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ON CONSTRAINED EXTREMA

THOMAS I. VOGEL

Assume that I and J are smooth functionals defined on a Hilbert space H. We derive sufficient conditions for I to have a local minimum at y subject to the constraint that J is constantly $J(y)$.

The first order necessary condition for I to have a constrained minimum at y is that for some constant λ, $I'_y + \lambda J'_y$ is identically zero. Here I'_y and J'_y are the Fréchet derivatives of I and J at y. For the rest of the paper, we assume that y in H satisfies this necessary condition.

A common misapprehension (upon which much of the stability results for capillary surfaces has been based) is to assume that if the quadratic form $I''_y + \lambda J''_y$ is positive definite on the kernel of J'_y then I has a local constrained minimum at y. This is not correct in a Hilbert space of infinite dimension; Finn [**1**] has supplied a counterexample in the unconstrained case, and the same difficulty will occur in the constrained case. In the unconstrained case, if (as often occurs in practice) the spectrum of I''_y is discrete and 0 is not a cluster point of the spectrum, then I''_y positive definite at a critical point y implies that I''_y is strongly positive, (i.e., there exists $k > 0$ such that $I''_y(x) \geq k\|x\|^2$ holds for all x), and this in turn *does* imply that y is a local minimum (see [**2**]). However, in the constrained case, things are not so easy. Even if $I''_y + \lambda J''_y$ has a nice spectrum (in some sense), it is not clear that $I''_y + \lambda J''_y$ being positive definite on the kernel of J'_y implies that this quadratic form is strongly positive on the kernel, nor that strong positivity implies that y is a local minimum.

In [**3**], Maddocks obtained sufficient conditions for $I''_y + \lambda J''_y$ to be positive definite on the kernel of J'_y. As Maddocks points out, this is not quite enough to say that I has a constrained minimum at y. Remarkably, essentially the same conditions as Maddocks obtained for positive definiteness do in fact imply that I has a strict local minimum at y subject to the constraint $J = J(y)$, as we shall see.

For any $h \in H$ we may say $J(y+h) - J(y) = J'_y(h) + \frac{1}{2}J''_y(h) + \epsilon_1(h)\|h\|^2$, where ϵ_1 goes to zero as $\|h\|$ goes to zero. If we consider an h for which $J(y+h) = J(y)$, then of course $0 = J'_y(h) + \frac{1}{2}J''_y(h) + \epsilon_1(h)\|h\|^2$. Now, for that h we have

$$
\begin{aligned}
\Delta I = I(y+h) - I(y) &= I'_y(h) + \frac{1}{2} I''_y(h) + \epsilon_2 \|h\|^2 \\
&= -\lambda J'_y(h) + \frac{1}{2} I''_y(h) + \epsilon_2 \|h\|^2 \qquad (1) \\
&= \frac{1}{2}(I''_y + \lambda J''_y)(h) + (\lambda\epsilon_1 + \epsilon_2)\|h\|^2.
\end{aligned}
$$

Since $I''_y + \lambda J''_y$ is a bilinear form, there is a linear operator A defined on H so that $(I''_y + \lambda J''_y)(u,v) = \langle u, Av\rangle$. Similarly there is some element of H, call it ∇J, so that J'_y applied to a vector h is $\langle h, \nabla J\rangle$. Let $\sigma(A)$ be the spectrum of A. There are three cases which often arise in practice:

Theorem 1. *If $\sigma(A) \cap (-\infty, c] = \emptyset$ for some $c > 0$, then I has a constrained minimum at y.*

Proof. From (1) we may write ΔI as $\langle h, Ah\rangle + (\lambda\epsilon_1 + \epsilon_2)\|h\|^2$. But $\langle h, Ah\rangle \geq c\|h\|^2$ (this is easily verified using the spectral theorem, see [**5**]), so for h sufficiently small, ΔI is positive. □

Theorem 2. *Suppose that $\sigma(A) \cap (-\infty, \epsilon]$ consists of a single negative eigenvalue λ_0 for some $\epsilon > 0$. Let ζ solve $A\zeta = \nabla J$. (A will be invertible.) I has a constrained minimum at y if $J'_y(\zeta) = \langle \zeta, A\zeta\rangle < 0$, and I does not have a constrained minimum at y if $J'_y(\zeta) = \langle \zeta, A\zeta\rangle > 0$.*

The proof of Theorem 2 will proceed in a series of steps.

Step 1. Assume that $\langle \zeta, A\zeta\rangle < 0$. Then $I''_y + \lambda J''_y$ is strongly positive on the kernel of J'_y.

Proof. Take x in the kernel of J'_y. As in [**4**], x may be written as $v + \alpha\zeta$, where v is perpendicular to φ_0, the eigenfunction corresponding to λ_0. (The key to this calculation is that $\langle \zeta, \varphi_0\rangle \neq 0$. But if ζ is orthogonal to φ_0, it can be shown that $\langle \zeta, A\zeta\rangle > 0$.) One can verify that $\langle x, Ax\rangle = \langle v, Av\rangle - \alpha^2\langle \zeta, A\zeta\rangle$, so that $\langle x, Ax\rangle \geq \langle v, Av\rangle$.

Let $\{E_\lambda\}$ be the spectral family associated with A, so that $A = \int_{-\infty}^{\infty} \lambda dE_\lambda$. By our assumption on $\sigma(A)$, $A = \lambda_0 E_{\lambda_0} + \int_\epsilon^\infty \lambda dE_\lambda$, where E_{λ_0} is orthogonal projection onto φ_0. Therefore,

$$\langle v, Av\rangle = \langle v, \lambda_0 E_{\lambda_0}(v)\rangle + \int_\epsilon^\infty \lambda d\|E_\lambda v\|^2.$$

The first term vanishes, so that

$$\langle v, Av\rangle \geq \epsilon \int_\epsilon^\infty d\|E_\lambda v\|^2 \geq \epsilon \int_{-\infty}^\infty d\|E_\lambda v\|^2 \geq \epsilon\|v\|^2.$$

Therefore, $\langle x, Ax \rangle \geq \epsilon \|v\|^2$.

To conclude the proof that $I''_y + \lambda J''_y$ is strongly positive on the kernel of J'_y, we need to show that $\|v\| \geq k\|x\|$ for some fixed positive constant k. Assume without loss of generality that $\|x\| = 1$. For any fixed x, $\|v\|$ is greater than or equal to the distance from x to the line $\{c\zeta : c \in \mathbf{R}\}$. Consider the projection of x onto ζ. Its length is $|\langle x, \zeta/\|\zeta\|\rangle|$. We may write ζ as $\beta\nabla J + \hat{\zeta}$, where $\hat{\zeta}$ is perpendicular to ∇J. We cannot have β equaling 0, since by assumption, $\langle \zeta, A\zeta \rangle = \langle \zeta, \nabla J \rangle < 0$.

Then the projection has length at most $\|x\|\|\hat{\zeta}\|/\|\zeta\|$. But $\|\hat{\zeta}\| < \|\zeta\|$ (since $\beta \neq 0$). Letting γ equal $\|\hat{\zeta}\|/\|\zeta\|$, we have $\gamma < 1$ and the length of the vector component of x perpendicular to ζ is greater than or equal to $\sqrt{1-\gamma^2}$. But $\|v\|$ is greater than or equal to the length of that component, so we get our k to be $\sqrt{1-\gamma^2}$, concluding step 1.

Step 2. If $\langle \zeta, A\zeta \rangle < 0$, then I has a minimum at y subject to the constraint $J = J(y)$.

Proof. Take an h for which $J(y+h) = J(y)$. Now h need not be in the kernel of J'_y, but we may write h as $h_1 + \alpha\zeta$, where h_1 is in the kernel of J'_y, by taking α to be $\langle h, \nabla J \rangle / \langle \zeta, \nabla J \rangle$. (Note that $\langle \zeta, \nabla J \rangle = \langle \zeta, A\zeta \rangle \neq 0$.) Substituting into equation (1),

$$\Delta I = \frac{1}{2}\langle h_1, Ah_1 \rangle + \alpha \langle h_1, A\zeta \rangle + \frac{1}{2}\alpha^2 \langle \zeta, A\zeta \rangle + (\lambda\epsilon_1 + \epsilon_2)\|h\|^2. \tag{2}$$

However, $\langle h_1, A\zeta \rangle = \langle h_1, \nabla J \rangle = 0$, causing this term to vanish. We have $0 = \Delta J = J'_y(h) + \epsilon_3\|h\|$, where ϵ_3 tends to 0 as $\|h\|$ tends to 0. Thus $\alpha^2 = \epsilon_3^2\|h\|^2$, and we conclude that

$$\Delta I = \frac{1}{2}\langle h_1, Ah_1 \rangle + \epsilon\|h\|^2$$

where ϵ tends to zero as $\|h\|$ tends to 0. From Step 1, A is strongly positive on the kernel of J'_y, so

$$\Delta I \geq \frac{k}{2}\|h_1\|^2 + \epsilon\|h\|^2.$$

Since $h = h_1 + \alpha\zeta$, with $\alpha = -\epsilon_3\|h\|$, it is easy to see that for $\|h\|$ sufficiently small there holds $\|h_1\| \geq \frac{1}{2}\|h\|$. Thus

$$\Delta I \geq \|h\| \left(\frac{k}{8} + \epsilon\right)$$

which must be greater than 0 for $\|h\|$ sufficiently small. Therefore I has a minimum at y subject to the constraint $J = J(y)$, concluding the proof of step 2 and the first half of Theorem 2.

Step 3. Suppose that $\langle \zeta, A\zeta \rangle > 0$. Then I does not have a minimum at y subject to the constraint $J = J(y)$.

Proof. First, $I''_y + \lambda J''_y$ is no longer positive definite on the kernel of J'. Indeed, $\eta = \varphi_0 + c\zeta$ is in the kernel of J'_y if $c = -\frac{\langle \varphi_0, \nabla J \rangle}{\langle \zeta, \nabla J \rangle} = -\frac{\langle \varphi_0, \nabla J \rangle}{\langle \zeta, A\zeta \rangle}$, but one can verify that $\langle \eta, A\eta \rangle < 0$.

Now consider $f(r,s) = J(y + r\eta + s\nabla J) - J(y)$, a differentiable function of r and s. Then $\nabla f(0,0) = (0, \|\nabla J\|^2)$, so the zero set of f is tangent to the r axis at the origin. From this we conclude that there is a function $s(r)$ so that $J(y + r\eta + s(r)\nabla J) - J(y) = 0$, with $\lim_{r\to 0} \frac{s(r)}{r} = 0$. From equation (1), for $h = r\eta + s(r)\nabla J$ we have

$$\Delta I = (I'' + \lambda J'')(r\eta + s(r)\nabla J) + (\lambda\epsilon_1 + \epsilon_2)\|r\eta + s(r)\nabla J\|^2$$

so that $\Delta I = r^2 \langle \eta, A\eta \rangle + o(r^2)$. Thus, for all r sufficiently small $\Delta I < 0$, indicating that we do not have a constrained minimum, concluding the proof of Theorem 2. $\square$

Theorem 3. *If $\sigma(A) \cap (-\infty, 0)$ consists of more than one point, I does not have a constrained minimum at y.*

Proof. Suppose that ν and μ are in $\sigma(A) \cap (-\infty, 0)$, with $\nu < \mu$. Let E_λ be the spectral decomposition of A, so that E_λ is not constant in any neighborhood of ν nor in any neighborhood containing μ. Take an $\epsilon > 0$ so that the two ϵ neighborhoods around ν and μ are disjoint and contained in $(-\infty, 0)$. Then $E_{\nu+\epsilon} - E_{\nu-\epsilon}$ is nonzero, i.e., is a nontrivial projection. Therefore there is some $\varphi_0 \neq 0$ so that $(E_{\nu+\epsilon} - E_{\nu-\epsilon})\,\varphi_0 = \varphi_0$. I claim that $\langle \varphi_0, A\varphi_0 \rangle < 0$.

Indeed, $\langle \varphi_0, A\varphi_0 \rangle = \langle \varphi_0, \int_{-\infty}^{\infty} \lambda dE_\lambda(\varphi_0) \rangle$, which is $\int_{-\infty}^{\infty} \lambda d\langle E_\lambda(\varphi_0), \varphi_0 \rangle$, where the latter just a Stieljes integral. But beyond $\nu + \epsilon$, $E_\lambda(\varphi_0) = \varphi_0$, so we only get a negative contribution. It is certainly strictly negative, since for $\lambda < \nu - \epsilon$, $E_\lambda(\varphi_0) = 0$.

Now find a φ_1 for μ in the same fashion. We need to show that $\langle \varphi_0, A\varphi_1 \rangle = 0$. But $\langle \varphi_0, A\varphi_1 \rangle = \int_{-\infty}^{\infty} \lambda d\langle \varphi_0, E_\lambda \varphi_1 \rangle$, and it is routine to show that $\langle \varphi_0, E_\lambda \varphi_1 \rangle = 0$ for all λ.

We may take c_0 and c_1, not both zero, so that $c_0\varphi_0 + c_1\varphi_1$ is perpendicular to ∇J. Then $\langle c_0\varphi_0 + c_1\varphi_1, Ac_0\varphi_0 + Ac_1\varphi_1 \rangle = c_0^2 \langle \varphi_0, A\varphi_0 \rangle + c_1^2 \langle \varphi_1, A\varphi_1 \rangle < 0$. The proof now proceeds as in Step 3 of Theorem 2. $\square$

Note. It often occurs in practice that the spectrum of A is discrete and may be written as $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$, with 0 not a cluster point of $\sigma(A)$. In this special case, the parts of the hypotheses of the above theorems which relate to $\sigma(A)$ are as follows. In Theorem 1 we require that $0 < \lambda_0$, in Theorem 2 we require that $\lambda_0 < 0 < \lambda_1$ (in addition to the hypotheses on ζ), and in Theorem 3 we require that $\lambda_0 < \lambda_1 < 0$.

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Texas A & M University
College Station, TX 77840
E-mail address: tvogel@math.tamu.edu