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NEW CONSTRUCTIONS OF MODELS FOR LINK INVARIANTS

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We study three types of statistical mechanical models for link invariants (vertex, IRF and spin models) and some relations between them when they exhibit certain symmetries described by an Abelian group. In particular we show the equivalence of three kinds of models: strongly conservative vertex models on an Abelian group X, doubly translation invariant IRF models on the same group X, and translation invariant spin models on the direct product $X \times X$. Some examples of constructions of spin models from vertex models are given (the associated link invariants are the generating function for the writhe of orientations, the Jones polynomial, and the number of Fox colourings). Then we introduce a composition of link invariants related to the decomposition of a link into its components, and we explore the above correspondence between vertex, IRF and spin models in connection with this operation. As a main consequence, we show that the link invariant associated with spin models recently constructed by K. Nomura from Hadamard matrices is a composition of two Jones polynomials.

1. Introduction.

Soon after the discovery of the Jones polynomial [**Jo1**] it was realized that some central concepts of statistical mechanics, namely those of model and partition function (see [**Bax**]), can be applied to link diagrams to construct invariants of links in 3-space (see for instance [**Jo2**], [**K1**], [**T**]). Some basic references on this topic are [**H1**], [**Jo3**], [**K2**], [**L**], [**WDA**], [**Wu2**].

There are three main classes of models which can be used to construct link invariants. The vertex models and the closely related IRF models are in a sense the most general and have been widely studied in connection with quantum groups [**D**]. Spin models are more exotic objects: very few link invariants seem to admit a spin model description, and no clear connection with quantum groups is known in general. However, recent progress on spin models has involved strong relations with algebraic combinatorics and in particular with association schemes (see for instance [**BB1**], [**Ja3**], [**Ja7**]). These developments are surveyed in [**Ban**], [**Ja4**], [**Ja6**].

The initial motivation for the present paper was the discovery by K. Nomura of spin models associated with Hadamard matrices [**N1**]. It was shown in [**Ja5**] that the value of the associated link invariant depends only on the order of the Hadamard matrix and not on its particular structure. Then we were able to express this link invariant in terms of the Jones polynomials of sublinks of the given link. It turned out that the most natural proof of this result involved a reformulation in terms of vertex models, and it led us to a wider exploration of some relations between the three classes of models for link invariants. These relations occur when the models exhibit certain symmetries described by an Abelian group.

This paper is organized as follows. Section 2 introduces the general setting. Section 3 relates certain types of vertex models and IRF models exhibiting Abelian group symmetry, and gives some examples (the generating function for the writhe of orientations, Kauffman's bracket polynomial, and Fox colourings) which will be used throughout the paper. Section 4 introduces doubly translation invariant IRF models and a convenient algebraic description of these objects. Section 5 deals with translation invariant spin models and relates these models to doubly translation invariant IRF models. This gives in particular a new derivation of the Potts model for the Jones polynomial when the number of spins is a square, and of known spin models based on the cycle of length four. We also obtain a new spin model for the number of Fox colourings which we relate in a special case with a model due to Goldschmidt and Jones [**GJ**]. This gives another proof of a topological interpretation by Przytycki [**Pr**] of the number of Fox colourings. In addition, we associate with every translation invariant spin model a dual spin model which defines the same link invariant. In Section 6 we study a composition of link invariants which is based on the decomposition of a link into its components. This composition has a natural counterpart for vertex models and we show that in some cases this extends to spin models. This is the key for the undersdanding of the partition function of Nomura's spin models. Section 7 concludes with some perspectives for further research.

2. Link invariants and models from statistical mechanics.

Following [**Jo3**], we shall consider here three ways of obtaining an invariant of oriented links as the partition function of a suitable model defined on link diagrams.

2.1. Link diagrams. For more details on this section the reader can refer to [**BZ**], [**CF**], [**K3**].

Let us recall first that an *oriented link* consists of a finite collection of disjoint simple closed curves smoothly embedded in $\mathbb{R}^3$ (these curves are the *components* of the link). Oriented links can be represented by oriented *link diagrams* which are "generic" plane projections together with some additional 3-dimensional information at crossing points. Considering oriented links up to a natural topological equivalence called *ambient isotopy* amounts to consider diagrams up to a combinatorial equivalence generated by elementary diagram deformations called *Reidemeister moves*. Thus, from the combinatorial point of view, a *link invariant* is a valuation of diagrams which is invariant under Reidemeister moves.

An oriented link diagram L will be considered as a directed graph embedded in the plane $\mathbb{R}^2$, with sets of vertices, edges and faces denoted by $V(L), E(L), F(L)$ respectively. The vertices of L correspond to the crossings, the edges are the connected components of $L - V(L)$, and the faces are the connected components of $\mathbb{R}^2 - L$. We must allow a special kind of edge called a *free loop* which is embedded as a simple closed curve disjoint from the remaining part of the graph.

The spatial structure of the link represented by the diagram is defined by a sign function $s : V(L) \to \{+, -\}$ whose interpretation is described in Figure 1. For any set A of vertices we write $s(A) = |s^{-1}(+) \cap A| - |s^{-1}(-) \cap A|$. In particular the *Tait number* (*or writhe*) of L is $T(L) = s(V(L))$. We shall also need the following convention. For every vertex v, the edges incident to v will be denoted by $e_i(v)$ $(i = 1, \dots, 4)$ and the faces incident to v will be denoted by $f_i(v)$ $(i = 1, \dots, 4)$ as shown on Figure 2. Note that the edges $e_i(v)$ need not be distinct, and similarly for the faces $f_i(v)$.

The following general terminology from statistical mechanics will be used. A *state* on a diagram L will be an assignment of values taken from a given finite set of *spins* to certain elements (edges or faces) of L. With each state will be associated a *local weight* at each vertex, belonging to some commutative ring. Then the *weight* of a state will be the product of local weights over all vertices. Finally the *partition function* will be the sum of weights of all states, multiplied by a suitable *normalization factor*.

In the sequel, the symbol Ω always stands for a commutative ring with identity 1.

2.2. Vertex models. We shall only be concerned with a special case of the models introduced in Definition 1.1 of [**Jo3**], namely those with no angle dependence (called "zero-field models" in [**HJ**] and "modèles à vertex sommaires" in [**H1**]). We shall also need a weaker version of the "type I property" stated in Definition 1.13 of [**Jo3**].

Definition 1. A *vertex model on X with modulus μ* is a 5-tuple $(X, w_+, w_-, \Omega, \mu)$, where X is a finite non empty set, μ is an invertible element of Ω, and w_+, w_- are mappings from X^4 to Ω which satisfy the following identities (where δ is the Kronecker symbol):

$$\sum_{a\in X} w_\pm(a,b,x,a) = \sum_{a\in X} w_\pm(b,a,a,x) = \mu^{\pm 1}\delta(x,b), \tag{1}$$

$$\sum_{b,y\in X} w_+(a,b,x,y)w_-(y,z,b,c) = \delta(a,c)\delta(x,z), \tag{2}$$

$$\sum_{b,y\in X} w_+(a,b,y,x)w_-(z,y,b,c) = \delta(a,c)\delta(x,z), \tag{3}$$

$$\sum_{b,y,s\in X} w_+(a,b,x,y)w_+(b,c,r,s)w_+(y,z,s,t) \tag{4}$$
$$= \sum_{b,y,s\in X} w_+(x,y,r,s)w_+(a,b,s,t)w_+(b,c,y,z).$$

The partition function associated with the vertex model $\nu = (X, w_+, w_-, \Omega, \mu)$ evaluated on the link diagram L is then defined as

$$Z^\nu(L) = \sum_{\sigma:E(L)\to X} \prod_{v\in V(L)} w_{s(v)}(\sigma(e_1(v)), \sigma(e_2(v)), \sigma(e_3(v)), \sigma(e_4(v))). \tag{5}$$

Here and later an empty product is equal to 1, and hence if L consists of k free loops, $Z^\nu(L) = |X|^k$. Note also that if L has connected components $L_1, \ldots, L_k$, $Z^\nu(L) = \prod_{j=1,\ldots,k} Z^\nu(L_i)$. We shall describe this property by saying that Z^ν is *multiplicative.*

Using (1)–(4) to analyze the behaviour of Z^ν under Reidemeister moves it is not difficult to prove that $\mu^{-T(L)}Z^\nu(L)$ defines an invariant of oriented links (see [**Jo3**], Theorem 1.12 and Corollary 1.15).

2.3. IRF models. The following definition is essentially equivalent to Definition 2.19 of [**Jo3**].

Definition 2. *An IRF model on X with modulus μ* is a 5-tuple $(X, w_+, w_-, \Omega, \mu)$ where X is a finite non empty set, μ is an invertible element of Ω, and w_+, w_- are mappings from X^4 to Ω which satisfy the following identities:

$$\sum_{x\in X} w_\pm(a,b,a,x) = \sum_{x\in X} w_\pm(a,x,a,b) = \mu^{\pm 1}, \tag{6}$$

$$\sum_{x\in X} w_+(a,b,x,d)w_-(x,b,e,d) = \delta(a,e), \tag{7}$$

$$\sum_{x\in X} w_+(d,x,b,e)w_-(b,x,d,a) = \delta(a,e), \tag{8}$$

$$\sum_{x\in X} w_+(a,b,x,f)w_+(b,c,d,x)w_+(x,d,e,f) \tag{9}$$
$$= \sum_{x\in X} w_+(b,c,x,a)w_+(a,x,e,f)w_+(x,c,d,e).$$

Remark. For reasons of coherence with Definition 1 (which will become clear in the proof of Proposition 1) we have replaced the identity (2.22) of [**Jo3**] by the equivalent identity (8).

The partition function associated with the IRF model $\imath = (X, w_+, w_-, \Omega, \mu)$ evaluated on L is

(10)
$$Z^\imath(L) = |X|^{-1} \sum_{\sigma: F(L)\to X} \prod_{v\in V(L)} w_{s(v)}(\sigma(f_1(v)), \sigma(f_2(v)), \sigma(f_3(v)), \sigma(f_4(v)))$$

and again one can show that $\mu^{-T(L)} Z^\imath(L)$ defines an invariant of oriented links (see [**Jo3**], Theorem 2.27). Note that if L consists of k free loops, $Z^\imath(L) = |X|^k$.

2.4. Spin models. The initial definition of [**Jo3**] using two symmetric weight functions was first generalized to non symmetric functions in [**KMW**] and then further generalized with the use of four functions in [**BB2**]. We shall work with this last generalization.

Definition 3. A *spin model on X with modulus μ and loop variable D* is a 8-tuple $(X, w_1, w_2, w_3, w_4, \Omega, \mu, D)$, where X is a finite non empty set, μ is an invertible element of Ω, D is a square root of $|X|$, and w_1, w_2, w_3, w_4 are mappings from X^2 to Ω which satisfy the following identities:

$$w_1(a,a) = \mu, \quad \sum_{x\in X} w_4(a,x) = \sum_{x\in X} w_4(x,a) = D\mu, \tag{11}$$

$$w_3(a,a) = \mu^{-1}, \quad \sum_{x\in X} w_2(a,x) = \sum_{x\in X} w_2(x,a) = D\mu^{-1}, \tag{12}$$

$$w_1(a,b)w_3(b,a) = 1 = w_2(a,b)w_4(b,a), \tag{13}$$

$$\sum_{x\in X} w_1(a,x)w_3(x,b) = |X|\delta(a,b) = \sum_{x\in X} w_2(a,x)w_4(x,b), \tag{14}$$

$$\sum_{x\in X} w_1(a,x)w_1(x,b)w_4(c,x) = Dw_1(a,b)w_4(c,a)w_4(c,b), \tag{15}$$

$$\sum_{x\in X} w_1(x,a)w_1(b,x)w_4(x,c) = Dw_1(b,a)w_4(a,c)w_4(b,c). \tag{16}$$

When $w_1 = w_2 = w_+$ and $w_3 = w_4 = w_-$ it can be shown that this reduces to the definition of [**KMW**] (see [**BB2**]). If moreover $w_+(x,y) = w_+(y,x)$ for all x, y in X, the definition of [**Jo3**] is recovered.

To define the (normalized) partition function $Z^\zeta(L)$ associated with the spin model $\zeta = (X, w_1, w_2, w_3, w_4, \Omega, \mu, D)$ evaluated on a connected diagram L, we first color the faces of L with two colors, black and white, in such a way that adjacent faces receive different colors and the unbounded face is colored white. Let $B(L)$ be the set of faces of L colored black. For every mapping $\sigma : B(L) \to X$, define the interaction weight $\langle v, \sigma\rangle$ of a vertex v with σ as shown on Figure 3. Then

$$Z^{\varsigma}(L) = D^{-|B(L)|} \sum_{\sigma : B(L) \to X} \prod_{v \in V(L)} \langle v, \sigma \rangle. \tag{17}$$

For a non connected diagram $L, Z^{\varsigma}(L)$ will be defined as the product of values of Z^{ς} on its connected components. In other words, we want Z^{ς} to be multiplicative. In particular if L consists of k free loops, $Z^{\varsigma}(L) = D^k$. Now one can show as before that $\mu^{-T(L)} Z^{\varsigma}(L)$ defines an invariant of oriented links [**BB2**].

2.5. A remark on normalization. The modulus μ which appears in Definitions 1, 2, 3, is introduced for reasons of convenience, but the following observations show that this parameter is redundant.

(i) If $(X, w_+, w_-, \Omega, \mu)$ is a vertex model, $(X, \mu^{-1}w_+, \mu w_-, \Omega, 1)$ is also a vertex model with the same associated link invariant.

(ii) If $(X, w_+, w_-, \Omega, \mu)$ is an IRF model, $(X, \mu^{-1}w_+, \mu w_-, \Omega, 1)$ is also an IRF model with the same associated link invariant.

(iii) If $(X, w_1, w_2, w_3, w_4, \Omega, \mu, D)$ is a spin model, $(X, \mu^{-1}w_1, \mu w_2, \mu w_3, \mu^{-1}w_4, \Omega, 1, D)$ is also a spin model with the same associated link invariant.

3. Vertex and IRF models on Abelian groups.

In this section we assume that X is an Abelian group (considered as left $\mathbb{Z}$-module).

3.1. Conservative vertex models and translation invariant IRF models. Let L be an oriented link diagram and consider a mapping $\sigma : F(L) \to X$. The *derivative* of σ is the mapping $\partial\sigma : E(L) \to X$ defined as follows. For any edge e, $(\partial\sigma)(e) = \sigma(f') - \sigma(f)$, where f (respectively: f') is the face lying on the left (respectively: right) of e (see Figure 4). Clearly, for every vertex v of L, $\varphi = \partial\sigma$ satisfies

$$\varphi(e_1(v)) + \varphi(e_3(v)) = \varphi(e_2(v)) + \varphi(e_4(v)). \tag{18}$$

Conversely it is well known (see for instance [**O**], Chapter 7) that if $\varphi : E(L) \to X$ satisfies (18) for every v (that is, in the terminology of graph theory, if φ is an X-valued flow on L), for any fixed value x in X there is a unique mapping $\sigma : F(L) \to X$ such that $\varphi = \partial\sigma$ and σ takes the value x on the unbounded face. We shall denote this mapping by $\omega_x\varphi$.

In order to extend this correspondence to a correspondence between vertex models and IRF models, we introduce the following definitions.

Definition 4. A mapping $w : X^4 \to \Omega$ is *conservative* if $w(a,b,c,d) = 0$ whenever $a + c \neq b + d$. A vertex model $(X, w_+, w_-, \Omega, \mu)$ is *conservative* if w_+ and w_- are conservative.

Definition 5. A mapping $w : X^4 \to \Omega$ is *translation invariant* if $w(a,b,c,d) = w(a+x, b+x, c+x, d+x)$ for every x in X. An IRF model $(X, w_+, w_-, \Omega, \mu)$ is *translation invariant* if w_+ and w_- are translation invariant.

For a mapping $w : X^4 \to \Omega$, we define the mappings $\partial^* w$ and $\omega^* w$ from X^4 to Ω by the following identities.

$$(\partial^* w)(a,b,c,d) = w(a-d, b-c, b-a, c-d), \tag{19}$$

$$(\omega^* w)(a,b,c,d) = \delta(a+c, b+d) w(0, c, c-b, -a). \tag{20}$$

Clearly $\partial^* w$ is translation invariant and $\omega^* w$ is conservative.

The proof of the following statement is an easy exercise.

(21) For a mapping $w : X^4 \to \Omega$, $\omega^*(\partial^*(w)) = w$ if and only if w is conservative, and $\partial^*(\omega^*(w)) = w$ if and only if w is translation invariant.

The following result is a version of a transformation which is well known in statistical mechanics (see [**Jo3**], Proposition 4.3, and [**FW**], [**Wu1**], [**KWe**]).

Proposition 1.

(i) *Let $\nu = (X, w_+, w_-, \Omega, \mu)$ be a conservative vertex model. Then $\partial^*(\nu) = (X, \partial^* w_+, \partial^* w_-, \Omega, \mu)$ is a translation invariant IRF model.*

(ii) *Let $\imath = (X, w_+, w_-, \Omega, \mu)$ be a translation invariant IRF model. Then $\omega^*(\imath) = (X, \omega^* w_+, \omega^* w_-, \Omega, \mu)$ is a conservative vertex model.*

(iii) *For every conservative vertex model ν and translation invariant IRF model $\imath$, $\omega^*(\partial^*(\nu)) = \nu$ and $\partial^*(\omega^*(\imath)) = \imath$.*

(iv) *For every conservative vertex model ν and translation invariant IRF model $\imath$, $Z^{\partial^*(\nu)} = Z^\nu$ and $Z^{\omega^*(\imath)} = Z^\imath$.*

Proof. First note that (iii) is immediate from (21).

Let us assume for the moment that (i) and (ii) hold and let us prove (iv). Consider a translation invariant IRF model $\imath = (X, w_+^\imath, w_-^\imath, \Omega, \mu)$ and an oriented diagram L. Recall from (10) that

$$Z^\imath(L) = |X|^{-1} \sum_{\sigma : F(L) \to X} \prod_{v \in V(L)} w^\imath_{s(v)}(\sigma(f_1(v)), \sigma(f_2(v)), \sigma(f_3(v)), \sigma(f_4(v))).$$

Since $\imath$ is translation invariant, for every vertex v,

$$\begin{aligned}
&w^{\imath}_{s(v)}(\sigma(f_1(v)),\sigma(f_2(v)),\sigma(f_3(v)),\sigma(f_4(v)))\\
&= w^{\imath}_{s(v)}(0,\sigma(f_2(v))-\sigma(f_1(v)),\sigma(f_2(v))-\sigma(f_1(v))\\
&\qquad +\sigma(f_3(v))-\sigma(f_2(v)),\sigma(f_4(v))-\sigma(f_1(v)))\\
&= w^{\imath}_{s(v)}(0,(\partial\sigma)(e_3(v)),(\partial\sigma)(e_3(v))-(\partial\sigma)(e_2(v)),-(\partial\sigma)(e_1(v)))\\
&= \left(\omega^* w^{\imath}_{s(v)}\right)((\partial\sigma)(e_1(v)),(\partial\sigma)(e_2(v)),(\partial\sigma)(e_3(v)),(\partial\sigma)(e_4(v)))
\end{aligned}$$

(by (20) and the fact that $\varphi = \partial\sigma$ satisfies (18)). On the other hand by (5),

$$Z^{\omega^*(\imath)}(L) = \sum_{\varphi:E(L)\to X}\ \prod_{v\in V(L)} \left(\omega^* w^{\imath}_{s(v)}\right)(\varphi(e_1(v)),\varphi(e_2(v)),\varphi(e_3(v)),\varphi(e_4(v))).$$

If φ contributes to this sum it must satisfy (18) for every v and we may associate with the term in $Z^{\omega^*(\imath)}(L)$ corresponding to φ the $|X|$ terms in $Z^{\imath}(L)$ corresponding to the $\omega_x\varphi$, $x \in X$. This shows that $Z^{\omega^*(\imath)} = Z^{\imath}$.

The other equality $Z^{\partial^*(\nu)} = Z^{\nu}$ then follows from (iii).

It is not difficult, but tedious, to check (i) and (ii) using the identities (1)–(4) and (6)–(9). However, it is simpler to proceed as in the proof of (iv). Indeed, each of the identities (1)–(4) and (6)–(9) is of the form $\mu^{-T(L_1)}Z'(L_1) = \mu^{-T(L_2)}Z'(L_2)$, where L_1 and L_2 are two diagrams related by a Reidemeister move and Z' is a local version of the partition function involving only the vertices concerned by this move. These local partition functions are sums over mappings σ from $E(L)$ (or $F(L)$) to X which have fixed values for edges (or faces) preserved by the move. It is then easy to see that the argument used above for the full partition function can be used for these local partition functions as well. □

3.2. Examples.

3.2.1. The first binary Lipson model ([**Li**], see also [**HJ**]). Let $X = \mathbb{Z}/2\mathbb{Z}$ and $\Omega = \mathbb{Z}[C, C^{-1}]$. Let

$$w_{\pm}(a,b,c,d) = \begin{cases} C^{\pm 1} & \text{if } a = c \neq b = d,\\ (C^{-1})^{\pm 1} & \text{if } a = d \neq b = c,\\ 0 & \text{otherwise.}\end{cases} \tag{22}$$

Then $\nu_1 = (X, w_+, w_-, \Omega, C^{-1})$ is a vertex model which is clearly conservative. The associated link invariant has a simple description in terms of linking numbers of sublinks with their complements, or equivalently as a generating function for the writhe of the reorientations of L [**LM**], and is a special evaluation of the Kauffman polynomial of [**K4**].

The corresponding IRF model $\imath_1 = \partial^*(\nu_1) = (X, w^\imath_+, w^\imath_-, \Omega, C^{-1})$ is defined by $w^\imath_\pm(a,b,c,d) = (\partial^* w_\pm)(a,b,c,d) = w_\pm(a-d, b-c, b-a, c-d)$, so that

$$w^\imath_\pm(a,b,c,d) = \begin{cases} C^{\pm 1} & \text{if } b = d \text{ and } c \neq a, \\ (C^{-1})^{\pm 1} & \text{if } a = c \text{ and } b \neq d, \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

3.2.2. The second binary Lipson model ([**Li**], see also [**HJ**]). Let $X = \mathbb{Z}/2\mathbb{Z}$ and $\Omega = \mathbb{Q}[C, C^{-1}]$. Let now

$$w_\pm(a,b,c,d) = \begin{cases} \frac{1}{2}\left(C^{\pm 1} - (C^{-1})^{\pm 1}\right) & \text{if } a = c \neq b = d, \\ -\frac{1}{2}\left(C^{\pm 1} - (C^{-1})^{\pm 1}\right) & \text{if } a = d \neq b = c, \\ \frac{1}{2}(C + C^{-1}) & \text{if } a = b = c = d, \\ -\frac{1}{2}(C + C^{-1}) & \text{if } a = b \neq c = d, \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

Then $\nu_2 = (X, w_+, w_-, \Omega, C^{-1})$ is a conservative vertex model. It is shown in [**Li**] that $Z^{\nu_1} = Z^{\nu_2}$ and we shall obtain soon a new explanation on this fact. The corresponding IRF model $\imath_2 = \partial^*(\nu_2) = (X, w^\imath_+, w^\imath_-, \Omega, C^{-1})$ is defined by

$$w^\imath_\pm(a,b,c,d) = \begin{cases} \frac{1}{2}\left(C^{\pm 1} - (C^{-1})^{\pm 1}\right) & \text{if } b = d \text{ and } c \neq a, \\ -\frac{1}{2}\left(C^{\pm 1} - (C^{-1})^{\pm 1}\right) & \text{if } a = c \text{ and } b \neq d, \\ \frac{1}{2}(C + C^{-1}) & \text{if } a = c \text{ and } b = d, \\ -\frac{1}{2}(C + C^{-1}) & \text{if } a \neq c \text{ and } b \neq d. \end{cases} \tag{25}$$

3.2.3. A model for Kauffman's bracket polynomial.

The following vertex model is an oriented version of a model by Lipson ([**Li**], see also [**HJ**], [**Wu3**]) and actually coincides with it when X is an elementary Abelian 2-group. Like Lipson's model, it can easily be derived from Kauffman's "bracket polynomial" model (see [**PW**] and [**Ja1**], Proposition 12). Let α be a complex root of the equation $\alpha^2 + \alpha^{-2} + |X| = 0$. Let

$$w_\pm(a,b,c,d) = \begin{cases} (\alpha^{-1})^{\pm 1} & \text{if } a + c = b + d = 0 \neq b - c, \\ \alpha^{\pm 1} & \text{if } b - c = d - a = 0 \neq a + c, \\ \alpha + \alpha^{-1} & \text{if } a + c = b + d = b - c = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{26}$$

Then $\nu_3 = (X, w_+, w_-, \mathbb{C}, -\alpha^3)$ is a conservative vertex model. The corresponding partition function gives the bracket polynomial of [**K1**], with α replacing Kauffman's variable A and with normalization chosen so that a free loop has value $-\alpha^2 - \alpha^{-2}$. Equivalently, the associated link invariant is the Jones polynomial $V(t)$ evaluated at $t = \alpha^4$ (with the notations of [**Jo1**]). The corresponding IRF model $\imath_3 = \partial^*(\nu_3) = (X, w^\imath_+, w^\imath_-, \mathbb{C}, -\alpha^3)$ is defined by

$$(27) \qquad w^\imath_\pm(a,b,c,d) = \begin{cases} (\alpha^{-1})^{\pm 1} & \text{if } b = d \text{ and } c \neq a, \\ \alpha^{\pm 1} & \text{if } a = c \text{ and } b \neq d, \\ \alpha + \alpha^{-1} & \text{if } a = c \text{ and } b = d, \\ 0 & \text{otherwise.} \end{cases}$$

Remark. The IRF models given by (23), (25), (27) already appear in [**Ja2**].

3.2.4. Fox colouring with orientations.

Let

$$(28) \qquad \begin{aligned} w_+(a,b,c,d) &= \delta(b,-a)\delta(d, c+2a), \\ w_-(a,b,c,d) &= \delta(b, 2c+a)\delta(d,-c). \end{aligned}$$

It is not difficult to check that $\nu_4 = (X, w_+, w_-, \mathbb{Z}, 1)$ is a conservative vertex model.

Remark. This model can be related to the model of [**HJ**] for Fox colourings (when X is an odd cyclic group) by the following trick. Color the Seifert circles of L in two colors, say black and white, in such a way that two Seifert circles meeting at some vertex have different colors. Then changing the sign of edge values for all edges belonging to black circles defines a weight-preserving bijection between the states of the model of de la Harpe-Jones and the states of the present model. Hence the two models have the same partition function.

It easy to verify that the corresponding IRF model $\imath_4 = \partial^*(\nu_4) = (X, w^\imath_+, w^\imath_-, \mathbb{Z}, 1)$ is defined by

$$(29) \quad w^\imath_+(a,b,c,d) = \delta(b-d, c-a), \quad w^\imath_-(a,b,c,d) = \delta(b-d, a-c).$$

4. Doubly translation invariant IRF models.

4.1. Generalities. It is clear from (23), (25), (27), (29) that the IRF models of the preceding section 3.2 share the following property:

$$(30) \qquad w^\imath_\pm(a,b,c,d) = w^\imath_\pm(a+x, b, c+x, d) = w^\imath_\pm(a, b+y, c, d+y)$$

for all x, y in X. An IRF model satisfying (30) will be said *doubly translation invariant.*

Define now a vertex model $(X, w_+^\nu, w_-^\nu, \Omega, \mu)$ to be *strongly conservative* if it is conservative and also satisfies the following property.

$$(31) \qquad w_\pm^\nu(a,b,c,d) = w_\pm^\nu(a+x, b-x, c-x, d+x) \quad \text{for all } x \text{ in } X.$$

Proposition 2. *A translation invariant IRF model $\imath$ is doubly translation invariant if and only if $\omega^*(\imath)$ is strongly conservative.*

Proof. Immediate from (20) and the above definitions. □

4.2. A simpler presentation. Suppose that $\imath = (X, w_+, w_-, \Omega, \mu)$ is a doubly translation invariant IRF model. Let us define $g_\pm : X^2 \to \Omega$ by $g_\pm(u,v) = w_\pm(0,0,u,v)$ for all u, v in X. Then it follows from (30) that

$$(32) \qquad w_\pm(a,b,c,d) = g_\pm(c-a, d-b).$$

One easily checks that the identities (6)–(9) reduce to

$$(33) \qquad \sum_{x\in X} g_\pm(0,x) = \mu^{\pm 1},$$

$$(34) \qquad \sum_{x\in X} g_+(x,b) g_-(a-x,b) = \delta(a,0),$$

$$(35) \qquad \sum_{x\in X} g_+(b,a+x) g_-(-b,x) = \delta(a,0),$$

$$(36) \qquad \sum_{x\in X} g_+(x-a, f-b) g_+(d-b, x-c) g_+(e-x, f-d) = \sum_{x\in X} g_+(x-b, a-c) g_+(e-a, f-x) g_+(d-x, e-c).$$

Thus any doubly translation invariant IRF model can be defined via (32) from mappings $g_\pm$ satisfying (33)–(36).

4.3. Algebraic reformulation. Let $\mathcal{A} = \Omega[X]$ be the group algebra with natural basis $\{A_x,\ x \in X\}$ such that $A_x A_y = A_{x+y}$ for all x, y in X. We introduce also on $\mathcal{A}$ the Hadamard product $\circ$ defined on the natural basis by $A_x \circ A_y = \delta(x,y) A_x$ for all x, y in X. We write I for the identity A_0 and J for $\sum_{x\in X} A_x$ (which is the identity for the Hadamard product). We denote by τ the linear map defined on the natural basis by $\tau(A_x) = A_{-x}$ for x in X.

We now introduce for each u in X four elements $H_u^\pm, V_u^\pm$ of $\mathcal{A}$ defined as follows.

$$(37) \qquad H_u^\pm = \sum_{x\in X} g_\pm(u,x) A_x,$$

$$V_u^{\pm} = \sum_{x \in X} g_{\pm}(x,u)A_x. \tag{38}$$

Proposition 3.

(i) *Each of the following identities is equivalent to the identity* (33) *:*

$$JH_0^{\pm} = \mu^{\pm 1}J, \tag{39}$$

$$I \circ \left(\sum_{u \in X} V_u^{\pm}\right) = \mu^{\pm 1}I. \tag{40}$$

(ii) *Each of the following identities is equivalent to the identity* (34) *:*

$$V_b^{+}V_b^{-} = I, \tag{41}$$

$$\sum_{x \in X} H_x^{+} \circ H_{a-x}^{-} = \delta(a,0)J. \tag{42}$$

(iii) *Each of the following identities is equivalent to the identity* (35) *:*

$$H_b^{+}\tau(H_{-b}^{-}) = I, \tag{43}$$

$$\sum_{x \in X} V_{a+x}^{+} \circ \tau(V_x^{-}) = \delta(a,0)J. \tag{44}$$

(iv) *The following identity is equivalent to the identity* (36) *:*

$$V_u^{+}(\tau(H_v^{+}) \circ (V_{u-v}^{+}A_i)) = \sum_{x \in X} \tau(H_x^{+}) \circ (V_{u-x}^{+}(\tau(H_{v-x}^{+}) \circ A_i)). \tag{45}$$

Proof. (i) (39) reads $J\left(\sum_{x \in X} g_{\pm}(0,x)A_x\right) = \mu^{\pm 1}J$. Since $JA_x = J$, this is equivalent to (33).

Since $I \circ A_x = A_0 \circ A_x = \delta(x,0)A_0$, $I \circ V_u^{\pm} = \sum_{x \in X} g_{\pm}(x,u)I \circ A_x = g_{\pm}(0,u)I$ and (40) is clearly equivalent to (33).

(ii) (41) reads $\left(\sum_{x \in X} g_{+}(x,b)A_x\right)\left(\sum_{y \in X} g_{-}(y,b)A_y\right) = A_0$. The left hand side can be expanded as

$$\sum_{x,y \in X} g_{+}(x,b)g_{-}(y,b)A_{x+y} = \sum_{x,a \in X} g_{+}(x,b)g_{-}(a-x,b)A_a$$

and the equivalence of (41) with (34) follows immediately.

Since

$$\begin{aligned} H_x^{+} \circ H_{a-x}^{-} &= \left(\sum_{b \in X} g_{+}(x,b)A_b\right) \circ \left(\sum_{b \in X} g_{-}(a-x,b)A_b\right) \\ &= \sum_{b \in X} g_{+}(x,b)g_{-}(a-x,b)A_b, \end{aligned}$$

we see that

$$\sum_{x\in X} H_x^+ \circ H_{a-x}^- = \sum_{b\in X}\left(\sum_{x\in X} g_+(x,b)g_-(a-x,b)\right) A_b$$

and the equivalence of (42) with (34) follows.

(iii) The proof is quite similar to that of (ii) and will be omitted.

(iv) We shall consider the following identity

(45′)

$$V_{f-b}^+(\tau(H_{d-b}^+)\circ(V_{f-d}^+A_{c-e})) = \sum_{x\in X}\tau(H_{x-b}^+)\circ(V_{f-x}^+(\tau(H_{d-x}^+)\circ A_{c-e}))$$

which is easily seen to be equivalent to (45) via the substitution $f-b\to u$, $d-b\to v$, $c-e\to i$, $x-b\to x$.

Let us expand the left hand side of (45′):

$$\begin{aligned}
&V_{f-d}^+A_{c-e} = \sum_{x\in X} g_+(x,f-d)A_{x+c-e},\\
&\tau(H_{d-b}^+)\circ(V_{f-d}^+A_{c-e})\\
&= \left(\sum_{x\in X} g_+(d-b,x)A_{-x}\right)\circ\left(\sum_{x\in X} g_+(x,f-d)A_{x+c-e}\right)\\
&= \left(\sum_{x\in X} g_+(d-b,x-c)A_{c-x}\right)\circ\left(\sum_{x\in X} g_+(e-x,f-d)A_{c-x}\right)\\
&= \sum_{x\in X} g_+(d-b,x-c)g_+(e-x,f-d)A_{c-x},\\
&V_{f-b}^+(\tau(H_{d-b}^+)\circ(V_{f-d}^+A_{c-e}))\\
&= \left(\sum_{y\in X} g_+(y,f-b)A_y\right)\left(\sum_{x\in X} g_+(d-b,x-c)g_+(e-x,f-d)A_{c-x}\right)\\
&= \sum_{x,y\in X} g_+(y,f-b)g_+(d-b,x-c)g_+(e-x,f-d)A_{y+c-x}\\
&= \sum_{x,a\in X} g_+(x-a,f-b)g_+(d-b,x-c)g_+(e-x,f-d)A_{c-a}.
\end{aligned}$$

Let us now expand the summand in the right hand side of (45′):

$$\tau(H_{d-x}^+)\circ A_{c-e} = \left(\sum_{y\in X} g_+(d-x,y)A_{-y}\right)\circ A_{c-e} = g_+(d-x,e-c)A_{c-e},$$

$$V_{f-x}^+(\tau(H_{d-x}^+)\circ A_{c-e}) = \left(\sum_{y\in X} g_+(y,f-x)A_y\right)(g_+(d-x,e-c)A_{c-e})$$

$$= \sum_{y \in X} g_+(y, f-x) g_+(d-x, e-c) A_{y+c-e}$$
$$= \sum_{a \in X} g_+(e-a, f-x) g_+(d-x, e-c) A_{c-a},$$
$$\tau(H^+_{x-b}) \circ (V^+_{f-x}(\tau(H^+_{d-x}) \circ A_{c-e}))$$
$$= \left(\sum_{y \in X} g_+(x-b, y) A_{-y} \right) \circ \left(\sum_{a \in X} g_+(e-a, f-x) g_+(d-x, e-c) A_{c-a} \right)$$
$$= \sum_{a \in X} g_+(x-b, a-c) g_+(e-a, f-x) g_+(d-x, e-c) A_{c-a}.$$

Comparison of the coefficients of A_{c-a} in both sides of (45′) now shows its equivalence with (36). □

4.4. Examples. Let us review the mappings $g_\pm$ and the elements $H^\pm_u, V^\pm_u$ of $\mathcal{A}$ corresponding to the examples of Section 3.2.

The first binary Lipson model (3.2.1):
$g_\pm(1,0) = C^{\pm 1}, g_\pm(0,1) = (C^{-1})^{\pm 1}$, and $g_\pm(1,1) = g_\pm(0,0) = 0$. Hence $H^\pm_0 = (C^{-1})^{\pm 1} A_1$, $H^\pm_1 = C^{\pm 1} A_0$, $V^\pm_0 = C^{\pm 1} A_1$, $V^\pm_1 = (C^{-1})^{\pm 1} A_0$.

The second binary Lipson model (3.2.2):

$$g_\pm(1,0) = \frac{1}{2}\left(C^{\pm 1} - (C^{-1})^{\pm 1}\right), \qquad g_\pm(0,1) = -\frac{1}{2}\left(C^{\pm 1} - (C^{-1})^{\pm 1}\right),$$
$$g_\pm(0,0) = \frac{1}{2}(C + C^{-1}), \qquad g_\pm(1,1) = -\frac{1}{2}(C + C^{-1}).$$

Hence

$$H^\pm_0 = \frac{1}{2}(C + C^{-1}) A_0 - \frac{1}{2}\left(C^{\pm 1} - (C^{-1})^{\pm 1}\right) A_1,$$
$$H^\pm_1 = \frac{1}{2}\left(C^{\pm 1} - (C^{-1})^{\pm 1}\right) A_0 - \frac{1}{2}(C + C^{-1}) A_1,$$
$$V^\pm_0 = \frac{1}{2}(C + C^{-1}) A_0 + \frac{1}{2}\left(C^{\pm 1} - (C^{-1})^{\pm 1}\right) A_1,$$
$$V^\pm_1 = -\frac{1}{2}\left(C^{\pm 1} - (C^{-1})^{\pm 1}\right) A_0 - \frac{1}{2}(C + C^{-1}) A_1.$$

Kauffman's bracket polynomial (3.2.3):
$g_\pm(u,0) = (\alpha^{-1})^{\pm 1}$ if $u \neq 0$, $g_\pm(0,u) = \alpha^{\pm 1}$ if $u \neq 0$, $g_\pm(0,0) = \alpha + \alpha^{-1}$, and $g_\pm(u,v) = 0$ if $u \neq 0 \neq v$. Consequently $H^\pm_0 = (\alpha^{-1})^{\pm 1} I + \alpha^{\pm 1} J$, $V^\pm_0 = \alpha^{\pm 1} I + (\alpha^{-1})^{\pm 1} J$, and $H^\pm_u = (\alpha^{-1})^{\pm 1} I$, $V^\pm_u = \alpha^{\pm 1} I$ whenever $u \neq 0$.

For colourings with orientations (3.2.4):
$g_+(u,v) = \delta(u,-v)$, $g_-(u,v) = \delta(u,v)$. Hence $H^+_u = V^+_u = A_{-u}$ and $H^-_u = V^-_u = A_u$.

In all these examples it is not difficult using Proposition 3 to check directly that we have indeed an IRF model.

5. Spin models and doubly translation invariant IRF models.

We assume now that Ω is the field of complex numbers (so that for instance in the binary Lipson models, C will be considered as a non-zero complex parameter). We shall identify $\mathcal{A}$ with the Bose–Mesner algebra of the association scheme of X (see [**BI**] for definitions) by identifying each basis element A_x with the X by X matrix whose (i,j) entry is $A_x[i,j] = \delta(j-i,x)$. Then τ is identified with the transposition map.

5.1. Translation invariant spin models. A spin model $(X, w_1, w_2, w_3, w_4, \mathbb{C}, \mu, D)$ (see Definition 3) will be said *translation invariant* if $w_i(a+x, b+x) = w_i(a,b)$ for every x in X and $i = 1, \dots, 4$. Then let

$$W_i = \sum_{x \in X} w_i(0,x) A_x \qquad \text{for } i = 1, \dots, 4. \tag{46}$$

Thus the (a,b) entry of W_i is $w_i(a,b)$.

Proposition 4.

(i) *The identity* (11) *is equivalent to:*

$$I \circ W_1 = \mu I, \quad JW_4 = D\mu J. \tag{47}$$

(ii) *The identity* (12) *is equivalent to:*

$$I \circ W_3 = \mu^{-1} I, \quad JW_2 = D\mu^{-1} J. \tag{48}$$

(iii) *The identity* (13) *is equivalent to:*

$$W_1 \circ \tau(W_3) = J = W_2 \circ \tau(W_4). \tag{49}$$

(iv) *The identity* (14) *is equivalent to:*

$$W_1 W_3 = |X| I = W_2 W_4. \tag{50}$$

(v) *Each of the identities* (15), (16) *is equivalent to:*

(51)
for every A in $\mathcal{A}$, $W_1(\tau(W_4) \circ (W_1 A)) = D\tau(W_4) \circ (W_1(\tau(W_4) \circ A))$.

Proof. (i)–(iv) are immediate.

Let us prove (v). The left hand side of (51) when $A = A_i$ is computed as follows.

$$W_1 A_i = \left(\sum_{x \in X} w_1(0,x) A_x \right) A_i = \sum_{x \in X} w_1(0,x) A_{x+i},$$

$$\tau(W_4)\circ(W_1A_i) = \left(\sum_{x\in X} w_4(0,-x)A_x\right)\circ\left(\sum_{x\in X} w_1(0,x-i)A_x\right)$$
$$= \sum_{x\in X} w_4(0,-x)w_1(0,x-i)A_x,$$
$$W_1(\tau(W_4)\circ(W_1A_i)) = \left(\sum_{y\in X} w_1(0,y)A_y\right)\left(\sum_{x\in X} w_4(0,-x)w_1(0,x-i)A_x\right)$$
$$= \sum_{x,y\in X} w_1(0,y)w_4(0,-x)w_1(0,x-i)A_{x+y}$$
$$= \sum_{x,z\in X} w_1(0,z-x)w_4(0,-x)w_1(0,x-i)A_z.$$

The right hand side when $A = A_i$ is computed similarly:

$$\tau(W_4)\circ A_i = \left(\sum_{x\in X} w_4(0,-x)A_x\right)\circ A_i = w_4(0,-i)A_i,$$
$$W_1(\tau(W_4)\circ A_i)$$
$$= \left(\sum_{x\in X} w_1(0,x)A_x\right)(w_4(0,-i)A_i) = \sum_{x\in X} w_1(0,x)w_4(0,-i)A_{x+i},$$
$$\tau(W_4)\circ(W_1(\tau(W_4)\circ A_i))$$
$$= \left(\sum_{z\in X} w_4(0,-z)A_z\right)\circ\left(\sum_{z\in X} w_1(0,z-i)w_4(0,-i)A_z\right)$$
$$= \sum_{z\in X} w_4(0,-z)w_1(0,z-i)w_4(0,-i)A_z.$$

Thus (51) is equivalent to the identity

(51 a)
$$\sum_{x\in X} w_1(0,z-x)w_4(0,-x)w_1(0,x-i) = Dw_4(0,-z)w_1(0,z-i)w_4(0,-i).$$

On the other hand, using translation invariance, the identity (15) becomes

$$\sum_{x\in X} w_1(0,x-a)w_1(0,b-x)w_4(0,x-c) = Dw_1(0,b-a)w_4(0,a-c)w_4(0,b-c)$$

which is equivalent to (51a) via the substitution $x \to c-x$, $b \to c-i$, $a \to c-z$.

Finally comparing (15) and (16) we see that interchanging the two variables in each occurence of w_1 or w_4 in (16) yields (15). This interchange corresponds via (46) to the transposition of matrices. Hence (16) is equivalent to

(51 b)
$$\text{for every } A \text{ in } \mathcal{A},\quad \tau(W_1)(W_4\circ(\tau(W_1)A)) = DW_4\circ(\tau(W_1)(W_4\circ A)).$$

This is equivalent to (51) via transposition since $\tau(AB) = \tau(A)\tau(B)$ and $\tau(A \circ B) = \tau(A) \circ \tau(B)$ for all A, B in $\mathcal{A}$. □

From now on we shall also represent translation invariant spin models as 7-tuples $(X, W_1, W_2, W_3, W_4, \mu, D)$, where the modulus μ is a non-zero complex number, $D^2 = |X|$, and W_1, W_2, W_3, W_4 are elements of $\mathcal{A}$ satisfying equations (47), (48), (49), (50), (51).

Remark. One can show (see [**BB2**]) that assuming (49), (50), (51), the four equations in (47), (48) hold for some non-zero complex number μ (so that we can define μ from any one of them).

5.2. Spin models for graphs and spin models for links. Let us consider a finite directed graph G (loops and multiple edges will be allowed) with vertex-set $V(G)$ and edge-set $E(G)$. The initial (respectively: terminal) end of an edge e of G will be denoted by $i(e)$ (respectively: $t(e)$). Let w be a mapping from $E(G)$ to $\mathcal{A}$. Then the *partition function of the spin model defined on the graph G by the system of weights w* is

$$Z(G, w) = \sum_{\sigma: V(G) \to X} \prod_{e \in E(G)} w(e)[\sigma(i(e)), \sigma(t(e))] \tag{52}$$

(this is the definition given in [**Ja5**]).

Let now $\zeta = (X, W_1, W_2, W_3, W_4, \mu, D)$ be a spin model as defined in Section 5.1 and consider a connected oriented link diagram L. Let us color the faces of L as in Section 2.4. Let $G(L)$ be the connected plane graph consisting of one vertex inside each black face and, for each crossing of L, one edge joining the two black faces incident with that crossing. Let us orient the edges of $G(L)$ and label them with matrices W_1, W_2, W_3, W_4 as shown on Figure 5. By comparing Figures 3 and 5, it is clear that

$$Z^{\zeta}(L) = D^{-|V(G(L))|} Z(G(L), w), \tag{53}$$

where w is the mapping from $E(G(L))$ to $\mathcal{A}$ which assigns to every edge its label.

Suppose now that in the definition of the partition function given in Section 2.4 we modify our convention for the coloring of faces of L and require that the unbounded face be colored black, every other convention remaining unchanged. Denote by $Z^{*\zeta}(L)$ the resulting partition function.

Then we can show exactly as in the proof of Proposition 2.14 of [**Jo3**] that

$$Z^{*\zeta}(L) = Z^{\zeta}(L). \tag{54}$$

To obtain another version of (53), we now stick to our original convention that the unbounded face should be colored white, and we define $G'(L)$, its orientation, and its labeling w' exactly as for $G(L)$, except that black faces are replaced by white faces (see Figure 6). Then, by (54),

$$Z^{\varsigma}(L) = D^{-|V(G'(L))|} Z(G'(L), w'). \tag{55}$$

5.3. Spin models and duality. A *duality* of $\mathcal{A}$ is a linear map Ψ from $\mathcal{A}$ to itself satisfying

$$\Psi^2 = |X|\tau, \tag{56}$$

$$\Psi(AB) = \Psi(A) \circ \Psi(B) \qquad \text{for all } A, B \text{ in } \mathcal{A}, \tag{57}$$

$$\Psi(A \circ B) = |X|^{-1}\Psi(A)\Psi(B) \qquad \text{for all } A, B \text{ in } \mathcal{A} \tag{58}$$

(given (56), the properties (57) and (58) are easily seen to be equivalent). It is well known that such dualities exist and may be defined on the natural basis of $\mathcal{A}$ by relations of the form

$$\Psi(A_i) = \sum_{j \in X} \chi_i(j) A_j \tag{59}$$

where χ_i, $i \in X$, are the characters of X, with indices chosen such that $\chi_i(j) = \chi_j(i)$ for all i, j in X.

For a connected directed plane G, we shall denote by G^* the plane dual graph of G, with edges directed as shown on Figure 7, where e and e^* represent dual edges. We shall identify each mapping w from $E(G)$ to $\mathcal{A}$ with the mapping from $E(G^*)$ to $\mathcal{A}$ which for every edge e of G assigns the value $w(e)$ to the edge e^* dual to e.

The following result is proved in [**Ja5**], Proposition 11 (see also [**Bi2**]).

Proposition 5. *For every connected directed plane graph G, and for every mapping w from $E(G)$ to $\mathcal{A}$, $Z(G, w) = |X|^{1-|V(G^*)|} Z(G^*, \Psi w)$.*

Then Proposition 5 applied to (55) gives

$$Z^{\varsigma}(L) = D^{-|V(G'(L))|} |X|^{1-|V(G'(L)^*)|} Z\left(G'(L)^*, \Psi w'\right).$$

Comparing Figures 5, 6 and 7 it is easy to see that $G'(L)^*$ is identical to $G(L)$ as an undirected graph, the orientations being different exactly on the edges labeled W_3 and W_4 in $G'(L)$. Also it is clear that a partition function of the form (52) is not modified if we reverse the orientation of one edge while transposing the corresponding matrix. Hence we obtain

$$Z^{\varsigma}(L) = D^{-|V(G'(L))|} |X|^{1-|V(G(L))|} Z(G(L), w''),$$

where w'' is obtained from $\Psi w'$ by transposing the matrices corresponding to edges labeled W_3 and W_4 in $G'(L)$.

In other words, if w' takes the value W_4 (respectively: W_3, W_2, W_1) on an edge of $G'(L)$ then w'' takes the value $\tau\Psi(W_4)$ (respectively: $\tau\Psi(W_3), \Psi(W_2), \Psi(W_1)$) on the corresponding edge of $G(L)$. Also, recall that by Euler's formula, $|V(G'(L))| = |F(G(L))| = |E(G(L))| - |V(G(L))| + 2$. Since $|X| = D^2$, we obtain

$$Z^{\zeta}(L) = D^{-|E(G(L))|-|V(G(L))|}Z(G(L), w'').$$

It is clear from (52) that the factor $D^{-|E(G(L))|}$ can be distributed on the edges of $G(L)$ to give

$$Z^{\zeta}(L) = D^{-|V(G(L))|}Z(G(L), D^{-1}w''). \tag{60}$$

This motivates the following definition. Given a translation invariant spin model $\zeta = (X, W_1, W_2, W_3, W_4, \mu, D)$, let

$$\begin{aligned} W_1^* &= D^{-1}\tau\Psi(W_4), \quad W_2^* = D^{-1}\tau\Psi(W_3), \\ W_3^* &= D^{-1}\Psi(W_2), \quad W_4^* = D^{-1}\Psi(W_1) \end{aligned} \tag{61}$$

and

$$\zeta^* = (X, W_1^*, W_2^*, W_3^*, W_4^*, \mu, D).$$

Proposition 6. *For every translation invariant spin model ζ, ζ^* is also a translation invariant spin model and $Z^{\zeta^*} = Z^{\zeta}$.*

Proof. The equality $Z^{\zeta^*}(L) = Z^{\zeta}(L)$ for any link diagram L follows immediately from (60), (53) and comparison of the definition of w'' with (61).

We now complete the proof by checking properties (47)–(51) for ζ^*. From (61), these properties are:

$$I \circ \tau\Psi(W_4) = D\mu I, \quad J\Psi(W_1) = |X|\mu J, \tag{47$'$}$$

$$I \circ \Psi(W_2) = D\mu^{-1}I, \quad J\tau\Psi(W_3) = |X|\mu^{-1}J, \tag{48$'$}$$

$$\tau\Psi(W_4) \circ \tau\Psi(W_2) = |X|J = \tau\Psi(W_3) \circ \tau\Psi(W_1), \tag{49$'$}$$

$$\tau\Psi(W_4)\Psi(W_2) = |X|^2 I = \tau\Psi(W_3)\Psi(W_1), \tag{50$'$}$$

$$\begin{aligned} &\tau\Psi(W_4)(\tau\Psi(W_1) \circ (\tau\Psi(W_4)A)) \\ &= D\tau\Psi(W_1) \circ (\tau\Psi(W_4)(\tau\Psi(W_1) \circ A)). \end{aligned} \tag{51$'$}$$

The equation (47$'$) (respectively: (48$'$), (49$'$), (50$'$)) follows by applying Ψ to (47) (respectively: (48), (50), (49)), using (57), (58), and the easily established formulas

$$\Psi(I) = J, \quad \Psi(J) = |X|I, \tag{62}$$

$$\Psi\tau = \tau\Psi. \tag{63}$$

By applying Ψ and using (56), (57), (58), equation (51′) becomes successively:

$$\begin{aligned}
&|X|W_4 \circ \Psi(\tau\Psi(W_1) \circ (\tau\Psi(W_4)A)) \\
&\qquad\qquad = D|X|^{-1}(|X|W_1)\Psi(\tau\Psi(W_4)(\tau\Psi(W_1) \circ A)), \\
&DW_4 \circ |X|^{-1}(|X|W_1\Psi(\tau\Psi(W_4)A)) = W_1(|X|W_4 \circ \Psi(\tau\Psi(W_1) \circ A)), \\
&\qquad W_4 \circ (W_1(|X|W_4 \circ \Psi(A))) = DW_1(W_4 \circ |X|^{-1}(|X|W_1\Psi(A))).
\end{aligned}$$

Upon replacement of the generic element $\Psi(A)$ of $\mathcal{A}$ by A, this becomes

$$DW_4 \circ (W_1(W_4 \circ A)) = W_1(W_4 \circ (W_1A)). \tag{51″}$$

Note that by exchanging a, b in (16) we get

$$\sum_{x\in X} w_1(x,b)w_1(a,x)w_4(x,c) = Dw_1(a,b)w_4(b,c)w_4(a,c) \tag{16′}$$

which can also be obtained from (15) by exchanging the two variables in each occurrence of w_4. Hence the same argument used in the proof of Proposition 4(v) to reformulate (15) as

$$W_1(\tau(W_4) \circ (W_1A)) = D\tau(W_4) \circ (W_1(\tau(W_4) \circ A)) \tag{51}$$

also shows the equivalence of (16′) with (51″). □

It is easily checked using (56) that $(\zeta^*)^* = \zeta$.

We shall call ζ^* the spin model *dual* to ζ.

5.4. Squares of spin models as IRF models. By (54) a partition function of a spin model on X can be computed by defining states either on black faces or on white faces. This allows us to compute the square of such a partition function as the partition function of an IRF model on X. This idea appears for instance in [**Jo3**], Proposition 4.1 and in [**KWa**] Theorem 6.1. The proof of the following similar result will be omitted.

Proposition 7. *Let $\zeta = (X, W_1, W_2, W_3, W_4, \mu, D)$ be a translation invariant spin model with $W_i = \sum_{x\in X} w_i(x)A_x$ for $i = 1, \dots, 4$.*

Let $g_+(u,v) = D^{-1}w_1(u)w_4(-v)$ and $g_-(u,v) = D^{-1}w_3(u)w_2(v)$ for all u, v in X. Then these mappings define via (32) a doubly translation invariant IRF model $\imath$ with $Z^\imath = (Z^\zeta)^2$.

5.5. Spin models from IRF models. Clearly not every doubly translation invariant IRF model can be associated with a spin model as in Proposition 7. However we still can decompose each state of such an IRF model as a product of a state on black faces with a state on white faces. Then we may decompose the partition function accordingly and use duality (Proposition 5) to convert an evaluation on white faces into an evaluation on black faces. This will reformulate the IRF partition function as a spin model partition function.

Let us formalize this idea. Consider a doubly translation invariant IRF model $\imath$ defined via (32) by mappings $g_\pm$. It follows from (10) that its partition function is given by

$$Z^\imath(L) = |X|^{-1} \sum_{\sigma:F(L)\to X} \prod_{v\in V(L)} g_{s(v)}(\sigma(f_3(v)) - \sigma(f_1(v)), \sigma(f_4(v)) - \sigma(f_2(v))).$$

In the sequel we assume that the diagram L is connected. Let us color its faces black or white as in Section 2.4. We also introduce the directed graph $G(L)$ as in Section 5.2, and its directed dual plane graph $G(L)^*$. So the vertices of $G(L)$ are the black faces of L and the vertices of $G(L)^*$ are the white faces of L. We identify each mapping $\sigma : F(L) \to X$ with the pair (σ^b, σ^w), where $\sigma^b : V(G(L)) \to X$ and $\sigma^w : V(G(L)^*) \to X$ are the restrictions of σ to the sets of black faces and white faces. Each vertex v of L corresponds to a pair of dual edges e and e^* in $G(L)$ and $G(L)^*$. Given $\sigma : F(L) \to X$, we now express in terms of e and e^* the corresponding contribution $g_{s(v)}(\sigma(f_3(v)) - \sigma(f_1(v)), \sigma(f_4(v)) - \sigma(f_2(v)))$ of v to $Z^\imath(L)$.

We distinguish two cases (see Figures 5, 7).

If $f_1(v)$ and $f_3(v)$ are black, this contribution is

$$g_{s(v)}(\sigma^b(t(e)) - \sigma^b(i(e)), \sigma^w(i(e^*)) - \sigma^w(t(e^*))).$$

If $f_1(v)$ and $f_3(v)$ are white, this contribution is

$$g_-(\sigma^w(t(e^*)) - \sigma^w(i(e^*)), \sigma^b(t(e)) - \sigma^b(i(e))) \quad \text{if } s(v) = -,$$

and

$$g_+(\sigma^w(i(e^*)) - \sigma^w(t(e^*)), \sigma^b(i(e)) - \sigma^b(t(e))) \quad \text{if } s(v) = +.$$

We introduce a mapping q_v from $X \times X$ to $\mathbb{C}$ defined as follows.

(64) If $f_1(v)$ and $f_3(v)$ are black, $q_v(x, y) = g_{s(v)}(x, -y)$.
If $f_1(v)$ and $f_3(v)$ are white, $q_v(x, y) = g_-(y, x)$ if $s(v) = -$,
and $q_v(x, y) = g_+(-y, -x)$ if $s(v) = +$.

We shall also write q_e or q_{e^*} for q_v when $v \in V(L)$ corresponds to $e \in E(G(L))$ and $e^* \in E(G(L)^*)$.

With this definition we see that for $\sigma = (\sigma^b, \sigma^w) = (\rho, \pi)$ the contribution $g_{s(v)}(\sigma(f_3(v)) - \sigma(f_1(v)), \sigma(f_4(v)) - \sigma(f_2(v)))$ of v to $Z^{\imath}(L)$ is

$$q_v(\rho(t(e)) - \rho(i(e)),\ \pi(t(e^*)) - \pi(i(e^*))).$$

Hence

$$Z^{\imath}(L) = |X|^{-1} \sum_{\rho:V(G(L))\to X} \sum_{\pi:V(G(L)^*)\to X} \prod_{e^*\in E(G(L)^*)} q_v(\rho(t(e)) - \rho(i(e)), \pi(t(e^*)) - \pi(i(e^*))).$$

With every mapping ρ from $V(G(L))$ to X we associate a mapping $q_\rho : E(G(L)^*) \to \mathcal{A}$ defined by $q_\rho(e^*) = \sum_{y\in X} q_e(\rho(t(e)) - \rho(i(e)), y)A_y$. Then

$$Z^{\imath}(L) = |X|^{-1} \sum_{\rho:V(G(L))\to X} \sum_{\pi:V(G(L)^*)\to X} \prod_{e^*\in E(G(L)^*)} q_\rho(e^*)[\pi(i(e^*)), \pi(t(e^*))]$$

or equivalently, using the definition (52):

$$Z^{\imath}(L) = |X|^{-1} \sum_{\rho:V(G(L))\to X} Z(G(L)^*, q_\rho).$$

Note that $(G(L)^*)^*$ is $G(L)$ with all edge orientations reversed, and that in the evaluation (52) the reversal of an edge can be compensated by the transposition of the corresponding matrix. Thus applying Proposition 5 we obtain:

$$\begin{aligned}
&Z^{\imath}(L) \\
&= |X|^{-|V(G(L))|} \sum_{\rho:V(G(L))\to X} Z(G(L), \tau\Psi q_\rho) \\
&= |X|^{-|V(G(L))|} \sum_{\rho:V(G(L))\to X} \sum_{\pi:V(G(L))\to X} \prod_{e\in E(G(L))} (\tau\Psi q_\rho)(e)[\pi(i(e)), \pi(t(e))] \\
&= |X|^{-|V(G(L))|} \sum_{(\rho,\pi):V(G(L))\to X\times X} \prod_{e\in E(G(L))} (\tau\Psi q_\rho)(e)[\pi(i(e)), \pi(t(e))].
\end{aligned}$$

In the sequel we identify $\mathcal{A} \otimes \mathcal{A}$ with the Bose–Mesner algebra of the association scheme of the direct product $X \times X$, the matrix $A_x \otimes A_y$ being identified with the matrix $A_{(x,y)}$. Thus for every A, B in $\mathcal{A}$, $A \otimes B$ is the Kronecker product of A and B, i.e. $(A \otimes B)[(i,j),(k,l)] = A[i,k]B[j,l]$ for all i, j, k, l in X.

We define the mapping w from $E(G(L))$ to $\mathcal{A} \otimes \mathcal{A}$ by

$$w(e) = \sum_{x,y\in X} q_e(x,y) A_x \otimes \tau\Psi(A_y) \quad \text{for every } e \text{ in } E(G(L)). \tag{65}$$

Note that $(\tau\Psi q_\rho)(e) = \sum_{y\in X} q_e(\rho(t(e)) - \rho(i(e)), y)\tau\Psi(A_y)$. Then one easily checks that

$$\begin{aligned}(\tau\Psi q_\rho)(e)[\pi(i(e)), \pi(t(e))] &= w(e)[(\rho(i(e)), \pi(i(e))), (\rho(t(e)), \pi(t(e)))] \\ &= w(e)[(\rho, \pi)(i(e)), (\rho, \pi)(t(e))].\end{aligned}$$

Hence

$$Z^\imath(L) = |X|^{-|V(G(L))|} \sum_{\sigma: V(G(L)) \to X\times X} \prod_{e\in E(G(L))} w(e)[\sigma(i(e)), \sigma(t(e))]$$

which, using the definition (52), becomes

$$Z^\imath(L) = |X|^{-|V(G(L))|} Z(G(L), w). \tag{66}$$

Thus we have expressed the partition function of the doubly translation invariant IRF model $\imath$ on the diagram L as a partition function of a spin model on the graph $G(L)$. This leads us to the following result.

Proposition 8. *Let $\imath$ be a doubly translation invariant IRF model on X with modulus μ defined via (32) by mappings $g_\pm$. For each u in X let $H_u^\pm = \sum_{x\in X} g_\pm(u,x)A_x$ and $V_u^\pm = \sum_{x\in X} g_\pm(x,u)A_x$ be the elements of $\mathcal{A}$ introduced in (37), (38). Define four elements W_i, $i = 1,2,3,4$ of $\mathcal{A}\otimes\mathcal{A}$ by*

$$W_1 = \sum_{u\in X} A_u \otimes \Psi(H_u^+) \tag{67}$$

$$W_2 = \sum_{u\in X} A_u \otimes \tau\Psi(V_u^-) \tag{68}$$

$$W_3 = \sum_{u\in X} A_u \otimes \Psi(H_u^-) \tag{69}$$

$$W_4 = \sum_{u\in X} A_u \otimes \Psi(V_{-u}^+) \tag{70}$$

Then $\zeta = (X \times X, W_1, W_2, W_3, W_4, \mu, |X|)$ is a translation invariant spin model and $Z^\zeta = Z^\imath$. Any translation invariant spin model on $X \times X$ is associated in this way with a doubly translation invariant IRF model on X.

Proof. We first establish, assuming ζ is a spin model, that $Z^\imath = Z^\zeta$. In view of (53) and (66), this amounts to check that the mapping w defined by (65) coincides via the definitions (67)–(70) with the mapping described on Figure 5. Note that (65) can be written as

$$(65') \qquad w(e) = \sum_{x\in X} A_x \otimes \sum_{y\in X} q_e(x,y)\tau\Psi(A_y) \quad \text{for every } e \text{ in } E(G(L)).$$

Then (64) gives the following identities (where v is the vertex of L corresponding to e).

(i) If $f_1(v)$ and $f_3(v)$ are black,

$$\begin{aligned} w(e) &= \sum_{x\in X} A_x \otimes \sum_{y\in X} g_{s(v)}(x,-y)\tau\Psi(A_y) \\ &= \sum_{x\in X} A_x \otimes \sum_{y\in X} g_{s(v)}(x,-y)\Psi\tau(A_y) \quad \text{(by (63))} \\ &= \sum_{x\in X} A_x \otimes \sum_{y\in X} g_{s(v)}(x,-y)\Psi(A_{-y}) \\ &= \sum_{x\in X} A_x \otimes \sum_{y\in X} g_{s(v)}(x,y)\Psi(A_y) = \sum_{x\in X} A_x \otimes \Psi\left(H_x^{s(v)}\right). \end{aligned}$$

Thus $w(e) = W_1$ if $s(v) = +$ and $w(e) = W_3$ if $s(v) = -$.

(ii) If $f_1(v)$ and $f_3(v)$ are white, and if $s(v) = -$,

$$\begin{aligned} w(e) &= \sum_{x\in X} A_x \otimes \sum_{y\in X} g_-(y,x)\tau\Psi(A_y) \\ &= \sum_{x\in X} A_x \otimes \tau\Psi(V_x^-) = W_2. \end{aligned}$$

(iii) If $f_1(v)$ and $f_3(v)$ are white, and if $s(v) = +$,

$$\begin{aligned} w(e) &= \sum_{x\in X} A_x \otimes \sum_{y\in X} g_+(-y,-x)\tau\Psi(A_y) \\ &= \sum_{x\in X} A_x \otimes \sum_{y\in X} g_+(-y,-x)\Psi\tau(A_y) \quad \text{(by (63))} \\ &= \sum_{x\in X} A_x \otimes \Psi\left(\sum_{y\in X} g_+(-y,-x)A_{-y}\right) = \sum_{x\in X} A_x \otimes \Psi(V_{-x}^+) = W_4. \end{aligned}$$

These identities together with Figure 5 complete the proof of the equality $Z^\imath = Z^\zeta$. It remains to prove that ζ is indeed a spin model of modulus μ. We shall check the equations (47)–(51) of Proposition 4. Recall that for every A, B, A', B' in $\mathcal{A}$, $(A\otimes B)(A'\otimes B') = AA'\otimes BB'$ and $(A\otimes B)\circ(A'\otimes B') = (A\circ A')\otimes(B\circ B')$. We still denote by I and J the identities for the ordinary and Hadamard product in $\mathcal{A}$. Thus the corresponding elements of $\mathcal{A}\otimes\mathcal{A}$ are $I\otimes I$ and $J\otimes J$.

In what follows we use the equalities (57), (58), (62), (63).

The first equalities of (47) and (48) become

$$(I\otimes I)\circ\left(\sum_{u\in X} A_u\otimes\Psi\left(H_u^{\pm}\right)\right)=\mu^{\pm 1}I\otimes I,$$

or equivalently

$$\sum_{u\in X}(I\circ A_u)\otimes\left(I\circ\Psi\left(H_u^{\pm}\right)\right)=\mu^{\pm 1}I\otimes I.$$

Since $I\circ A_u=\delta(u,0)I=\delta(u,0)A_0$ this reduces to $I\circ\Psi\left(H_0^{\pm}\right)=\mu^{\pm 1}I$, which follows by applying Ψ to (39).

The second equalities of (47) and (48) become

$$\sum_{u\in X} JA_u\otimes J(\Psi(V_{-u}^{+}))=|X|\mu J\otimes J,$$
$$\sum_{u\in X} JA_u\otimes J(\tau\Psi(V_u^{-}))=|X|\mu^{-1}J\otimes J.$$

Since $JA_u=J$ this reduces to

$$J\Psi\left(\sum_{u\in X}V_{-u}^{+}\right)=|X|\mu J,$$
$$J\tau\Psi\left(\sum_{u\in X}V_u^{-}\right)=|X|\mu^{-1}J$$

which follow by applying Ψ or $\tau\Psi$ to (40).

Since $\tau(W_3)=\sum_{u\in X}A_{-u}\otimes\tau\Psi(H_u^{-})=\sum_{u\in X}A_u\otimes\Psi\tau(H_{-u}^{-})$ the first equality of (49) becomes

$$\left(\sum_{u\in X}A_u\otimes\Psi(H_u^{+})\right)\circ\left(\sum_{u\in X}A_u\otimes\Psi\tau(H_{-u}^{-})\right)=J\otimes J,$$

or equivalently

$$\sum_{u\in X}A_u\otimes\left(\Psi(H_u^{+})\circ\Psi\tau(H_{-u}^{-})\right)=J\otimes J.$$

Now by (43) $\Psi(H_u^{+})\circ\Psi\tau(H_{-u}^{-})=\Psi(H_u^{+}\tau(H_{-u}^{-}))=\Psi(I)=J$ and we are done.

Similarly the second equality of (49) becomes

$$\left(\sum_{u\in X} A_u \otimes \tau\Psi(V_u^-)\right) \circ \left(\sum_{u\in X} A_{-u} \otimes \tau\Psi(V_{-u}^+)\right) = J \otimes J$$

or equivalently

$$\sum_{u\in X} A_u \otimes (\tau\Psi(V_u^-) \circ \tau\Psi(V_u^+)) = J \otimes J$$

which follows by applying $\tau\Psi$ to (41).

The first equality of (50) reads

$$\left(\sum_{u\in X} A_u \otimes \Psi(H_u^+)\right) \left(\sum_{v\in X} A_v \otimes \Psi(H_v^-)\right) = |X|^2 I \otimes I.$$

The left–hand side is

$$\begin{aligned}
&\sum_{u\in X}\sum_{v\in X} A_{u+v} \otimes \Psi(H_u^+)\Psi(H_v^-) \\
&= \sum_{a\in X}\sum_{u\in X} A_a \otimes \Psi(H_u^+)\Psi(H_{a-u}^-) \\
&= \sum_{a\in X} A_a \otimes \left(\sum_{u\in X} |X|\Psi(H_u^+ \circ H_{a-u}^-)\right)
\end{aligned}$$

which by (42) is equal to

$$|X| \sum_{a\in X} A_a \otimes \Psi(\delta(a,0)J) = |X|A_0 \otimes \Psi(J) = |X|^2 I \otimes I$$

as required.

Similarly the second equality of (50) reads

$$\left(\sum_{u\in X} A_u \otimes \tau\Psi(V_u^-)\right) \left(\sum_{v\in X} A_v \otimes \Psi(V_{-v}^+)\right) = |X|^2 I \otimes I.$$

The left–hand side is, using (44),

$$\begin{aligned}
&\sum_{u\in X}\sum_{v\in X} A_{u+v} \otimes \tau\Psi(V_u^-)\Psi(V_{-v}^+) \\
&= \sum_{a\in X}\sum_{u\in X} A_a \otimes \Psi(V_{u-a}^+)\Psi\tau(V_u^-) \\
&= \sum_{a\in X} A_a \otimes \sum_{u\in X} |X|\Psi(V_{u-a}^+ \circ \tau(V_u^-))
\end{aligned}$$

$$= |X| \sum_{a \in X} A_a \otimes \Psi(\delta(-a,0)J)$$

and the result follows as before.

Finally let us consider (51). It will be enough to check this identity on the basis $\{A_i \otimes \Psi(A_j),\ i \in X,\ j \in X\}$ of $\mathcal{A} \otimes \mathcal{A}$.

First we compute the left–hand side.

$$\begin{aligned} &W_1(A_i \otimes \Psi(A_j)) \\ &= \sum_{u \in X} A_{u+i} \otimes \Psi(H_u^+)\Psi(A_j) \\ &= |X| \sum_{u \in X} A_{u+i} \otimes \Psi(H_u^+ \circ A_j). \end{aligned}$$

$$\begin{aligned} &\tau(W_4) \circ (W_1(A_i \otimes \Psi(A_j))) \\ &= |X| \left(\sum_{u \in X} A_{-u} \otimes \tau\Psi(V_{-u}^+) \right) \circ \left(\sum_{u \in X} A_{u+i} \otimes \Psi(H_u^+ \circ A_j) \right) \\ &= |X| \sum_{u \in X} A_{u+i} \otimes \left(\Psi\tau(V_{u+i}^+) \circ \Psi(H_u^+ \circ A_j)\right) \\ &= |X| \sum_{u \in X} A_{u+i} \otimes \Psi(\tau(V_{u+i}^+)(H_u^+ \circ A_j)). \end{aligned}$$

$$\begin{aligned} &W_1(\tau(W_4) \circ (W_1(A_i \otimes \Psi(A_j)))) \\ &= |X| \left(\sum_{v \in X} A_v \otimes \Psi(H_v^+) \right) \left(\sum_{u \in X} A_{u+i} \otimes \Psi(\tau(V_{u+i}^+)(H_u^+ \circ A_j)) \right) \\ &= |X| \sum_{v \in X} \sum_{u \in X} A_{v+u+i} \otimes \Psi(H_v^+)\Psi(\tau(V_{u+i}^+)(H_u^+ \circ A_j)) \\ &= |X|^2 \sum_{v \in X} \sum_{u \in X} A_{v+u+i} \otimes \Psi(H_v^+ \circ (\tau(V_{u+i}^+)(H_u^+ \circ A_j))) \\ &= |X|^2 \sum_{y \in X} A_{y+i} \otimes \sum_{v \in X} \Psi(H_v^+ \circ (\tau(V_{y-v+i}^+)(H_{y-v}^+ \circ A_j))). \end{aligned}$$

Let us now compute the right–hand side of (51).

$$\begin{aligned} &\tau(W_4) \circ (A_i \otimes \Psi(A_j)) \\ &= \left(\sum_{u \in X} A_{-u} \otimes \tau\Psi(V_{-u}^+) \right) \circ (A_i \otimes \Psi(A_j)) \\ &= A_i \otimes \left(\Psi\tau(V_i^+) \circ \Psi(A_j)\right) = A_i \otimes \Psi(\tau(V_i^+)A_j). \end{aligned}$$

$$\begin{aligned}
&W_1(\tau(W_4)\circ(A_i\otimes\Psi(A_j)))\\
&=\left(\sum_{u\in X}A_u\otimes\Psi(H_u^+)\right)(A_i\otimes\Psi(\tau(V_i^+)A_j))\\
&=\sum_{u\in X}A_{u+i}\otimes\Psi(H_u^+)\Psi(\tau(V_i^+)A_j)\\
&=|X|\sum_{u\in X}A_{u+i}\otimes\Psi(H_u^+\circ(\tau(V_i^+)A_j)).
\end{aligned}$$

$$\begin{aligned}
&|X|\tau(W_4)\circ(W_1(\tau(W_4)\circ(A_i\otimes\Psi(A_j))))\\
&=|X|^2\left(\sum_{y\in X}A_y\otimes\tau\Psi(V_y^+)\right)\circ\left(\sum_{u\in X}A_{u+i}\otimes\Psi(H_u^+\circ(\tau(V_i^+)A_j))\right)\\
&=|X|^2\sum_{y\in X}A_{y+i}\otimes\left(\Psi\tau(V_{y+i}^+)\circ\Psi(H_y^+\circ(\tau(V_i^+)A_j))\right)\\
&=|X|^2\sum_{y\in X}A_{y+i}\otimes\Psi(\tau(V_{y+i}^+)(H_y^+\circ(\tau(V_i^+)A_j))).
\end{aligned}$$

By comparing the expressions for the two sides of (51) we see that this identity reduces to the identity

$$\sum_{v\in X}H_v^+\circ(\tau(V_{y-v+i}^+)(H_{y-v}^+\circ A_j))=\tau(V_{y+i}^+)(H_y^+\circ(\tau(V_i^+)A_j)).$$

This is equivalent to

$$\sum_{x\in X}\tau(H_x^+)\circ(V_{u-x}^+(\tau(H_{v-x}^+)\circ A_i))=V_u^+(\tau(H_v^+)\circ(V_{u-v}^+A_i)) \tag{45}$$

via the application of the substitution $x\to v$, $u\to y+i$, $v\to y$, $i\to -j$ followed by the application of τ.

Finally if $\zeta=(X\times X,W_1,W_2,W_3,W_4,\mu,|X|)$ is a translation invariant spin model, we may clearly use formulas (67)–(70) to define for each u in X the elements $H_u^\pm$ and $V_u^\pm$ of $\mathcal{A}$. The same arguments as above will show that equations (39)–(45) are satisfied. Then we may use (37), (38) to define mappings $g_\pm$ which, by Proposition 3, will give via (32) the required doubly translation invariant IRF model on X. □

Remarks.

(i) A slight variation in the above construction would yield a spin model where each matrix W_i is replaced by its image under the "flip" automorphism of $\mathcal{A}\otimes\mathcal{A}$ which for every A,B in $\mathcal{A}$ sends $A\otimes B$ to $B\otimes A$.

A more significant variation consists in applying the same ideas to express the IRF partition function as a spin model partition function evaluated on the white faces. Then it is not difficult to check that the resulting spin model is the dual of the previous one with respect to the duality $\Psi \otimes \Psi$ of $\mathcal{A} \otimes \mathcal{A}$ which for every A, B in $\mathcal{A}$ sends $A \otimes B$ to $\Psi(A) \otimes \Psi(B)$.

(ii) It follows from (50) that when $W_1 = W_2$ we have also $W_3 = W_4$ and then setting $W_+ = W_1 = W_2$, $W_- = W_3 = W_4$, we obtain a spin model in the sense of [**KMW**]. By (67), (68), this is the case if and only if $H_u^+ = \tau(V_u^-)$ for every u in X.

5.6. Examples. We now apply Proposition 8 to the examples of Section 4.4. It is easy to see that in all cases $H_u^+ = \tau(V_u^-)$ for every u in X, and thus by the above remark we shall obtain a spin model in the sense of [**KMW**], for which we shall only compute the matrix $W_+ = W_1 = \sum_{u \in X} A_u \otimes \Psi(H_u^+)$. Note also that when X is an elementary 2–group, the corresponding spin model is symmetric.

The first binary Lipson model (3.2.1):
Recall that $H_0^+ = C^{-1}A_1$, $H_1^+ = CA_0$. Also, $\Psi(A_0) = \Psi(I) = J = A_0 + A_1$ and $\Psi(A_1) = \Psi(J - I) = 2I - J = A_0 - A_1$. Hence

$$\begin{aligned} W_+ &= A_0 \otimes \Psi(H_0^+) + A_1 \otimes \Psi(H_1^+) \\ &= A_0 \otimes \Psi(C^{-1}A_1) + A_1 \otimes \Psi(CA_0) \\ &= C^{-1}A_0 \otimes (A_0 - A_1) + CA_1 \otimes (A_0 + A_1). \end{aligned}$$

This is the well–known one–parameter family of spin models belonging to the Bose–Mesner algebra of the cycle of length 4 (see [**Jo3**], [**Ja3**], [**H2**]). Thus we have a direct proof that the partition function of this spin model gives the special evaluation of the Kauffman polynomial corresponding to Lipson's model.

The second binary Lipson model (3.2.2):
Setting $E_0 = \frac{1}{2}(A_0 + A_1)$, $E_1 = \frac{1}{2}(A_0 - A_1)$, we may write

$$H_0^+ = C^{-1}E_0 + CE_1, \qquad H_1^+ = -C^{-1}E_0 + CE_1.$$

Since $\Psi(E_0) = A_0$ and $\Psi(E_1) = A_1$, we obtain

$$\begin{aligned} W_+ &= A_0 \otimes \Psi(H_0^+) + A_1 \otimes \Psi(H_1^+) \\ &= A_0 \otimes \Psi(C^{-1}E_0 + CE_1) + A_1 \otimes \Psi(-C^{-1}E_0 + CE_1) \\ &= A_0 \otimes (C^{-1}A_0 + CA_1) + A_1 \otimes (-C^{-1}A_0 + CA_1). \end{aligned}$$

Clearly this matrix is obtained from the previous one by application of the flip automorphism. Thus we have a direct proof that the two binary Lipson models give the same partition function.

Kauffman's bracket polynomial (3.2.3):
We have found $H_0^+ = \alpha^{-1}I + \alpha J$ and $H_u^+ = \alpha^{-1}I$ whenever $u \neq 0$. Also recall that $|X| = -\alpha^2 - \alpha^{-2}$.

Hence

$$\begin{aligned} W_+ &= A_0 \otimes \Psi(H_0^+) + \sum_{u \in X - \{0\}} A_u \otimes \Psi(H_u^+) \\ &= I \otimes \Psi(\alpha^{-1}I + \alpha J) + \sum_{u \in X - \{0\}} A_u \otimes \Psi(\alpha^{-1}I) \\ &= I \otimes (\alpha^{-1}J + \alpha(-\alpha^2 - \alpha^{-2})I) + (J - I) \otimes \alpha^{-1}J \\ &= -\alpha^3 I \otimes I + \alpha^{-1}(I \otimes (J - I) + (J - I) \otimes J) \\ &= -\alpha^3 I \otimes I + \alpha^{-1}(J \otimes J - I \otimes I). \end{aligned}$$

This is the well-known "Potts" model for the Jones polynomial (see [**Jo3**], [**Ja3**], [**H2**], [**HJ**]). Thus we have a new construction of this model when the number of spins is a square.

Fox colourings with orientations (3.2.4):
We have $H_u^+ = A_{-u}$. Thus

$$W_+ = \sum_{u \in X} A_u \otimes \Psi(H_u^+) = \sum_{i \in X} A_i \otimes \Psi(A_{-i}).$$

We may assume that the duality Ψ is given by

$$\Psi(A_i) = \sum_{j \in X} \chi_i(j) A_j \tag{59}$$

where χ_i, $i \in X$, are the characters of X, with indices chosen such that $\chi_i(j) = \chi_j(i)$ for all i, j in X. Then the corresponding spin model matrix is

$$W_+ = \sum_{i,j \in X} \chi_{-i}(j) A_i \otimes A_j.$$

Remark. It is easy to check that this spin model belongs to the class described in Proposition 23 of [**Ja5**] (this class generalizes the one discovered in [**BB3**] and is related as shown in [**BBJ**] to the models of [**KWa**]).

Assume now that $X = \mathbb{Z}/n\mathbb{Z}$ with n odd. Let $\chi_i(j) = \omega^{ij}$, where ω is a primitive n-th root of unity.

Thus

$$W_+ = \sum_{(i,j) \in X \times X} \omega^{-ij} A_{(i,j)}, \quad W_- = \sum_{(i,j) \in X \times X} \omega^{ij} A_{(i,j)}.$$

Consider a link diagram L. Using (52), (53), we write the partition function as

$$Z(L) = |X|^{-|V(G(L))|} \sum_{\sigma:V(G(L))\to X\times X} \prod_{e\in E(G(L))} w(e)[\sigma(i(e)), \sigma(t(e))],$$

where w is a suitable mapping from $E(G(L))$ to $\mathcal{A}\otimes\mathcal{A}$ which takes the values W_+, W_-.

Let π be an automorphism of $X \times X$. Then

$$Z(L) = |X|^{-|V(G(L))|} \sum_{\sigma:V(G(L))\to X\times X} \prod_{e\in E(G(L))} w(e)[\pi\sigma(i(e)), \pi\sigma(t(e))],$$

and hence in the evaluation of Z we may replace W_+, W_- by

$$W'_+ = \sum_{(i,j)\in X\times X} \omega^{-ij} A_{\pi^{-1}(i,j)}, \quad W'_- = \sum_{(i,j)\in X\times X} \omega^{ij} A_{\pi^{-1}(i,j)}.$$

Let us define π by $\pi(i,j) = (j-i, i+j)$ for every i, j in X. Then

$$W'_+ = \sum_{(i,j)\in X\times X} \omega^{i^2-j^2} A_{(i,j)}, \quad W'_- = \sum_{(i,j)\in X\times X} \omega^{j^2-i^2} A_{(i,j)}.$$

Thus setting

$$W''_+ = \sum_{i\in X} \omega^{i^2} A_i, \qquad W''_- = \sum_{i\in X} \omega^{-i^2} A_i,$$

we have

$$W'_+ = W''_+ \otimes \overline{W''_+} \qquad \text{and} \qquad W'_- = W''_- \otimes \overline{W''_-}.$$

It is known (see [**BB3**], [**GJ**], [**Jo3**]) that W''_+, W''_- define a spin model on X (in the sense of [**Jo3**]). It easily follows that, denoting by Z'' the partition function of this model,

$$Z(L) = Z''(L)\overline{Z''(L)} = |Z''(L)|^2.$$

When n is prime, it follows from [**GJ**], Section 7, that for every link diagram L, $|Z''(L)| = (\sqrt{n})^{d(L)+1}$, where $d(L)$ is the dimension of the first homology with coefficients modulo n of the 2-fold cyclic cover of S^3 branched over the link represented by L. Then $Z(L) = n^{d(L)+1}$.

This expression for the number of Fox colourings of L is established by a different method in [**Pr**].

6. Composition of link invariants and Nomura's Hadamard spin models.

6.1. Composition of link invariants and vertex models. Let $f_1, \ldots, f_p$ be p invariants of oriented links which take their values in the same commutative ring Ω, and let λ be an invertible element of Ω. Let $\mathcal{L}$ be a link with set of components K. For any subset S of K we denote by $\mathcal{L}_S$ the link consisting of the components of $\mathcal{L}$ belonging to S (each of these components retaining the previous embedding in 3-space). We shall allow the empty link $\mathcal{L}_\varnothing$ with no components and assume that the invariants f_i are defined on the empty link. Let $\mathrm{lk}(C_1, C_2)$ denote the linking number of the components C_1, C_2 of $\mathcal{L}$. For any mapping $\gamma : K \to \{1, \ldots, p\}$ we denote by $\mathrm{lk}_\gamma(\mathcal{L})$ the sum of linking numbers $\mathrm{lk}(C_1, C_2)$ over all ordered pairs (C_1, C_2) of components such that $\gamma(C_1) \neq \gamma(C_2)$. Recall that if L is any diagram representing $\mathcal{L}$ with sign function s, $\mathrm{lk}(C_1, C_2) = \frac{1}{2}s(V_{12})$, where V_{12} is the set of vertices corresponding to crossings of C_1 with C_2. Thus $\mathrm{lk}_\gamma(\mathcal{L}) = s(V_\gamma)$, where V_γ is the set of vertices corresponding to crossings of components with different values of γ. Then we assign to $\mathcal{L}$ the value

$$(f_1, \ldots, f_p)_\lambda(\mathcal{L}) = \sum_{\gamma : K \to \{1, \ldots, p\}} \lambda^{\mathrm{lk}_\gamma(\mathcal{L})} \prod_{i=1,\ldots,p} f_i\left(\mathcal{L}_{\gamma^{-1}(i)}\right). \tag{71}$$

Clearly this defines an invariant $(f_1, \ldots, f_p)_\lambda$ of oriented links with values in Ω which we call the λ-*composition* of $f_1, \ldots, f_p$.

Remark.

(i) Related notions appear in remark 4.1 of [**PT**] and in [**Y**].

(ii) It is easy to check that the λ-composition of link invariants is associative, that is, $(f_1, (f_2, f_3)_\lambda)_\lambda = ((f_1, f_2)_\lambda, f_3)_\lambda = (f_1, f_2, f_3)_\lambda$. Thus we could restrict our attention to the case $p = 2$. The λ-composition is also clearly commutative.

Consider now p vertex models $\nu_i = (X, w_+^i, w_-^i, \Omega, \mu)$ with associated link invariants f_i, $i = 1, \ldots, p$. Let $X_p = \{1, \ldots, p\} \times X$ and define the mappings $w_\pm$ from X_p^4 to Ω as follows:

(72)

For every i, j, k, l in $\{1, \ldots, p\}$ and a, b, c, d in X,

$$w_\pm((i,a),(j,b),(k,c),(l,d)) = \delta(i,j)\delta(k,l)(\delta(i,k)w_\pm^i(a,b,c,d) + (1-\delta(i,k))\lambda^{\pm 1}\delta(a,b)\delta(c,d)).$$

Proposition 9. *$\nu = (X_p, w_+, w_-, \Omega, \mu)$ is a vertex model with associated* *link invariant* $(f_1, \dots, f_p)_{\lambda\mu^{-1}}$.

Proof. It is not difficult to check directly that ν satisfies the identities (1)–(4), although a simpler argument will be given below.

Consider a diagram L. Let us represent every mapping $\sigma : E(L) \to X_p$ as a pair (γ, η), where $\gamma : E(L) \to \{1, \dots, p\}$ and $\eta : E(L) \to X$ are such that $\sigma(e) = (\gamma(e), \eta(e))$ for every e in $E(L)$. Then, by (72), if $\sigma = (\gamma, \eta)$ contributes to the sum

$$Z^\nu(L) = \sum_{\sigma: E(L) \to X} \prod_{v \in V(L)} w_{s(v)}(\sigma(e_1(v)), \sigma(e_2(v)), \sigma(e_3(v)), \sigma(e_4(v)))$$

we must have $\gamma(e_1(v)) = \gamma(e_2(v))$ and $\gamma(e_3(v)) = \gamma(e_4(v))$ for every vertex v. This means that we may identify γ with a mapping from K to $\{1, \dots, p\}$, where K is the set of components of the link $\mathcal{L}$ represented by L. Indeed each such component C can be identified with a cycle of L, and γ takes only one value on the edges of this cycle (see Figure 2). We shall call $\gamma(C)$ this value. Thus

$$Z^\nu(L) = \sum_{\gamma: K \to \{1, \dots, p\}} \sum_{\eta: E(L) \to X} \prod_{v \in V(L)} w^\gamma_{s(v)}(\eta(e_1(v)), \eta(e_2(v)), \eta(e_3(v)), \eta(e_4(v))),$$

where $w^\gamma_{s(v)}(a, b, c, d)$ equals $w^i_{s(v)}(a, b, c, d)$ if γ assigns the same value i to the two (possibly identical) components crossing at v, and equals $\delta(a, b)\delta(c, d)\lambda^{s(v)}$ if γ assigns different values to these components.

Let us now consider γ as fixed and study the corresponding summand

$$S_\gamma = \sum_{\eta: E(L) \to X} \prod_{v \in V(L)} w^\gamma_{s(v)}(\eta(e_1(v)), \eta(e_2(v)), \eta(e_3(v)), \eta(e_4(v)))$$

in the above expression for $Z^\nu(L)$.

For each i in $\{1, \dots, p\}$ the edges e such that $\gamma(e) = i$ together with the incident vertices form a subgraph $L_{\gamma^{-1}(i)}$ of L. All vertices of $L_{\gamma^{-1}(i)}$ are of degree 4 or 2, and if we erase each vertex of degree 2 (merging the two incident edges) we obtain (with the obvious sign function) a diagram L^i representing the link $\mathcal{L}_{\gamma^{-1}(i)}$.

Then if η contributes to the sum S_γ, for each i in $\{1, \dots, p\}$ its restriction η_i to the edges of $L_{\gamma^{-1}(i)}$ takes the same value on any two edges meeting at a vertex of degree 2. Hence η_i can be identified with a mapping from $E(L^i)$ to X. In this case the summand

$$\prod_{v \in V(L)} w^\gamma_{s(v)}(\eta(e_1(v)), \eta(e_2(v)), \eta(e_3(v)), \eta(e_4(v)))$$

of S_γ can be written

$$\lambda^{s(V\gamma)} \prod_{i\in\{1,\dots,p\}} \prod_{v\in V(L^i)} w^i_{s(v)}(\eta_i(e_1(v)),\eta_i(e_2(v)),\eta_i(e_3(v)),\eta_i(e_4(v))).$$

Hence

$$\begin{aligned}
S_\gamma = &\sum_{\eta_1:E(L^1)\to X} \cdots \sum_{\eta_p:E(L^p)\to X} \lambda^{s(V\gamma)} \prod_{i\in\{1,\dots,p\}} \\
&\prod_{v\in V(L^i)} w^i_{s(v)}(\eta_i(e_1(v)),\eta_i(e_2(v)),\eta_i(e_3(v)),\eta_i(e_4(v))) \\
= &\lambda^{s(V\gamma)} \prod_{i\in\{1,\dots,p\}} \sum_{\eta_i:E(L^i)\to X} \\
&\prod_{v\in V(L^i)} w^i_{s(v)}(\eta_i(e_1(v)),\eta_i(e_2(v)),\eta_i(e_3(v)),\eta_i(e_4(v))) \\
= &\lambda^{s(V\gamma)} \prod_{i\in\{1,\dots,p\}} Z^{\nu_i}(L^i).
\end{aligned}$$

It follows that

$$\begin{aligned}
\mu^{-T(L)}Z^\nu(L) &= \sum_{\gamma:K\to\{1,\dots,p\}} \mu^{-s(V(L))}\lambda^{s(V_\gamma)} \prod_{i\in\{1,\dots,p\}} Z^{\nu_i}(L^i) \\
&= \sum_{\gamma:K\to\{1,\dots,p\}} \mu^{-s(V\gamma)}\lambda^{s(V_\gamma)} \prod_{i\in\{1,\dots,p\}} \mu^{-s(V(L^i))}Z^{\nu_i}(L^i).
\end{aligned}$$

This means that the link invariant associated with ν is the $\lambda\mu^{-1}$-composition of the link invariants associated with $\nu_1,\dots,\nu_p$.

A local version of the above argument can also be used to show that ν satisfies the identities (1)–(4). □

6.2. Application to IRF models. We keep the notations of the preceding section and we now assume that X is an Abelian group. We also endow $Y=\{1,\dots,p\}$ with an Abelian group structure. Thus $X_p = Y\times X$ is now an Abelian group. It is clear from (72) that if each of the p vertex models ν_i is conservative, the vertex model ν is also conservative.

By (19) and (72),

$$\begin{aligned}
&(\partial^* w_\pm)((i,a),(j,b),(k,c),(l,d)) \\
&= \delta(i-l,j-k)\delta(j-i,k-l)(\delta(i-l,j-i)w_\pm^{i-l}(a-d,b-c,b-a,c-d) \\
&\quad + (1-\delta(i-l,j-i))\lambda^{\pm1}\delta(a-d,b-c)\delta(b-a,c-d)) \\
&= \delta(i+k,j+l)(\delta(i,k)(\partial^* w_\pm^{i-l})(a,b,c,d) + (1-\delta(i,k))\lambda^{\pm1}\delta(a+c,b+d)).
\end{aligned}$$

Consider now p translation invariant IRF models $\imath_i = (X, w_+^i, w_-^i, \Omega, \mu)$ with associated link invariants f_i, $i \in Y$. Define the mappings $w_\pm$ from $(Y \times X)^4$ to Ω as follows:

$$\begin{aligned}
&\text{For every } i,j,k,l \text{ in } Y \text{ and } a,b,c,d \text{ in } X,\\
&w_\pm((i,a),(j,b),(k,c),(l,d))\\
&= \delta(i+k, j+l)(\delta(i,k)w_\pm^{i-l}(a,b,c,d)\\
&\qquad + (1-\delta(i,k))\lambda^{\pm 1}\delta(a+c, b+d)).
\end{aligned} \tag{73}$$

The following result is now an immediate consequence of Propositions 1, 9.

Proposition 10. *With the notations of* (73), $\imath = (Y \times X, w_+, w_-, \Omega, \mu)$ *is a translation invariant IRF model whose associated link invariant is the* $\lambda\mu^{-1}$*-composition of the* f_i, $i \in Y$.

6.3. Application to spin models. We are now interested in the case where the IRF model $\imath$ of Proposition 10 is doubly translation invariant. It is clear from (73) that this will be true if both X and Y are elementary Abelian 2-groups and all IRF models $\imath_i$ are equal to the same doubly translation invariant IRF model. We assume these properties now and we define mappings $g_\pm : X^2 \to \Omega$ by $g_\pm(u,v) = w_\pm^i(0,0,u,v)$ for all u,v in X and i in Y.

Then it follows from (32) and (73) that for every i,j,k,l in Y and a,b,c,d in X,

$$\begin{aligned}
&w_\pm((i,a),(j,b),(k,c),(l,d))\\
&= \delta(i+k, j+l)(\delta(i,k)g_\pm(c-a, d-b) + (1-\delta(i,k))\lambda^{\pm 1}\delta(a+c, b+d)).
\end{aligned}$$

Thus the IRF model $\imath$ is defined via (32) by the mappings $g_\pm^Y : (Y \times X)^2 \to \Omega$ such that

$$g_\pm^Y((i,u),(j,v)) = \delta(i,j)(\delta(i,0)g_\pm(u,v) + (1-\delta(i,0))\lambda^{\pm 1}\delta(u,v))$$

for all i,j in Y and u,v in X.

We now take $\Omega = \mathbb{C}$. Let $\mathcal{A}$ be the Bose-Mesner algebra of X with natural basis $\{A_x,\ x \in X\}$ and let $\mathcal{B}$ be the Bose-Mesner algebra of Y with natural basis $\{B_y,\ y \in Y\}$. We consider again for each u in X the elements $H_u^\pm = \sum_{x\in X} g_\pm(u,x)A_x$ and $V_u^\pm = \sum_{x \in X} g_\pm(x,u)A_x$ of $\mathcal{A}$ introduced in (37), (38). The corresponding elements of $\mathcal{B} \otimes \mathcal{A}$ for the mappings $g_\pm^Y$ are (for i in Y and u in X)

$$H_{(i,u)}^\pm = \sum_{k \in Y}\sum_{x \in X} g_\pm^Y((i,u),(k,x))B_k \otimes A_x$$

$$= \sum_{k\in Y}\sum_{x\in X} \delta(i,k)(\delta(i,0)g_\pm(u,x) + (1-\delta(i,0))\lambda^{\pm 1}\delta(u,x))B_k \otimes A_x$$
$$= \sum_{x\in X}(\delta(i,0)g_\pm(u,x) + (1-\delta(i,0))\lambda^{\pm 1}\delta(u,x))B_i \otimes A_x$$

and

$$V^\pm_{(i,u)} = \sum_{k\in Y}\sum_{x\in X} g^Y_\pm((k,x),(i,u))B_k \otimes A_x$$
$$= \sum_{k\in Y}\sum_{x\in X} \delta(k,i)(\delta(k,0)g_\pm(x,u) + (1-\delta(k,0))\lambda^{\pm 1}\delta(x,u))B_k \otimes A_x$$
$$= \sum_{x\in X}(\delta(i,0)g_\pm(x,u) + (1-\delta(i,0))\lambda^{\pm 1}\delta(x,u))B_i \otimes A_x.$$

Hence $H^\pm_{(0,u)} = I_Y \otimes H^\pm_u$, $V^\pm_{(0,u)} = I_Y \otimes V^\pm_u$ (where $I_Y = B_0$ is the identity of $\mathcal{B}$ for the ordinary matrix product), and for $i \neq 0$, $H^\pm_{(i,u)} = V^\pm_{(i,u)} = \lambda^{\pm 1} B_i \otimes A_u$. Let now Ψ_X be a duality of $\mathcal{A}$ and Ψ_Y be a duality of $\mathcal{B}$, so that $\Psi_Y \otimes \Psi_X$ is a duality Ψ of $\mathcal{B} \otimes \mathcal{A}$. Note that the transposition map of $\mathcal{B} \otimes \mathcal{A}$ is the identity. Applying Proposition 8 to the doubly translation invariant IRF model $\imath$, we obtain a translation invariant spin model

$$\zeta = ((Y\times X)x(Y\times X), W_1^Y, W_2^Y, W_3^Y, W_4^Y, \mu, |Y|x|X|) \quad \text{with} \quad Z^\zeta = Z^\imath.$$

The matrices W_i^Y, $i = 1, \dots, 4$, are the following elements of $\mathcal{B}\otimes\mathcal{A}\otimes\mathcal{B}\otimes\mathcal{A}$.

$$W_1^Y = \sum_{(i,u)\in Y\times X} B_i \otimes A_u \otimes \Psi\left(H^+_{(i,u)}\right)$$
$$= \sum_{u\in X} I_Y \otimes A_u \otimes \Psi\left(I_Y \otimes H_u^+\right) + \sum_{(i,u)\in Y\times X,\ i\neq 0} B_i \otimes A_u \otimes \Psi(\lambda B_i \otimes A_u)$$
$$= \sum_{u\in X} I_Y \otimes A_u \otimes J_Y \otimes \Psi_X\left(H_u^+\right)$$
$$+ \lambda \sum_{(i,u)\in Y\times X,\ i\neq 0} B_i \otimes A_u \otimes \Psi_Y(B_i) \otimes \Psi_X(A_u),$$

where J_Y is the identity for the Hadamard product in $\mathcal{B}$.

Similarly

$$W_2^Y = \sum_{u\in X} I_Y \otimes A_u \otimes J_Y \otimes \Psi_X(V_u^-)$$
$$+ \lambda^{-1} \sum_{(i,u)\in Y\times X,\ i\neq 0} B_i \otimes A_u \otimes \Psi_Y(B_i) \otimes \Psi_X(A_u),$$

$$W_3^Y = \sum_{u\in X} I_Y \otimes A_u \otimes J_Y \otimes \Psi_X(H_u^-) \\ + \lambda^{-1} \sum_{(i,u)\in Y\times X,\ i\neq 0} B_i \otimes A_u \otimes \Psi_Y(B_i) \otimes \Psi_X(A_u),$$

and

$$W_4^Y = \sum_{u\in X} I_Y \otimes A_u \otimes J_Y \otimes \Psi_X(V_u^+) \\ + \lambda \sum_{(i,u)\in Y\times X,\ i\neq 0} B_i \otimes A_u \otimes \Psi_Y(B_i) \otimes \Psi_X(A_u).$$

We now consider the images $W_i'^Y$, $i = 1, \dots, 4$, of these matrices under the isomorphism from $\mathcal{B}\otimes\mathcal{A}\otimes\mathcal{B}\otimes\mathcal{A}$ to $\mathcal{B}\otimes\mathcal{B}\otimes\mathcal{A}\otimes\mathcal{A}$ which exchanges the second and third factor of the tensor product. We denote by W_i, $i = 1, \dots, 4$, the matrices associated by Proposition 8 with the doubly translation IRF model corresponding to the mappings $g_\pm$. In other words,

$$W_1 = \sum_{u\in X} A_u \otimes \Psi_X(H_u^+), \qquad W_2 = \sum_{u\in X} A_u \otimes \Psi_X(V_u^-),$$
$$W_3 = \sum_{u\in X} A_u \otimes \Psi_X(H_u^-), \text{ and } W_4 = \sum_{u\in X} A_u \otimes \Psi_X(V_u^+).$$

Then

$$W_1'^Y = \sum_{u\in X} I_Y \otimes J_Y \otimes A_u \otimes \Psi_X(H_u^+) \\ + \lambda \sum_{(i,u)\in Y\times X,\ i\neq 0} B_i \otimes \Psi_Y(B_i) \otimes A_u \otimes \Psi_X(A_u).$$

Let

$$M_Y = \sum_{i\in Y} B_i \otimes \Psi_Y(B_i), \quad M_X = \sum_{u\in X} A_u \otimes \Psi_X(A_u). \tag{74}$$

Then

$$W_1'^Y = I_Y \otimes J_Y \otimes W_1 + \lambda(M_Y - (I_Y \otimes J_Y)) \otimes M_X \tag{75}$$

and similarly

$$W_2'^Y = I_Y \otimes J_Y \otimes W_2 + \lambda^{-1}(M_Y - (I_Y \otimes J_Y)) \otimes M_X, \tag{76}$$
$$W_3'^Y = I_Y \otimes J_Y \otimes W_3 + \lambda^{-1}(M_Y - (I_Y \otimes J_Y)) \otimes M_X, \tag{77}$$

and

$$W_4'^Y = I_Y \otimes J_Y \otimes W_4 + \lambda(M_Y - (I_Y \otimes J_Y)) \otimes M_X. \tag{78}$$

Thus using Proposition 8 and 10 we obtain the following result.

Proposition 11. *Let X, Y be elementary Abelian 2-groups and*

$$\zeta = (X \times X, W_1, W_2, W_3, W_4, \mu, |X|)$$

be a translation invariant spin model with associated link invariant f. Then, with the notations of (74) – (78),

$$\zeta' = \left(Y \times Y \times X \times X, W_1'^Y, W_2'^Y, W_3'^Y, W_4'^Y, \mu, |Y|x|X|\right)$$

is also a translation invariant spin model whose associated link invariant is the $\lambda\mu^{-1}$-composition of $|Y|$ link invariants f.

We first observe that

$$M_X \text{ and } M_Y \text{ are Hadamard matrices.} \tag{79}$$

Indeed let us consider for instance $M_X = \sum_{u \in X} A_u \otimes \Psi_X(A_u)$. By applying Ψ_X to the equation $A_u^2 = I_X$ we see that $\Psi_X(A_u)$ has entries ± 1. Then clearly the same holds for M_X and for $M'_X = \sum_{u \in X} \Psi_X(A_u) \otimes A_u$. Applying $\Psi_X \otimes \Psi_X$ to the equation $M'_X \circ M'_X = J_X \otimes J_X$ gives

$$|X|^{-2}((\Psi_X \otimes \Psi_X)(M'_X))^2 = |X|^2 I_X \otimes I_X.$$

Since $(\Psi_X \otimes \Psi_X)(M'_X) = |X| M_X$ and M_X is symmetric, (79) is proved.

Remark. It is easy to show that M_X and M_Y are equivalent (up to permutations of rows and columns) to Sylvester matrices.

Let us now take a closer look at the simplest case $Y = \mathbb{Z}/2\mathbb{Z}$.

Note first that by (74),

$$M_Y = B_0 \otimes \Psi_Y(B_0) + B_1 \otimes \Psi_Y(B_1) = I_Y \otimes J_Y + (J_Y - I_Y) \otimes (2I_Y - J_Y).$$

Then, with the notations of (74)-(78),

$$W_i'^Y = \begin{bmatrix} W_i & W_i & \lambda^\varepsilon H & -\lambda^\varepsilon H \\ W_i & W_i & -\lambda^\varepsilon H & \lambda^\varepsilon H \\ \lambda^\varepsilon H & -\lambda^\varepsilon H & W_i & W_i \\ -\lambda^\varepsilon H & \lambda^\varepsilon H & W_i & W_i \end{bmatrix} \tag{80}$$

for $i = 1, \dots, 4$, where $H = M_X$ is a Hadamard matrix and ε equals 1 if $i = 1, 4$ and -1 if $i = 2, 3$.

Related constructions are considered in [**N2**].

6.4. Nomura's Hadamard spin models and the Jones polynomial. A Hadamard graph is a distance-regular graph of diameter 4 on a set of $n = 16m$ vertices (m a positive integer) with intersection array $\{4m, 4m-1, 2m, 1; 1, 2m, 4m-1, 4m\}$ (see [**Bi1**], [**BI**], [**BCN**] for definitions). K. Nomura [**N1**] has recently associated with every Hadamard graph Γ on the vertex-set $V(\Gamma)$ of size $16m$ some spin models $(V(\Gamma), w_1, w_2, w_3, w_4, \mathbb{C}, t_0, 4\sqrt{m})$ which are defined as follows.

Let t_0, t_1, s be complex numbers satisfying $s^2 + 2(2m-1)s + 1 = 0$, $t_0^2 = \dfrac{2\sqrt{m}}{(4m-1)s+1}$, $t_1^4 = 1$. For x, y in $V(\Gamma)$, $w_1(x,y)$ equals t_0 (respectively $t_1, st_0, -t_1, t_0$) if the distance of x and y in Γ is 0 (respectively 1, 2, 3, 4), $w_2(x,y) = w_1(x,y)$ and $w_3(x,y) = w_4(x,y) = (w_1(x,y))^{-1}$.

Let α be a complex number such that $\alpha^2 + \alpha^{-2} + 2t_1^2\sqrt{m} = 0$. Then $s^2 + (\alpha^4 + \alpha^{-4})s + 1 = 0$, so that we can take without loss of generality $s = -\alpha^{-4}$. Now we find $t_0^2 = \alpha^6 t_1^2$ and we may choose without loss of generality $t_0 = -\alpha^3 t_1$.

We shall need the following result.

Proposition 12. *Let $\zeta = (X, w_1, w_2, w_3, w_4, \mathbb{C}, \mu, D)$ be a spin model and ω be a complex 4th root of unity. Then $\zeta' = (X, \omega w_1, \omega w_2, \omega^{-1} w_3, \omega^{-1} w_4, \mathbb{C}, \omega\mu, \omega^2 D)$ is also a spin model, and replacing ζ by ζ' amounts to multiply the corresponding link invariant by a factor $(\omega^2)^{c(L)}$, where $c(L)$ denotes the number of components of the link L.*

Proof. It is easy to check that ζ' satisfies the identities (11)–(16). By (17), the corresponding invariant of oriented links is defined (for connected diagrams L) by $(\omega\mu)^{-T(L)} Z^{\zeta'}(L) = \omega^{-T(L)-2|B(L)|+K(L)} \mu^{-T(L)} Z^{\zeta}(L)$, where $K(L) = v_1(L) + v_2(L) - v_3(L) - v_4(L)$ and, for $i = 1, \ldots, 4$, $v_i(L)$ is the number of vertices for which the evaluation of interaction weights given in Figure 3 makes use of the mapping w_i.

Note that $T(L) = v_1(L) - v_2(L) - v_3(L) + v_4(L)$. Hence replacing ζ by ζ' amounts to multiply the corresponding link invariant by a factor $(\omega^2)^{v_2(L)-v_4(L)-|B(L)|}$. Thus if we assign to every connected diagram L the value $(\omega^2)^{v_2(L)-v_4(L)-|B(L)|}$ this defines a unique multiplicative invariant of oriented links. Clearly this value is not modified if we change the spatial structure of any crossing. Performing such changes until we obtain a trivial link, we see that the value of this invariant is $(\omega^2)^{c(L)}$, where $c(L)$ is the number of components of the link represented by L. □

Using Proposition 12 we shall now restrict our attention to the spin model

$$\zeta(\Gamma, \alpha) = (V(\Gamma), t_1^{-1} w_1, t_1^{-1} w_2, t_1 w_3, t_1 w_4, \mathbb{C}, t_1^{-1} t_0, 4t_1^{-2}\sqrt{m}).$$

Note that $t_1^{-1}t_0 = -\alpha^3$ and $4t_1^{-2}\sqrt{m} = 2(-\alpha^2 - \alpha^{-2})$. Moreover $t_1^{-1}w_1(x,y)$ equals $-\alpha^3$ (respectively $1, \alpha^{-1}, -1, -\alpha^3$) if the distance of x and y in Γ is 0 (respectively 1, 2, 3, 4).

It is shown in [**Ja5**], Proposition 22, that $Z^{\varsigma(\Gamma,\alpha)}$ is not modified if the Hadamard graph Γ is replaced by another one with the same number of vertices. We shall need the following more precise result.

Proposition 13. *With every diagram L is associated a one variable rational function Q_L such that $Z^{\varsigma(\Gamma,\alpha)}(L) = Q_L(\alpha)$ for every Hadamard graph Γ on $16m = 4(-\alpha^2 - \alpha^{-2})^2$ vertices.*

Sketch of proof. Let us present briefly the "matrix–free" approach introduced in [**Ja5**] for the computation of $Z^{\varsigma(\Gamma,\alpha)}(L)$. Let $\mathcal{H}$ be the Bose–Mesner algebra of the Hadamard graph Γ, with basis of Hadamard idempotents $\{A_i,\ i \in \{0,\dots,4\}\}$, labeled so that for any two vertices x, y of Γ at distance $d(x,y)$, $A_i[x,y] = \delta(d(x,y), i)$. For every graph G and mapping $w : E(G) \to \mathcal{H}$, let

$$Z(G,w) = \sum_{\sigma:V(G)\to V(\Gamma)} \prod_{e\in E(G)} w(e)[\sigma(i(e)), \sigma(t(e))]. \tag{81}$$

We consider a connected diagram L and the connected plane graph $G(L)$ defined in Section 5.2 (but orientations of edges are not significant). We write

$$Z^{\varsigma(\Gamma,\alpha)}(L) = (2(-\alpha^2 - \alpha^{-2}))^{-|B(L)|} Z(G(L), w_L), \tag{82}$$

where w_L is a mapping from $E(G(L))$ to $\mathcal{H}$ which can take only two values

$$\begin{aligned} W_+ &= -\alpha^3 A_0 + A_1 + \alpha^{-1} A_2 - A_3 - \alpha^3 A_4, \\ W_- &= -\alpha^{-3} A_0 + A_1 + \alpha A_2 - A_3 - \alpha^{-3} A_4. \end{aligned}$$

The map $Z_G : w \to Z(G,w)$ given by (81) is multilinear in the components $w(e)$ of w. This leads us to introduce a vector space $\mathcal{H}_G$ which is a tensor product of copies of $\mathcal{H}$, one copy for each edge of G. Then each mapping w from $E(G) = \{e_1,\dots,e_n\}$ to $\mathcal{H}$ is represented by the element $w(e_1) \otimes \cdots \otimes w(e_n)$ of $\mathcal{H}_G$, and Z_G is identified with a linear form on $\mathcal{H}_G$. The vector space $\mathcal{H}_G$ has a natural basis $\{A_{i_1} \otimes \cdots \otimes A_{i_n} / i_1,\dots,i_n \in \{0,\dots,4\}\}$, and the coordinates of w_L with respect to this basis are clearly powers of α up to sign. Thus it will be enough to show that the values of the linear form $Z_{G(L)}$ on elements of the same basis are given by rational functions of α.

It is known that every connected plane graph can be reduced to the trivial graph with one vertex and no edge by a finite number of elementary local transformations of the following kind: deletion of a loop, contraction of a pendant edge, deletion of an edge parallel to another edge, contraction of an edge in series with another edge, and *star–triangle transformations*, that is, replacement of a triangle by a "star" (one vertex incident with three edges) or replacement of a star by a triangle. It is shown in [**Ja5**] that when two graphs G, G' are related by such an elementary transformation, the corresponding linear forms $Z_G, Z_{G'}$ are also related in a simple way. For instance when e_1, e_2 are two parallel edges in G, we may compute $Z(G,w)$ by first deleting e_1, thus obtaining the graph G', and then replacing $w(e_2)$ by the Hadamard product $w(e_1) \circ w(e_2)$, thus obtaining the mapping w'. The equality of $Z(G,w)$ and $Z(G',w')$ for arbitrary w is conveniently expressed by the equation $Z_G = Z_{G'}(\mu^* \otimes Id)$, where the map $\mu^* \otimes Id$ from $\mathcal{H}_G$ to $\mathcal{H}_{G'}$ acts as the Hadamard product $\mu^* : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ on the factors of $\mathcal{H}_G$ corresponding to e_1, e_2, and acts as the identity on the other factors. The fact that a similar procedure also works for star–triangle transformations is a special property of $\mathcal{H}$ which is established in [**Ja5**] using some results of [**N1**].

In this way we obtain (see [**Ja5**], Proposition 6) that for every connected plane graph G, the linear form Z_G on $\mathcal{H}_G$ is a composition $\rho_0\rho_1 \dots \rho_k$, where ρ_0 is scalar multiplication by $|V(\Gamma)| = 16m$, and each of $\rho_1, \dots, \rho_k$ corresponds to the action of one of the maps $\theta, \theta^*, \mu, \mu^*, \kappa, \kappa^*$ on some factors of a tensor product of copies of $\mathcal{H}$. Here θ, θ^* are linear forms which give the (constant) diagonal element and the (constant) row sum of a matrix in $\mathcal{H}$, μ, μ^* are linear maps from $\mathcal{H} \otimes \mathcal{H}$ to $\mathcal{H}$ which correspond to the ordinary matrix product and Hadamard product, and κ, κ^* are certain linear maps from $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$ to itself associated with star–triangle transformations. It is easily checked (see for instance [**N1**]) that the matrices of $\theta, \theta^*, \mu, \mu^*$ with respect to the bases $\{1\}$, $\{A_i,\, i \in \{0, \dots, 4\}\}$, $\{A_i \otimes A_j,\, i,j \in \{0, \dots, 4\}\}$ of $\mathbb{C}$, $\mathcal{H}$, $\mathcal{H} \otimes \mathcal{H}$, have entries given by polynomials in m (these polynomials are of degree 0 for θ, μ^* and of degree 1 for θ^*, μ). The matrix of the map κ, as defined in equation (47) of [**Ja5**], with respect to the basis $\{A_i \otimes A_j \otimes A_k,\, i,j,k \in \{0, \dots, 4\}\}$ of $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$, has non-zero entries of the form $K(ijk/uvw)$, or $P_{ijk}(u,v,w)$ in the notations of [**N1**], where these parameters are expressed as polynomials of degree 1 in m. Finally, to deal with the map κ^* defined in equation (53) of [**Ja5**], we shall show that $\kappa^* = (16m)^{-4}(\Psi \otimes \Psi \otimes \Psi)\kappa(\Psi \otimes \Psi \otimes \Psi)$,where Ψ is some duality map on $\mathcal{H}$. This follows from Proposition 18 of [**Ja5**] and the fact that $\mathcal{H}$ satisfies the *planar duality property*, which means that Proposition 5 of the present paper holds with $\mathcal{A}$ replaced by $\mathcal{H}$. The simplest way to establish this last fact when $m > 1$ is to use Proposition 12 of [**Ja5**]. Indeed in this case it is easy to see that the coefficients of W_+ with respect to A_1, A_2, A_3, A_4 are all distinct and the spin model defined by W_+, W_- strongly generates $\mathcal{H}$. On the other hand, when $m = 1$, the Hadamard graph Γ is isomorphic to the 4-cube and we may apply Proposition 11 of [**Ja5**] or equivalently Proposition 5 of the present paper. It is easy to check (see [**Ja5**], Section 7.3) that the matrix of Ψ in the basis $\{A_i,\ i \in \{0, \dots, 4\}\}$ of $\mathcal{H}$ has entries given by polynomials of degree at most 2 in the variable $\sqrt{m}$. Hence the entries of the matrix of κ^* with respect to the basis $\{A_i \otimes A_j \otimes A_k,\ i, j, k \in \{0, \dots, 4\}\}$ of $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$ are also given by rational functions of α. □

We are now ready to prove

Proposition 14. *For every Hadamard graph Γ on $16m = 4(-\alpha^2 - \alpha^{-2})^2$ vertices, the link invariant associated with $\zeta(\Gamma, \alpha)$ is the $(-\alpha^{-3})$-composition of two Jones polynomials evaluated at $t = \alpha^4$.*

Proof. Let X be an elementary Abelian 2-group, and let $\zeta = (X \times X, W_1, W_2, W_3, W_4, \mu, |X|)$ be the spin model for the Jones polynomial described in Section 5.6. Thus $|X| = -\alpha^2 - \alpha^{-2}$, $\mu = -\alpha^3$,

$$W_1 = W_2 = -\alpha^3 I_X \otimes I_X + \alpha^{-1}(J_X \otimes J_X - I_X \otimes I_X),$$
$$W_3 = W_4 = -\alpha^{-3} I_X \otimes I_X + \alpha(J_X \otimes J_X - I_X \otimes I_X),$$

and the associated link invariant is the Jones polynomial evaluated at $t = \alpha^4$.

It is easy to see that when the Hadamard graph Γ comes from the Hadamard matrix $H = M_X$ as explained in Theorem 1.8.1 of [**BCN**], using the above matrices $W_i (i = 1, \dots, 4)$ and $\lambda = 1$ in the matrices (80), the spin model

$$\zeta' = \Big((\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \times X \times X, W_1'^Y, W_2'^Y, W_3'^Y, W_4'^Y, -\alpha^3, 2|X|\Big)$$

of Proposition 11 can be identified with $\zeta(\Gamma, \alpha)$. Hence in this case Proposition 14 follows from Proposition 11. In view of Proposition 13 this implies that for any link represented by a diagram L and for every number α such that $-\alpha^2 - \alpha^{-2}$ is a power of 2, the $(-\alpha^{-3})$-composition of two Jones polynomials evaluated at $t = \alpha^4$ equals $(-\alpha^3)^{-T(L)} Q_L(\alpha)$. The equality of the corresponding rational functions follows. □

7. Conclusion.

The classification problem for spin models seems to be hopelessly difficult in general (see [**BJS**], [**N3**] for some recent contributions). Even for translation invariant spin models, the problem is solved only for a restricted class of spin models in the sense of [**KMW**] satisfying a so-called *modular invariance property* [**BBJ**]. We have proposed new operations on the class of translation invariant spin models: dualization (Proposition 6) and composition (Proposition 11). These operations should be taken into account as well as the tensor product construction of [**H2**] and the twisted extension construction of [**N2**] in the study of the classification problem. They should also provide new examples of four-weight spin models in the sense of [**BB2**].

For groups of the form $X \times X$, we have shown that translation invariant spin models are essentially equivalent to doubly translation invariant IRF models on X, or to strongly conservative vertex models on X (Propositions 8, 2). As shown in Proposition 14 this can lead to a better understanding of the corresponding link invariants. Moreover one may hope that this could establish some relations between the study of spin models and the theory of quantum groups since these algebraic structures are closely connected with vertex models.

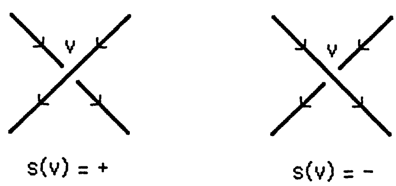

Figure 1.

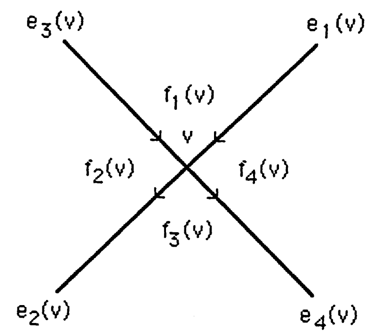

Figure 2.

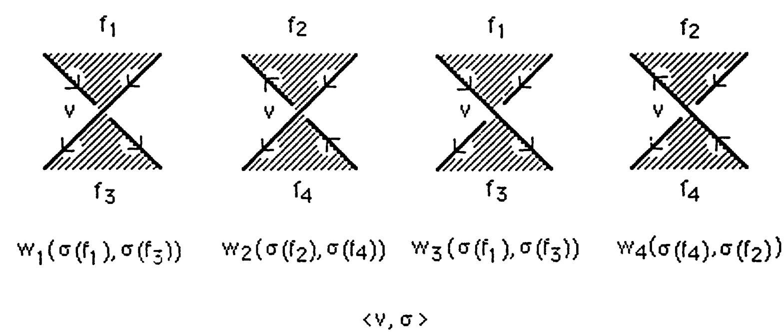

Figure 3.

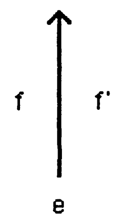

Figure 4.

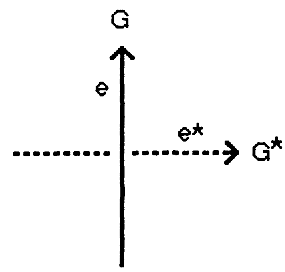

Figure 7.

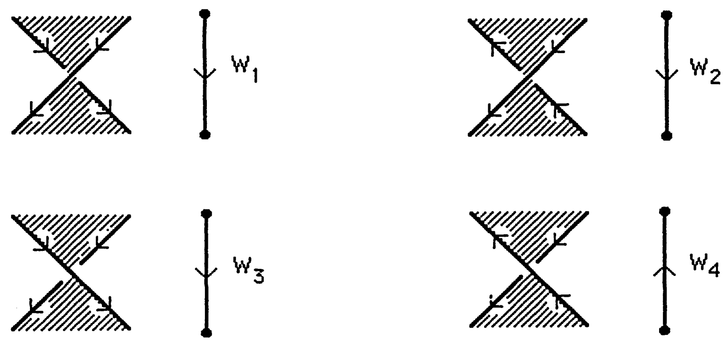

Figure 5.

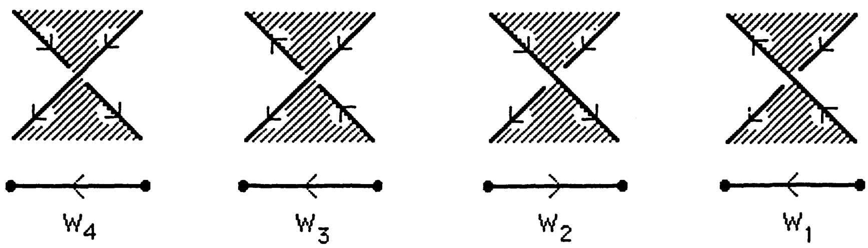

Figure 6.

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