

# BOUNDARY LIMITS OF MONOTONE SOBOLEV FUNCTIONS FOR DOUBLE PHASE FUNCTIONALS

YOSHIHIRO MIZUTA and TETSU SHIMOMURA

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## Abstract

Our aim in this paper is to deal with boundary limits of monotone Sobolev functions for the double phase functional  $\Phi_{p,q}(x, t) = t^p + (b(x)t)^q$  in the unit ball  $\mathbf{B}$  of  $\mathbb{R}^n$ , where  $1 < p < q < \infty$  and  $b(\cdot)$  is a non-negative bounded function on  $\mathbf{B}$  which is Hölder continuous of order  $\theta \in (0, 1]$ .

## 1. Introduction

We say that a continuous function  $u$  is monotone in the unit ball  $\mathbf{B}$  of  $\mathbb{R}^n$  ( $n \geq 2$ ) in the sense of Lebesgue, if both

$$\max_D u(x) = \max_{\partial D} u(x) \quad \text{and} \quad \min_D u(x) = \min_{\partial D} u(x)$$

hold for every relatively compact open set  $D$  with the closure  $\bar{D} \subset \mathbf{B}$  (see [15]). Clearly, harmonic functions are monotone, and more generally, solutions of elliptic partial differential equations of second order and weak solutions for variational problems may be monotone. A function  $u \in W^{1,p}(\mathbf{B})$  is  $\mathcal{A}$ -harmonic if it is a weak solution of equation

$$\operatorname{div}(\mathcal{A}(x, \nabla u)) = 0,$$

where  $\mathcal{A}(x, \xi) \cdot \xi \approx |\xi|^p$  for some fixed  $p \in (1, \infty)$ ,  $\xi \in \mathbb{R}^n$  (see [12]). Note that  $\mathcal{A}$ -harmonic functions and hence coordinate functions of quasiregular mappings are monotone (see [8, 12, 14, 24, 26, 28]), and thus the class of monotone functions is considerably wide. For these facts, see Gilbarg-Trudinger [9], Heinonen-Kilpeläinen-Martio [12], Serrin [27], Vuorinen [29, 30] and [7].

Regarding regularity theory of differential equations, Baroni, Colombo and Mingione [1, 2, 5, 6] studied a double phase functional:

$$\Phi(x, t) = \Phi_{p,q}(x, t) = t^p + (b(x)t)^q,$$

where  $1 < p < q < \infty$  and  $b(\cdot)$  is a non-negative bounded function on  $\mathbf{B}$  which is Hölder continuous of order  $\theta \in (0, 1]$ . Harjulehto, Hästö and Karppinen [10] studied local higher integrability of the gradient of a quasiminimizer of the double phase functional. We refer to [5] for the minimization problem and [4] for the eigenvalue problem for the double phase functional. See also [11] for the boundedness of maximal operators.

We say that a function  $u$  on  $\mathbf{B}$  has a nontangential limit  $\ell$  at  $\xi \in \partial\mathbf{B}$  if

$$\lim_{x \in \Gamma(\xi, a) \rightarrow \xi} u(x) = \ell$$

for all  $a > 0$ , where  $\Gamma(\xi, a) = \{x \in \mathbf{B} : |x - \xi| \leq a(1 - |x|)\}$ .

If  $0 < \alpha < n$ ,  $1 < p < \infty$ ,  $G$  is an open set in  $\mathbb{R}^n$  and  $E \subset G$ , then the relative  $(\alpha, p)$ -capacity is defined by

$$C_{\alpha,p}(E; G) = \inf \int_G f(y)^p dy,$$

where the infimum is taken over all nonnegative measurable functions  $f$  on  $G$  such that

$$\int_G |x - y|^{\alpha-n} f(y) dy \geq 1 \quad \text{for every } x \in E;$$

see [17] and [22] for the basic properties of  $(1, p)$ -capacity. For simplicity, we write  $C_{\alpha,p}(E) = 0$  if  $C_{\alpha,p}(E \cap G; G) = 0$  for every bounded open set  $G \subset \mathbb{R}^n$ .

In [21], we studied boundary limits of monotone Sobolev functions on  $\mathbf{B}$  satisfying

$$(1.1) \quad \int_{\mathbf{B}} |\nabla u(x)|^p dx < \infty,$$

where  $\nabla$  denotes the gradient and  $1 < p < \infty$ . Manfredi-Villamor [16] proved the following result concerning the existence of nontangential limits for monotone functions.

**Theorem A** ([16, Theorem 1]). *Let  $n - 1 < p < n$ . If  $u$  is a monotone function on  $\mathbf{B}$  satisfying (1.1), then  $u$  has a nontangential limit at  $\xi \in \partial\mathbf{B} \setminus E$ , where  $C_{1,p}(E) = 0$ .*

The nontangential limits for harmonic functions have been discussed by many authors; see e.g. Carleson [3], Mizuta [19] and Wallin [31]. For  $\mathcal{A}$ -harmonic functions, we refer to Koskela-Manfredi-Villamor [14]. Note from [19, 20] that Theorem A is the best possible as to the size of the exceptional set.

Our aim in this paper is to study boundary limits of monotone Sobolev functions for the double phase functional  $\Phi(x, t)$ , as an extension of Theorem A ([16]) and [14].

**Theorem 1.1.** *Let  $n - 1 < p < n$ ,  $1/p - 1/q = \theta/n$  and*

$$\frac{n - (1 + \theta)p}{p} = \frac{n - q}{q} > 0.$$

*If  $u$  is a monotone function on  $\mathbf{B}$  satisfying*

$$(1.2) \quad \int_{\mathbf{B}} \Phi(x, |\nabla u(x)|) dx < \infty,$$

*then  $bu$  has a nontangential limit at  $\xi \in \partial\mathbf{B} \setminus E$ , where  $C_{1,q}(E) = 0$ .*

**EXAMPLE 1.2.** Examples of  $b(x)$  are  $|x - x_0|^\theta$  with  $x_0 \in \partial\mathbf{B}$  and  $(1 - |x|)^\theta$ , where  $0 < \theta \leq 1$ .

**REMARK 1.3.** In Theorem 1.1, as will be seen in Remark 4.1 given below, condition (1.2) can not be replaced by the single condition

$$\int_{\mathbf{B}} (b(x)|\nabla u(x)|)^q dx < \infty.$$

Hence Theorem 1.1 deeply depends on the double phase functional.

The key lemma to prove Theorem 1.1 is Lemma 2.1 below. We discuss the sharpness of condition (1.2) in the last section (Remark 4.1).

It is well known that a coordinate function  $u = f_i$  of a quasiregular mapping  $f = (f_1, \dots, f_n) : \mathbf{B} \rightarrow \mathbb{R}^n$  is  $\mathcal{A}$ -harmonic (see [12, Theorem 14.39]) and monotone in  $\mathbf{B}$ , so that Theorem 1.1 gives the following corollary.

**Corollary 1.4.** *Let  $n - 1 < p < n$  and  $1/p - 1/q = \theta/n$ . If  $u$  is a coordinate function of a quasiregular mapping on  $\mathbf{B}$  satisfying (1.2), then  $bu$  has a nontangential limit at  $\xi \in \partial\mathbf{B} \setminus E$ , where  $C_{1,q}(E) = 0$ .*

Miklyukov [18] studied the nontangential limits for quasiregular mappings with finite Dirichlet integral.

Throughout this paper, let  $C$  denote various constants independent of the variables in question.

### 2. Preliminary lemmas

We use the notation  $B(x, r)$  to denote the open ball centered at  $x$  of radius  $r$ . Let us begin with the key fact for monotone functions.

**Lemma 2.1.** *If  $u$  is a monotone function on  $B(x_0, 2r)$  and  $p > n - 1$ , then*

$$|u(x) - u(y)|^p \leq Cr^{p-n} \int_{B(x_0, 2r)} |\nabla u(y)|^p dy$$

whenever  $x, y \in B(x_0, r)$ .

For this, see e.g. [13, Lemma 7.1], [16, Remark, p. 9] and, for the case  $p = n$ , [30, Section 16].

**Lemma 2.2.** *For a nonnegative measurable function  $f$  on  $\mathbf{B}$  such that*

$$\int_{\mathbf{B}} \Phi(y, f(y)) dy < \infty,$$

set

$$E_1 = \{x_0 \in \mathbb{R}^n : \int_{\mathbf{B}} |x_0 - y|^{1+\theta-n} f(y) dy = \infty\},$$

$$E_2 = \{x_0 \in \mathbb{R}^n : \int_{\mathbf{B}} |x_0 - y|^{1-n} b(y) f(y) dy = \infty\},$$

$$E_3 = \{x_0 \in \mathbb{R}^n : \limsup_{r \rightarrow 0} r^{(1+\theta)p-n} \int_{B(x_0, 2r)} f(y)^p dy > 0\}$$

and

$$E_4 = \{x_0 \in \mathbb{R}^n : \limsup_{r \rightarrow 0} r^{q-n} \int_{B(x_0, 2r)} (b(y)f(y))^q dy > 0\}.$$

Then  $C_{1,q}(E_1 \cup E_2 \cup E_3 \cup E_4) = 0$ .

Proof. By [23, Section 7], we find  $C_{1+\theta,p}(E_1) = 0$ ,  $C_{1,q}(E_2) = 0$ ,  $C_{1+\theta,p}(E_3) = 0$  and  $C_{1,q}(E_4) = 0$ . Hence, in view of [22, Section 5.2, Theorem 2.4], we have  $C_{1,q}(E_1 \cup E_2 \cup$

$E_3 \cup E_4 = \emptyset$  since  $n - (1 + \theta)p = (p/q)(n - q) < n - q$ . □

**Lemma 2.3.** *Let  $u$  be a continuous function on  $\mathbf{B}$  such that*

$$(2.1) \quad \int_0^1 r^\theta |\nabla u(r\eta)| \, dr < \infty$$

and

$$(2.2) \quad \int_0^1 b(r\eta) |\nabla u(r\eta)| \, dr < \infty$$

for  $\eta \in \partial\mathbf{B}$ . Then  $\lim_{r \rightarrow 0} b(r\eta)u(r\eta)$  exists and is finite.

Proof. Let  $u$  be a continuous function on  $\mathbf{B}$  such that (2.1) and (2.2). Let  $0 < r_0 < 1$ . Note that

$$u(x) = u(r_0\eta) - \int_{|x|}^{r_0} \frac{d}{dr} u(r\eta) \, dr$$

for  $x = |x|\eta$ . Hence

$$\begin{aligned} b(x)u(x) &= b(x)u(r_0\eta) - b(x) \int_{|x|}^{r_0} \frac{d}{dr} u(r\eta) \, dr \\ &= b(x)u(r_0\eta) - \int_{|x|}^{r_0} (b(x) - b(r\eta)) \frac{d}{dr} u(r\eta) \, dr - \int_{|x|}^{r_0} b(r\eta) \frac{d}{dr} u(r\eta) \, dr. \end{aligned}$$

By (2.1) and (2.2), we have

$$\int_{|x|}^1 \left| (b(x) - b(r\eta)) \frac{d}{dr} u(r\eta) \right| \, dr \leq C \int_0^1 r^\theta |\nabla u(r\eta)| \, dr < \infty$$

and

$$\int_{|x|}^1 \left| b(r\eta) \frac{d}{dr} u(r\eta) \right| \, dr \leq C \int_0^1 b(r\eta) |\nabla u(r\eta)| \, dr < \infty.$$

Consequently we see from dominated convergence theorem that  $\lim_{r \rightarrow 0} b(r\eta)u(r\eta)$  exists and is finite. □

### 3. Proof of Theorem 1.1

Proof of Theorem 1.1. Let  $u$  be a monotone function on  $\mathbf{B}$  satisfying (1.2). Suppose  $\xi \in \partial\mathbf{B} \setminus E_1$ . Then

$$\int_{\mathbf{B}} |\xi - y|^{1+\theta-n} |\nabla u(y)| \, dy \geq \int_{\mathbf{B} \cap \partial B(\xi, 1)} \left( \int_0^1 r^\theta |\nabla u((1-r)\xi + r\eta)| \, dr \right) dS(\eta),$$

so that

$$(3.1) \quad \int_0^1 r^\theta |\nabla u((1-r)\xi + r\eta)| \, dr < \infty$$

for every  $\eta \in \mathbf{B} \cap \partial B(\xi, 1)$  except for a set of the  $(n - 1)$ -dimensional measure zero. Similarly, if  $\xi \in \mathbf{B} \cap \partial B(\xi, 1) \setminus E_2$ , then

$$\begin{aligned} & \int_{\mathbf{B}} |\xi - y|^{1-n} b(y) |\nabla u(y)| dy \\ & \geq \int_{\mathbf{B} \cap \partial B(\xi, 1)} \left( \int_0^1 b((1-r)\xi + r\eta) |\nabla u((1-r)\xi + r\eta)| dr \right) dS(\eta), \end{aligned}$$

so that

$$(3.2) \quad \int_0^1 b((1-r)\xi + r\eta) |\nabla u((1-r)\xi + r\eta)| dr < \infty$$

for every  $\eta \in \mathbf{B} \cap \partial B(\xi, 1)$  except for a set of the  $(n - 1)$ -dimensional measure zero.

Set  $L(\xi, \eta) = \{(1 - r)\xi + r\eta; 0 < r < 1\}$ . Then we see from (3.1), (3.2) and Lemma 2.3 that  $\lim_{x^* \in L(\xi, \eta) \rightarrow \xi} b(x^*)u(x^*)$  exists and is finite for every  $\eta \in \mathbf{B} \cap \partial B(\xi, 1)$  except for a set of the  $(n - 1)$ -dimensional measure zero.

In view of Lemma 2.1, we find for  $x \in B(x^*, r)$  with  $x^* \in L(\xi, \eta)$  and  $B(x^*, 2r) \subset \mathbf{B}$

$$\begin{aligned} |b(x)u(x) - b(x^*)u(x^*)| & \leq |b(x) - b(x^*)||u(x) - u(x^*)| + |b(x) - b(x^*)||u(x^*)| \\ & \leq C \left( r^{(1+\theta)p-n} \int_{B(x^*, 2r)} |\nabla u(y)|^p dy \right)^{1/p} + Cr^\theta |u(x^*)| \end{aligned}$$

and

$$\begin{aligned} |b(x^*)u(x) - b(x^*)u(x^*)| & \leq Cb(x^*) \left( r^{p-n} \int_{B(x^*, 2r)} |\nabla u(y)|^p dy \right)^{1/p} \\ & \leq C \left( r^{p-n} \int_{B(x^*, 2r)} |b(x^*) - b(y)|^p |\nabla u(y)|^p dy \right)^{1/p} \\ & \quad + C \left( r^{p-n} \int_{B(x^*, 2r)} (b(y)|\nabla u(y)|)^p dy \right)^{1/p} \\ & \leq C \left( r^{(1+\theta)p-n} \int_{B(x^*, 2r)} |\nabla u(y)|^p dy \right)^{1/p} \\ & \quad + C \left( r^{q-n} \int_{B(x^*, 2r)} (b(y)|\nabla u(y)|)^q dy \right)^{1/q} \end{aligned}$$

by Jensen's inequality, so that

$$\begin{aligned} |b(x)u(x) - b(x^*)u(x^*)| & \leq C \left( r^{(1+\theta)p-n} \int_{B(x^*, 2r)} |\nabla u(y)|^p dy \right)^{1/p} + Cr^\theta |u(x^*)| \\ & \quad + C \left( r^{q-n} \int_{B(x^*, 2r)} (b(y)|\nabla u(y)|)^q dy \right)^{1/q}. \end{aligned}$$

Consequently, as in the proof of Theorem 2 in [21], we insist that  $b(x)u(x)$  has a nontangential limit at  $\xi \in \partial \mathbf{B} \setminus (E_1 \cup E_2 \cup E_3 \cup E_4)$ . □

#### 4. Sharpness

REMARK 4.1. We can find a monotone function  $u$  on  $\mathbf{B}$  and a non-negative bounded  $\theta$ -Hölder continuous function  $b$  on  $\mathbf{B}$  satisfying

- (1)  $\int_{\mathbf{B}} (b(x)|\nabla u(x)|)^q dx < \infty$ ; and
- (2)  $\limsup_{x \in \Gamma(\xi, 1) \rightarrow \xi} b(x)u(x) = \infty$  for all  $\xi \in L = \{(\xi', 0) : |\xi'| = 1\}$ .

Note here that  $C_{1,q}(L) > 0$  when  $n - q < n - 2$  or  $q > n - 1 \geq 2$ .

For this, set

$$\varphi(t) = \begin{cases} 0 & (t < -2), \\ t + 2 & (-2 \leq t < -1), \\ 1 & (-1 \leq t < 1), \\ 2 - t & (1 \leq t < 2), \\ 0 & (t \geq 2). \end{cases}$$

For  $\beta > 0, 0 < \theta \leq 1$  and  $a > 1$ , consider

$$u(x) = (1 - |x'| + |x_n|)^{-\beta} \quad \text{and} \quad b(x) = \sum_{j=1}^{\infty} (1 - |x'|)^{\theta a} \varphi((2^{-j} - (1 - |x'|))2^{ja}),$$

where  $x = (x_1, \dots, x_{n-1}, x_n) = (x', x_n)$ . Then  $u$  is monotone in  $\mathbf{B}$ .

If  $|2^{-j} - (1 - |x'|)|2^{ja} \leq 2$  and  $|2^{-j} - (1 - |y'|)|2^{ja} \leq 2$ , then

$$\begin{aligned} |b(x) - b(y)| &\leq |(1 - |x'|)^{\theta a} - (1 - |y'|)^{\theta a}| \varphi((2^{-j} - (1 - |x'|))2^{ja}) \\ &\quad + (1 - |y'|)^{\theta a} |\varphi((2^{-j} - (1 - |x'|))2^{ja}) - \varphi((2^{-j} - (1 - |y'|))2^{ja})| \\ &\leq C |(1 - |x'|)^{\theta} - (1 - |y'|)^{\theta}| + (1 - |y'|)^{\theta a} ||x'| - |y'|| 2^{ja} \\ &\leq C |x' - y'|^{\theta} + C (2^{-j})^{\theta a} ||x'| - |y'||^{\theta} (2^{-ja})^{1-\theta} 2^{ja} \\ &\leq C |x' - y'|^{\theta} \leq C |x - y|^{\theta}, \end{aligned}$$

which implies that  $b$  is  $\theta$ -Hölder in  $\mathbf{B}$ .

Set  $\mathbf{B}'_j = \{(x', 0) : 2^{-j} - 2^{-ja+1} < 1 - |x'| < 2^{-j} + 2^{-ja+1}\}$  and  $\mathbf{B}' = \bigcup_{j=1}^{\infty} \mathbf{B}'_j$ . Moreover,

$$\begin{aligned} \int_{\mathbf{B}} (b(x)|\nabla u(x)|)^q dx &\leq C \int_{\mathbf{B}} (b(x)(1 - |x'| + |x_n|)^{-\beta-1})^q dx \\ &\leq C \int_{\mathbf{B}'} b(x)^q \left( \int_{|x_n| \leq \sqrt{1-|x'|^2}} (1 - |x'| + |x_n|)^{-(\beta+1)q} dx_n \right) dx' \\ &\leq C \int_{\mathbf{B}'} b(x)^q (1 - |x'|)^{-(\beta+1)q+1} dx' \\ &\leq C \sum_{j=1}^{\infty} 2^{-j(\theta a q - (\beta+1)q+1)} 2^{-ja} < \infty \end{aligned}$$

when  $\theta a q - (\beta + 1)q + 1 + a > 0$ .

Hence, letting  $a = n - 1$  and

$$\theta(n - 1) < \beta < \theta(n - 1) - 1 + n/q,$$

then we see that (1) and (2) hold.

Finally note :

$$(3) \int_{\mathbf{B}} |\nabla u(x)|^p dx = \infty \text{ when } \beta \geq -1 + 2/p \text{ and } p > n - 1 \geq 2.$$

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Yoshihiro Mizuta  
Department of Mathematics  
Graduate School of Science  
Hiroshima University  
Higashi-Hiroshima 739–8521  
Japan  
e-mail: yomizuta@hiroshima-u.ac.jp

Tetsu Shimomura  
Department of Mathematics  
Graduate School of Education  
Hiroshima University  
Higashi-Hiroshima 739–8524  
Japan  
e-mail: tshimo@hiroshima-u.ac.jp