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LOCAL DIFFERENTIAL GEOMETRY OF CUSPIDAL EDGE AND SWALLOWTAIL

Toshizumi FUKUI

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Abstract

We investigate the local differential geometric invariants of cuspidal edge and swallowtail from the view point of singularity theory. We introduce finite type invariants of such singularities (see Remark 1.5 and Theorem 2.11) based on certain normal forms for cuspidal edge and swallowtail. Then we discuss several geometric aspects based on our normal form. We also present several asymptotic formulas concerning our invariants with respect to Gauss curvature and mean curvature.

Typical examples of wave fronts are parallel surfaces of a regular surface in the 3-dimensional Euclidean space, and it is well-known that such surfaces may have several singularities like cuspidal edge and swallowtail. Singularity types of parallel surfaces are investigated in [3], and the next interest is to investigate local differential geometries of such singularities. There are several attempts to describe them. For instance, K. Saji, M. Umehara, and K. Yamada ([12]) defined the notion of singular curvature κ_s and normal curvature κ_ν of cuspidal edge, and, later, K. Saji and L. Martins ([7]) described all invariants up to order 3. It is clear that there are more differential geometric invariants in higher order terms, and to describe all such invariants up to finite order is one motivation of the paper.

Since Gauss curvature and mean curvature are often diverge at singularities and we are interested in their asymptotic behaviors near a singularity in terms of our invariants. We are going to describe their asymptotic behaviors of our local differential geometric invariants of cuspidal edge near swallowtail.

An ideas of singularity theory is to reduce a given map-germ $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ to certain normal form (see [9], for example). Their normal forms are obtained up to $\mathcal{A}$-equivalence where $\mathcal{A}$ is the group of coordinate changes of the source and the target. In that context, we reduce a given map-germ to one of normal forms in the list there, composing certain coordinate changes of the source and the target. For differential geometric purpose, general coordinate changes of the target are too rough, since they do not preserve differential geometric properties, and we should restrict the coordinate change of the target to the motion group. From this point, we will consider the product group of coordinate change of the source with the motion group of the target (the rotation group when we consider map-germs) and we introduce a normal form for cuspidal edge (see (1.1)) and swallowtail (Theorem 2.4) by the equivalence relation defined by this group. We believe that this is a powerful method to investigate singular surfaces, since this unable us to describe all differential geometric

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Dedicated to Professor Takashi Nishimura on the occasion of his 60th birthday.

properties in terms of them. The purpose of the paper is to investigate them in a reasonably complete form for cuspidal edge and swallowtail.

The paper is organized as follows. In §1, we investigate cuspidal edge as moving cusps with introducing a normal form (1.1) with conditions (i), (ii), (iii) there. We describe the first fundamental form and the second fundamental form, and conclude an asymptotic formula (Theorem 1.9) of Gauss curvature, the mean curvature and thus the principal curvatures. We also investigate the singularity of asymptotic lines at a non parabolic point (subsection 1.5) and curvature lines (subsection 1.6) in a generic context. In §2, we investigate swallowtail with introducing a normal form (Theorem 2.4). We describe the first fundamental form and the second fundamental form in terms of this normal form, and conclude an asymptotic formula (Theorem 2.20) of Gauss curvature, the mean curvature and the principal curvatures. We also investigate the singularity of asymptotic lines (subsection 2.4) and curvature lines (subsection 2.5) in a generic context. Asymptotic behaviors of several invariants of cuspidal edge nearby swallowtail is also investigated in subsection 2.7. In Appendix A, we quickly review several basic notions of a surface in the 3-dimensional Euclidean space for convenience of reference. In Appendix B, we review criteria of singularity types.

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Contents

Throughout the paper, we use the following notation

$$\boldsymbol{e}_1 = (1,0,0),\ \boldsymbol{e}_2 = (0,1,0),\ \boldsymbol{e}_3 = (0,0,1),$$

which form a basis of the 3-dimensional Euclidean space $\mathbb{R}^3$. We sometimes (in §2) express elements in $\mathbb{R}^3$ using column vectors to shorten the expressions.

By custom, one writes $f(u,v) = O(g(u,v))$, if and only if there exist positive numbers δ and M such that $|f(u,v)| \leq M|g(u,v)|$ when $|(u,v)| < \delta$. For shortness, one also writes $f(u,v) = O(p)$ when $f(u,v) = O(|(u,v)|^p)$.

1. Cuspidal edge

1.1. Cuspidal edge as moving cusps. Let $\gamma : (\mathbb{R},0) \to (\mathbb{R}^3,0)$, $s \mapsto \gamma(s)$, be a regular curve with arc length parameter s. Let $\boldsymbol{t}$, $\boldsymbol{n}$, $\boldsymbol{b}$ denote its Frenet-Serre frame. We consider a map-germ $f : (\mathbb{R}^2,0) \to (\mathbb{R}^3,0)$ as a singular surface with the following conditions: There is a sequence $\{\boldsymbol{f}_k : (\mathbb{R},0) \to (\mathbb{R}^3,0), s \mapsto \boldsymbol{f}_k(s)\}_{k=1,2,\ldots}$ of C^∞-maps so that

(o) for any positive integer m we have

$$f(s,t) = \gamma(s) + \sum_{k=1}^{m} \boldsymbol{f}_k(s)\frac{t^k}{k!} + O(t^{m+1}), \tag{1.1}$$

(i) the singular set $\Sigma(f) = \{t = 0\}$.
(ii) $\langle \boldsymbol{f}_k(s), \boldsymbol{t}(s)\rangle = 0$ for $k = 1,2,\ldots$, and
(iii) $t^2/2$ is an arc length parameter of the section of the plane spanned by $\boldsymbol{n}$ and $\boldsymbol{b}$, that is, $\langle f_t(s,t), f_t(s,t)\rangle = t^2$.

Remark that $\boldsymbol{f}_k(s) = \frac{\partial^k f}{\partial t^k}|_{t=0}$, $\langle f_s, f_s\rangle|_{t=0} = 1$, $\langle f_s(s,0), f_t(s,t)\rangle = 0$, and $\langle f_t, f_t\rangle = t^2$. The condition (ii) implies that t is a parameter of the singular curves which are sections of the surface with the planes spanned by $\boldsymbol{n}(s)$ and $\boldsymbol{b}(s)$. If these curves are of multiplicity 2, we can take parameter t with the condition (iii). We remark that

$$f_s = \boldsymbol{t} + \sum_{k=1}^{m-1} \boldsymbol{f}'_k\frac{t^k}{k!} + O(t^m), \qquad f_t = \sum_{k=0}^{m-1} \boldsymbol{f}_{k+1}\frac{t^k}{k!} + O(t^m),$$

and $f_s|_{t=0} = \boldsymbol{t}$, $f_t|_{t=0} = \boldsymbol{f}_1(s)$. By the condition (iii), we have $\langle f_t(s,0), f_t(s,0)\rangle = 0$, and conclude that $\boldsymbol{f}_1(s) = 0$. We remark that $\eta = \partial_t$ represents a null vector on $\Sigma(f)$, i.e., $df(\eta) = 0$ on $\Sigma(f)$.

Throughout this section we consider the map $f : (\mathbb{R}^2,0) \to (\mathbb{R}^3,0)$ with the properties above.

We here recall the notion of multiplicities of curves γ ([4]). We say that $\gamma : (\mathbb{R},0) \to (\mathbb{R}^3,0)$ is of **multiplicity** m **at** $t = 0$ if there is a C^∞-map $\tilde{\gamma} : (\mathbb{R},0) \to \mathbb{R}^3$ with the following property:

$$\gamma(t) = \frac{t^m}{m}\tilde{\gamma}(t), \qquad \tilde{\gamma}(0) \neq 0.$$

Remark 1.1. A typical singularity of a map with the conditions above is cuspidal edge, a map $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ which is $\mathcal{A}$-equivalent to the map represented by

$$(u, v) \mapsto (u, v^2, v^3). \tag{1.2}$$

Another example is cuspidal crosscap, a map which is $\mathcal{A}$-equivalent to the map represented by

$$(u, v) \mapsto (u, v^2, uv^3). \tag{1.3}$$

Remark 1.2. S. Shiba and M. Umehara ([14]) has analyzed $(2, 3)$ cusp ($3/2$-cusp, in their terminology) in the plane $\mathbb{R}^2$ using the square root of an arc length parameter as a parameter (they call it the half-arclength parameter). For a curve with multiplicity 2 in $\mathbb{R}^n$, there exists a parameter t so that $t^2/2$ is an arc length parameter ([4, Theorem 1.1]).

When the curvature κ of γ is not zero, we have the following Frenet-Serret formula for γ:

$$\begin{pmatrix} \boldsymbol{t}' \\ \boldsymbol{n}' \\ \boldsymbol{b}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{pmatrix} \tag{1.4}$$

where $'$ denote derivative by the arc length parameter s. Let us define θ $(0 \leq \theta \leq \pi)$ and b_k by

$$\cos\theta = |\boldsymbol{t}\ \boldsymbol{f}_2\ \boldsymbol{b}|, \quad b_k = |\boldsymbol{t}\ \boldsymbol{f}_2\ \boldsymbol{f}_k|. \tag{1.5}$$

We use the orthnormal frame defined by $\boldsymbol{a}_1 = \boldsymbol{t}$, $\boldsymbol{a}_2 = \boldsymbol{f}_2$, and $\boldsymbol{a}_3 = \boldsymbol{t} \times \boldsymbol{f}_2$. When we write $\boldsymbol{a}_2 = \cos\theta\,\boldsymbol{n} - \sin\theta\,\boldsymbol{b}$, and $\boldsymbol{a}_3 = \boldsymbol{\gamma}' \times \boldsymbol{f}_2 = \sin\theta\,\boldsymbol{n} + \cos\theta\,\boldsymbol{b}$, we have $\cos\theta = |\boldsymbol{t}\ \boldsymbol{f}_2\ \boldsymbol{b}| = \langle \boldsymbol{a}_3, \boldsymbol{b}\rangle$, and thus

$$\boldsymbol{n}(s) = \cos\theta \boldsymbol{a}_2(s) + \sin\theta \boldsymbol{a}_3(s), \quad \boldsymbol{b}(s) = -\sin\theta \boldsymbol{a}_2(s) + \cos\theta \boldsymbol{a}_3(s), \tag{1.6}$$

and

$$\cos\theta = \langle \boldsymbol{a}_3, \boldsymbol{b}\rangle = \left.\frac{\langle f_s \times f_t, f_s \times f_{ss}\rangle}{|f_s \times f_t||f_s \times f_{tt}|}\right|_{t=0} = \left.\frac{\langle f_s, f_s\rangle\langle f_t, f_{ss}\rangle - \langle f_s, f_t\rangle\langle f_s, f_{ss}\rangle}{|f_s \times f_t||f_s \times f_{ss}|}\right|_{t=0}.$$

Lemma 1.3. *Assume that $\kappa \neq 0$. We have*

$$\begin{pmatrix} \boldsymbol{a}_1' \\ \boldsymbol{a}_2' \\ \boldsymbol{a}_3' \end{pmatrix} = \begin{pmatrix} 0 & \kappa\cos\theta & \kappa\sin\theta \\ -\kappa\cos\theta & 0 & \tau - \theta' \\ -\kappa\sin\theta & \theta' - \tau & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix}.$$

Proof. Since $\begin{pmatrix} \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{b} \end{pmatrix}$, we have

$$\begin{pmatrix} \boldsymbol{a}_2' \\ \boldsymbol{a}_3' \end{pmatrix} = \theta' \begin{pmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix} \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{b} \end{pmatrix} + \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \boldsymbol{n}' \\ \boldsymbol{b}' \end{pmatrix}$$

$$
\begin{aligned}
&= -\theta' \begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix} \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{b} \end{pmatrix} + \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{pmatrix} \\
&= \theta' \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix} + \begin{pmatrix} -\kappa\cos\theta & 0 & \tau \\ -\kappa\sin\theta & -\tau & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix} \\
&= -\kappa \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \boldsymbol{a}_1 + (\tau - \theta') \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix}.
\end{aligned}
$$

□

We can write $\boldsymbol{f}_k$ as a linear combination of $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$:

(1.7) $$\boldsymbol{f}_k = a_k \boldsymbol{a}_2 + b_k \boldsymbol{a}_3, \quad a_k = \langle \boldsymbol{f}_k, \boldsymbol{a}_2 \rangle,$$

and we have $\boldsymbol{f}_3 = |\boldsymbol{a}_1\ \boldsymbol{a}_2\ \boldsymbol{f}_3|\boldsymbol{a}_3$ (i.e., $a_3 = 0$). Remark that $f = \gamma(s) + a\boldsymbol{a}_2 + b\boldsymbol{a}_3$ where

$$
\begin{aligned}
a =& a(s,t) = \frac{t^2}{2} + \sum_{k=3}^{m} a_k(s)\frac{t^k}{k!} + O(t^{m+1}), \\
b =& b(s,t) = \sum_{k=3}^{m} b_k(s)\frac{t^k}{k!} + O(t^{m+1}).
\end{aligned}
$$

Lemma 1.4. *The coefficient a_k ($k \geq 3$) are determined by the lower order terms inductively. Precisely speaking, a_k is determined by $b_2, b_3, \ldots, b_{k-1}$.*

Proof. Under the condition (i) we have

$$t^2 = \langle f_t, f_t \rangle = \sum_k t^k \sum_{i+j=k} \frac{\langle \boldsymbol{f}_{i+1}, \boldsymbol{f}_{j+1} \rangle}{i!\,j!},$$

and we obtain that $|\boldsymbol{f}_2| = 1$, $\langle \boldsymbol{f}_2, \boldsymbol{f}_3 \rangle = 0$, $\frac{1}{3}\langle \boldsymbol{f}_2, \boldsymbol{f}_4 \rangle + \frac{1}{4}\langle \boldsymbol{f}_3, \boldsymbol{f}_3 \rangle = 0$,

$$\tfrac{1}{24}\langle \boldsymbol{f}_2, \boldsymbol{f}_5 \rangle + \tfrac{1}{12}\langle \boldsymbol{f}_3, \boldsymbol{f}_4 \rangle = 0, \quad \frac{2\langle \boldsymbol{f}_2, \boldsymbol{f}_k \rangle}{(k-1)!} + \sum_{i=2}^{k-2} \frac{\langle \boldsymbol{f}_{i+1}, \boldsymbol{f}_{k-i+1} \rangle}{i!(k-i)!} = 0 \ (k \geq 6).$$

Since $a_k = \langle \boldsymbol{f}_2, \boldsymbol{f}_k \rangle$, a_k ($k \geq 3$) are determined by $b_2, b_3, \ldots, b_{k-1}$. □

REMARK 1.5. It is clear that $(d^i b_k/ds^i)(0)$ ($k \geq 3$) are invariants of the maps, under the actions by orientation preserving diffeomorphisms of the source preserving the singular curves with their orientation and rotations of $\mathbb{R}^3$.

Proposition 1.6. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a map as in the first paragraph in this section. We have that*

- *the singularity of f is cuspidal edge if $b_3(0) \neq 0$, and*
- *the singularity of f is cuspidal cross-cap if $b_3(0) = 0$, $b_3'(0) \neq 0$,*

where b_3 is the invariant defined in (1.5)

Proof. See Appendix B.1. □

1.2. The first order derivatives and the first fundamental form. Since

$$f_t = a_t \boldsymbol{a}_2 + b_t \boldsymbol{a}_3 = \Big(t + \sum_{k=3}^{m-1} a_{k+1}\frac{t^k}{k!} + O(t^m)\Big)\boldsymbol{a}_2 + \Big(\sum_{k\geq 2} b_{k+1}\frac{t^k}{k!} + O(t^m)\Big)\boldsymbol{a}_3, \tag{1.8}$$

$$\begin{aligned} f_s =& \boldsymbol{a}_1 + a_s\boldsymbol{a}_2 + a\boldsymbol{a}_2' + b_s\boldsymbol{a}_3 + b\boldsymbol{a}_3{}' \\ =& \boldsymbol{a}_1 + a_s\boldsymbol{a}_2 + b_s\boldsymbol{a}_3 + a(-\kappa\cos\theta\boldsymbol{a}_1 + (\tau-\theta')\boldsymbol{a}_3) + b(-\kappa\sin\theta\boldsymbol{a}_1 + (\theta'-\tau)\boldsymbol{a}_2) \\ =& (1-\kappa(a\cos\theta + b\sin\theta))\boldsymbol{a}_1 + (a_s + b(\theta'-\tau))\boldsymbol{a}_2 + (b_s + a(\tau-\theta'))\boldsymbol{a}_3, \end{aligned} \tag{1.9}$$

we obtain the following expressions of the first fundamental quantities:

$$\begin{aligned} \langle f_s, f_s\rangle =& (1-\kappa(a\cos\theta + b\sin\theta))^2 + (a_s + b(\theta'-\tau))^2 + (b_s + a(\tau-\theta'))^2 \\ =& 1 - (\kappa\cos\theta)t^2 - \frac{b_3\kappa\sin\theta}{3}t^3 + O(t^4), \\ \langle f_s, f_t\rangle =& a_t(a_s + b(\theta'-\tau)) + b_t(b_s + a(\tau-\theta')) = \frac{b_3}{2}(\tau-\theta')\frac{t^3}{6} + O(t^4), \\ \langle f_t, f_t\rangle =& a_t^2 + b_t^2 = t^2. \end{aligned}$$

The last relation is expressed by

$$\begin{aligned} t^2 =& (t + \sum_{i\geq 3} a_{i+1}t^i/i!)^2 + (\sum_{j\geq 2} b_{j+1}t^j/j!)^2, \text{ and thus} \\ 1 =& (1 + \sum_{i\geq 2} a_{i+2}t^i/i!)^2 + (\sum_{j\geq 1} b_{j+2}t^j/j!)^2. \end{aligned}$$

Comparing the coefficients of t^k in both sides, we easily see that a_k is determined by $a_3, \ldots, a_{k-1}, b_3, \ldots, b_{k-1}$ and b_k. By induction, we conclude that a_k is determined by $b_2, \ldots, b_k$. We also remark that

$$\langle f_s, f_s\rangle\langle f_t, f_t\rangle - \langle f_s, f_t\rangle^2 = t^2(1 - t^2\kappa\cos\theta + \cdots).$$

1.3. Unit normal vector $\boldsymbol{\nu}$.

Lemma 1.7. *We have the following asymptotic expansion of the unit normal vector $\boldsymbol{\nu}$:*

$$\boldsymbol{\nu} = \frac{f_s \times \frac{f_t}{t}}{|f_s \times \frac{f_t}{t}|} = \Big[(\theta'-\tau)\frac{t^2}{2} + O(t^3)\Big]\boldsymbol{a}_1 + \Big[-\frac{b_3}{2}t + O(t^3)\Big]\boldsymbol{a}_2 + \Big[1 - \frac{b_3^2}{8}t^2 + O(t^3)\Big]\boldsymbol{a}_3.$$

Proof. Since

$$f_s \times \frac{f_t}{t} = (\boldsymbol{a}_1 + \boldsymbol{a}_2'\frac{t^2}{2} + \cdots) \times (\boldsymbol{a}_2 + \boldsymbol{f}_3\frac{t}{2} + \cdots) = \boldsymbol{a}_3 - b_3\boldsymbol{a}_2\frac{t}{2} + (\boldsymbol{a}_2' \times \boldsymbol{a}_2)\frac{t^2}{2} + O(t^3),$$

we have

$$|f_s \times (f_t/t)|^{-1} = 1 - \tfrac{1}{8}(b_3t)^2 + O(t^3).$$

Since

$$\boldsymbol{a}_2' \times \boldsymbol{a}_2 = (-\kappa\cos\theta\,\boldsymbol{a}_1 + (\tau-\theta')\boldsymbol{a}_3) \times \boldsymbol{a}_2 = -\kappa\cos\theta\,\boldsymbol{a}_3 - (\tau-\theta')\boldsymbol{a}_1,$$

we have

$$f_s \times (f_t/t) = \boldsymbol{a}_3 - (b_3/2)\boldsymbol{a}_2 t - (\kappa\cos\theta\,\boldsymbol{a}_3 + (\tau-\theta')\boldsymbol{a}_1)(t^2/2) + O(t^3),$$

and we obtain the expression of $\boldsymbol{\nu}$. □

Lemma 1.8. *The map* $(f,\boldsymbol{\nu}) : (\mathbb{R}^2, 0) \to (\mathbb{R}^3 \times \mathbb{R}^3, (0, \boldsymbol{\nu}(0)))$ *is an embedding germ, if and only if* $b_3 \neq 0$.

Proof. This is a consequence of the following:

$$df(s,0) = \begin{pmatrix} \boldsymbol{a}_1 \\ 0 \end{pmatrix}, \qquad d\boldsymbol{\nu}(s,0) = \begin{pmatrix} -\kappa\sin\theta & \theta'-\tau & 0 \\ 0 & -b_3/2 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix}.$$

□

1.4. The second order derivatives and the second fundamental form. Let us first compute Christoffel symbols Γ^s_{ss}, Γ^t_{ss}, Γ^s_{st}, Γ^t_{st}, Γ^s_{tt}, Γ^t_{tt} defined by

$$f_{ss} = \Gamma^s_{ss} f_s + \Gamma^t_{ss} f_t + L\boldsymbol{\nu}, \qquad f_{st} = \Gamma^s_{st} f_s + \Gamma^t_{st} f_t + M\boldsymbol{\nu}, \qquad f_{tt} = \Gamma^s_{tt} f_s + \Gamma^t_{tt} f_t + N\boldsymbol{\nu}.$$

Since

$$\begin{aligned} \langle f_s, f_s\rangle_s &= 2\langle f_{ss}, f_s\rangle, & \langle f_s, f_t\rangle_s &= \langle f_{ss}, f_t\rangle + \langle f_{st}, f_t\rangle, & \langle f_s, f_s\rangle_t &= 2\langle f_{st}, f_s\rangle, \\ \langle f_s, f_s\rangle_t &= 2\langle f_{st}, f_s\rangle, & \langle f_s, f_t\rangle_t &= \langle f_{st}, f_t\rangle + \langle f_s, f_{tt}\rangle, & \langle f_t, f_t\rangle_t &= 2\langle f_{tt}, f_t\rangle, \end{aligned}$$

we obtain

$$\begin{aligned} \frac{1}{2}\begin{pmatrix} \langle f_s, f_s\rangle & \langle f_s, f_t\rangle \\ \langle f_t, f_s\rangle & \langle f_t, f_t\rangle \end{pmatrix} \begin{pmatrix} \Gamma^s_{ss} & \Gamma^s_{st} & \Gamma^s_{tt} \\ \Gamma^t_{ss} & \Gamma^t_{st} & \Gamma^t_{tt} \end{pmatrix} &= \begin{pmatrix} \langle f_{ss}, f_s\rangle & \langle f_{st}, f_s\rangle & \langle f_{tt}, f_s\rangle \\ \langle f_{ss}, f_t\rangle & \langle f_{st}, f_t\rangle & \langle f_{tt}, f_t\rangle \end{pmatrix} \\ &= \begin{pmatrix} \langle f_s, f_s\rangle_s & \langle f_s, f_s\rangle_t & 2\langle f_s, f_t\rangle_t - \langle f_t, f_t\rangle_s \\ 2\langle f_s, f_t\rangle_s - \langle f_s, f_s\rangle_t & \langle f_t, f_t\rangle_s & \langle f_t, f_t\rangle_t \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \langle f_s, f_s\rangle_s & \langle f_s, f_s\rangle_t & 2\langle f_s, f_t\rangle_t \\ 2\langle f_s, f_t\rangle_s - \langle f_s, f_s\rangle_t & 0 & t \end{pmatrix}, \end{aligned}$$

and we obtain that

$$f_{ss} = O(t^2)\boldsymbol{a}_1 + [\kappa\cos\theta + O(t^2)]\boldsymbol{a}_2 - [\kappa\sin\theta + O(t^2)]\boldsymbol{a}_3, \tag{1.10}$$

$$f_{st} = [(-\kappa\cos\theta)t + O(t^2)]\,\boldsymbol{a}_1 + [(\tau-\theta')t + O(t^2)]\boldsymbol{a}_2 + (\tau-\theta')t\boldsymbol{a}_3, \tag{1.11}$$

$$f_{tt} = O(t^2)\boldsymbol{a}_1 + [1 + O(t^2)]\boldsymbol{a}_2 + [b_3 t + O(t^2)]\boldsymbol{a}_3. \tag{1.12}$$

We thus obtain the following expressions of the second fundamental quantities:

$$\begin{aligned} \langle f_{ss}, \boldsymbol{\nu}\rangle &= \kappa\sin\theta - \frac{b_3\kappa\cos\theta}{2}t + O(t^2), \\ \langle f_{st}, \boldsymbol{\nu}\rangle &= (\tau-\theta')t + \frac{b_3'}{2}t^2 + O(t^3), \\ \langle f_{tt}, \boldsymbol{\nu}\rangle &= \frac{b_3}{2}t + \frac{b_4}{3}t^2 + (a_5 - \frac{a_3^3}{2})\frac{t^3}{8} + O(t^4). \end{aligned}$$

Theorem 1.9. *We consider a map* $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ *as in the first paragraph of this section. The asymptotic expansions of Gauss curvature* K *and the mean curvature* H *are* *expressed as follows:*

$$K = \frac{1}{t}\Big(\frac{b_3\kappa\sin\theta}{2} + \Big[\kappa\Big(\frac{b_4\sin\theta}{3} - \frac{b_3^2\cos\theta}{4}\Big) - (\tau-\theta')^2\Big]t + O(t^2)\Big), \quad \textit{and}$$
$$H = \frac{1}{t}\Big(\frac{b_3}{4} + \Big(\frac{b_4}{6} + \frac{\kappa\sin\theta}{2}\Big)t + O(t^2)\Big)$$

where κ and τ are the curvature and the torsion of γ defined in (1.4)*, and θ, b_k are the invariants defined in* (1.5)*. If the singularity of f is cuspidal edge (i.e., $b_3 \neq 0$), then the principal curvatures are given by*

$$\kappa_1 = \kappa\sin\theta - \frac{b_3^2\kappa\cos\theta + 4(\tau-\theta_s)^2}{2b_3}t + O(t^2), \quad \kappa_2 = \frac{1}{t}\Big(\frac{b_3}{2} + \frac{b_4}{3}t + \frac{2(\tau-\theta_s)^2}{b_3}t^2 + O(t^3)\Big).$$

Proof. Since

$$\langle f_{ss}, \boldsymbol{\nu}\rangle\langle f_{tt}, \boldsymbol{\nu}\rangle - \langle f_{st}, \boldsymbol{\nu}\rangle^2 = \frac{b_3\kappa\sin\theta}{2}t + \Big(\frac{b_4}{3}\kappa\sin\theta - \frac{b_3^2}{4}\kappa\cos\theta - (\tau-\theta')^2\Big)t^2 + O(t^3),$$

we obtain the expression for K. Since

$$\langle f_s, f_s\rangle\langle f_{tt}, \boldsymbol{\nu}\rangle - 2\langle f_s, f_t\rangle\langle f_{st}, \boldsymbol{\nu}\rangle + \langle f_t, f_t\rangle\langle f_{ss}, \boldsymbol{\nu}\rangle = \frac{b_3}{2}t + \Big(\frac{b_4}{3} + \kappa\sin\theta\Big)t^2 + O(t^3),$$

we obtain the expression for H. The assertion for principal curvatures are obtained by solving the equation $\lambda^2 - 2H\lambda + K = 0$. □

We assume that $b_3 \neq 0$, that is, $t \mapsto f(s,t)$ define a (2, 3)-cusp. Then we have the following:

- If $\kappa\sin\theta \neq 0$, then one side of the singular locus is hyperbolic (i.e., $K < 0$) and the other side of the singular locus is elliptic (i.e., $K > 0$) near the singular locus.
- If $\kappa \neq 0$ and $\theta \equiv 0 \bmod \pi$, then $K = -\kappa b_3^2/4 - (\tau-\theta')^2 + O(2)$.

Remark 1.10. Several geometric invariants for cuspidal edge were already defined. Here is a list for these invariants:

- normal curvature κ_ν and singular curvature κ_s in [12],
- cuspidal curvature κ_c in [8], and
- cusp-directional torsion κ_t and edge inflectional curvature κ_i in [7].

We express them in terms of §1:

$$\kappa_s = |f_s\ f_{ss}\ \boldsymbol{a}_3|_{t=0} = \kappa\cos\theta, \qquad \kappa_\nu = f_{ss}\cdot\boldsymbol{\nu}|_{t=0} = \kappa\sin\theta, \qquad \kappa_c = |f_s\ f_{tt}\ f_{ttt}|_{t=0} = b_3,$$
$$\kappa_t = |f_s\ f_{tt}\ f_{stt}|_{t=0} = \tau - \theta', \qquad \kappa_i = |f_s\ f_{tt}\ f_{sss}|_{t=0} = \kappa\tau\cos\theta + \kappa'\sin\theta.$$

To check them we need to look the mid terms closely, using (1.9), (1.8), (1.10), (1.11), (1.12), and

$$f_{stt} = -\kappa(\cos\theta + a_{tt}\sin\theta)\boldsymbol{a}_1 + (\theta'-\tau)\boldsymbol{a}_2 + (\tau-\theta')\boldsymbol{a}_3,\ f_{ttt} = b_3\boldsymbol{a}_3, \quad \text{on} \quad t = 0.$$

1.5. Asymptotic lines. The equation for asymptotic directions is defined by

$$\Big[\kappa\sin\theta - \frac{b_3\kappa\cos\theta}{2}t + O(t^2)\Big]ds^2 + 2[(\tau-\theta')t + O(t^2)]ds\,dt + \Big[\frac{b_3}{2}t + O(t^2)\Big]dt^2 = 0 \tag{1.13}$$

in the region defined by $K = \frac{b_3\kappa\sin\theta}{t} + \cdots \leq 0$. Assume that the singularity of f is cuspidal edge (i.e., $b_3 \neq 0$). We say that a point in cuspidal edge (i.e., a point in the locus defined by $t = 0$) is **parabolic** if it is in the closure of the set of parabolic points in the regular locus. Theparabolic cuspidal edge is defined by $\kappa \sin\theta$=0 in the generic context, that is, $\kappa \sin\theta$ is not identically zero (see the end of Appendix A). If $\kappa \sin\theta > 0$ (or < 0), the equation (1.13) defines asymptotic directions in the region $t \leq 0$ (or $t \geq 0$), and there is a homeomorphism of $(\mathbb{R}^2, 0)$ which sends solution curves of (1.13) to that of folded regular point (see Appendex B.2). The singularities of asymptotic curves near a parabolic cuspidal edge point (i.e., $t = \kappa \sin\theta = 0$) are degenerate, and we do not consider them here.

1.6. Curvature lines. The equation for principal directions is

$$\begin{vmatrix} 1-(\kappa\cos\theta)t^2+O(t^3) & \kappa\sin\theta-\frac{b_3\kappa\cos\theta}{2}t+O(t^2) & dt^2 \\ O(t^3) & (\tau-\theta')t+O(t^2) & -ds\,dt \\ t^2 & \frac{b_3}{2}t+O(t^2) & ds^2 \end{vmatrix} = 0.$$

This reduces to

$$\Big[(\tau-\theta')+\frac{t}{2}b_3'+\cdots\Big]ds^2+\Big[\frac{b_3}{2}+t\Big(\frac{b_4}{3}-\kappa\sin\theta\Big)+\cdots\Big]ds\,dt-\Big[t^2(\tau-\theta')+\cdots\Big]dt^2=0.$$

Assume that $(f, \boldsymbol{\nu})$ is an embedding (i.e., $b_3 \neq 0$). This defines two nonsingular transverse flows at any point near $t = 0$. This fact is already recognized in [10, Lemma 1.3]. The author thanks the referee to let him know this paper.

1.7. Ridge and subparabolic lines. By the equation for principal directions in the previous subsection, we obtain the following expression of the principal vectors near cuspidal edge.

$$\boldsymbol{v}_1=\Big(1-\frac{2(\tau-\theta')^2t^2}{b_3^2}+O(t^3)\Big)\partial_s+\Big(\frac{2(\theta'-\tau)}{b_3}+O(t)\Big)\partial_t,$$

$$\boldsymbol{v}_2=\Big(\frac{2(\tau-\theta')t}{2}+O(t^2)\Big)\partial_s+\Big(\frac{1}{t}-\frac{2(\tau-\theta')^2t}{b_3^2}+O(t^2)\Big)\partial_t.$$

So the ridge lines are defined by zero of

$$\boldsymbol{v}_1\kappa_1=\frac{b_3^2\,(\kappa'\sin\theta+\kappa\tau\cos\theta)+4(\tau-\theta')^3}{b_3^2}+O(t),\qquad\text{or}\qquad \boldsymbol{v}_2\kappa_2=-\frac{b_3}{2t^3}+O(t^0).$$

Similarly the subparabolic lines are defined by zero of

$$\boldsymbol{v}_2\kappa_1=-\frac{b_3^2\kappa\cos\theta+4(\tau-\theta')^2}{2tb_3}+O(t^0),\qquad\text{or}\qquad \boldsymbol{v}_1\kappa_2=\frac{\tau-\theta'}{t^2}+O(t^{-1}).$$

1.8. Moving cusps along a straight line. Since Lemma 1.3 requires the assumption $\kappa \neq 0$, we need to consider separately the case that the curvature κ is identically zero. At this case $\gamma(s)$ is a part of line, and $\boldsymbol{a}_1 = \boldsymbol{t} = \gamma'$, $\boldsymbol{a}_2 = \boldsymbol{f}_2$, $\boldsymbol{a}_3 = \boldsymbol{t} \times \boldsymbol{f}_2$ form an orthonormal frame. One can define $\bar{\kappa}$ by

$$\begin{pmatrix}\boldsymbol{a}_1'\\ \boldsymbol{a}_2'\\ \boldsymbol{a}_3'\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&\bar{\kappa}\\0&-\bar{\kappa}&0\end{pmatrix}\begin{pmatrix}\boldsymbol{a}_1\\ \boldsymbol{a}_2\\ \boldsymbol{a}_3\end{pmatrix}.$$

For $f(s,t)=\gamma(s)+a\boldsymbol{a}_2+b\boldsymbol{a}_3$, $a=t^2/2-b_3^2t^4/32+O(t^5)$, $b=b_3t^3/6+b_4t^4/24+O(t^4)$, we have $f_s=\boldsymbol{a}_1+(a_s-b\bar\kappa)\boldsymbol{a}_2+(b_s+a\bar\kappa)\boldsymbol{a}_3$, $f_t=a_t\boldsymbol{a}_2+b_t\boldsymbol{a}_3$, and

$$\langle f_s,f_s\rangle=1+\bar\kappa^2t^4/4+O(t^5),\quad \langle f_s,f_t\rangle=b_3t/12+O(t^5),\quad \langle f_t,f_t\rangle=t^2.$$

Since $f_s\times f_t=(a_sb_t-a_tb_s-(aa_t+bb_t)\bar\kappa)\boldsymbol{a}_1-b_t\boldsymbol{a}_2+a_t\boldsymbol{a}_3$, we have

$$\boldsymbol{\nu}=(-\bar\kappa(t^2/2)+O(t^3))\boldsymbol{a}_1+(-b_3t/2-b_4(t^2/6)+O(t^3))\boldsymbol{a}_2+(1-b_3^2(t^2/8)+O(t^3))\boldsymbol{a}_3.$$

The vector $\eta=\partial_t$ represents a null vector along $\Sigma(f)$. Since $\lambda=\det(f_s\ f_t\ \boldsymbol{\nu})=t+O(t^3)$, $\psi=\det(\boldsymbol{t}\ \eta\boldsymbol{\nu}\ \boldsymbol{\nu})=-\frac{b_3}{2}-\frac{b_4}{4}t+O(t^2)$, the singularity of f at $(0,0)$ is cuspidal edge (resp. a cuspidal crosscap) if $b_3(0)\neq 0$ (resp. $b_3(0)=0$ and $b_3'(0)\neq 0$). We also remark that $(f,\boldsymbol{\nu})\colon(\mathbb{R}^2,0)\to\mathbb{R}^3\times\mathbb{R}^3$ is an embedding if $b_3(0)\neq 0$.

Moreover, we have $f_{ss}=(a_{ss}-2b_s\bar\kappa-a\bar\kappa^2-b\bar\kappa')\boldsymbol{a}_2+(b_{ss}+\bar\kappa(2a_s-b\bar\kappa)+a\bar\kappa')\boldsymbol{a}_3$, $f_{st}=(a_{st}-b_t\bar\kappa)\boldsymbol{a}_2+(b_{st}+a_t\bar\kappa)\boldsymbol{a}_3$, $f_{tt}=a_{tt}\boldsymbol{a}_2+b_{tt}\boldsymbol{a}_3$, and

$$\langle f_{ss},\boldsymbol{\nu}\rangle=\bar\kappa'\frac{t^2}{2}+O(t^3),\quad \langle f_{st},\boldsymbol{\nu}\rangle=\bar\kappa t+b_3'\frac{t^2}{2}+O(t^3),\quad \langle f_{tt},\boldsymbol{\nu}\rangle=\frac{b_3}{2}t+\frac{b_4}{3}t^3+O(t^3).$$

We thus conclude the asymptotic expansions of Gauss curvature K and the mean curvature H as follows:

$$K=-\bar\kappa^2+\frac{1}{4}(b_3\bar\kappa'-4b_3'\bar\kappa)t+O(t^2),\qquad H=\frac{1}{t}\Big(\frac{b_3}{4}+\frac{1}{6}b_4t+O(t^2)\Big).$$

Moreover, we obtain the asymptotic expansions of the principal curvatures:

$$\frac{1}{t}\Big(\frac{b_3}{2}+\frac{b_4}{3}t+O(t^2)\Big),\quad t\Big(\frac{2\bar\kappa^2}{b_3}+(\frac{\bar\kappa'}{2}-\frac{b_3'\bar\kappa}{b_3}+\frac{4b_4\bar\kappa^2}{3b_3^2})t+O(t^2)\Big).$$

The configuration of asymptotic lines is folded regular point if $b_3(0)\neq 0$ and $\bar\kappa'(0)\neq 0$. The equation for principal directions is

$$\Big(\bar\kappa+\frac{b_3'}{2}t+O(t^2)\Big)ds^2+\Big(\frac{b_3}{2}+\frac{b_4}{3}t+O(t^2)\Big)ds\,dt+(-\bar\kappa t^2+O(t^3))dt^2=0,$$

which defines two transverse directions whenever $b_3(0)\neq 0$.

2. Swallowtails

2.1. Normal form theorem. Throughout this section, we consider a C^∞-map

$$f\colon(\mathbb{R}^2,0)\to(\mathbb{R}^3,0),\ (u_1,v_1)\mapsto f(u_1,v_1),$$

with the following conditions:

(i) The singular locus $\Sigma(f)=\{v_1=0\}$.
(ii) $f(\Sigma(f))$ is a curve of multiplicity 2 at $u_1=0$ with an arc length parameter $(u_1)^2/2$.
(iii) The Jacobi matrix of $f|_{\Sigma(f)}$ is of rank 1.

REMARK 2.1. A typical singularity of a map with these conditions is swallowtail, a map $f\colon(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ which is $\mathcal{A}$-quivalent to

$$(u,v)\mapsto(3u^4+u^2v,4u^3+2uv,v). \tag{2.1}$$

We are going to change f a normal form under the action of the product group of coordinate change of the source with the rotation group as we explained in Introduction.

We can assume that there is a sequence $\{\boldsymbol{g}_k : (\mathbb{R},0) \to (\mathbb{R}^3,0)), u_1 \mapsto \boldsymbol{g}_k(u_1)\}_{k=0,1,2,\dots}$ of C^∞-maps so that

$$f(u_1,v_1) = \sum_{k=0}^{m} \boldsymbol{g}_k(u_1)\frac{(v_1)^k}{k!} + O(v_1{}^{m+1}) \quad \text{for any positive integer } m. \tag{2.2}$$

We express Taylor expansions of $\boldsymbol{g}_k$ as follows.

$$\boldsymbol{g}_k(u_1) = \sum_{i=2}^{m-k} \begin{pmatrix} a_{k,i} \\ b_{k,i} \\ c_{k,i} \end{pmatrix} \frac{(u_1)^i}{i!} + O(u_1^{m-k+1}) \quad (k = 1,2,\dots,m).$$

Lemma 2.2. *Without loss of generality, we can assume the following condition:*

(iv) $\boldsymbol{g}_0'(u_1) = u_1\boldsymbol{g}_1(u_1)$ *and* $|\boldsymbol{g}_1(u_1)| = 1$.

(v) $\boldsymbol{g}_0$ *and* $\boldsymbol{g}_1$ *satisfy the following:*

$$\boldsymbol{g}_1(u_1) = \begin{pmatrix} 1 + a_{1,1}u_1 \\ b_{1,1}u_1 \\ 0 \end{pmatrix} + \sum_{i=2}^{m} \begin{pmatrix} a_{1,i} \\ b_{1,i} \\ c_{1,i} \end{pmatrix} \frac{(u_1)^i}{i!} + O(u_1^{m+1}), \tag{2.3}$$

$$\boldsymbol{g}_0(u_1) = \begin{pmatrix} (u_1)^2/2 \\ b_{1,1}(u_1)^3/3 \\ 0 \end{pmatrix} + \begin{pmatrix} -b_{1,1}^2 \\ b_{1,2} - 2a_{1,1}b_{1,1} \\ c_{1,2} \end{pmatrix} \frac{(u_1)^4}{8} + O((u_1)^5). \tag{2.4}$$

Proof. Since $df(u_1,0) = (\boldsymbol{g}_0'(u_1), \boldsymbol{g}_1(u_1))$, the condition (i) implies $\boldsymbol{g}_0'(u_1)$ and $\boldsymbol{g}_1(u_1)$ are linearly dependent. By (ii), $\boldsymbol{g}_0'(0) = 0$ and the condition (iii) implies $\boldsymbol{g}_1(0) \neq 0$. So there is a function $g(u_1)$ with $\boldsymbol{g}_0'(u_1) = \boldsymbol{g}_1(u_1)g(u_1)$, $g(0) = 0$, $g'(0) \neq 0$. Setting $(u_1,v_1) = (u, v/|\boldsymbol{g}_1(u)|)$, we have

$$f(u_1,v_1) = \sum_{k=1}^{m} \boldsymbol{g}_k(u_1)\frac{(v_1)^2}{k!} + O(v_1{}^{m+1}) = \sum_{k=1}^{m} \frac{\boldsymbol{g}_k(u)}{|\boldsymbol{g}_1(u)|^k}\frac{v^k}{k!} + O(v^{m+1}).$$

So we can assume that $|\boldsymbol{g}_1(u_1)| = 1$.

Rotating $f(u_1,v_1)$ in $\mathbb{R}^3$, if necessary, we may assume (2.3). Since $\sigma = (u_1)^2/2$ is an arc length parameter of the curve $u_1 \mapsto \boldsymbol{g}_0(u_1)$,

$$|g(u_1)| = |g(u_1)||\boldsymbol{g}_1(u_1)| = \left|\frac{d\boldsymbol{g}_0}{du_1}\right| = \left|\frac{d\boldsymbol{g}_0}{d\sigma}\right|\left|\frac{d\sigma}{du_1}\right| = |u_1|,$$

and we conclude that $g(u_1) = \pm u_1$. We assume that $g(u_1) = u_1$. Then we have (2.4). □

We can assume that $b_{1,1} \geq 0$ changing the sign of u, if necessary.

Since the 1-jet of $\tilde{\boldsymbol{\nu}} = f_{u_1} \times f_{v_1}$ is $(0,0,b_{1,1}v_1)$, $\boldsymbol{\nu} = \tilde{\boldsymbol{\nu}}/|\tilde{\boldsymbol{\nu}}|$ is extendible continuously to $(u_1,v_1) = (0,0)$, if $b_{1,1} \neq 0$.

REMARK 2.3. If $b_{1,1} = 0$, then the singularity of f cannot be swallowtail. In fact, when $b_{1,1} = 0$, the coefficient of uv in the Taylor expansion of f is zero. But a map, which is swallowtail has non zero uv term whenever its 1-jet is $v\boldsymbol{e}_1$.

Theorem 2.4. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a map as in the first paragraph of this section with conditions* (iv) *and* (v) *of* Lemma 2.2. *If $b_{1,1} \neq 0$, then there is a coordinate change, $(u, v) \mapsto (u_1, v_1) = h(u, v) = (h_1(u, v), h_2(u, v))$, of the source so that*

(i) $\Sigma(f\circ h) = \{v = 0\}$,
(ii) *$f(\Sigma(f\circ h))$ is a curve of multiplicity 2 at $u = 0$ with an arc length parameter $u^2/2$,*
(iii) *the Jacobi matrix of $f\circ h|_{\Sigma(f\circ h)}$ is of rank 1, and*
(iv) *$\langle (f\circ h)_u, (f\circ h)_u\rangle|_{v=0} = u^2$, $\langle (f\circ h)_u, (f\circ h)_v\rangle = u + O(p)$, and $\langle (f\circ h)_v, (f\circ h)_v\rangle = 1 + O(p)$ for any positive integer p.*

2.2. Proof of Theorem 2.4. The key of the proof of Theorem 2.4 is the following

Theorem 2.5. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a map as in the first paragraph of this section with conditions (iv) and (v) of* Lemma 2.2. *Let k be a positive integer and $b_{1,1} > 0$. There is a coordinate system (u_k, v_k) so that $u_1 = u_k + v_k\widetilde{P}_{k-1}$, $v_1 = v_k(1 + \widetilde{Q}_k)$, where $\widetilde{P}_k$ and $\widetilde{Q}_k$ are polynomials in (u_k, v_k) of degrees $k - 1$ and k, respectively, and*

$$\langle f_{u_k}, f_{u_k}\rangle = u_k^2 + b_{1,1}^2 v_k^2 + v_k\, O(2),\ \langle f_{u_k}, f_{v_k}\rangle = u_k + v_k\, O(k-1),\ \langle f_{v_k}, f_{v_k}\rangle = 1 + O(k).$$

For the coordinate system (u_k, v_k), we easily see the following conditions:

(i) $\Sigma(f) = \{v_k = 0\}$;
(ii) $f(\Sigma(f))$ is a curve of multiplicity 2 at $u_k = 0$ with an arc length parameter $(u_k)^2/2$;
(iii) The Jacobi matrix of $f|_{\Sigma(f)}$ is of rank 1.

REMARK 2.6. If $\langle f_u, f_v\rangle = u$ and $\langle f_v, f_v\rangle = 1$, then the curves $v \mapsto f(u, v)$ present geodesics, since $\langle f_{vv}, f_v\rangle = \frac{1}{2}\langle f_v, f_v\rangle_v = 0$, and $\langle f_{vv}, f_u\rangle = \langle f_u, f_v\rangle_v - \frac{1}{2}\langle f_v, f_v\rangle_u = 0$. This is a strong evidence to expect the existence of a geodesic which reaches swallowtail singularity.

Corollary 2.7. *Under the same assumption to the previous theorem, there exists a C^∞-coordinate system (u, v) so that the Taylor expansions of $\langle f_u, f_u\rangle$, $\langle f_u, f_v\rangle$ and $\langle f_v, f_v\rangle$ are given by $u^2 + b_{1,1}^2 v^2 + vO(2)$, u, and* 1, *respectively.*

Proof. Consequence of the previous theorem and Bott's theorem ([11, §1.5]). □

Lemma 2.8. *Assume that $b_{1,1} \neq 0$. If*

$$p = -\frac{b_{2,0}}{2b_{1,1}},\quad q_0 = -a_{1,1} - \frac{b_{2,0}}{2b_{1,1}},\quad q_1 = -\frac{a_{2,0}}{2} + \frac{a_{1,1}b_{2,0}}{2b_{1,1}} - \frac{b_{2,0}^2}{b_{1,1}^2}, \tag{2.5}$$

then $j^2 f(0) = \begin{pmatrix} \frac{u^2}{2} + v \\ b_{1,1}uv \\ c_{2,0}\frac{v^2}{2} \end{pmatrix}$ *where $u_1 = u + pv$, $v_1 = v + v(q_0 u + q_1 v)$.*

Proof. Taylor expansion of f is

$$\boldsymbol{g}_0(u_1) + \boldsymbol{g}_1(u_1)v_1 + \boldsymbol{g}_2(u_1)\frac{(v_1)^2}{2} + \sum_{k\geq 3} \boldsymbol{g}_k(u_1)\frac{(v_1)^k}{k!}$$

$$= \begin{pmatrix} (u_1)^2/2 \\ 0 \\ 0 \end{pmatrix} + v_1 \begin{pmatrix} 1 + a_{1,1}u_1 \\ b_{1,1}u_1 \\ 0 \end{pmatrix} + \frac{(v_1)^2}{2} \begin{pmatrix} a_{2,0} \\ b_{2,0} \\ c_{2,0} \end{pmatrix} + \sum_{i+j\geq 3} \begin{pmatrix} a_{i,j} \\ b_{i,j} \\ c_{i,j} \end{pmatrix} \frac{(u_1)^j (v_1)^i}{i!\,j!}$$

$$
\begin{aligned}
&= \begin{pmatrix} (u+pv)^2/2 \\ 0 \\ 0 \end{pmatrix} + v(1+q_0u+q_1v) \begin{pmatrix} 1+a_{1,1}(u+pv) \\ b_{1,1}(u+pv) \\ 0 \end{pmatrix} + \frac{v^2(1+q_0u+q_1v)^2}{2} \begin{pmatrix} a_{2,0} \\ b_{2,0} \\ c_{2,0} \end{pmatrix} \\
&\quad + \sum_{i+j\geq 3} \begin{pmatrix} a_{i,j} \\ b_{i,j} \\ c_{i,j} \end{pmatrix} \frac{(u+pv)^j(v+v(q_0u+q_1v))^i}{i!j!} \\
&= \begin{pmatrix} v+\frac{u^2}{2} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} a_{1,1}+p+q_0 \\ b_{1,1} \\ 0 \end{pmatrix} uv + \frac{1}{2} \begin{pmatrix} a_{2,0}+2a_{1,1}p+p^2+2q_1 \\ b_{2,0}+2b_{1,1}p \\ c_{2,0} \end{pmatrix} v^2 + O(3).
\end{aligned}
$$

By (2.5), we have

$$
a_{1,1}+p+q_0=0, \quad a_{2,0}+2a_{1,1}p+p^2+2q_0=0, \quad b_{2,0}+2b_{1,1}p=0,
$$

and we conclude the result. □

By the lemma, we have

$$
j^2f(0) = \begin{pmatrix} \frac{u^2}{2}+v \\ b_{1,1}uv \\ c_{2,0}\frac{v^2}{2} \end{pmatrix}, \quad j^1f_u(0) = \begin{pmatrix} u \\ b_{1,1}v \\ 0 \end{pmatrix}, \quad j^1f_v(0) = \begin{pmatrix} 1 \\ b_{1,1}u \\ c_{2,0}v \end{pmatrix},
$$

$$
\langle f_u, f_u\rangle = u^2 + b_{1,1}^2v^2 + v\,O(2), \quad \langle f_u, f_v\rangle = u + v\,O(1), \quad \langle f_v, f_v\rangle = 1 + O(2).
$$

This shows Theorem 2.5 when $k=2$.

Lemma 2.9. *Set $f(u,v) = \sum_{k=1}^{p} \boldsymbol{g}_k(u)v^k/k! + O(v^{p+1})$. Assume that $\boldsymbol{g}_0'(u) = g(u)\boldsymbol{g}_1(u)$. If $\langle f_v, f_v\rangle = 1 + O(u^k)v + O(v^2)$, then*

$$
\langle f_u, f_v\rangle = u + O(u^k)v + O(v^2).
$$

Proof. Since $f_v = \sum_{k=0}^{p-1} \boldsymbol{g}_{k+1}(u)v^k/k! + O(v^p)$,

$$
\begin{aligned}
\text{(2.6)} \qquad \langle f_v, f_v\rangle &= \Big\langle \sum_{i=0}^{p-1} \boldsymbol{g}_{i+1}\frac{v^i}{i!}, \sum_{j=0}^{p-1} \boldsymbol{g}_{j+1}\frac{v^j}{j!} \Big\rangle + O(v^p) = \sum_{k=0}^{p-1}\sum_{i+j=k} \langle \boldsymbol{g}_{i+1}, \boldsymbol{g}_{j+1}\rangle \frac{v^k}{i!j!} + O(v^p) \\
&= \langle \boldsymbol{g}_1, \boldsymbol{g}_1\rangle + 2\langle \boldsymbol{g}_1, \boldsymbol{g}_2\rangle v + (\langle \boldsymbol{g}_2, \boldsymbol{g}_2\rangle + \langle \boldsymbol{g}_1, \boldsymbol{g}_3\rangle)v^2 + O(v^3).
\end{aligned}
$$

Since $\langle \boldsymbol{g}_1, \boldsymbol{g}_1\rangle = 1$, and $\langle \boldsymbol{g}_1, \boldsymbol{g}_2\rangle = O(u^k)$, we have

$$
\langle \boldsymbol{g}_1', \boldsymbol{g}_1\rangle + \langle \boldsymbol{g}_0', \boldsymbol{g}_2\rangle = \frac{1}{2}\langle \boldsymbol{g}_1, \boldsymbol{g}_1\rangle_u + g(u)\langle \boldsymbol{g}_1, \boldsymbol{g}_2\rangle = O(u^k)
$$

Since $u^2 = \langle \boldsymbol{g}_0', \boldsymbol{g}_0'\rangle = g(u)^2\langle \boldsymbol{g}_1, \boldsymbol{g}_1\rangle = g(u)^2$, we may assume that $g(u) = u$, and

$$
\langle \boldsymbol{g}_0', \boldsymbol{g}_1\rangle = g(u)\langle \boldsymbol{g}_1, \boldsymbol{g}_1\rangle = u.
$$

Since $f_u = \sum_{k=0}^{p-1} \boldsymbol{g}_k'(u)\frac{v^k}{k!} + O(v^p)$,

$$\langle f_u, f_v\rangle = \Big\langle \sum_{i=0}^{p-1} \boldsymbol{g}'_i \frac{v^i}{i!}, \sum_{j=0}^{p-1} \boldsymbol{g}_{j+1} \frac{v^j}{j!} \Big\rangle + O(v^p) = \sum_{k=0}^{p-1} \sum_{i+j=k} \langle \boldsymbol{g}'_i, \boldsymbol{g}_{j+1}\rangle \frac{v^k}{i!j!} + O(v^p) \tag{2.7}$$
$$= \langle \boldsymbol{g}'_0, \boldsymbol{g}_1\rangle + (\langle \boldsymbol{g}'_0, \boldsymbol{g}_2\rangle + \langle \boldsymbol{g}'_1, \boldsymbol{g}_1\rangle)v + O(v^2) = u + O(u^k)v + O(v^2).$$

□

We are looking for a coordinate system (u, v) with

$$\langle f_u, f_u\rangle = u^2 + b_{1,1}^2 v^2 + v\, O(2), \quad \langle f_u, f_v\rangle = u + O(k), \quad \langle f_v, f_v\rangle = 1 + O(k)$$

where k is a positive integer. We consider tuples $H_i(u, v)$ of homogeneous polynomials of degree i in (u, v) so that $f(u, v) = \sum_{i=1}^{k} H_i(u, v) + O(k + 1)$. We have $H_1(u, v) = v\boldsymbol{e}_1$, and

$$H_2(u, v) = \begin{pmatrix} u^2/2 \\ b_{1,1}uv \\ c_{2,0}v^2/2 \end{pmatrix}, \quad H_3(u, v) = \begin{pmatrix} 0 \\ b_{1,1} \\ 0 \end{pmatrix} \frac{u^3}{3} + \begin{pmatrix} -b_{1,1}^2 \frac{u^2v}{2} - c_{2,0}^2 \frac{v^3}{6} \\ b_{1,2}\frac{u^2v}{2} + b_{2,1}^* \frac{uv^2}{2} + b_{3,0}^* \frac{v^3}{6} \\ c_{1,2}\frac{u^2v}{2} + c_{2,1}^* \frac{uv^2}{2} + c_{3,0}^* \frac{v^3}{6} \end{pmatrix}.$$

where $b_{2,1}^* = \frac{c_{2,0}}{b_{1,1}}(c_{2,0} - c_{1,2})$, $c_{2,1}^* = c_{2,1} + \frac{b_{2,0}(c_{2,0}-c_{1,2})}{b_{1,1}}$, $b_{3,0}^* = \frac{c_{2,0}c_{2,1}}{b_{1,1}} - \frac{b_{2,0}c_{2,0}(c_{2,0}-c_{1,2})}{b_{1,1}^2}$, $c_{3,0}^* = c_{3,0} - 3a_{2,0}c_{2,0} - \frac{3b_{2,0}c_{2,1}}{2b_{1,1}} + \frac{3b_{2,0}(x_{1,2}-c_{2,0})}{4b_{1,1}^2}$. So $b_{1,1}$ and $c_{2,0}$ are invariants of order 2, and $b_{1,2}$, $c_{1,2}$, $c_{2,1}^*$, and $c_{3,0}^*$ are invariants of order 3.

Lemma 2.10. *Let (u_k, v_k) be a coordinate system so that*

$$f = v_k\boldsymbol{e}_1 + \sum_{i=2}^{k+1} H_i(u_k, v_k) + b_{1,1}v_k^2 P_{k-2}(u_k, v_k)\boldsymbol{e}_2 + O(k + 2), \tag{2.8}$$
$$\langle f_{u_k}, f_{u_k}\rangle = u_k^2 + b_{1,1}^2 v_k^2 + v_k\, O(2),$$
$$\langle f_{u_k}, f_{v_k}\rangle = u_k + v_k A_{k-1} + b_{1,1}v_k^2 P_{k-2}(u_k, v_k) + v_k\, O(k),$$
$$\langle f_{v_k}, f_{v_k}\rangle = 1 + B_k + O(k + 1),$$

where A_{k-1} and B_k are homogeneous polynomials in (u_k, v_k) of degrees $k-1$ and k, respectively. Setting $u_k = u_{k+1} + v_{k+1}P_{k-1}(u_{k+1}, v_{k+1})$, $v_k = v_{k+1}(1 + Q_k(u_{k+1}, v_{k+1}))$ where $P_{k-1}(u, v)$ and $Q_k(u, v)$ are homogeneous polynomials of degrees $k-1$ and k in (u, v), respectively, we have

$$f = v_{k+1}\boldsymbol{e}_1 + \sum_{i=2}^{k+1} H_i(u_{k+1}, v_{k+1}) + b_{1,1}v_{k+1}^2 P_{k-2}(u_{k+1}, v_{k+1})\boldsymbol{e}_2$$
$$+ \begin{pmatrix} uvP_{k-1}(u_{k+1}, v_{k+1}) + vQ_k(u_{k+1}, v_{k+1}) \\ b_{1,1}v^2 P_{k-1}(u_{k+1}, v_{k+1}) \\ c_{2,0}vQ_k(u_{k+1}, v_{k+1}) \end{pmatrix} + O(k + 2),$$

and, for a suitable choice of P_{k-2} and Q_k, we conclude that

$$\langle f_{u_{k+1}}, f_{u_{k+1}}\rangle = u_{k+1}^2 + b_{1,1}^2 v_{k+1}^2 + v_{k+1}\, O(2),$$
$$\langle f_{u_{k+1}}, f_{v_{k+1}}\rangle = u_{k+1} + v_{k+1}\, O(k),$$
$$\langle f_{v_{k+1}}, f_{v_{k+1}}\rangle = 1 + O(k + 1).$$

Proof. Setting $u_k = u + vP_{k-1}$, $v_k = v(1 + Q_k)$, we have $v_k\boldsymbol{e}_1 = v(1 + Q_k)\boldsymbol{e}_1$,

$$H_2(u_k, v_k) = \begin{pmatrix} u_k^2/2 \\ b_{1,1}u_kv_k \\ c_{2,0}v_k^2/2 \end{pmatrix} = \begin{pmatrix} u^2/2 \\ b_{1,1}uv \\ c_{2,0}v^2/2 \end{pmatrix} + \begin{pmatrix} uvP_{k-1} \\ b_{1,1}v^2P_{k-1} \\ c_{2,0}vQ_k \end{pmatrix} + O(k+2).$$

Since

$$(u + vP_{k-1})^i v^j (1 + Q_k)^j = \sum_{s=0}^{i}\sum_{t=0}^{j}\binom{i}{s}\binom{j}{t} u^{i-s}v^sP_{k-1}^s v^j Q_k^t,$$

we have $H_i(u_k, v_k) = H_i(u, v) + O(k + 2)$, $i = 3, 4, \ldots, k + 1$. We thus have

$$\text{(2.9)}\quad f = v\boldsymbol{e}_1 + \begin{pmatrix} u^2/2 \\ b_{1,1}uv \\ c_{2,0}v^2/2 \end{pmatrix} + \sum_{i=3}^{k+1} H_i(u,v) + b_{1,1}v^2P_{k-2}(u,v)\boldsymbol{e}_2 + \begin{pmatrix} uvP_{k-1} + vQ_k \\ b_{1,1}v^2P_{k-1} \\ c_{2,0}vQ_k \end{pmatrix} + O(k+2).$$

Then we obtain that

$$f_u = \begin{pmatrix} u \\ b_{1,1}v \\ 0 \end{pmatrix} + \sum_{i=3}^{k+1}(H_i)_u + b_{1,1}(v^2P_{k-2})_u\boldsymbol{e}_2 + \begin{pmatrix} (uvP_{k-1} + vQ_k)_u \\ b_{1,1}v^2(P_{k-1})_u \\ c_{2,0}(vQ_k)_u \end{pmatrix} + O(k+1),$$

$$f_v = \boldsymbol{e}_1 + \begin{pmatrix} 0 \\ b_{1,1}u \\ c_{2,0}v \end{pmatrix} + \sum_{i=3}^{k+1}(H_i)_v + b_{1,1}(v^2P_{k-2})_v\boldsymbol{e}_2 + \begin{pmatrix} (uvP_{k-1} + vQ_k)_v \\ b_{1,1}(v^2P_{k-1})_v \\ c_{2,0}(vQ_k)_v \end{pmatrix} + O(k+1).$$

Remark that the homogeneous part of degree k of $\langle f_v, f_v\rangle$ is

$$\text{(2.10)}\qquad 2(uvP_{k-1} + vQ_k)_v + 2b_{1,1}^2u(v^2P_{k-2})_v + \sum_{i=2}^{k}\langle (H_i)_v, (H_{k+2-i})_v\rangle + 2\langle (H_{k+1})_v, \boldsymbol{e}_1\rangle.$$

We choose a homogeneous polynomial R_{k+1} of degree $k + 1$ so that

$$(R_{k+1})_v = \frac{1}{2}\sum_{i=2}^{k}\langle (H_i)_v, (H_{k+2-i})_v\rangle, \quad (R_{k+1} + \langle H_{k+1}, \boldsymbol{e}_1\rangle)|_{v=0} = 0.$$

Since $R_{k+1} + \langle H_{k+1}, \boldsymbol{e}_1\rangle$ is divisible by v, we can choose a homogeneous polynomial Q_k of degree k so that

$$uvP_{k-1} + vQ_k + b_{1,1}^2uv^2P_{k-2} + R_{k+1} + \langle H_{k+1}, \boldsymbol{e}_1\rangle = 0.$$

Then (2.10) is zero and the first component of (2.9) does not depend on P_{k-1}. Moreover, we have that the degree k-part of $\langle f_u, f_v\rangle$ is equal to

$$\begin{aligned} &\sum_{i=2}^{k}\langle (H_i)_u, (H_{k+2-i})_v\rangle + (uvP_{k-1} + vQ_k)_u + \langle (H_{k+1})_u, \boldsymbol{e}_1\rangle + b_{1,1}^2kv^2P_{k-2} \\ \text{(2.11)}\qquad &= \sum_{i=2}^{k}\langle (H_i)_u, (H_{k+2-i})_v\rangle - (R_{k+1})_u + b_{1,1}^2v^2[kP_{k-2} - (uP_{k-2})_u], \end{aligned}$$

since $(uvP_{k-1} + vQ_k)_u + b_{1,1}^2(uv^2P_{k-2})_u + (R_{k+1})_u + \langle (H_{k+1})_u, \boldsymbol{e}_1\rangle = 0$. We finish the proof if we choose P_{k-2} so that (2.11) is zero. Setting $P_{k-2} = \sum_{i=0}^{k-2} p_iu^iv^{k-i-2}$, the equation becomes

$$\sum_{i=2}^{k}\langle (H_i)_u, (H_{k+2-i})_v\rangle - (R_{k+1})_u + b_{1,1}^2 \sum_{i=0}^{k-2}(k-i)p_i u^i v^{k-i} = 0,$$

which is possible to solve inductively by Lemma 2.9. □

2.3. Computation based on the normal form. From now on, we assume that the C^∞-map

$$f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0),\ (u, v) \mapsto f(u, v),$$

as in the first paragraph of this section with conditions (iv) and (v) of Lemma 2.2. Let

$$H_k(u, v) = \sum_{i+j=k} \begin{pmatrix} a_{i,j} \\ b_{i,j} \\ c_{i,j} \end{pmatrix} \frac{u^j v^i}{i!\,j!} \quad (k = 1, 2, \dots)$$

be homogeneous polynomials with $f(u, v) = \sum_{k=1}^{p} H_k(x, y) + O(p+1)$ for any positive integer p. Remark that

$$H_1 = \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix},\ H_2 = \begin{pmatrix} u^2/2 \\ b_{1,1}uv \\ c_{2,0}v^2/2 \end{pmatrix},\ \text{and } H_3 = \begin{pmatrix} 0 \\ b_{1,1} \\ 0 \end{pmatrix}\frac{u^3}{3} + \begin{pmatrix} a_{1,2}\frac{u^2v}{2} + a_{2,1}\frac{uv^2}{2} + a_{3,0}\frac{v^3}{6} \\ b_{1,2}\frac{u^2v}{2} + b_{2,1}\frac{uv^2}{2} + b_{3,0}\frac{v^3}{6} \\ c_{1,2}\frac{u^2v}{2} + c_{2,1}\frac{uv^2}{2} + c_{3,0}\frac{v^3}{6} \end{pmatrix}.$$

We first see the following

Theorem 2.11. *Let f be as in the previous paragraph. The coefficients $a_{i,j}$, $b_{i,j}$, $c_{i,j}$ are invariants under the action of orientation preserving diffeomorphisms of the source preserving the singular curves with their orientation.*

Proof. Assume that there is another coordinate (u', v') with conditions (i)–(iv). We can assume that $u' = u + v\psi_1(u, v)$ and $v' = v(1 + \psi_2(u, v))$, by (i) and (ii). It is enough to show that both $\psi_1(u, v)$, $\psi_2(u, v)$ are flat functions, that is, all partial derivatives, including higher order's, are zero at 0. Let us assume the contrary. Then there exist $\phi_1(u, v)$, $\phi_2(u, v)$ homogeneous polynomials (possibly zero) of degree $k-1$, k, respectively, so that $\phi_1 \neq 0$ or $\phi_2 \neq 0$ and $\psi_1 = \phi_1 + O(k)$, $\psi_2 = \phi_2 + O(k+1)$. We can assume that $k \geq 2$. Since

$$f_u = (1 + (v\psi_1)_u)f_{u'} + (v\psi_2)_u f_{v'}, \qquad f_v = (v\psi_1)_v f_{u'} + (1 + (v\psi_2)_v)f_{v'},$$

we obtain

$$\begin{aligned}\langle f_u, f_v\rangle =& (1 + (v\psi_1)_u)(v\psi_1)_v\langle f_{u'}, f_{u'}\rangle + [(1 + (v\psi_1)_u)(1 + (v\psi_2)_v) + (v\psi_2)_u(v\psi_1)_v]\langle f_{u'}, f_{v'}\rangle \\ &+ (v\psi_2)_u(1 + (v\psi_2)_v)\langle f_{v'}, f_{v'}\rangle, \\ \langle f_v, f_v\rangle =& (v\psi_1)_v^2\langle f_{u'}, f_{u'}\rangle + 2(v\psi_1)_v(1 + (v\psi_2)_v)\langle f_{u'}, f_{v'}\rangle + (1 + (v\psi_2)_v)^2\langle f_{v'}, f_{v'}\rangle.\end{aligned}$$

Comparing degree k parts of them, we obtain that

$$0 = (v\phi_1)_u u + v\phi_1 + (v\phi_2)_v, \quad (v\phi_1)_v u + (v\phi_2)_v = 0$$

and thus $(v\phi_1)_v u = (v\phi_1)_u u + v\phi_1$. When $\phi_1 = \sum_{i=0}^{k-1} a_i u^i v^{k-1-i}$, we have

$$\sum_{i=0}^{k-1}(k-i)a_i u^{i+1}v^{k-i-1} = \sum_{i=0}^{k-1}(i+1)a_i u^i v^{k-i}.$$

This implies $\phi_1 = 0$, and $\phi_2 = 0$ also. $\square$

Remark that we have $b_{2,1} = c_{2,0}(c_{2,0} - c_{1,2})/b_{1,1}$, $b_{3,0} = -c_{2,0}c_{2,1}/(2b_{1,1})$ when $b_{1,1} \neq 0$. Actually, we have the following

Proposition 2.12. *The coefficients $a_{i,j}$ $(i \geq 1)$, $b_{i,j}$ $(i \geq 2)$ are determined by the lower order terms and $c_{p,i+j-p}$, $0 \leq p \leq i+j$, inductively, whenever $b_{1,1} \neq 0$. Precisely speaking, $a_{1,k}, a_{2,k-1}, \dots, a_{k+1,0}, b_{2,k-2}, b_{3,k-3}, \dots, b_{k,0}$ are determined by $b_{1,1}, b_{1,2}, \dots, b_{1,k-1}$, and $c_{p,q}$ $(p+q \leq k)$.*

REMARK 2.13. This proposition implies that the coefficients in the first components of Taylor expansions of $\boldsymbol{g}_i(u)$ $(i \geq 1)$ and the coefficients in the second components of Taylor expansions of $\boldsymbol{g}_i(u)$ $(i \geq 2)$ are determined by the lower order terms. Remark that the orthogonal projection of the singular curve $\boldsymbol{g}_0(u)$ to y-axis (the principal normal line of $\boldsymbol{g}_0(u)$ at $u = 0$) determines $b_{1,j}$ and the orthogonal projection of $f(u, v)$ to z-axis (the binormal line of $\boldsymbol{g}_0(u)$ at $u = 0$), determines $c_{i,j}$. By Proposition 2.12, these informations determine all our finite order invariants.

Proof of Proposition 2.12. By (2.4), we obtain

$$\begin{aligned}1 =&\langle \boldsymbol{g}_1, \boldsymbol{g}_1\rangle = 1 + 2a_{1,1}u + (a_{1,2} + a_{1,1}^2 + b_{1,1}^2)u^2\\ &+ \sum_{k=3}^{p}\Big(\frac{2a_{1,k}}{k!} + \sum_{i=2}^{k-2}\frac{a_{1,i}a_{1,k-i} + b_{1,i}b_{1,k-i} + c_{1,i}c_{1,k-i}}{i!(k-i)!}\Big)u^k + O(u^{p+1}),\end{aligned}$$

and $a_{1,k}$ is determined by $b_{1,1}, b_{1,2}, \dots, b_{1,k-1}, c_{1,1}, c_{1,2}, \dots, c_{1,k-1}$. Since

$$f_u = \begin{pmatrix} u \\ b_{1,1}v \\ 0 \end{pmatrix} + \sum_{i=3}^{p}(H_i)_u + O(p+1), \qquad f_v = \boldsymbol{e}_1 + \begin{pmatrix} 0 \\ b_{1,1}u \\ c_{2,0}v \end{pmatrix} + \sum_{i=3}^{p}(H_i)_v + O(p+1),$$

for $k \geq 2$, the conditions imply that

$$\begin{aligned}
0 =&\text{the degree } k\text{-part of } \langle f_u, f_v\rangle = \sum_{i=2}^{k}\langle (H_i)_u, (H_{k+2-i})_v\rangle + \langle (H_{k+1})_u, \boldsymbol{e}_1\rangle\\
=&\left\langle \begin{pmatrix} u \\ b_{1,1}v \\ 0 \end{pmatrix}, (H_k)_v \right\rangle + \sum_{i=3}^{k-1}\langle (H_i)_u, (H_{k+2-i})_v\rangle + \left\langle (H_k)_u, \begin{pmatrix} 0 \\ b_{1,1}u \\ c_{2,0}v \end{pmatrix} \right\rangle + \langle (H_{k+1})_u, \boldsymbol{e}_1\rangle\\
=&u\langle \boldsymbol{e}_1, (H_k)_v\rangle + \sum_{i=3}^{k-1}\langle (H_i)_u, (H_{k+2-i})_v\rangle + c_{2,0}v\langle (H_k)_u, \boldsymbol{e}_3\rangle + b_{1,1}k\langle \boldsymbol{e}_2, H_k\rangle + \langle (H_{k+1})_u, \boldsymbol{e}_1\rangle,\\
0 =&\text{the degree } k\text{-part of } \langle f_v, f_v\rangle = \sum_{i=2}^{k}\langle (H_i)_v, (H_{k+2-i})_v\rangle + 2\langle (H_{k+1})_v, \boldsymbol{e}_1\rangle\\
=&\sum_{i=3}^{k-1}\langle (H_i)_v, (H_{k+2-i})_v\rangle + 2\left\langle \begin{pmatrix} 0 \\ b_{1,1}u \\ c_{2,0}v \end{pmatrix}, (H_k)_v \right\rangle + 2\langle (H_{k+1})_v, \boldsymbol{e}_1\rangle\\
=&\sum_{i=3}^{k-1}\langle (H_i)_v, (H_{k+2-i})_v\rangle + 2c_{2,0}v\langle \boldsymbol{e}_3, (H_k)_v\rangle + 2b_{1,1}u\langle \boldsymbol{e}_2, (H_k)_v\rangle + 2\langle (H_{k+1})_v, \boldsymbol{e}_1\rangle.
\end{aligned}$$

In other words, we have

$$b_{1,1}k\langle \boldsymbol{e}_2, H_k\rangle + \langle (H_{k+1})_u, \boldsymbol{e}_1\rangle = -\, u\langle \boldsymbol{e}_1, (H_k)_v\rangle - \sum_{i=3}^{k-1}\langle (H_i)_u, (H_{k+2-i})_v\rangle - c_{2,0}v\langle (H_k)_u, \boldsymbol{e}_3\rangle,$$

$$b_{1,1}u\langle \boldsymbol{e}_2, (H_k)_v\rangle + \langle (H_{k+1})_v, \boldsymbol{e}_1\rangle = -\,\frac{1}{2}\sum_{i=3}^{k-1}\langle (H_i)_v, (H_{k+2-i})_v\rangle - c_{2,0}v\langle \boldsymbol{e}_3, (H_k)_v\rangle.$$

These equations can be written in the following forms:

$$\sum_{i+j=k}(b_{1,1}kb_{j,i} + a_{j,i+1})\frac{u^i v^j}{i!j!} = \sum_{i+j=k} p_{j,i}\frac{u^i v^j}{i!j!},$$

$$\sum_{i+j=k}(b_{1,1}b_{j+1,i-1} + a_{j+1,i})\frac{u^i v^j}{i!j!} = \sum_{i+j=k} q_{j,i}\frac{u^i v^j}{i!j!}.$$

Setting $b = b_{1,1}$, we have

$$\begin{pmatrix} 0 & 1 & \cdots & 0 & bk & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & bk \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots & b & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & b \end{pmatrix} \begin{pmatrix} a_{k+1,0} \\ a_{k,1} \\ \vdots \\ a_{2,k-1} \\ b_{k,0} \\ b_{k-1,1} \\ \vdots \\ b_{2,k-2} \end{pmatrix} = \begin{pmatrix} p_{k,0} \\ p_{k-1,1} \\ \vdots \\ p_{2,k-2} \\ q_{k,0} \\ q_{k-1,1} \\ \vdots \\ q_{1,k-1} \end{pmatrix}$$

and this determines $a_{k+1,0}, a_{k,1}, \ldots a_{2,k-1}, b_{k,0}, b_{k-1,1}, \ldots, b_{2,k-2}$. □

From now on, we assume that $b_{1,1} \neq 0$.

Remark 2.14. Let us assume that the coordinate (u, v) satisfies that

$$\langle f_u, f_u\rangle = u^2 + vO(2), \quad \langle f_u, f_v\rangle = u + vO(k-1), \quad \langle f_v, f_v\rangle = 1 + O(k),$$

for any k. Since $\langle \boldsymbol{g}_1, \boldsymbol{g}_1\rangle = 1$, we have $\langle f_u, f_v\rangle = u + v^2 O(k)$ for any positive integer k by (2.7). Since

$$\langle f_u, f_u\rangle = u^2 + 2u\langle \boldsymbol{g}_1, \boldsymbol{g}_1'\rangle v + O(v^2) = u^2 + O(v^2),$$

we obtain $\langle f_u, f_u\rangle = u^2 + v^2\varphi^2$ where φ is a non-zero function whose Taylor expansion is the same as that of $|(f_u \times f_v)/v|$. The first few terms of Taylor expansion of φ is given by

$$\begin{aligned} \varphi = b_{1,1} + b_{1,2}u + \tfrac{c_{2,0}(c_{2,0}-c_{1,2})}{2b_{1,1}}v + \big(\tfrac{c_{1,2}^2 - c_{2,0}^2}{b_{1,1}} + b_{1,3} + b_{1,1}^3\big)\tfrac{u^2}{2} \\ + \big(\tfrac{b_{1,2}c_{2,0}(c_{1,2}-c_{2,0})}{b_{1,1}^2} - \tfrac{2c_{1,3}c_{2,0} + 2c_{1,2}c_{2,1} - 7c_{2,0}c_{2,1}}{4b_{1,1}}\big)uv \\ + \big(-\tfrac{c_{2,0}^2(c_{1,2}-c_{2,0})^2}{6b_{1,1}^3} + \tfrac{b_{1,2}c_{2,0}c_{2,1}}{6b_{1,1}^2} + \tfrac{1}{12}c_{2,1}^2 - \tfrac{1}{6}c_{2,0}c_{2,2} - \tfrac{1}{3}c_{1,2}c_{3,0} + \tfrac{1}{2}c_{2,0}c_{3,0} - \tfrac{1}{3}c_{2,0}^2 b_{1,1}\big)\tfrac{v^2}{2} + O(3). \end{aligned}$$

Lemma 2.15. *A unit normal $\boldsymbol{\nu}$ is expressed by*

$$\boldsymbol{\nu} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{c_{1,2}-c_{2,0}}{b_{1,1}} \\ 0 \end{pmatrix} u + \begin{pmatrix} c_{2,0} \\ \frac{c_{2,1}}{2b_{1,1}} \\ 0 \end{pmatrix} v + \begin{pmatrix} 2c_{2,0} - c_{1,2} \\ \frac{2b_{1,2}(c_{2,0}-c_{1,2})}{b_{1,1}^2} + \frac{c_{1,3}-2c_{2,1}}{b_{1,1}} \\ \frac{(c_{1,2}-c_{2,0})^2}{b_{1,1}^2} \end{pmatrix} \frac{u^2}{2}$$
$$+ \begin{pmatrix} c_{2,1}/2 \\ \frac{c_{2,0}(c_{1,2}-c_{2,0})^2}{2b_{1,1}^3} - \frac{b_{1,2}c_{2,1}}{2b_{1,1}^2} + \frac{c_{2,2}-c_{3,0}}{2b_{1,1}} + c_{2,0}b_{1,1} \\ \frac{(c_{1,2}-c_{2,0})c_{2,1}}{2b_{1,1}^2} \end{pmatrix} uv + \begin{pmatrix} c_{3,0} \\ \frac{c_{2,0}c_{2,1}(c_{1,2}-c_{2,0})}{2b_{1,1}^3} + \frac{c_{3,1}}{3b_{1,1}} \\ \frac{c_{2,1}^2}{4b_{1,1}^2} + c_{2,0}^2 \end{pmatrix} \frac{v^2}{2} + O(3).$$

In particular, $(f, \boldsymbol{\nu})$ is an embedding, if and only if $c_{2,0} \neq c_{1,2}$.

Proof. Since

$$(f_u \times f_v)/v = (0, 0, -b_{1,1}) + (b_{1,1}c_{2,0}, (c_{1,2} - c_{2,0})u + c_{2,1}v/2, -b_{1,2}u - c_{2,0}(c_{2,0} - c_{1,2})v/2) + O(2),$$

we obtain $|(f_u \times f_v)/v|^{-1/2} = \frac{1}{b_{1,1}} - \frac{b_{1,2}u + b_{2,1}v/2}{b_{1,1}^2} + O(2)$, and we conclude the formula up to order 2. The second order part is obtained similarly. The last assertion is a consequence of the following:

$$d(f, \boldsymbol{\nu})(0,0) = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{c_{1,2}-c_{2,0}}{b_{1,1}} & 0 \\ 1 & 0 & 0 & c_{2,0} & \frac{c_{2,1}}{2b_{1,1}} & 0 \end{pmatrix}.$$

□

Thus the initial terms of the second fundamental quantities are given by

$$L = \langle f_{uu}, \boldsymbol{\nu} \rangle = (c_{2,0} - c_{1,2})v - c_{2,0}u^2 + \Big(\frac{b_{1,2}(c_{1,2}-c_{2,0})}{b_{1,1}} + \frac{3}{2}c_{2,1} - c_{1,3}\Big)uv + \Big(\frac{b_{1,2}c_{2,1}}{2b_{1,1}} - \frac{1}{2}c_{2,2} + \frac{1}{2}c_{3,0} - c_{2,0}b_{1,1}^2\Big)v^2 + O(3), \tag{2.12}$$

$$M = \langle f_{uv}, \boldsymbol{\nu} \rangle = -c_{2,0}u - c_{2,1}v/2 - c_{2,1}u^2 - \Big(\frac{c_{2,0}(c_{2,0}-c_{1,2})^2}{2b_{1,1}^2} + \frac{c_{3,0}+c_{2,2}}{2}\Big)uv + \Big(\frac{c_{2,0}c_{2,1}(c_{2,0}-c_{1,2})}{4b_{1,1}^2} - \frac{c_{3,1}}{3}\Big)v^2 + O(3), \tag{2.13}$$

$$N = \langle f_{vv}, \boldsymbol{\nu} \rangle = -c_{2,0} - c_{2,1}u - c_{3,0}v - \Big(\frac{c_{2,0}(c_{2,0}-c_{1,2})^2}{b_{11}^2} + c_{2,2}\Big)\frac{u^2}{2} + \Big(\frac{c_{2,0}c_{2,1}(c_{2,0}-c_{1,2})}{2b_{11}^2} - c_{3,1}\Big)uv - \Big(\frac{c_{2,0}c_{2,1}^2}{8b_{1,1}^2} + \frac{1}{2}(c_{2,0}^3 + c_{4,0})\Big)v^2 + O(3). \tag{2.14}$$

We will use Christoffel symbols $\Gamma^u_{uu}, \Gamma^v_{uu}, \Gamma^u_{uv}, \Gamma^v_{uv}, \Gamma^u_{vv}, \Gamma^v_{vv}$ defined by

$$\begin{aligned} f_{uu} &= \Gamma^u_{uu} f_u + \Gamma^v_{uu} f_v + L\boldsymbol{\nu}, \\ f_{uv} &= \Gamma^u_{uv} f_u + \Gamma^v_{uv} f_v + M\boldsymbol{\nu}, \\ f_{vv} &= \Gamma^u_{vv} f_u + \Gamma^v_{vv} f_v + N\boldsymbol{\nu}. \end{aligned} \tag{2.15}$$

Lemma 2.16. *For any positive integer p, we have*

$$\begin{pmatrix} \Gamma^u_{uu} & \Gamma^u_{uv} & \Gamma^u_{vv} \\ \Gamma^v_{uu} & \Gamma^v_{uv} & \Gamma^v_{vv} \end{pmatrix} = \frac{1}{v\varphi} \begin{pmatrix} u\varphi + uv\varphi_v + v\varphi_u & \varphi + v\varphi_v & 0 \\ \begin{pmatrix} v\varphi - u^2\varphi - u^2 v\varphi_v \\ -uv\varphi_u - v^2\varphi^3 - v^2\varphi^2\varphi_v \end{pmatrix} & -u(\varphi + v\varphi_v) & 0 \end{pmatrix} + O(p).$$

Proof. Since

$$\begin{aligned}
&2(u+v^2\varphi\varphi_u)+O(p)=\langle f_u,f_u\rangle_u=2\langle f_{uu},f_u\rangle, && 2(v\varphi^2+v^2\varphi\varphi_v)+O(p)=\langle f_u,f_u\rangle_v=2\langle f_{uv},f_u\rangle\\
&1+O(p)=\langle f_u,f_v\rangle_u=\langle f_{uu},f_v\rangle+\langle f_u,f_{uv}\rangle, && O(p)=\langle f_u,f_v\rangle_v=\langle f_{uv},f_v\rangle+\langle f_u,f_{vv}\rangle,\\
&O(p)=\langle f_v,f_v\rangle_u=2\langle f_{uv},f_v\rangle, && O(p)=\langle f_v,f_v\rangle_v=2\langle f_{vv},f_v\rangle,
\end{aligned}$$

we obtain

$$\begin{aligned}
&\langle f_{uu},f_u\rangle=u+v^2\varphi\varphi_u+O(p), && \langle f_{uv},f_u\rangle=v(\varphi^2+v\varphi\varphi_v)+O(p), && \langle f_{vv},f_u\rangle=O(p),\\
&\langle f_{uu},f_v\rangle=1-v(\varphi^2+v\varphi\varphi_v)+O(p), && \langle f_{uv},f_v\rangle=O(p), && \langle f_{vv},f_v\rangle=O(p).
\end{aligned}$$

Since

$$\begin{aligned}
&\begin{pmatrix}\langle f_u,f_u\rangle & \langle f_u,f_v\rangle\\ \langle f_u,f_v\rangle & \langle f_v,f_v\rangle\end{pmatrix}\begin{pmatrix}\Gamma^u_{uu} & \Gamma^u_{uv} & \Gamma^u_{vv}\\ \Gamma^v_{uu} & \Gamma^v_{uv} & \Gamma^v_{vv}\end{pmatrix}\\
&\qquad=\frac{1}{2}\begin{pmatrix}\langle f_u,f_u\rangle_u & \langle f_u,f_u\rangle_v & 2\langle f_u,f_v\rangle_v-\langle f_v,f_v\rangle_u\\ 2\langle f_u,f_v\rangle_u-\langle f_u,f_u\rangle_v & \langle f_v,f_v\rangle_u & \langle f_v,f_v\rangle_v\end{pmatrix},
\end{aligned}$$

we have

$$\begin{aligned}
&\begin{pmatrix}u^2+v^2\varphi^2+O(p) & u+O(p)\\ u+O(p) & 1+O(p)\end{pmatrix}\begin{pmatrix}\Gamma^u_{uu} & \Gamma^u_{uv} & \Gamma^u_{vv}\\ \Gamma^v_{uu} & \Gamma^v_{uv} & \Gamma^v_{vv}\end{pmatrix}\\
&\qquad=\begin{pmatrix}u+v^2\varphi_u\varphi & v\varphi^2+v^2\varphi\varphi_v & 0\\ 1-v\varphi^2-v^2\varphi\varphi_v & 0 & 0\end{pmatrix}+O(p),
\end{aligned}$$

and thus

$$\begin{aligned}
\begin{pmatrix}\Gamma^u_{uu} & \Gamma^u_{uv} & \Gamma^u_{vv}\\ \Gamma^v_{uu} & \Gamma^v_{uv} & \Gamma^v_{vv}\end{pmatrix}&=\frac{1}{v^2\varphi^2}\begin{pmatrix}1 & -u\\ -u & u^2+v^2\varphi^2\end{pmatrix}\begin{pmatrix}u+v^2\varphi_u\varphi & v\varphi^2+v^2\varphi\varphi_v & 0\\ 1-v\varphi^2-v^2\varphi\varphi_v & 0 & 0\end{pmatrix}+O(p)\\
&=\frac{1}{v\varphi}\begin{pmatrix}u\varphi+uv\varphi_v+v\varphi_u & \varphi+v\varphi_v & 0\\ \begin{pmatrix}v\varphi-u^2\varphi-u^2v\varphi_v\\ -uv\varphi_u-v^2\varphi^3-v^3\varphi^2\varphi_v\end{pmatrix} & -u(\varphi+v\varphi_v) & 0\end{pmatrix}+O(p).
\end{aligned}$$

□

Lemma 2.17. *The Gauss curvature K is given by*

$$K=-\frac{2\varphi_v+v\varphi_{vv}}{v\varphi}+O(p).$$

Proof. Since $A=(\langle f_u,f_u\rangle\langle f_v,f_v\rangle-\langle f_u,f_v\rangle^2)^{1/2}=|v|\varphi+O(p)$, we have

$$\begin{aligned}
K&=\frac{1}{A}\Big[\Big(\frac{A\Gamma^u_{vv}}{\langle f_v,f_v\rangle}\Big)_u-\Big(\frac{A\Gamma^u_{uv}}{\langle f_v,f_v\rangle}\Big)_v\Big]=\frac{1}{|v|\varphi}[(0)_u-(\pm(\varphi+v\varphi_v))_v]+O(p)\\
&=-\frac{2\varphi_v+v\varphi_{vv}}{v\varphi}+O(p).
\end{aligned}$$

□

Remark 2.18. The formula in the previous lemma is equivalent to Gauss's equations. Minardi-Coddazi equations $\langle (f_{uu})_v,\boldsymbol{\nu}\rangle=\langle (f_{uv})_u,\boldsymbol{\nu}\rangle$, $\langle (f_{uv})_v,\boldsymbol{\nu}\rangle=\langle (f_{vv})_u,\boldsymbol{\nu}\rangle$ are stated as follows:

$$L_v - M_u + \tfrac{L-2uM+(u^2+\varphi^2/v^2)N}{v\varphi}(\varphi + v\varphi_v) + \tfrac{M\varphi_u+N(\varphi-u\varphi_u)}{\varphi} = O(p),$$
$$M_v - N_u + \tfrac{M-uN}{v\varphi}(\varphi + v\varphi_v) = O(p).$$

Proposition 2.19. *The singularity of f is swallowtail, if*

$$(f,\nu) : (\mathbb{R}^2, 0) \to (\mathbb{R}^3 \times \mathbb{R}^3, (0, \nu(0)))$$

is an embedding (*i.e.*, $c_{2,0} \neq c_{1,2}$).

Proof. See Appendix B.1. □

Theorem 2.20. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a map as in the first paragraph of this section with conditions (iv) and (v) of Lemma 2.2. If the singularity of f is swallowtail (i.e., $c_{2,0} \neq c_{1,2}$), then the asymptotic expansions of Gauss curvature K and the mean curvature H are given by*

$$\begin{aligned} K =& \frac{1}{v}\Big[\frac{c_{2,0}(c_{2,0}-c_{1,2})}{b_{1,1}^2} + \Big(\frac{3c_{2,0}b_{1,2}(c_{2,0}-c_{1,2})}{b_{11}^3} + \frac{c_{2,0}c_{1,3} + c_{2,1}c_{1,2} - \frac{7}{2}c_{2,0}c_{2,1}}{b_{1,1}^2}\Big)u \\ &-\Big(\frac{c_{2,0}^2(c_{2,0}-c_{1,2})^2}{b_{1,1}^4} - \frac{b_{1,2}c_{2,0}c_{2,1}}{2b_{1,1}^3} - \frac{c_{2,1}^2 - 2c_{2,0}c_{2,2} - 4c_{1,2}c_{3,0} + 6c_{2,0}c_{3,0}}{4b_{1,1}^2} + c_{2,0}^2\Big)v + O(2)\Big], \\ H =& \frac{1}{v}\Big[\frac{c_{2,0}-c_{1,2}}{2b_{1,1}^2} + \Big(\frac{3b_{1,2}(c_{1,2}-c_{2,0})}{2b_{1,1}^3} + \frac{5c_{2,1}-2c_{1,3}}{4b_{1,1}^2}\Big)u \\ &+\Big(\frac{-c_{2,0}(c_{1,2}-c_{2,0})^2}{2b_{1,1}^4} + \frac{b_{1,2}c_{2,1}}{4b_{1,1}^3} + \frac{c_{3,0}-c_{2,2}}{4b_{1,1}^2} - c_{2,0}\Big)v + O(2)\Big]. \end{aligned}$$

If the singularity of f is swallowtail (i.e., $c_{2,0} \neq c_{1,2}$), then the asymptotic expansions of the principal curvatures are $\kappa_1 = -c_{2,0} - c_{2,1}u + \big(\frac{c_{2,1}^2}{4(c_{1,2}-c_{2,0})} - c_{3,0}\big)v + O(2)$, and

$$\kappa_2 = \frac{1}{v}\Big[\frac{c_{1,2}-c_{2,0}}{b_{1,1}^2} + \Big(\frac{3b_{1,2}(c_{1.2}-c_{2,0})}{b_{1,1}^2} + \frac{5c_{2,1}-2c_{1,3}}{2b_{1,1}}\Big)u + \Big(\frac{b_{1,2}c_{2,1}}{2b_{1,1}^3} - \frac{c_{2,2}-c_{3,0}}{2b_{1,1}^2} - c_{2,0}\Big)v + O(2)\Big].$$

Proof. The assertions for K and H are followed by (A.1). The assertion for principal curvatures is obtained by solving the equation $\lambda^2 - 2H\lambda + K = 0$. □

REMARK 2.21. In [8], L. Martins, K. Saji, M. Umehara, and K. Yamda define the limiting normal curvature κ_ν, the normalized cuspidal curvature μ_c, and the limiting singular curvature τ_s for swallowtail. We have that

$$\kappa_\nu = -c_{2,0}, \quad \mu_c = \frac{c_{1,2}-c_{2,0}}{b_{1,1}^2}, \quad \tau_s = 2b_{1,1}.$$

The first equality is from (2.2) in [8]. We obtain the second comparing (4.6) in [8] with the expression of H in Theorem 2.20. The last one is from the definition of τ_s (the last line of the page 272 in [8]) and the fact that $\kappa_s = \kappa\cos\theta$ combining with (2.16) and (2.18) below. The referee kindly informed the author that a normal form theorem, similar to us, also appeared in K. Saji's recent paper ([13]). He described the configurations of asymptotic lines and curvature lines, for example. We see below that one can recover such results in our computation.

2.4. Asymptotic lines. The equation of the asymptotic directions is

$$((c_{2,0}-c_{1,2})v+O(2))du^2-(2c_{2,0}u-c_{2,1}v+O(2))du\,dv-(c_{2,0}+c_{2,1}u+c_{3,0}v+O(2))dv^2=0.$$

We observe that the coefficient of $u^2\,du^2$ is $c_{2,0}$.

When the singularity of f is swallowtail (i.e., $c_{1,2}-c_{2,0}\neq 0$), we conclude that there is a homeomorphism of $(\mathbb{R}^2,0)$ which sends solution curves of the equation above to that of folded saddle (resp. folded node, folded focus), if $c_{2,0}(3c_{2,0}+c_{1,2})>0$, (resp. $\frac{1}{8}(c_{2,0}-c_{1,2})^2<c_{2,0}(3c_{2,0}+c_{1,2})<0$, $c_{2,0}(3c_{2,0}+c_{1,2})<\frac{1}{8}(c_{2,0}-c_{1,2})^2$). Use Lemma B.2 in subsection B.2, to show this assertion.

2.5. Curvature lines. Since the equation of the principal directions is

$$\begin{vmatrix} u^2+v^2\varphi^2 & L & dv^2 \\ u & M & -du\,dv \\ 1 & N & du^2 \end{vmatrix}=0,$$

we have

$$v\Big[(O(1))du^2+(c_{1,2}-c_{2,0}+O(1))du\,dv+\Big(\frac{c_{2,1}}{2}+O(1)\Big)dv^2\Big]=0.$$

It defines two transvese directions in the source in the region $v\neq 0$ and it extends on $v=0$ as two transvese directions, when the singularity of f is swallowtail (i.e., $c_{1,2}-c_{2,0}\neq 0$).

2.6. Ridge and subparabolic lines. We show here computational experiences. Since principal vectors, on $v\neq 0$, are represented by

$$\begin{aligned}
\boldsymbol{v}_1 &=\Big(\frac{c_{2,1}}{2c_{2,0}-c_{1,2}}+O(1)\Big)\partial_u+\Big(1-\frac{c_{2,1}u}{2(c_{2,0}-c_{1,2})}+O(2)\Big)\partial_v\\
\boldsymbol{v}_2 &=\Big(\frac{1}{b_{1,1}v}+O(1)\Big)\partial_u+\Big(-\frac{u}{b_{1,1}v}+O(2)\Big)\partial_v,
\end{aligned}$$

we obtain

$$\begin{aligned}
\boldsymbol{v}_1\kappa_1 &=\frac{3}{4}\frac{c_{2,1}^2}{c_{1,2}-c_{2,0}}-c_{3,0}+O(1),\\
\boldsymbol{v}_2\kappa_1 &=\frac{1}{v}\Big[-\frac{c_{2,1}}{b_{1,1}}+O(1)\Big],\\
\boldsymbol{v}_1\kappa_2 &=\frac{1}{v^2}\Big[\frac{c_{1,2}-c_{2,0}}{b_{1,1}^2}+O(1)\Big],\\
\boldsymbol{v}_2\kappa_2 &=\frac{1}{v^3}\Big[\frac{1}{b_{1,1}^3}\Big((c_{2,0}-c_{1,2})u-\Big[(c_{1,3}-\tfrac{5}{2}c_{2,1})+\frac{3b_{1,2}(c_{2,0}-c_{1,2})}{b_{1,1}^4}\Big]v\Big)+O(1)\Big].
\end{aligned}$$

Thus we have the following:

- A $\boldsymbol{v}_1$-ridge line is arriving at swallowtail, only if $\frac{3}{4}\frac{c_{2,1}^2}{c_{1,2}-c_{2,0}}=c_{3,0}$.
- A $\boldsymbol{v}_2$-subparabolic line is arriving at swallowtail, only if $c_{2,1}=0$.
- No $\boldsymbol{v}_1$-subparabolic line is arriving at swallowtail.
- Exactly one $\boldsymbol{v}_2$-ridge line is arriving at swallowtail.

2.7. Cuspidal edge nearby swallowtail. Suppose that there is a coordinate (u, v) with

$$\langle f_u, f_u\rangle = u^2 + v^2\varphi^2 + O(p),\ \langle f_u, f_v\rangle = u + O(p),\ \langle f_v, f_v\rangle = 1 + O(p),$$

for any p. The goal of this sebsection is to obtain asymptotic expansions of differential geometric invariants of cuspidal edge. They are functions on $\Sigma(f) \setminus \{(0,0)\}$ near $(0,0)$, that is, as meromorphic functions in u. Here u is a parameter of the singular curve $\Sigma(f)$ so that $u^2/2$ is an arc length parameter of $\Sigma(f)$. The statements of asymptotic expansions of differential geometric invariants of cuspidal edge, defined in (1.4) and (1.5), are as follows.

Theorem 2.22. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a map as in the first paragraph of this section with conditions (iv) and (v) of Lemma 2.2. Assume that $b_{1,1} \neq 0$. The asymptotic expansions of κ, τ, θ, b_3 are given as follows:*

$$\kappa = \frac{1}{|u|}\Big[b_{1,1} + b_{1,2}u + \Big(b_{1,3} + b_{1,1}^3 + \tfrac{c_{1,2}^2}{b_{1,1}}\Big)\frac{u^2}{2} + O(u^3)\Big], \tag{2.16}$$

$$\begin{aligned}\tau = &\frac{1}{u}\Big[\frac{c_{1,2}}{b_{1,1}} + \frac{b_{1,1}c_{1,3} - 2b_{1.2}c_{1,2}}{2b_{1,1}^2}u \\ &+\Big(\frac{2c_{1,2}(3b_{1,2}^2 - c_{1,2}^2)}{b_{1,1}^3} - \frac{3(b_{1,3}c_{1,2} + b_{1,2}c_{1,3})}{b_{1,1}^2} + \frac{c_{1,4}}{b_{1,1}} - 2b_{1,1}c_{1,2}\Big)\frac{u^2}{2} + O(u^3)\Big],\end{aligned} \tag{2.17}$$

$$\cos\theta = -1 + \frac{c_{2,0}^2}{b_{1,1}^2}\frac{u^2}{2} - \frac{c_{2,0}(b_{1,2}c_{2,0} - b_{1,1}c_{2,1})}{b_{1,1}^3}u^3 + O(u^4), \tag{2.18}$$

$$b_3 = \frac{-1}{|b_{1,1}u|^{\frac{1}{2}}}\Big(\frac{2(c_{1,2} - c_{2,0})}{b_{1,1}} + \Big(\frac{5c_{2,1} - 2c_{1,3}}{b_{1,1}} + \frac{b_{1,2}(c_{2,0} - c_{1,2})}{b_{1,1}^2}\Big)u + O(u^2)\Big). \tag{2.19}$$

Proof of (2.16), (2.17). Since $s = u^2/2$ is an arc length parameter of $\Sigma(f)$, we have

$$\frac{d\boldsymbol{g}_0}{ds} = \frac{d\boldsymbol{g}_1/du}{ds/du} = \frac{1}{u}\frac{d\boldsymbol{g}_0}{du} = \boldsymbol{g}_1, \qquad \frac{d^2\boldsymbol{g}_0}{ds^2} = \frac{d\boldsymbol{g}_1}{ds} = \frac{1}{u}\frac{d\boldsymbol{g}_1}{du}.$$

We thus obtain an asymptotic expansion of $\boldsymbol{n}$, κ, τ as follows:

$$\boldsymbol{n} = \frac{d\boldsymbol{g}_1/du}{|d\boldsymbol{g}_1/du|} = \begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}-b_{1,1}\\0\\\frac{c_{1,2}}{b_{1,1}}\end{pmatrix}u + \begin{pmatrix}-b_{1,3}\\-b_{1,1}^2 - \frac{c_{1,2}^2}{b_{1,1}^2}\\\frac{b_{1,1}c_{1,3} - 2b_{1,2}c_{1,2}}{b_{1,1}^2}\end{pmatrix}\frac{u^2}{2} + O(u^3), \tag{2.20}$$

$$\kappa = \Big|\frac{d\boldsymbol{g}_1}{ds}\Big| = \frac{1}{|u|}\Big|\frac{d\boldsymbol{g}_1}{du}\Big| = \frac{1}{|u|}\Big(b_{1,1} + b_{1,2}u + \Big(b_{1,3} + b_{1,1}^3 + \tfrac{c_{1,2}^2}{b_{1,1}}\Big)\frac{u^2}{2} + O(u^3)\Big),$$

$$\begin{aligned}\tau = &\frac{|\boldsymbol{g}_0'\ \boldsymbol{g}_0''\ \boldsymbol{g}_0'''|}{|\boldsymbol{g}_0' \times \boldsymbol{g}_0''|^2} = \frac{|\boldsymbol{g}_1\ \boldsymbol{g}_1'\ \boldsymbol{g}_1''|}{u|\boldsymbol{g}_1 \times \boldsymbol{g}_1'|^2} = \frac{1}{u}\Big[\frac{c_{1,2}}{b_{1,1}} + \frac{b_{1,1}c_{1,3} - 2b_{1.2}c_{1,2}}{2b_{1,1}^2}u \\ &+\Big(\frac{2c_{1,2}(3b_{1,2}^2 - c_{1,2}^2)}{b_{1,1}^3} - \frac{3(b_{1,3}c_{1,2} + b_{1,2}c_{1,3})}{b_{1,1}^2} + \frac{c_{1,4}}{b_{1,1}} - 2b_{1,1}c_{1,2}\Big)\frac{u^2}{2} + O(u^3)\Big].\end{aligned}$$

□

Set $\Phi = (f_u - uf_v)/v$. We have $|\Phi| = \varphi$, since

$$|f_u - uf_v|^2 = \langle f_u - uf_v, f_u - uf_v\rangle = \langle f_u, f_u\rangle - 2u\langle f_u, f_v\rangle + u^2\langle f_v, f_v\rangle = v^2\varphi^2.$$

Since $\langle f_u, f_u - uf_v\rangle = \langle f_u, f_u\rangle - u\langle f_u, f_v\rangle = v^2\varphi^2$,

$$\cos\angle(f_u,\Phi) = \frac{\langle f_u, f_u - uf_v\rangle}{|f_u||f_u - uf_v|} = \frac{|v|\varphi}{(u^2+v^2\varphi^2)^{1/2}} + O(p) \to 0\ (v\to 0),$$

whenever $u \neq 0$. Thus we conclude that the three vectors

$$\boldsymbol{a}_1 = \lim_{v\to 0}\frac{f_u}{u},\quad \boldsymbol{a}_2 = -\lim_{v\to 0}\frac{\Phi}{\varphi},\quad \boldsymbol{a}_3 = -\lim_{v\to 0}\frac{f_u\times\Phi}{u\varphi} = \lim_{v\to 0}\frac{f_u\times f_v}{v\varphi}$$

form an orthonormal frame along $\Sigma(f)$. We also have

$$\boldsymbol{a}_1 = \frac{f_u(u,0)}{u} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\b_{1,1}\\0\end{pmatrix}u + \begin{pmatrix}-b_{1,1}^2\\b_{1,2}\\c_{1,2}\end{pmatrix}\frac{u^2}{2} + \begin{pmatrix}-b_{1,1}b_{1,3}\\b_{1,3}\\c_{1,3}\end{pmatrix}\frac{u^3}{6} + O(u^4).$$

Since

$$\lim_{v\to 0}\Phi = \begin{pmatrix}0\\b_{1,1}\\0\end{pmatrix} + \begin{pmatrix}-b_{1,1}^2\\b_{1,2}\\c_{1,2}-c_{2,0}\end{pmatrix}u + \begin{pmatrix}3b_{1,1}b_{1,2}\\b_{1,3}+\frac{2(c_{1,2}-c_{2,0})c_{2,0}}{b_{1,1}}\\c_{1,3}-2c_{2,1}\end{pmatrix}\frac{u^2}{2} + O(u^3), \tag{2.21}$$

and by $b_{1,1} > 0$, we obtain

$$\begin{aligned}\boldsymbol{a}_2 &= -\frac{f_u - uf_v}{|f_u - uf_v|}(u,0)\\ &= \begin{pmatrix}0\\-1\\0\end{pmatrix} + \begin{pmatrix}b_{1,1}\\0\\-\frac{c_{1,2}-c_{2,0}}{b_{1,1}}\end{pmatrix}u + \begin{pmatrix}b_{1,2}\\b_{1,1}^2+\frac{(c_{1,2}-c_{2,0})^2}{b_{1,1}^2}\\-2\frac{b_{1,2}}{b_{1,1}^2}(c_{2,0}-c_{1,2})-\frac{c_{1,3}-2c_{2,1}}{b_{1,1}^2}\end{pmatrix}\frac{u^2}{2} + O(u^3),\end{aligned} \tag{2.22}$$

$$\boldsymbol{a}_3 = \boldsymbol{a}_1\times\boldsymbol{a}_2 = \begin{pmatrix}0\\0\\-1\end{pmatrix} - \begin{pmatrix}0\\\frac{c_{2,0}-c_{1,2}}{b_{1,1}}\\0\end{pmatrix}u - \begin{pmatrix}c_{1,2}-2c_{2,0}\\\frac{2b_{1,2}(c_{1,2}-c_{2,0})}{b_{1,1}^2}+\frac{c_{2,1}-c_{1,3}}{b_{1,1}}\\\frac{(c_{2,0}-c_{1,2})^2}{b_{1,1}^2}\end{pmatrix}\frac{u^2}{2} + O(u^3). \tag{2.23}$$

Proof of (2.18). A Direct computation based on (2.20) and (2.22) shows

$$\cos\theta = \langle\boldsymbol{n},\boldsymbol{a}_2\rangle = -1 + \frac{c_{2,0}^2}{b_{1,1}^2}\frac{u^2}{2} - \frac{c_{2,0}(b_{1,2}c_{2,0}-b_{1,1}c_{2,1})}{b_{1,1}^3}u^3 + O(u^4),$$

which completes the proof. □

Set $u = p_0 + tP$, $v = tQ$ where $P = \sum_{i=1}^m p_i(s)t^{i-1}/i!$, $Q = \sum_{i=0}^m q_i(s)t^i/i!$. We take P, Q so that

$$\langle f_s(s,0), f_t(s,t)\rangle = 0, \qquad \langle f_t(s,t), f_t(s,t)\rangle = t^2.$$

When we set $f(s,t) = \sum_{k=0}^m \boldsymbol{f}_k(s)t^k/k! + O(m+1)$, we have

$$f_s(s,0) = \boldsymbol{f}_0'(s),\quad f_t(s,t) = \sum_{k=0}^{m-1}\boldsymbol{f}_{k+1}(s)\frac{t^k}{k!} + O(t^m).$$

We obtain

$$\boldsymbol{f}_1(s) = 0,\ \boldsymbol{f}_2(s) = \boldsymbol{a}_2(s),\ \langle\boldsymbol{f}_0'(s),\boldsymbol{f}_k(s)\rangle = 0. \tag{2.24}$$

Since $\boldsymbol{f}_k = a_k\boldsymbol{a}_2 + b_k\boldsymbol{a}_3$, we have $a_k = \langle\boldsymbol{f}_k,\boldsymbol{a}_2\rangle$, $b_k = \langle\boldsymbol{f}_k,\boldsymbol{a}_3\rangle$, and

$$\langle \boldsymbol{a}_2, \boldsymbol{f}_3\rangle = 0,\ \tfrac{1}{3}\langle \boldsymbol{a}_2, \boldsymbol{f}_4\rangle + \tfrac{1}{4}\langle \boldsymbol{f}_3, \boldsymbol{f}_3\rangle = 0,\ \tfrac{1}{12}\langle \boldsymbol{a}_2, \boldsymbol{f}_5\rangle + \tfrac{1}{6}\langle \boldsymbol{f}_3, \boldsymbol{f}_4\rangle = 0, \tag{2.25}$$

$$\frac{2\langle \boldsymbol{a}_2(s), \boldsymbol{f}_k(s)\rangle}{(k-1)!} + \sum_{i=2}^{k-2} \frac{\langle \boldsymbol{f}_{i+1}(s), \boldsymbol{f}_{k-i+1}(s)\rangle}{i!(k-i)!} = 0\ (k \geq 6). \tag{2.26}$$

We obtain that $a_3 = 0$, $a_4/3 + b_3^2/4 = 0$, $a_5/2 + b_3 b_4 = 0$, ans so on. Since $\langle \boldsymbol{f}_k, \boldsymbol{a}_1\rangle$ is degree one in q_k and $a_k = \langle \boldsymbol{f}_k, \boldsymbol{a}_2\rangle$ is degree one in p_{k-1}, the conditions (2.24) and (2.25), (2.26) determine p_{k-1}, q_k inductively.

Lemma 2.23. *We have the following:*

$$\begin{aligned}
\boldsymbol{f}_0 &= f(p_0, 0),\\
\boldsymbol{f}_1 &= p_1 f_u(p_0, 0) + q_1 f_v(p_0, 0),\\
\boldsymbol{f}_2 &= p_2 f_u(p_0, 0) + q_2 f_v(p_0, 0) + p_1^2 f_{uu}(p_0, 0) + 2p_1 q_1 f_{uv}(p_0, 0) + q_1^2 f_{vv}(p_0, 0),\\
\boldsymbol{f}_3 &= p_3 f_u(p_0, 0) + q_3 f_v(p_0, 0) + 3[p_1 p_2 f_{uu}(p_0, 0) + (p_1 q_2 + q_1 p_2) f_{uv}(p_0, 0) + q_1 q_2 f_{vv}(p_0, 0)]\\
&\quad + p_1^3 f_{uuu}(p_0, 0) + 3p_1^2 q_1 f_{uuv}(p_0, 0) + 3p_1 q_1^2 f_{uvv}(p_0, 0) + q_1^3 f_{vvv}(p_0, 0).
\end{aligned}$$

Proof. Consequences of the following identity:

$$\begin{aligned}
\sum_{k=0}^{m-1} \boldsymbol{f}_{k+1}(s)\frac{t^k}{k!} + O(m) &= f_t(s,t) = \frac{\partial u}{\partial t} f_u(u,v) + \frac{\partial v}{\partial t} f_v(u,v)\\
&= \sum_{i=0}^{m-1} p_{i+1}(s)\frac{t^i}{i!} f_u(u,v) + \sum_{i=0}^{m-1} q_{i+1}(s)\frac{t^i}{i!} f_v(u,v) + O(m).
\end{aligned}$$

□

Lemma 2.24. *We have that* $p_1 = |p_0\varphi(p_0, 0)|^{-1/2}$*, and* $q_1 = -p_0|p_0\varphi(p_0, 0)|^{-1/2}$.

Proof. We show that $p_1 p_0 + q_1 = 0$ and $p_1 q_1 \varphi(p_0, 0) + 1 = 0$. Since

$$\begin{aligned}
0 = \boldsymbol{f}_1 &= (p_1 f_u + q_1 f_v)(p_0, 0) = \lim_{u\to p_0, v\to 0}\Big[\Big(p_1 + \frac{q_1}{u}\Big) f_u(p_0, v) - \frac{v q_1}{u}\Big(\frac{f_u - u f_v}{v}\Big)(p_0, v)\Big]\\
&= \lim_{u\to p_0, v\to 0}\Big[(p_1 u + q_1)\boldsymbol{a}_1 + \frac{v q_1 \varphi}{u}\boldsymbol{a}_2\Big] = (p_1 p_0 + q_1)\boldsymbol{a}_1,
\end{aligned}$$

we have $p_1 p_0 + q_1 = 0$. By Lemma 2.23,

$$\begin{aligned}
\boldsymbol{f}_2 &= [p_2 f_u + q_2 f_v + p_1^2 f_{uu} + 2p_1 q_1 f_{uv} + q_1^2 f_{vv}](p_0, 0)\\
&= \lim_{u\to p_0, v\to 0}\Big[\Big(p_2 + \frac{q_2}{u}\Big) f_u - \frac{v q_2}{u}\Big(\frac{f_u - u f_v}{v}\Big) + p_1^2 f_{uu} + 2p_1 q_1 f_{uv} + q_1^2 f_{vv}\Big]\\
&= (p_2 p_0 + q_2)\boldsymbol{a}_1 + [p_1^2 f_{uu} + 2p_1 q_1 f_{uv} + q_1^2 f_{vv}](p_0, 0).
\end{aligned} \tag{2.27}$$

By Lemma 2.25 below, we have

$$\begin{aligned}
f_{uu}(p_0, 0) &= \boldsymbol{a}_1 - p_0\varphi(p_0, 0)\boldsymbol{a}_2 + L(p_0, 0)\boldsymbol{a}_3,\\
f_{uv}(p_0, 0) &= -\varphi(p_0, 0)\boldsymbol{a}_2 + M(p_0, 0)\boldsymbol{a}_3,\\
f_{vv}(p_0, 0) &= N(p_0, 0)\boldsymbol{a}_3,
\end{aligned}$$

and we conclude that

$$1 = \langle \boldsymbol{f}_2, \boldsymbol{a}_2\rangle = -p_1^2 p_0\varphi(p_0, 0) - 2p_1q_1\varphi(p_0, 0) = -\varphi(p_0, 0)p_1(p_1p_0 + 2q_1) = -\varphi(p_0, 0)p_1q_1.$$

□

Let us consider a frame

$$\boldsymbol{A}_1 = \frac{f_u}{u}, \quad \boldsymbol{A}_2 = -\frac{\Phi}{\varphi}, \quad \boldsymbol{A}_3 = \boldsymbol{A}_1 \times \boldsymbol{A}_2 = \frac{f_u \times f_v}{v\varphi} = \boldsymbol{\nu}$$

defined on $u \neq 0$. These are extensions of $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{a}_3$. We remark that

$$\begin{pmatrix} \boldsymbol{A}_1 \\ \boldsymbol{A}_2 \\ \boldsymbol{A}_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{u} & 0 & 0 \\ -\frac{1}{v\varphi} & \frac{u}{v\varphi} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_u \\ f_v \\ \boldsymbol{\nu} \end{pmatrix}, \quad \begin{pmatrix} f_u \\ f_v \\ \boldsymbol{\nu} \end{pmatrix} = \begin{pmatrix} u & 0 & 0 \\ 1 & \frac{v\varphi}{u} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{A}_1 \\ \boldsymbol{A}_2 \\ \boldsymbol{A}_3 \end{pmatrix}.$$

Since $f_u - uf_v = v\Phi$, we have

$$L - uM = vL_1, \qquad M - uN = vM_1,$$

where $L_1 = \langle \Phi_u, \boldsymbol{\nu}\rangle$ and $M_1 = \langle \Phi_v, \boldsymbol{\nu}\rangle$. By (2.12) and (2.13), we have that

(2.28)
$$L_1 = c_{2,0} - c_{1,2} + (2c_{2,1} - c_{1,3} - \tfrac{b_{1,2}(c_{1,2} - c_{2,0})}{b_{1,1}})u + (\tfrac{b_{1,2}c_{2,1}}{b_{1,1}} + c_{3,0} - c_{2,2} - c_{2,0}b_{1,1}^2)v + O(2).$$

We observe that

$$\begin{aligned}
\langle \boldsymbol{A}_1, \boldsymbol{A}_1\rangle &= \tfrac{\langle f_u, f_u\rangle}{u^2} = 1 + \tfrac{v^2\varphi^2}{u^2} + O(p-2), \\
\langle \boldsymbol{A}_1, \boldsymbol{A}_2\rangle &= -\tfrac{\langle f_u, f_u\rangle - u\langle f_u, f_v\rangle}{uv\varphi} = -\tfrac{u^2 + v^2\varphi^2 - u^2 + O(p)}{uv\varphi} = -\tfrac{v\varphi}{u} + O(p-1), \\
\langle \boldsymbol{A}_2, \boldsymbol{A}_2\rangle &= \tfrac{|\Phi|^2}{\varphi^2} = 1.
\end{aligned}$$

Lemma 2.25. *We have*

$$\begin{aligned}
f_{uu} &= [1 - v\varphi(\varphi + v\varphi_v)]\boldsymbol{A}_1 - [v(\varphi_u - \tfrac{\varphi}{u}) + (u + \tfrac{v^2\varphi^2}{u})(\varphi + v\varphi_v)]\boldsymbol{A}_2 + L\boldsymbol{A}_3, \\
f_{uv} &= -(\varphi + v\varphi_v)\boldsymbol{A}_2 + M\boldsymbol{A}_3, \\
f_{vv} &= N\boldsymbol{A}_3.
\end{aligned}$$

Proof. Since $f_u = u\boldsymbol{A}_1$, $f_v = \boldsymbol{A}_1 + \frac{v\varphi}{u}\boldsymbol{A}_2$, we obtain that

$$\begin{aligned}
f_{uu} &= \Gamma^u_{uu}f_u + \Gamma^v_{uu}f_v + L\boldsymbol{\nu} = (u\Gamma^u_{uu} + \Gamma^v_{uu})\boldsymbol{A}_1 + \tfrac{v\varphi}{u}\Gamma^v_{uu}\boldsymbol{A}_2 + L\boldsymbol{\nu}, \\
f_{uv} &= \Gamma^u_{uv}f_u + \Gamma^v_{uv}f_v + M\boldsymbol{\nu} = (u\Gamma^u_{uu} + \Gamma^v_{uv})\boldsymbol{A}_1 + \tfrac{v\varphi}{u}\Gamma^v_{uv}\boldsymbol{A}_2 + M\boldsymbol{\nu}, \\
f_{vv} &= \Gamma^u_{vv}f_u + \Gamma^v_{vv}f_v + N\boldsymbol{\nu} = (u\Gamma^u_{vv} + \Gamma^v_{vv})\boldsymbol{A}_1 + \tfrac{v\varphi}{u}\Gamma^v_{vv}\boldsymbol{A}_2 + N\boldsymbol{\nu}.
\end{aligned}$$

We then conclude the lemma, by (2.15). □

Proof of (2.19). The asymptotic expansion of b_3 is obtained by (2.28) and (2.14), because of the following Proposition 2.26. □

Proposition 2.26. $b_3 = -u^{-1/2}\varphi^{-3/2}(2L_1 - uN_u + u^2N_v)(p_0, 0)$.

Before the proof of this proposition, we need some preliminary

Lemma 2.27. *We have*

$$
\begin{aligned}
\Phi_u &= -\varphi(\varphi + v\varphi_v)\boldsymbol{A}_1 - \big(\varphi_u + \tfrac{v\varphi^2}{u}(\varphi + v\varphi_v)\big)\boldsymbol{A}_2 + L_1\boldsymbol{A}_3,\\
\Phi_v &= -\varphi_v\boldsymbol{A}_2 + \langle \Phi_v, \boldsymbol{\nu}\rangle \boldsymbol{A}_3,\\
(\boldsymbol{A}_1)_u &= -\tfrac{v\varphi(\varphi+v\varphi_v)}{u}\boldsymbol{A}_1 - \big((1 + \tfrac{v^2\varphi^2}{u^2})(\varphi + v\varphi_v) + \tfrac{v(u\varphi_u-\varphi)}{u^2}\big)\boldsymbol{A}_2 + \tfrac{L}{u}\boldsymbol{A}_3,\\
(\boldsymbol{A}_2)_u &= (\varphi + v\varphi_v)(\boldsymbol{A}_1 + \tfrac{v\varphi}{u}\boldsymbol{A}_2) - \tfrac{L_1}{\varphi}\boldsymbol{A}_3,\\
(\boldsymbol{A}_3)_u &= -M\boldsymbol{A}_1 + (\tfrac{L_1}{\varphi} - \tfrac{vM\varphi}{u})\boldsymbol{A}_2,\\
(\boldsymbol{A}_1)_v &= -\tfrac{\varphi+v\varphi_v}{u}\boldsymbol{A}_2 + \tfrac{M}{u}\boldsymbol{A}_3,\\
(\boldsymbol{A}_2)_v &= -\tfrac{M_1}{\varphi}\boldsymbol{A}_3,\\
(\boldsymbol{A}_3)_v &= -N\boldsymbol{A}_1 + (\tfrac{M_1}{\varphi} - \tfrac{vN\varphi}{u})\boldsymbol{A}_2.
\end{aligned}
$$

Proof. The formula for Φ_u and Φ_v can be concluded as follows:

$$
\begin{aligned}
\Phi_u &= \big(\tfrac{f_u-uf_v}{v}\big)_u = \tfrac{f_{uu}-uf_{uv}-f_v}{v}\\
&= \tfrac{1}{v}\big[(1 - v\varphi(\varphi + v\varphi_v))\boldsymbol{A}_1 - \big(v(\varphi_u - \tfrac{\varphi}{u}) + (u + \tfrac{v^2\varphi^2}{u})(\varphi + v\varphi_v)\big)\boldsymbol{A}_2 + L\boldsymbol{A}_3\big]\\
&\quad - \tfrac{u}{v}[-(\varphi + v\varphi_v)\boldsymbol{A}_2 + M\boldsymbol{A}_3] - \tfrac{1}{v}\boldsymbol{A}_1 - \tfrac{\varphi}{u}\boldsymbol{A}_2\\
&= -\varphi(\varphi + v\varphi_v)\boldsymbol{A}_1 - \big(\varphi_u + \tfrac{v\varphi^2}{u}(\varphi + v\varphi_v)\big)\boldsymbol{A}_2 + L_1\boldsymbol{A}_3,\\
\Phi_v &= \big(\tfrac{f_u-uf_v}{v}\big)_v = \tfrac{f_{uv}-uf_{vv}}{v} - \tfrac{f_u-uf_v}{v^2} = \tfrac{f_{uv}-f_{vv}}{v} - \tfrac{\Phi}{v}\\
&= -\tfrac{\varphi+v\varphi_v}{v}\boldsymbol{A}_2 + \langle \Phi_v, \nu\rangle \boldsymbol{A}_3 + \tfrac{\varphi}{v}\boldsymbol{A}_2 = -\varphi_v\boldsymbol{A}_2 + \langle \Phi_v, \boldsymbol{\nu}\rangle \boldsymbol{A}_3.
\end{aligned}
$$

We compute the differentials of $\boldsymbol{A}_1$, $\boldsymbol{A}_2$ as follows:

$$
\begin{aligned}
(\boldsymbol{A}_1)_u &= (f_u/u)_u = \tfrac{f_{uu}}{u} - \tfrac{f_u}{u^2}\\
&= \tfrac{1}{u}\Big[(1 - v\varphi(\varphi + v\varphi_v))\boldsymbol{A}_1 - \big(v(\varphi_u - \tfrac{\varphi}{u}) + (u + \tfrac{v^2\varphi^2}{u})(\varphi + v\varphi_v)\big)\boldsymbol{A}_2 + L\boldsymbol{A}_3\Big] - \tfrac{1}{u}\boldsymbol{A}_1\\
&= -\tfrac{v\varphi(\varphi+v\varphi_v)}{u}\boldsymbol{A}_1 - \big((1 + \tfrac{v^2\varphi^2}{u^2})(\varphi + v\varphi_v) + \tfrac{v(u\varphi_u-\varphi)}{u^2}\big)\boldsymbol{A}_2 + \tfrac{L}{u}\boldsymbol{A}_3,\\
(\boldsymbol{A}_2)_u &= -\big(\tfrac{\Phi}{\varphi}\big)_u = -\tfrac{\Phi_u}{\varphi} + \tfrac{\varphi_u\Phi}{\varphi^2}\\
&= \tfrac{1}{\varphi}\big[\varphi(\varphi + v\varphi_v)\boldsymbol{A}_1 + \big(\varphi_u + \tfrac{v\varphi^2}{u}(\varphi + v\varphi_v)\big)\boldsymbol{A}_2 - L_1\boldsymbol{A}_3\big] - \tfrac{\varphi_u}{\varphi}\boldsymbol{A}_2\\
&= (\varphi + v\varphi_v)\boldsymbol{A}_1 + \tfrac{v\varphi}{u}(\varphi + v\varphi_v)\boldsymbol{A}_2 - \tfrac{L_1}{\varphi}\boldsymbol{A}_3,\\
(\boldsymbol{A}_1)_v &= \big(\tfrac{f_u}{u}\big)_v = \tfrac{f_{uv}}{u} = -\tfrac{\varphi+v\varphi_v}{u}\boldsymbol{A}_2 + \tfrac{M}{u}\boldsymbol{A}_3,\\
(\boldsymbol{A}_2)_v &= -\big(\tfrac{\Phi}{\varphi}\big)_v = -\tfrac{\Phi_v}{\varphi} + \tfrac{\varphi_v}{\varphi^2}\Phi = \tfrac{\varphi_v}{\varphi}\boldsymbol{A}_2 + \tfrac{\langle \Phi_v, \boldsymbol{\nu}\rangle}{\varphi}\boldsymbol{A}_3 - \tfrac{\varphi_v}{\varphi}\boldsymbol{A}_2 = -\tfrac{\langle \Phi_v, \boldsymbol{\nu}\rangle}{\varphi}\boldsymbol{A}_3.
\end{aligned}
$$

Setting $(\boldsymbol{A}_3)_u = s_{1,1}\boldsymbol{A}_1 + s_{1,2}\boldsymbol{A}_2$, $(\boldsymbol{A}_3)_v = s_{2,1}\boldsymbol{A}_1 + s_{2,2}\boldsymbol{A}_2$, we have

$$
\begin{aligned}
0 &= \langle \boldsymbol{A}_1, \boldsymbol{A}_3\rangle_u = \langle (\boldsymbol{A}_1)_u, \boldsymbol{A}_3\rangle + \langle \boldsymbol{A}_1, (\boldsymbol{A}_3)_u\rangle = \tfrac{L}{u} + s_{1,1}(1 + \tfrac{v^2\varphi^2}{u^2}) - s_{1,2}\tfrac{v\varphi}{u},\\
0 &= \langle \boldsymbol{A}_2, \boldsymbol{A}_3\rangle_u = \langle (\boldsymbol{A}_2)_u, \boldsymbol{A}_3\rangle + \langle \boldsymbol{A}_2, (\boldsymbol{A}_3)_u\rangle = -\tfrac{L_1}{\varphi} - s_{1,1}\tfrac{v\varphi}{u} + s_{1,2},\\
0 &= \langle \boldsymbol{A}_1, \boldsymbol{A}_3\rangle_v = \langle (\boldsymbol{A}_1)_v, \boldsymbol{A}_3\rangle + \langle \boldsymbol{A}_1, (\boldsymbol{A}_3)_v\rangle = \tfrac{M}{u} + s_{2,1}(1 + \tfrac{v^2\varphi^2}{u^2}) - s_{2,2}\tfrac{v\varphi}{u},\\
0 &= \langle \boldsymbol{A}_2, \boldsymbol{A}_3\rangle_v = \langle (\boldsymbol{A}_2)_v, \boldsymbol{A}_3\rangle + \langle \boldsymbol{A}_2, (\boldsymbol{A}_3)_v\rangle = -\tfrac{M_1}{\varphi} - s_{2,1}\tfrac{v\varphi}{u} + s_{2,2}.
\end{aligned}
$$

Solving the equation

$$\begin{pmatrix} s_{1,1} & s_{1,2} \\ s_{2,1} & s_{2,2} \end{pmatrix} \begin{pmatrix} 1 + \frac{v^2\varphi^2}{u^2} & -v\varphi/u \\ -v\varphi/u & 1 \end{pmatrix} = \begin{pmatrix} -L/u & L_1/\varphi \\ -M/u & M_1/\varphi \end{pmatrix},$$

we have

$$\begin{aligned} \begin{pmatrix} s_{1,1} & s_{1,2} \\ s_{2,1} & s_{2,2} \end{pmatrix} &= \begin{pmatrix} -L/u & L_1/\varphi \\ -M/u & M_1/\varphi \end{pmatrix} \begin{pmatrix} 1 & v\varphi/u \\ v\varphi/u & 1 + \frac{v^2\varphi^2}{u^2} \end{pmatrix} \\ &= \begin{pmatrix} \langle -f_{uu} + v\Phi_u, \boldsymbol{\nu}\rangle & \langle -\frac{v\varphi}{u^2} f_{uu} + (\frac{1}{\varphi} + \frac{v^2\varphi}{u^2})\Phi_u, \boldsymbol{\nu}\rangle \\ \langle -f_{uv} + v\Phi_v, \boldsymbol{\nu}\rangle & \langle -\frac{v\varphi}{u^2} f_{uv} + (\frac{1}{\varphi} + \frac{v^2\varphi}{u^2})\Phi_v, \boldsymbol{\nu}\rangle \end{pmatrix} \\ &= \begin{pmatrix} -M & \langle -\frac{v\varphi}{u} f_{uv} + \frac{1}{\varphi}\Phi_u, \boldsymbol{\nu}\rangle \\ -N & \langle -\frac{v\varphi}{u} f_{vv} + \frac{1}{\varphi}\Phi_v, \boldsymbol{\nu}\rangle \end{pmatrix}, \end{aligned}$$

and we obtain the result. □

Thus we conclude that, on $v = 0$,

$$\begin{aligned} f_{uuu} &= -u(u^2N^2 + \varphi^2)\boldsymbol{a}_1 + (\tfrac{u^2L_1N}{\varphi} - 2\varphi - u\varphi_u)\boldsymbol{a}_2 + u(L_1 + 3N + uN_u)\boldsymbol{a}_3, \\ f_{uvu} &= -(u^2N^2 + \varphi^2)\boldsymbol{a}_1 + (\tfrac{uL_1N}{\varphi} - \varphi_u)\boldsymbol{a}_2 + (L_1 + N + uN_u)\boldsymbol{a}_3, \\ f_{vvu} &= -uN^2\boldsymbol{a}_1 + \tfrac{L_1N}{\varphi}\boldsymbol{a}_2 + N_u\boldsymbol{a}_3, \\ f_{vvv} &= -N^2\boldsymbol{a}_1 + \tfrac{M_1N}{\varphi}\boldsymbol{a}_2 + N_v\boldsymbol{a}_3. \end{aligned}$$

Proof of Proposition 2.26. By Lemma 2.23, we have

$$\begin{aligned} \boldsymbol{f}_3 =& [p_3 f_u + q_3 f_v + 3(p_1p_2 f_{uu} + (p_1q_2 + q_1p_2) f_{uv} + q_1q_2 f_{vv})](p_0, 0) \\ &+ [p_1^3 f_{uuu} + 3p_1^2 q_1 f_{uuv} + 3p_1 q_1^2 f_{uvv} + q_1^3 f_{vvv}](p_0, 0) \\ =& \lim_{u\to p_0, v\to 0} [(p_3 + \tfrac{q_3}{u}) f_u - \tfrac{vq_2}{u} \tfrac{f_u - u f_v}{v}] \\ &+ \begin{pmatrix} 3(p_1p_2 f_{uu} + (p_1q_2 + q_1p_2) f_{uv} + q_1q_2 f_{vv}) \\ +p_1^3 f_{uuu} + 3p_1^2 q_1 f_{uuv} + 3p_1 q_1^2 f_{uvv} + q_1^3 f_{vvv} \end{pmatrix}(p_0, 0) \\ =& (p_3 u + q_3)\boldsymbol{a}_1 + [3(p_1p_2 f_{uu} + (p_1q_2 + q_1p_2) f_{uv} + q_1q_2 f_{vv}) \\ &+ p_1^3 f_{uuu} + 3p_1^2 q_1 f_{uuv} + 3p_1 q_1^2 f_{uvv} + q_1^3 f_{vvv}](p_0, 0). \end{aligned} \tag{2.29}$$

We remark that

$$\begin{aligned} \langle f_{uu}, \boldsymbol{a}_3\rangle &= p_0^2 N, \\ \langle f_{uv}, \boldsymbol{a}_3\rangle &= p_0 N, \\ \langle f_{vv}, \boldsymbol{a}_3\rangle &= N, \\ \langle f_{uuu}, \boldsymbol{a}_3\rangle &= p_0(L_1 + 3N + p_0 N_u), \\ \langle f_{uuv}, \boldsymbol{a}_3\rangle &= L_1 + N + p_0 N_u, \\ \langle f_{uvv}, \boldsymbol{a}_3\rangle &= N_u, \text{ and} \\ \langle f_{vvv}, \boldsymbol{a}_3\rangle &= N_v \end{aligned}$$

on $\{v = 0\}$. We thus have

$$b_3 = \langle \boldsymbol{f}_3, \boldsymbol{a}_3\rangle = 3N[p_1p_2p_0^2 + (p_1q_2 + q_1p_2)p_0 + q_1q_2]$$

$$+ p_1^3 p_0(L_1 + 3N + p_0 N_u) + 3p_1^2 q_1(L_1 + N + p_0 N_u) + 3p_1 q_1^2 N_u + q_1^3 N_v|_{u=p_0}$$
$$=3N(p_1 p_0 + q_1)(p_2 p_0 + q_2) + (p_1 p_0 + 3q_1)p_1^2 L_1 + 3p_1^2(p_1 p_0 + q_1)N$$
$$+ p_1(p_1^2 p_0^2 + 3p_0 p_1 q_1 + 3q_1^2)N_u + q_1^3 N_v.$$

When $q_1 = -p_1 p_0$, we have

$$b_3 = -2p_1^3 p_0(2L_1 - p_0 N_u + p_0^2 N_v)(p_0, 0).$$

Since $p_1 = |p_0\varphi(p_0, 0)|^{-1/2}$, we obtain the result. □

Appendix A. A quick review of surfaces in $\mathbb{R}^3$

Since a surface in $\mathbb{R}^3$ is locally expressed as the image of a C^∞-map $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$, it is possible to investigate surfaces as a subject of singularity theory. We describe this idea briefly. The first fundamental quantities E, F, G are defined by

$$E = \langle f_u, f_u\rangle,\ F = \langle f_u, f_v\rangle,\ G = \langle f_v, f_v\rangle.$$

The singular point of f is exactly defined by $EG - F^2 = 0$. A unit normal vector $\boldsymbol{\nu}$ is defined by $\boldsymbol{\nu} = f_u \times f_v / |f_u \times f_v|$ whenever the denominator is not zero. The second fundamental quantities L, M, N are defined by

$$L = \langle f_{uu}, \boldsymbol{\nu}\rangle = \frac{|f_{uu}\ f_u\ f_v|}{|f_u \times f_v|},\ M = \langle f_{uv}, \boldsymbol{\nu}\rangle = \frac{|f_{uv}\ f_u\ f_v|}{|f_u \times f_v|},\ N = \langle f_{vv}, \boldsymbol{\nu}\rangle = \frac{|f_{vv}\ f_u\ f_v|}{|f_u \times f_v|}.$$

A **principal curvature** κ is a solution to

$$\begin{vmatrix} L - \kappa E & M - \kappa F \\ M - \kappa F & N - \kappa G \end{vmatrix} = 0.$$

If this equation defines two principal curvatures κ_1 and κ_2, then the kernel fields of the matrix $\begin{pmatrix} L - \kappa_i E & M - \kappa_i F \\ M - \kappa_i F & N - \kappa_i G \end{pmatrix}$ represent the **principal directions** with respect to κ_i, $i = 1, 2$. The principal directions are also described by the solutions to

$$\begin{vmatrix} E & L & dv^2 \\ F & M & -du\,dv \\ G & N & du^2 \end{vmatrix} = 0.$$

A **principal vector** $\boldsymbol{v}_i$ is a unit vector which represents the principal direction with respect to κ_i. Integral curves of principal vectors are called **curvature lines**.

We say a point is $\boldsymbol{v}_i$**-ridge** if $\boldsymbol{v}_i\kappa_i = 0$ at this point. We say a point is $\boldsymbol{v}_i$**-subparabolic** if $\boldsymbol{v}_i\kappa_j = 0$ at this point where $j \neq i$.

Asymptotic directions are represented by the solutions to $L\,du^2 + 2M\,du\,dv + N\,dv^2 = 0$. Their integral curves are called by **asymptotic lines**.

The **Gauss curvature** K and the **mean curvature** H are defined by

$$K = \kappa_1\kappa_2 = \frac{LN - M^2}{EG - F^2},\quad H = \frac{\kappa_1 + \kappa_2}{2} = \frac{EN + GL - 2FM}{2(EG - F^2)}. \tag{A.1}$$

A **parabolic point** is defined by $K = 0$ whenever $EG - F^2 \neq 0$ (i.e., f is nonsingular).

We consider Taylor expansion of f: $f(u,v) = \sum_{j=0}^{m} \boldsymbol{h}_j(u)\frac{v^j}{j!} + O(v^{m+1})$. We have $\Sigma(f) = \{v = 0\}$ if $\operatorname{rank}(\boldsymbol{h}_0' \ \boldsymbol{h}_1) < 2$ and $|\boldsymbol{h}_0' \times \boldsymbol{h}_2 + \boldsymbol{h}_1' \times \boldsymbol{h}_1| \neq 0$. The later condition also implies that $\operatorname{rank}(\boldsymbol{h}_0' \ \boldsymbol{h}_1) = 1$. The normal vector $\boldsymbol{\nu} = \frac{f_u \times f_v}{|f_u \times f_v|}$ is continuously extendible on $\Sigma(f)$ in this case.

Since

$$K = \frac{LN - M^2}{EG - F^2} = \frac{|f_{uu}\ f_u\ f_v||f_{vv}\ f_u\ f_v| - |f_{uv}\ f_u\ f_v|^2}{(EG - F^2)^{3/2}},$$

the notion of parabolic point is extended by $p(u) = 0$ on $\Sigma(f) = \{v = 0\}$ when

$$|f_{uu}\ f_u\ f_v||f_{vv}\ f_u\ f_v| - |f_{uv}\ f_u\ f_v|^2 = p(u)|v|^m + O(|v|^{m+1}), \quad p(u) \not\equiv 0.$$

We have $m = 1$ in generic context, that is, $|\boldsymbol{h}_0''\ \boldsymbol{h}_0'\ \boldsymbol{h}_1||\boldsymbol{h}_2\ \boldsymbol{h}_0'\ \boldsymbol{h}_1| - |\boldsymbol{h}_1'\ \boldsymbol{h}_0'\ \boldsymbol{h}_1|^2$ is not identically zero.

Appendix B. Criteria of singularity types

B.1. Criteria of singularity types of singular surfaces. Assume that $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$, $(u, v) \mapsto f(u, v)$, has rank one singularity at 0 and an unit normal vector is extended to $\boldsymbol{\nu}$ on the singular locus. Set $\lambda = \det(f_u\ f_v\ \boldsymbol{\nu})$, $\psi = \det(\boldsymbol{t}\ \eta\boldsymbol{\nu}\ \boldsymbol{\nu})$, where $\boldsymbol{t}$ is a unit tanget vector, and η is a vector field whose restriction is null to the singular locus. We have that $(f, \boldsymbol{\nu}) : (\mathbb{R}^2, 0) \to (\mathbb{R}^3 \times \mathbb{R}^3, (0, \boldsymbol{\nu}(0)))$ is an embedding, if and only if $\psi(0) \neq 0$.

Lemma B.1. *The singularity of f is*

- *cuspidal edge, if $\psi(0) \neq 0$, $\eta\lambda(0) \neq 0$;*
- *swallowtail, if $\psi(0) \neq 0$, $\eta\lambda(0) = 0$, $\eta^2\lambda(0) \neq 0$;*
- *cuspidal cross-cap, if $\psi(0) = 0$, $\eta\lambda(0) \neq 0$, $\psi'(0) \neq 0$.*

Proof. See [6, §1–2] and [2, §1]. □

Proof of Proposition 1.6. In the notation in §1, $\eta = \partial_t$. Setting $f_t = t\boldsymbol{v}$, $\lambda = |f_s\ f_t\ \boldsymbol{\nu}| = |f_s\ t\boldsymbol{v}\ \boldsymbol{\nu}| = t \times (\text{unit})$,

$$\psi(s, 0) = \det(\boldsymbol{t}, \eta\boldsymbol{\nu}, \boldsymbol{\nu})(s, 0) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -b_3/2 & 0 \end{vmatrix} = \frac{b_3}{2}.$$

So the criteria above shows the proposition. □

Proof of Proposition 2.19. In the notation in §2, we have $\eta = \partial_u - u\partial_v$,

$$\lambda = |f_u\ f_v\ \boldsymbol{\nu}| = |f_u/u\ uf_v - f_u\ \boldsymbol{\nu}| = v|\boldsymbol{A}_1\ \boldsymbol{A}_2\ \boldsymbol{\nu}| = v,$$

$\eta\lambda = (\partial_u - u\partial_v)v = u$, and $\eta^2\lambda = (\partial_u - u\partial_v)u = 1$. So f is swallowtail if $(f, \boldsymbol{\nu}) : (\mathbb{R}^2, 0) \to (\mathbb{R}^3 \times \mathbb{R}^3, (0, \boldsymbol{\nu}(0)))$ is an embedding (i.e., $c_{2,0} \neq c_{1,2}$). □

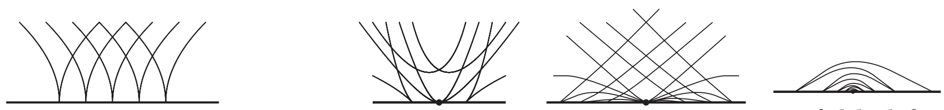
B.2. Criteria of singularity types of differential equations. We consider the binary differential equation:

$$A\,dx^2 + 2B\,dx\,dy + C\,dy^2 = 0, \tag{B.1}$$

where $A = \sum_{0\le i+j\le 2} a_{i,j}\dfrac{x^i y^j}{i!j!} + O(3)$, $B = \sum_{0\le i+j\le 2} b_{i,j}\dfrac{x^i y^j}{i!j!} + O(3)$, $C = \sum_{0\le i+j\le 2} c_{i,j}\dfrac{x^i y^j}{i!j!} + O(3)$.
The solution curves of this equation is investigated by Davidov ([1]).

Lemma B.2. *We assume that* $a_{0,0} = b_{0,0} = 0$*,* $c_{0,0} \neq 0$*. Then there is a homeomorphism of* $(\mathbb{R}^2, 0)$ *which sends the solution curves of the equation* (B.1) *to*

- *that of* $u\,du^2 + dv^2 = 0$*, folded regular point, if* $a_{1,0} \neq 0$*;*
- *that of* $(\lambda u^2 + v)du^2 + dv^2 = 0$ *with* $\lambda < 0$ *(resp.* $0 < \lambda < 1/16$*,* $1/16 < \lambda$*), folded saddle (resp. folded node, folded focus), if* $a_{1,0} = 0$*,* $a_{0,1} \neq 0$*,* $\lambda < 0$ *(resp.* $0 < \lambda < 1/16$*,* $1/16 < \lambda$*) where* $\lambda = (a_{2,0}c_{0,0} - a_{0,1}b_{1,0} - 2b_{1,0}^2)/2a_{0,1}^2$*.*



folded regular point　folded saddle　folded node　folded focus

Proof (Sketch). First we remark that the gradient of the discriminant $B^2 - AC$ is $-c_{0,0}(a_{1,0}, a_{0,1})$ at $(x, y) = (0, 0)$. This implies the discriminant defines a nonsingular curve near $(0, 0)$. Set

$$\phi = \sum_{i=0}^{2} \phi_{i,j} u^i v^j + O(3),\ x = \sum_{i+j=1}^{3} p_{i,j} u^i v^j + O(4),\ y = q_{0,1} v + \sum_{i+j=2}^{3} c_{i,j} u^i v^j + O(4).$$

When $a_{1,0} \neq 0$, we can choose $\phi_{i,j}$ $(0 \le i + j \le 2)$, $p_{i,j}$ $(1 \le i + j \le 3)$, $q_{i,j}$ $(2 \le i + j \le 3,\ i \neq 0)$ so that

$$\phi(A\,dx^2 + 2B\,dx\,dy + C\,dy^2) = (u + O(3))du^2 + O(3)du\,dv + (1 + O(3))dv^2.$$

We then see the contact form $\omega = dv - p\,du$ defines a regular curves on the surface S : $u + p^2 + O(3) = 0$ in (u, v, p)-space. The projection $S \to \mathbb{R}^2$ defines a 2-1 map on the set defined by $B^2 - AC > 0$, and we conclude the proof in the case $a_{1,0} \neq 0$.

The case that $a_{1,0} = 0$ and $a_{0,1} \neq 0$ is similar. In this case, we can choose $\phi_{i,j}$ $(0 \le i + j \le 2)$, $p_{2,0}$, $p_{1,1}$, $q_{i,j}$ $(1 \le i + j \le 3,\ i \neq 0)$ so that

$$\phi(A\,dx^2 + 2B\,dx\,dy + C\,dy^2) = (\lambda u^2 + v + O(3))du^2 + O(3)du\,dv + (1 + O(3))dv^2.$$

On the surface $\lambda u^2 + v + p^2 + O(3) = 0$ in (u, v, p)-space, we see the contact form $\omega = dv - p\,du$ defines a saddle (resp. a node, a focus), if $\lambda < 0$ (resp. $0 < \lambda < 1/16$, $\lambda > 1/16$). □

NOTE ADDED IN PROOF. The author thanks to K. Saji for informing that computation in subsection 1.6 can be continued using Proposition 3.2 in the following paper: K. Saji, On pairs of geometric foliations on a cuspidal edge, Advanced Studies in Pure Mathematics, 78, 2018, Singularities in Generic Geometry, 411–429.
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Department of Mathematics
Saitama University
255, Shimo-Okubo
Saitama, 338–8570
Japan
e-mail: tfukui@rimath.saitama-u.ac.jp