. Fourier (Grenoble), **29** (1979), 91-97.

[Wo] B. Wong, *Characterization of the ball in $\mathbb{C}^n$ by its automorphism group*, Invent. Math., **41** (1977), 253-257.

[Yu] J. Yu, *Ph. D. Dissertation*, Washington Univ. in St. Louis, St. Louis, Missouri, 1993.

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