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SOLVABILITY OF SOME INTEGRO-DIFFERENTIAL EQUATIONS WITH DRIFT

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Abstract

We prove the existence in the sense of sequences of solutions for some integro-differential type equations involving the drift term in the appropriate H^2 spaces using the fixed point technique when the elliptic problems contain second order differential operators with and without Fredholm property. It is shown that, under the reasonable technical conditions, the convergence in L^1 of the integral kernels yields the existence and convergence in H^2 of solutions.

1. Introduction

We recall that a linear operator L acting from a Banach space E into another Banach space F satisfies the Fredholm property if its image is closed, the dimension of its kernel and the codimension of its image are finite. Consequently, the equation $Lu = f$ is solvable if and only if $\phi_i(f) = 0$ for a finite number of functionals ϕ_i from the dual space F^*. These properties of Fredholm operators are widely used in many methods of linear and nonlinear analysis.

Elliptic equations in bounded domains with a sufficiently smooth boundary satisfy the Fredholm property if the ellipticity condition, proper ellipticity and Lopatinskii conditions are satisfied (see e.g. [1], [6], [9], [10]). This is the main result of the theory of linear elliptic equations. In the case of unbounded domains, these conditions may not be sufficient and the Fredholm property may not be satisfied. For example, Laplace operator, $Lu = \Delta u$, in $\mathbb{R}^d$ fails to satisfy the Fredholm property when considered in Hölder spaces, $L : C^{2+\alpha}(\mathbb{R}^d) \to C^{\alpha}(\mathbb{R}^d)$, or in Sobolev spaces, $L : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.

Linear elliptic equations in unbounded domains satisfy the Fredholm property if and only if, in addition to the conditions cited above, limiting operators are invertible (see [11]). In some simple cases, limiting operators can be explicitly constructed. For example, if

$$Lu = a(x)u'' + b(x)u' + c(x)u, \quad x \in \mathbb{R},$$

where the coefficients of the operator have limits at infinity,

$$a_{\pm} = \lim_{x\to\pm\infty} a(x), \quad b_{\pm} = \lim_{x\to\pm\infty} b(x), \quad c_{\pm} = \lim_{x\to\pm\infty} c(x),$$

the limiting operators are:

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$$L_{\pm}u = a_{\pm}u'' + b_{\pm}u' + c_{\pm}u.$$

Since the coefficients are constants, the essential spectrum of the operator, that is the set of complex numbers λ for which the operator $L - \lambda$ does not satisfy the Fredholm property, can be explicitly found by means of the Fourier transform:

$$\lambda_{\pm}(\xi) = -a_{\pm}\xi^2 + b_{\pm}i\xi + c_{\pm}, \quad \xi \in \mathbb{R}.$$

Invertibility of limiting operators is equivalent to the condition that the essential spectrum does not contain the origin.

In the case of general elliptic equations, the same assertions hold true. The Fredholm property is satisfied if the essential spectrum does not contain the origin or if the limiting operators are invertible. However, these conditions may not be explicitly written.

In the case of non-Fredholm operators the usual solvability conditions may not be applicable and solvability conditions are, in general, not known. There are some classes of operators for which solvability conditions are obtained. We illustrate them with the following example. Consider the equation

$$Lu \equiv \Delta u + au = f \tag{1.1}$$

in $\mathbb{R}^d$, where a is a positive constant. The operator L coincides with its limiting operators. The homogeneous equation has a nonzero bounded solution. Hence the Fredholm property is not satisfied. However, since the operator has constant coefficients, we can apply the Fourier transform and find the solution explicitly. Solvability conditions can be formulated as follows. If $f \in L^2(\mathbb{R}^d)$ and $xf \in L^1(\mathbb{R}^d)$, then there exist a solution of this equation in $H^2(\mathbb{R}^d)$ if and only if

$$\left(f(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\right)_{L^2(\mathbb{R}^d)} = 0, \quad p \in S^d_{\sqrt{a}} \quad a.e.$$

(see [18]). Here and further down S^d_r denotes the sphere in $\mathbb{R}^d$ of radius r centered at the origin. Therefore, though the operator fails to satisfy the Fredholm property, solvability conditions are formulated in a similar way. However, this similarity is only formal since the range of the operator is not closed.

In the case of the operator with a potential,

$$Lu \equiv \Delta u + a(x)u = f,$$

Fourier transform is not directly applicable. Nevertheless, solvability conditions in $\mathbb{R}^3$ can be obtained by a rather sophisticated application of the theory of self-adjoint operators (see [15]). As before, solvability conditions are formulated in terms of orthogonality to solutions of the homogeneous adjoint equation. There are several other examples of linear elliptic non Fredholm operators for which solvability conditions can be obtained (see [11], [12], [15], [17], [18]).

Solvability conditions play an important role in the analysis of nonlinear elliptic equations. In the case of non-Fredholm operators, in spite of some progress in understanding of linear problems, there exist only few examples where nonlinear non-Fredholm operators are analyzed (see [5], [16], [18], [19]). In the present article we consider another class of stationary nonlinear equations, for which the Fredholm property may not be satisfied:

$$(1.2)\quad \frac{d^2u}{dx^2}+b\frac{du}{dx}+au+\int_\Omega G(x-y)F(u(y),y)dy=0,\quad a\ge 0,\quad b\in\mathbb{R},\quad b\neq 0,\quad x\in\Omega.$$

For the simplicity of presentation we restrict ourselves to the one dimensional case (the multidimensional case is more technical). Hence Ω is a domain on the real line. In population dynamics the integro-differential equations describe models with intra-specific competition and nonlocal consumption of resources (see e.g. [2], [3]). Let us use the explicit form of the solvability conditions and study the existence of solutions of such nonlinear equation. The studies on the solutions of the integro-differential equations with the drift term are relevant to the understanding of the emergence and propagation of patterns in the theory of speciation (see [13]). The solvability of the linear equation involving the Laplace operator with the drift term was treated in [17], see also [4]. In the case of the vanishing drift term, namely when $b=0$, the equation analogous to (1.2) was treated in [16] and [19].

2. Formulation of the results

The nonlinear part of equation (1.2) will satisfy the following regularity conditions.

Assumption 1. *Function $F(u,x):\mathbb{R}\times\Omega\to\mathbb{R}$ is satisfying the Caratheodory condition (see* [8]*), such that*

$$(2.1)\qquad |F(u,x)|\le k|u|+h(x)\quad for\quad u\in\mathbb{R},\ x\in\Omega$$

with a constant $k>0$ and $h(x):\Omega\to\mathbb{R}^+$, $h(x)\in L^2(\Omega)$. Moreover, it is a Lipschitz continuous function, such that

$$(2.2)\qquad |F(u_1,x)-F(u_2,x)|\le l|u_1-u_2|\quad for\quad any\quad u_{1,2}\in\mathbb{R},\quad x\in\Omega$$

with a constant $l>0$.

For the purpose of the study of the existence of solutions of (1.2), we introduce the auxiliary problem

$$(2.3)\qquad -\frac{d^2u}{dx^2}-b\frac{du}{dx}-au=\int_\Omega G(x-y)F(v(y),y)dy.$$

Let us denote $(f_1(x),f_2(x))_{L^2(\Omega)}:=\int_\Omega f_1(x)\bar{f}_2(x)dx$, with a slight abuse of notations when these functions are not square integrable, like for instance those involved in orthogonality relation (5.4) below. In the first part of the article we treat the case of the whole real line, $\Omega=\mathbb{R}$, such that the appropriate Sobolev space is equipped with the norm

$$(2.4)\qquad \|u\|^2_{H^2(\mathbb{R})}:=\|u\|^2_{L^2(\mathbb{R})}+\left\|\frac{d^2u}{dx^2}\right\|^2_{L^2(\mathbb{R})}.$$

The main issue for the problem above is that in the absence of the drift term we were dealing with the self-adjoint, non Fredholm operator

$$-\frac{d^2u}{dx^2}-a:H^2(\mathbb{R})\to L^2(\mathbb{R}),\ a\ge 0,$$

which was the obstacle to solve our equation. The similar situations but in linear problems, both self- adjoint and non self-adjoint involving non Fredholm differential operators have been treated extensively in recent years (see [11], [12], [15], [17], [18]). However, the situation is different when the constant in the drift term $b \neq 0$. The operator

$$L_{a,\,b} := -\frac{d^2}{dx^2} - b\frac{d}{dx} - a: \quad H^2(\mathbb{R}) \to L^2(\mathbb{R}) \tag{2.5}$$

with $a \geq 0$ and $b \in \mathbb{R},\ b \neq 0$ involved in the left side of problem (2.3) is non-selfadjoint. By means of the standard Fourier transform, it can be easily verified that the essential spectrum of the operator $L_{a,\,b}$ is given by

$$\lambda_{a,\,b}(p) = p^2 - a - ibp, \quad p \in \mathbb{R}.$$

Clearly, when $a > 0$ the operator $L_{a,\,b}$ is Fredholm, since its essential spectrum stays away from the origin. But for $a = 0$ our operator $L_{a,\,b}$ fails to satisfy the Fredholm property since the origin belongs to its essential spectrum. We manage to show that under the reasonable technical conditions equation (2.3) defines a map $T_{a,\,b} : H^2(\mathbb{R}) \to H^2(\mathbb{R}),\ a \geq 0,\ b \in \mathbb{R},\ b \neq 0$, which is a strict contraction.

Theorem 1. *Let* $\Omega = \mathbb{R},\ G(x) : \mathbb{R} \to \mathbb{R},\ G(x) \in L^1(\mathbb{R})$ *and* Assumption 1 *holds.*
I) *When* $a > 0$ *we assume that* $2\sqrt{\pi}N_{a,\,b}l < 1$ *with* $N_{a,\,b}$ *defined by* (5.3) *below. Then the map* $v \mapsto T_{a,b}v = u$ *on* $H^2(\mathbb{R})$ *defined by equation* (2.3) *has a unique fixed point* $v_{a,b}$, *which is the only solution of equation* (1.2) *in* $H^2(\mathbb{R})$.
II) *When* $a = 0$ *we assume that* $xG(x) \in L^1(\mathbb{R})$, *orthogonality relation* (5.4) *holds and* $2\sqrt{\pi}N_{0,\,b}l < 1$. *Then the map* $T_{0,b}v = u$ *on* $H^2(\mathbb{R})$ *defined by equation* (2.3) *admits a unique fixed point* $v_{0,b}$, *which is the only solution of problem* (1.2) *with* $a = 0$ *in* $H^2(\mathbb{R})$.
In both cases I and II the fixed point $v_{a,b},\ a \geq 0,\ b \in \mathbb{R},\ b \neq 0$ *is nontrivial provided the intersection of supports of the Fourier transforms of functions* $supp\widehat{F(0,x)} \cap supp\widehat{G}$ *is a set of nonzero Lebesgue measure in* $\mathbb{R}$.

Note that in the case of $a > 0$ of the theorem above, as distinct from part I) of Theorem 1 of [16], the orthogonality conditions are not needed. Related to problem (1.2) on the real line, we consider the sequence of approximate equations with $m \in \mathbb{N}$

$$\frac{d^2u_m}{dx^2} + b\frac{du_m}{dx} + au_m + \int_{-\infty}^{\infty} G_m(x-y)F(u_m(y), y)dy = 0, \quad a \geq 0, \quad b \in \mathbb{R}, \quad b \neq 0. \tag{2.6}$$

The sequence of kernels $\{G_m(x)\}_{m=1}^{\infty}$ converges to $G(x)$ as $m \to \infty$ in the appropriate function spaces discussed below. We will establish that, under the certain technical conditions, each of problems (2.6) admits a unique solution $u_m(x) \in H^2(\mathbb{R})$, the limiting equation (1.2) possesses a unique solution $u(x) \in H^2(\mathbb{R})$, and $u_m(x) \to u(x)$ in $H^2(\mathbb{R})$ as $m \to \infty$, which is the so-called *existence of solutions in the sense of sequences*. In this case, the solvability conditions can be formulated for the iterated kernels G_m. They yield the convergence of the kernels in terms of the Fourier transforms (see the Appendix) and, consequently, the convergence or the solutions (Theorems 2, 4). Similar ideas in the sense of standard Schrödinger type operators were used in [14]. Our second main result is as follows.

Theorem 2. *Let* $\Omega = \mathbb{R},\ m \in \mathbb{N},\ G_m(x) : \mathbb{R} \to \mathbb{R},\ G_m(x) \in L^1(\mathbb{R})$ *are such that* $G_m(x) \to G(x)$ *in* $L^1(\mathbb{R})$ *as* $m \to \infty$. *Let* Assumption 1 *hold.*
I) *Let* $a > 0$. *Assume that*

$$2\sqrt{\pi}N_{a,\,b,\,m}l \leq 1 - \varepsilon \tag{2.7}$$

for all $m \in \mathbb{N}$ *with some fixed* $0 < \varepsilon < 1$. *Then each equation* (2.6) *admits a unique solution* $u_m(x) \in H^2(\mathbb{R})$, *and limiting equation* (1.2) *has a unique solution* $u(x) \in H^2(\mathbb{R})$.
II) *Let* $a = 0$. *Assume that* $xG_m(x) \in L^1(\mathbb{R})$, $xG_m(x) \to xG(x)$ *in* $L^1(\mathbb{R})$ *as* $m \to \infty$, *orthogonality condition*

$$(G_m(x), 1)_{L^2(\mathbb{R})} = 0, \ m \in \mathbb{N} \tag{2.8}$$

holds and

$$2\sqrt{\pi}N_{0,\ b,\ m}l \leq 1 - \varepsilon \tag{2.9}$$

for all $m \in \mathbb{N}$ *with some* $0 < \varepsilon < 1$. *Then each equation* (2.6) *possesses a unique solution* $u_m(x) \in H^2(\mathbb{R})$, *and limiting problem* (1.2) *has a unique solution* $u(x) \in H^2(\mathbb{R})$.
In both cases I and II, we have $u_m(x) \to u(x)$ *in* $H^2(\mathbb{R})$ *as* $m \to \infty$.
The unique solution $u_m(x)$ *of each equation* (2.6) *is nontrivial provided that the intersection of supports of the Fourier transforms of functions* $\operatorname{supp}\widehat{F(0,x)} \cap \operatorname{supp}\widehat{G_m}$ *is a set of nonzero Lebesgue measure in* $\mathbb{R}$. *Analogously, the unique solution* $u(x)$ *of limiting problem* (1.2) *does not vanish identically if* $\operatorname{supp}\widehat{F(0,x)} \cap \operatorname{supp}\widehat{G}$ *is a set of nonzero Lebesgue measure in* $\mathbb{R}$.

In the second part of the article we treat the analogous equation on the finite interval with periodic boundary conditions, i.e. $\Omega = I := [0,\ 2\pi]$ and the appropriate functional space is

$$H^2(I) = \{u(x) : I \to \mathbb{R} \mid u(x), u''(x) \in L^2(I), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi)\}.$$

For the technical purposes, we introduce the following auxiliary constrained subspace

$$H_0^2(I) = \{u(x) \in H^2(I) \mid (u(x), 1)_{L^2(I)} = 0\}, \tag{2.10}$$

which is a Hilbert spaces as well (see e.g. Chapter 2.1 of [7]). Let us establish that problem (2.3) in this situation defines a map $\tau_{a,b}$, $a \geq 0$, $b \in \mathbb{R}$, $b \neq 0$ on the above mentioned spaces which will be a strict contraction under the given technical conditions.

Theorem 3. *Let* $\Omega = I$, $G(x) : I \to \mathbb{R}$, $G(x) \in L^\infty(I)$, $G(0) = G(2\pi)$, $F(u,0) = F(u,2\pi)$ *for* $u \in \mathbb{R}$ *and* Assumption 1 *holds.*
I) *When* $a > 0$ *we assume that* $2\sqrt{\pi}\mathcal{N}_{a,\ b}l < 1$ *with* $\mathcal{N}_{a,\ b}$ *defined by* (5.21) *below. Then the map* $v \mapsto \tau_{a,b}v = u$ *on* $H^2(I)$ *defined by equation* (2.3) *has a unique fixed point* $v_{a,b}$, *the only solution of problem* (1.2) *in* $H^2(I)$.
II) *When* $a = 0$ *assume that orthogonality relation* (5.22) *holds and* $2\sqrt{\pi}\mathcal{N}_{0,\ b}l < 1$. *Then the map* $\tau_{0,b}v = u$ *on* $H_0^2(I)$ *defined by equation* (2.3) *has a unique fixed point* $v_{0,b}$, *the only solution of problem* (1.2) *in* $H_0^2(I)$.
In both cases I and II the fixed point $v_{a,b}$, $a \geq 0$, $b \in \mathbb{R}$, $b \neq 0$ *is nontrivial provided the Fourier coefficients* $G_nF(0,x)_n \neq 0$ *for some* $n \in \mathbb{Z}$.

Remark 1. We use the constrained subspace $H_0^2(I)$ in cases II) of the theorem, such that the operator $-\frac{d^2}{dx^2} - b\frac{d}{dx} : H_0^2(I) \to L^2(I)$, which possesses the Fredholm property, has the empty kernel.

To study the existence in the sense of sequences of solutions for our integro- differential problem on the interval I, we consider the sequence of iterated equations, similarly to the whole real line case

$$\frac{d^2u_m}{dx^2} + b\frac{du_m}{dx} + au_m + \int_0^{2\pi} G_m(x-y)F(u_m(y), y)dy = 0, \quad a \geq 0, \quad b \in \mathbb{R}, \quad b \neq 0, \tag{2.11}$$

where $m \in \mathbb{N}$. Our final main result is as follows.

Theorem 4. *Let $\Omega = I$, $m \in \mathbb{N}$, $G_m(x) : I \to \mathbb{R}$, $G_m(x) \in L^\infty(I)$ are such that $G_m(x) \to G(x)$ in $L^\infty(I)$ as $m \to \infty$, $G_m(0) = G_m(2\pi)$, $F(u, 0) = F(u, 2\pi)$ for $u \in \mathbb{R}$. Let Assumption 1 hold.*
I) *Let $a > 0$. Assume that*

$$2\sqrt{\pi}\mathcal{N}_{a,\, b,\, m}l \leq 1 - \varepsilon \tag{2.12}$$

for all $m \in \mathbb{N}$ with some fixed $0 < \varepsilon < 1$. Then each equation (2.11) possesses a unique solution $u_m(x) \in H^2(I)$ and limiting problem (1.2) admits a unique solution $u(x) \in H^2(I)$.
II) *Let $a = 0$. Assume that the orthogonality relations*

$$(G_m(x), 1)_{L^2(I)} = 0, \quad m \in \mathbb{N} \tag{2.13}$$

hold and

$$2\sqrt{\pi}\mathcal{N}_{0,\, b,\, m}\, l \leq 1 - \varepsilon \tag{2.14}$$

for all $m \in \mathbb{N}$ with some $0 < \varepsilon < 1$. Then each equation (2.11) admits a unique solution $u_m(x) \in H_0^2(I)$ and limiting problem (1.2) has a unique solution $u(x) \in H_0^2(I)$.
In both cases I *and* II, *$u_m(x) \to u(x)$ as $m \to \infty$ in the norms in $H^2(I)$ and $H_0^2(I)$ respectively. The unique solution $u_m(x)$ of each equation (2.11) is nontrivial provided that the Fourier coefficients $G_{m,n}F(0, x)_n \neq 0$ for a certain $n \in \mathbb{Z}$. Analogously, the unique solution $u(x)$ of limiting equation (1.2) does not vanish identically if $G_nF(0, x)_n \neq 0$ for some $n \in \mathbb{Z}$.*

Remark 2. Note that in the article we work with real valued functions by means of the assumptions on $F(u, x), G_m(x)$ and $G(x)$ involved in the nonlocal terms of the iterated and limiting equations discussed above.

Remark 3. The significance of Theorems 2 and 4 is the continuous dependence of solutions with respect to the integral kernels.

3. The Whole Real Line Case

Proof of Theorem 1. First we suppose that in the case of $\Omega = \mathbb{R}$ for some $v \in H^2(\mathbb{R})$ there exist two solutions $u_{1,2} \in H^2(\mathbb{R})$ of equation (2.3). Then their difference $w(x) := u_1(x) - u_2(x) \in H^2(\mathbb{R})$ will satisfy the homogeneous equation

$$-\frac{d^2w}{dx^2} - b\frac{dw}{dx} - aw = 0.$$

Since the operator $L_{a,\, b}$ defined in (2.5) acting on the whole real line does not have any nontrivial square integrable zero modes, $w(x)$ vanishes on $\mathbb{R}$.

Let $v(x) \in H^2(\mathbb{R})$ be arbitrary. We apply the standard Fourier transform (5.1) to both sides of (2.3) and obtain

$$\widehat{u}(p) = \sqrt{2\pi}\frac{\widehat{G}(p)\widehat{f}(p)}{p^2 - a - ibp} \tag{3.1}$$

with $\widehat{f}(p)$ standing for the Fourier image of $F(v(x), x)$. Obviously, we have the bounds from above

$$|\widehat{u}(p)| \le \sqrt{2\pi}N_{a,\,b}|\widehat{f}(p)| \quad \textit{and} \quad |p^2\widehat{u}(p)| \le \sqrt{2\pi}N_{a,\,b}|\widehat{f}(p)|,$$

where $N_{a,\,b} < \infty$ by virtue of Lemma A1 of the Appendix without any orthogonality conditions when $a > 0$ and under orthogonality relation (5.4) for $a = 0$. This allows us to estimate the norm

$$\|u\|^2_{H^2(\mathbb{R})} = \|\widehat{u}(p)\|^2_{L^2(\mathbb{R})} + \|p^2\widehat{u}(p)\|^2_{L^2(\mathbb{R})} \le 4\pi N^2_{a,\,b}\|F(v(x), x)\|^2_{L^2(\mathbb{R})},$$

which is finite by virtue of (2.1) of Assumption 1 since $v(x)$ is square integrable. Therefore, for an arbitrary $v(x) \in H^2(\mathbb{R})$ there exists a unique solution $u(x) \in H^2(\mathbb{R})$ of equation (2.3) with its Fourier image given by (3.1) and the map $T_{a,b} : H^2(\mathbb{R}) \to H^2(\mathbb{R})$ is well defined. This enables us to choose arbitrarily $v_{1,2}(x) \in H^2(\mathbb{R})$ such that their images $u_{1,2} = T_{a,b}v_{1,2} \in H^2(\mathbb{R})$ and estimate

$$|\widehat{u}_1(p) - \widehat{u}_2(p)| \le \sqrt{2\pi}N_{a,\,b}|\widehat{f}_1(p) - \widehat{f}_2(p)|, \quad |p^2\widehat{u}_1(p) - p^2\widehat{u}_2(p)| \le \sqrt{2\pi}N_{a,\,b}|\widehat{f}_1(p) - \widehat{f}_2(p)|,$$

where $\widehat{f}_1(p)$ and $\widehat{f}_2(p)$ denote the Fourier images of $F(v_1(x), x)$ and $F(v_2(x), x)$ respectively. For the appropriate norms of functions this gives us

$$\|u_1 - u_2\|^2_{H^2(\mathbb{R})} \le 4\pi N^2_{a,\,b}\|F(v_1(x), x) - F(v_2(x), x)\|^2_{L^2(\mathbb{R})}.$$

Note that $v_{1,2}(x) \in H^2(\mathbb{R}) \subset L^\infty(\mathbb{R})$ due to the Sobolev embedding. By means of condition (2.2) we easily obtain

$$\|T_{a,b}v_1 - T_{a,b}v_2\|_{H^2(\mathbb{R})} \le 2\sqrt{\pi}N_{a,\,b}l\|v_1 - v_2\|_{H^2(\mathbb{R})}$$

and the constant in the right side of this upper bound is less than one as assumed. Therefore, by virtue of the Fixed Point Theorem, there exists a unique function $v_{a,b} \in H^2(\mathbb{R})$ with the property $T_{a,b}v_{a,b} = v_{a,b}$, which is the only solution of equation (1.2) in $H^2(\mathbb{R})$. Suppose $v_{a,b}(x) = 0$ identically on the real line. This will contradict to the assumption that the Fourier images of $G(x)$ and $F(0, x)$ do not vanish on a set of nonzero Lebesgue measure in $\mathbb{R}$. □

Then we turn our attention to establishing the existence in the sense of sequences of the solution for our integro-differential equation on the real line.

Proof of Theorem 2. By means of the result of Theorem 1 above, each equation (2.6) admits a unique solution $u_m(x) \in H^2(\mathbb{R})$, $m \in \mathbb{N}$. Limiting equation (1.2) has a unique solution $u(x) \in H^2(\mathbb{R})$ by virtue of Lemma A2 below and Theorem 1. By applying the standard Fourier transform (5.1) to both sides of (1.2) and (2.6), we arrive at

$$\widehat{u}(p) = \sqrt{2\pi}\frac{\widehat{G}(p)\widehat{\varphi}(p)}{p^2 - a - ibp}, \quad \widehat{u}_m(p) = \sqrt{2\pi}\frac{\widehat{G}_m(p)\widehat{\varphi}_m(p)}{p^2 - a - ibp}, \quad m \in \mathbb{N}, \tag{3.2}$$

where $\widehat{\varphi}(p)$ and $\widehat{\varphi}_m(p)$ denote the Fourier images of $F(u(x), x)$ and $F(u_m(x), x)$ respectively. Evidently,

$$|\widehat{u}_m(p)-\widehat{u}(p)| \le \sqrt{2\pi}\left\|\frac{\widehat{G}_m(p)}{p^2-a-ibp}-\frac{\widehat{G}(p)}{p^2-a-ibp}\right\|_{L^\infty(\mathbb{R})}|\widehat{\varphi}(p)|+$$

$$+\sqrt{2\pi}\left\|\frac{\widehat{G}_m(p)}{p^2-a-ibp}\right\|_{L^\infty(\mathbb{R})}|\widehat{\varphi}_m(p)-\widehat{\varphi}(p)|.$$

Thus

$$\|u_m-u\|_{L^2(\mathbb{R})} \le \sqrt{2\pi}\left\|\frac{\widehat{G}_m(p)}{p^2-a-ibp}-\frac{\widehat{G}(p)}{p^2-a-ibp}\right\|_{L^\infty(\mathbb{R})}\|F(u(x),x)\|_{L^2(\mathbb{R})}+$$

$$+\sqrt{2\pi}\left\|\frac{\widehat{G}_m(p)}{p^2-a-ibp}\right\|_{L^\infty(\mathbb{R})}\|F(u_m(x),x)-F(u(x),x)\|_{L^2(\mathbb{R})}.$$

Inequality (2.2) of Assumption 1 yields

$$\|F(u_m(x),x)-F(u(x),x)\|_{L^2(\mathbb{R})} \le l\|u_m(x)-u(x)\|_{L^2(\mathbb{R})}. \tag{3.3}$$

Let us note that $u_m(x),\ u(x) \in H^2(\mathbb{R}) \subset L^\infty(\mathbb{R})$ by means of the Sobolev embedding. Hence, we obtain

$$\|u_m(x)-u(x)\|_{L^2(\mathbb{R})}\left\{1-\sqrt{2\pi}\left\|\frac{\widehat{G}_m(p)}{p^2-a-ibp}\right\|_{L^\infty(\mathbb{R})} l\right\} \le$$

$$\le \sqrt{2\pi}\left\|\frac{\widehat{G}_m(p)}{p^2-a-ibp}-\frac{\widehat{G}_m(p)}{p^2-a-ibp}\right\|_{L^\infty(\mathbb{R})}\|F(u(x),x)\|_{L^2(\mathbb{R})}.$$

By means of (2.7) for $a > 0$ and of (2.9) when $a = 0$, we arrive at

$$\|u_m(x)-u(x)\|_{L^2(\mathbb{R})} \le \frac{\sqrt{2\pi}}{\varepsilon}\left\|\frac{\widehat{G}_m(p)}{p^2-a-ibp}-\frac{\widehat{G}(p)}{p^2-a-ibp}\right\|_{L^\infty(\mathbb{R})}\|F(u(x),x)\|_{L^2(\mathbb{R})}.$$

By virtue of inequality (2.1) of Assumption 1, we have $F(u(x),x) \in L^2(\mathbb{R})$ for $u(x) \in H^2(\mathbb{R})$. This yields

$$u_m(x) \to u(x), \quad m \to \infty \tag{3.4}$$

in $L^2(\mathbb{R})$ via the result of Lemma A2 of the Appendix. Obviously,

$$p^2\widehat{u}(p) = \sqrt{2\pi}\frac{p^2\widehat{G}(p)\widehat{\varphi}(p)}{p^2-a-ibp}, \quad p^2\widehat{u}_m(p) = \sqrt{2\pi}\frac{p^2\widehat{G}_m(p)\widehat{\varphi}_m(p)}{p^2-a-ibp}, \quad m \in \mathbb{N}.$$

Thus

$$|p^2\widehat{u}_m(p)-p^2\widehat{u}(p)| \le \sqrt{2\pi}\left\|\frac{p^2\widehat{G}_m(p)}{p^2-a-ibp}-\frac{p^2\widehat{G}(p)}{p^2-a-ibp}\right\|_{L^\infty(\mathbb{R})}|\widehat{\varphi}(p)|+$$

$$+\sqrt{2\pi}\left\|\frac{p^2\widehat{G}_m(p)}{p^2-a-ibp}\right\|_{L^\infty(\mathbb{R})}|\widehat{\varphi}_m(p)-\widehat{\varphi}(p)|.$$

By means of (3.3), we derive

$$\left\| \frac{d^2 u_m}{dx^2} - \frac{d^2 u}{dx^2} \right\|_{L^2(\mathbb{R})} \le \sqrt{2\pi} \left\| \frac{p^2 \widehat{G}_m(p)}{p^2 - a - ibp} - \frac{p^2 \widehat{G}(p)}{p^2 - a - ibp} \right\|_{L^\infty(\mathbb{R})} \|F(u(x), x)\|_{L^2(\mathbb{R})} +$$

$$+ \sqrt{2\pi} \left\| \frac{p^2 \widehat{G}_m(p)}{p^2 - a - ibp} \right\|_{L^\infty(\mathbb{R})} l \|u_m(x) - u(x)\|_{L^2(\mathbb{R})}.$$

By virtue of the result of Lemma A2 of the Appendix, we obtain $\frac{d^2 u_m}{dx^2} \to \frac{d^2 u}{dx^2}$ in $L^2(\mathbb{R})$ as $m \to \infty$. Definition (2.4) of the norm yields $u_m(x) \to u(x)$ in $H^2(\mathbb{R})$ as $m \to \infty$.

Let us assume that the solution $u_m(x)$ of problem (2.6) studied above vanishes on the real line for some $m \in \mathbb{N}$. This will contradict to our assumption that the Fourier images of $G_m(x)$ and $F(0, x)$ are nontrivial on a set of nonzero Lebesgue measure in $\mathbb{R}$. The analogous argument holds for the solution $u(x)$ of limiting problem (1.2). □

4. The Problem on the Finite Interval

Proof of Theorem 3. We demonstrate the proof of the theorem in case I) and when $a = 0$ the ideas will be similar, using the constrained subspace (2.10) instead of $H^2(I)$. The operator involved in the left side of problem (2.3)

$$\mathcal{L}_{a,\,b} := -\frac{d^2}{dx^2} - b\frac{d}{dx} - a : \quad H^2(I) \to L^2(I) \tag{4.1}$$

is Fredholm, non-selfadjoint, its set of eigenvalues is given by

$$\lambda_{a,\,b}(n) = n^2 - a - ibn, \quad n \in \mathbb{Z} \tag{4.2}$$

and its eigenfunctions are the standard Fourier harmonics $\frac{e^{inx}}{\sqrt{2\pi}}$, $n \in \mathbb{Z}$. Note that the eigenvalues of the the operator $\mathcal{L}_{a,\,b}$ are simple, as distinct from the analogous situation without the drift term, when the eigenvalues corresponding to $n \neq 0$ have the multiplicity of two (see [16]).

Let us first suppose that for a certain $v(x) \in H^2(I)$ there are two solutions $u_{1,2}(x) \in H^2(I)$ of equation (2.3) with $\Omega = I$. Then the function $w(x) := u_1(x) - u_2(x) \in H^2(I)$ will satisfy the equation

$$-\frac{d^2 w}{dx^2} - b\frac{dw}{dx} - aw = 0.$$

But the operator $\mathcal{L}_{a,\,b} : H^2(I) \to L^2(I)$ discussed above does not have nontrivial zero modes. Therefore, $w(x)$ vanishes in I.

Let us choose an arbitrary $v(x) \in H^2(I)$ and apply the Fourier transform (5.19) to equation (2.3) considered on the interval I. This gives us

$$u_n = \sqrt{2\pi}\frac{G_n f_n}{n^2 - a - ibn}, \quad n^2 u_n = \sqrt{2\pi}\frac{n^2 G_n f_n}{n^2 - a - ibn}, \quad n \in \mathbb{Z}, \tag{4.3}$$

where $f_n := F(v(x), x)_n$. This allows us to estimate

$$|u_n| \le \sqrt{2\pi}\mathcal{N}_{a,\,b}|f_n|, \quad |n^2 u_n| \le \sqrt{2\pi}\mathcal{N}_{a,\,b}|f_n|,$$

with $\mathcal{N}_{a,\,b} < \infty$ under the given technical conditions by means of Lemma A3 of the Appendix. Hence, we obtain

$$\|u\|^2_{H^2(I)} = \sum_{n=-\infty}^{\infty} |u_n|^2 + \sum_{n=-\infty}^{\infty} |n^2 u_n|^2 \leq 4\pi \mathcal{N}^2_{a,\,b} \|F(v(x), x)\|^2_{L^2(I)} < \infty$$

by means of (2.1) of Assumption 1 for a square integrable $v(x)$. Therefore, for an arbitrary $v(x) \in H^2(I)$ there exists a unique $u(x) \in H^2(I)$ satisfying equation (2.3) with its Fourier image given by (4.3) and the map $\tau_{a,b} : H^2(I) \to H^2(I)$ in case I) of the theorem is well defined.

We consider any $v_{1,2}(x) \in H^2(I)$ with their images under the map mentioned above $u_{1,2} = \tau_{a,b} v_{1,2} \in H^2(I)$. Using Fourier transform (5.19), we easily arrive at

$$u_{1,n} = \sqrt{2\pi} \frac{G_n f_{1,n}}{n^2 - a - ibn}, \quad u_{2,n} = \sqrt{2\pi} \frac{G_n f_{2,n}}{n^2 - a - ibn}, \quad n \in \mathbb{Z},$$

where $f_{j,n} := F(v_j(x), x)_n,\ j = 1, 2$. Thus,

$$|u_{1,n} - u_{2,n}| \leq \sqrt{2\pi} \mathcal{N}_{a,\,b} |f_{1,n} - f_{2,n}|, \quad |n^2(u_{1,n} - u_{2,n})| \leq \sqrt{2\pi} \mathcal{N}_{a,\,b} |f_{1,n} - f_{2,n}|,$$

such that

$$\|u_1 - u_2\|^2_{H^2(I)} = \sum_{n=-\infty}^{\infty} |u_{1,n} - u_{2,n}|^2 + \sum_{n=-\infty}^{\infty} |n^2(u_{1,n} - u_{2,n})|^2 \leq$$

$$\leq 4\pi \mathcal{N}^2_{a,\,b} \|F(v_1(x), x) - F(v_2(x), x)\|^2_{L^2(I)}.$$

Evidently, $v_{1,2}(x) \in H^2(I) \subset L^\infty(I)$ by virtue of the Sobolev embedding. By means of (2.2) we easily derive

$$\|\tau_{a,b} v_1 - \tau_{a,b} v_2\|_{H^2(I)} \leq 2\sqrt{\pi} \mathcal{N}_{a,\,b}\, l \|v_1 - v_2\|_{H^2(I)},$$

where the constant in the right side of this upper bound is less than one as assumed. Therefore, the Fixed Point Theorem gives us the existence and uniqueness of a function $v_{a,b} \in H^2(I)$ satisfying $\tau_{a,b} v_{a,b} = v_{a,b}$, which is the only solution of equation (1.2) in $H^2(I)$. Suppose $v_{a,b}(x)$ vanishes in I. This yields the contradiction to our assumption that $G_n F(0, x)_n \neq 0$ for some $n \in \mathbb{Z}$. Let us note that in the case of $a > 0$ of the theorem the argument does not require any orthogonality conditions. □

Let us proceed to establishing the final main result of the article.

Proof of Theorem 4. Let us note that the limiting kernel $G(x)$ is also periodic on the interval I (see the argument of Lemma A4 of the Appendix). Each equation (2.11) admits a unique solution $u_m(x),\ m \in \mathbb{N}$ belonging to $H^2(I)$ in case I and to $H^2_0(I)$ in case II by means of Theorem 3 above. Limiting problem (1.2) has a unique solution $u(x)$, which belongs to $H^2(I)$ in case I and to $H^2_0(I)$ in case II by virtue of Lemma A4 of the Appendix and of Theorem 3 above.

We apply Fourier transform (5.19) to both sides of problems (1.2) and (2.11) and arrive at

(4.4) $$u_n = \sqrt{2\pi}\frac{G_n\varphi_n}{n^2 - a - ibn}, \quad u_{m,n} = \sqrt{2\pi}\frac{G_{m,n}\varphi_{m,n}}{n^2 - a - ibn}, \quad n \in \mathbb{Z}, \quad m \in \mathbb{N},$$

where φ_n and $\varphi_{m,n}$ are the Fourier images of $F(u(x), x)$ and $F(u_m(x), x)$ respectively under transform (5.19). We easily estimate from above

$$|u_{m,n} - u_n| \le \sqrt{2\pi}\left\|\frac{G_{m,n}}{n^2 - a - ibn} - \frac{G_n}{n^2 - a - ibn}\right\|_{l^\infty}|\varphi_n| + \sqrt{2\pi}\left\|\frac{G_{m,n}}{n^2 - a - ibn}\right\|_{l^\infty}|\varphi_{m,n} - \varphi_n|.$$

Thus,

$$\|u_m - u\|_{L^2(I)} \le \sqrt{2\pi}\left\|\frac{G_{m,n}}{n^2 - a - ibn} - \frac{G_n}{n^2 - a - ibn}\right\|_{l^\infty}\|F(u(x), x)\|_{L^2(I)}$$

$$+\sqrt{2\pi}\left\|\frac{G_{m,n}}{n^2 - a - ibn}\right\|_{l^\infty}\|F(u_m(x), x) - F(u(x), x)\|_{L^2(I)}.$$

By means of bound (2.2) of Assumption 1, we have

(4.5) $$\|F(u_m(x), x) - F(u(x), x)\|_{L^2(I)} \le l\|u_m(x) - u(x)\|_{L^2(I)}.$$

Note that $u_m(x), u(x) \in H^2(I) \subset L^\infty(I)$ due to the Sobolev embedding. Apparently,

$$\|u_m - u\|_{L^2(I)}\left\{1 - \sqrt{2\pi}l\left\|\frac{G_{m,n}}{n^2 - a - ibn}\right\|_{l^\infty}\right\}$$

$$\le \sqrt{2\pi}\left\|\frac{G_{m,n}}{n^2 - a - ibn} - \frac{G_n}{n^2 - a - ibn}\right\|_{l^\infty}\|F(u(x), x)\|_{L^2(I)}.$$

Using inequalities (2.12) and (2.14) in cases I and II respectively, we derive

$$\|u_m - u\|_{L^2(I)} \le \frac{\sqrt{2\pi}}{\varepsilon}\left\|\frac{G_{m,n}}{n^2 - a - ibn} - \frac{G_n}{n^2 - a - ibn}\right\|_{l^\infty}\|F(u(x), x)\|_{L^2(I)}.$$

Obviously, $F(u(x), x) \in L^2(I)$ for $u(x) \in H^2(I)$ by virtue of estimate (2.1) of Assumption 1. Lemma A4 below yields

(4.6) $$u_m(x) \to u(x), \quad m \to \infty$$

in $L^2(I)$. Apparently,

$$|n^2u_{m,n} - n^2u_n| \le \sqrt{2\pi}\left\|\frac{n^2G_{m,n}}{n^2 - a - ibn} - \frac{n^2G_n}{n^2 - a - ibn}\right\|_{l^\infty}|\varphi_n| + \sqrt{2\pi}\left\|\frac{n^2G_{m,n}}{n^2 - a - ibn}\right\|_{l^\infty}|\varphi_{m,n} - \varphi_n|.$$

Using (4.5), we derive

$$\left\|\frac{d^2u_m}{dx^2} - \frac{d^2u_m}{dx^2}\right\|_{L^2(I)} \le \sqrt{2\pi}\left\|\frac{n^2G_{m,n}}{n^2 - a - ibn} - \frac{n^2G_n}{n^2 - a - ibn}\right\|_{l^\infty}\|F(u(x), x)\|_{L^2(I)}$$

$$+\sqrt{2\pi}\left\|\frac{n^2G_{m,n}}{n^2 - a - ibn}\right\|_{l^\infty} l\|u_m(x) - u(x)\|_{L^2(I)}.$$

By virtue of Lemma A4 and (4.6), we have $\frac{d^2u_m}{dx^2} \to \frac{d^2u}{dx^2}$ as $m \to \infty$ in $L^2(I)$. Therefore,

$u_m(x) \to u(x)$ in the $H^2(I)$ norm as $m \to \infty$.

Let us suppose that $u_m(x)$ vanishes in the interval I for a certain $m \in \mathbb{N}$. This yields a contradiction to our assumption that $G_{m,n}F(0,x)_n \neq 0$ for some $n \in \mathbb{Z}$. The analogous argument holds for the solution $u(x)$ of limiting problem (1.2). □

5. Appendix

Let $G(x)$ be a function, $G(x) : \mathbb{R} \to \mathbb{R}$, for which we denote its standard Fourier transform using the hat symbol as

$$\widehat{G}(p) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(x) e^{-ipx} dx, \quad p \in \mathbb{R}, \tag{5.1}$$

such that

$$\|\widehat{G}(p)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2\pi}} \|G\|_{L^1(\mathbb{R})} \tag{5.2}$$

and $G(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{G}(q) e^{iqx} dq,\ x \in \mathbb{R}$. For the technical purposes we define the auxiliary quantities

$$N_{a,\,b} := \max\left\{ \left\| \frac{\widehat{G}(p)}{p^2 - a - ibp} \right\|_{L^\infty(\mathbb{R})}, \quad \left\| \frac{p^2 \widehat{G}(p)}{p^2 - a - ibp} \right\|_{L^\infty(\mathbb{R})} \right\} \tag{5.3}$$

for $a \geq 0,\ b \in \mathbb{R},\ b \neq 0$.

Lemma A1. *Let $G(x) : \mathbb{R} \to \mathbb{R},\ G(x) \in L^1(\mathbb{R})$.*
a) *If $a > 0,\ b \in \mathbb{R},\ b \neq 0$ then $N_{a,\,b} < \infty$.*
b) *If $a = 0,\ b \in \mathbb{R},\ b \neq 0$ and in addition $xG(x) \in L^1(\mathbb{R})$ then $N_{0,\,b} < \infty$ if and only if*

$$(G(x), 1)_{L^2(\mathbb{R})} = 0 \tag{5.4}$$

holds.

Proof. First of all, we observe that in both cases a) and b) of the lemma the boundedness of $\frac{\widehat{G}(p)}{p^2 - a - ibp}$ implies the boundedness of $\frac{p^2\widehat{G}(p)}{p^2 - a - ibp}$. Indeed, we can express $\frac{p^2\widehat{G}(p)}{p^2 - a - ibp}$ as the following sum

$$\widehat{G}(p) + a\frac{\widehat{G}(p)}{p^2 - a - ibp} + ib\frac{p\widehat{G}(p)}{p^2 - a - ibp}. \tag{5.5}$$

Obviously, the first term in (5.5) is bounded via inequality (5.2) since $G(x) \in L^1(\mathbb{R})$ due to the one of our assumptions. The third term in (5.5) can be estimated from above in the absolute value using (5.2) as

$$\frac{|b||p||\widehat{G}(p)|}{\sqrt{(p^2 - a)^2 + b^2p^2}} \leq \frac{1}{\sqrt{2\pi}} \|G(x)\|_{L^1(\mathbb{R})} < \infty.$$

Therefore, $\frac{\widehat{G}(p)}{p^2 - a - ibp} \in L^\infty(\mathbb{R})$ yields $\frac{p^2\widehat{G}(p)}{p^2 - a - ibp} \in L^\infty(\mathbb{R})$. To establish the result of the part a) of the lemma, we need to estimate

(5.6) $$\frac{|\widehat{G}(p)|}{\sqrt{(p^2-a)^2+b^2p^2}}.$$

Evidently, the numerator of (5.6) can be bounded from above by means of (5.2) and the denominator in (5.6) can be trivially estimated below by a finite, positive constant, such that

$$\frac{\widehat{G}(p)}{p^2-a-ibp} \le C\|G(x)\|_{L^1(\mathbb{R})} < \infty.$$

Here and below C will stand for a finite, positive constant. This implies that under the given conditions, when $a > 0$ we have $N_{a,\,b} < \infty$. In the case of $a = 0$, we express

$$\widehat{G}(p) = \widehat{G}(0) + \int_0^p \frac{d\widehat{G}(s)}{ds}ds,$$

such that

(5.7) $$\frac{\widehat{G}(p)}{p^2-ibp} = \frac{\widehat{G}(0)}{p(p-ib)} + \frac{\int_0^p \frac{d\widehat{G}(s)}{ds}ds}{p(p-ib)}.$$

By virtue of definition (5.1) of the standard Fourier transform, we easily estimate

$$\left|\frac{d\widehat{G}(p)}{dp}\right| \le \frac{1}{\sqrt{2\pi}}\|xG(x)\|_{L^1(\mathbb{R})}.$$

Hence, we obtain

$$\left|\frac{\int_0^p \frac{d\widehat{G}(s)}{ds}ds}{p(p-ib)}\right| \le \frac{\|xG(x)\|_{L^1(\mathbb{R})}}{\sqrt{2\pi}|b|} < \infty$$

due to the one of our assumptions. Therefore, the expression in the left side of (5.7) is bounded if and only if $\widehat{G}(0)$ vanishes, which equivalent to orthogonality relation (5.4). □

For the purpose of the study of equations (2.6), we introduce the following auxiliary expressions

(5.8) $$N_{a,\,b,\,m} := \max\left\{\left\|\frac{\widehat{G}_m(p)}{p^2-a-ibp}\right\|_{L^\infty(\mathbb{R})},\quad \left\|\frac{p^2\widehat{G}_m(p)}{p^2-a-ibp}\right\|_{L^\infty(\mathbb{R})}\right\}$$

with $a \ge 0,\ b \in \mathbb{R},\ b \ne 0$ and $m \in \mathbb{N}$. We have the following technical proposition.

Lemma A2. *Let the conditions of* Theorem 2 *hold,* $a \ge 0,\ b \in \mathbb{R},\ b \ne 0$. *Then*

(5.9) $$\frac{\widehat{G}_m(p)}{p^2-a-ibp} \to \frac{\widehat{G}(p)}{p^2-a-ibp}, \quad m \to \infty,$$

(5.10) $$\frac{p^2\widehat{G}_m(p)}{p^2-a-ibp} \to \frac{p^2\widehat{G}(p)}{p^2-a-ibp}, \quad m \to \infty$$

in $L^\infty(\mathbb{R})$, *such that*

(5.11) $$\left\|\frac{\widehat{G}_m(p)}{p^2-a-ibp}\right\|_{L^\infty(\mathbb{R})} \to \left\|\frac{\widehat{G}(p)}{p^2-a-ibp}\right\|_{L^\infty(\mathbb{R})}, \quad m \to \infty,$$

(5.12) $$\left\|\frac{p^2\widehat{G}_m(p)}{p^2-a-ibp}\right\|_{L^\infty(\mathbb{R})} \to \left\|\frac{p^2\widehat{G}(p)}{p^2-a-ibp}\right\|_{L^\infty(\mathbb{R})}, \quad m\to\infty.$$

Furthermore,

(5.13) $$2\sqrt{\pi}N_{a,\,b}\,l \le 1-\varepsilon.$$

Proof. Evidently,

(5.14) $$\|\widehat{G}_m(p)-\widehat{G}(p)\|_{L^\infty(\mathbb{R})} \le \frac{1}{\sqrt{2\pi}}\|G_m(x)-G(x)\|_{L^1(\mathbb{R})} \to 0, \quad m\to\infty,$$

due to the one of our assumptions. Let us show that (5.9) yields (5.10). Indeed, the expression $\frac{p^2[\widehat{G}_m(p)-\widehat{G}(p)]}{p^2-a-ibp}$ can be written as the sum

(5.15) $$[\widehat{G}_m(p)-\widehat{G}(p)]+a\left[\frac{\widehat{G}_m(p)}{p^2-a-ibp}-\frac{\widehat{G}_m(p)}{p^2-a-ibp}\right]+ibp\frac{[\widehat{G}_m(p)-\widehat{G}(p)]}{p^2-a-ibp}.$$

The first term in (5.15) tends to zero as $m\to\infty$ in the $L^\infty(\mathbb{R})$ norm by means of (5.14). The third term in (5.15) can be estimated from above in the absolute value as

$$|b|\frac{|p||\widehat{G}_m(p)-\widehat{G}(p)|}{\sqrt{(p^2-a)^2+b^2p^2}} \le \|\widehat{G}_m(p)-\widehat{G}(p)\|_{L^\infty(\mathbb{R})},$$

hence it converges to zero as $m\to\infty$ in the $L^\infty(\mathbb{R})$ norm due to (5.14) as well. Therefore, the statement of (5.9) implies (5.10). Let us note that (5.11) and (5.12) will follow from the statements of (5.9) and (5.10) respectively by means of the triangle inequality.

Let us first prove (5.9) in the case of $a>0$. Then we need to estimate

(5.16) $$\frac{|\widehat{G}_m(p)-\widehat{G}(p)|}{\sqrt{(p^2-a)^2+b^2p^2}}.$$

Clearly, the denominator in fraction (5.16) can be bounded from below by a positive constant and the numerator in (5.16) can be estimated from above via (5.14). This yields the result of (5.9) when the constant a is positive.

Then we turn our attention to proving (5.9) when $a=0$. In this case we have orthogonality conditions (2.8). Let us show that the analogous statement will hold in the limit. Indeed,

$$|(G(x),1)_{L^2(\mathbb{R})}| = |(G(x)-G_m(x),1)_{L^2(\mathbb{R})}| \le \|G_m(x)-G(x)\|_{L^1(\mathbb{R})} \to 0$$

as $m\to\infty$ as assumed. Thus,

(5.17) $$(G(x),1)_{L^2(\mathbb{R})}=0.$$

Let us express

$$\widehat{G}(p)=\widehat{G}(0)+\int_0^p \frac{d\widehat{G}(s)}{ds}ds, \quad \widehat{G}_m(p)=\widehat{G}_m(0)+\int_0^p \frac{d\widehat{G}_m(s)}{ds}ds, \quad m\in\mathbb{N}.$$

By virtue of (5.17) and (2.8), we have

$$\widehat{G}(0)=0, \quad \widehat{G}_m(0)=0, \quad m\in\mathbb{N}.$$

Hence, we obtain

(5.18)
$$\left|\frac{\widehat{G}_m(p)}{p^2 - ibp} - \frac{\widehat{G}(p)}{p^2 - ibp}\right| = \left|\frac{\int_0^p \left[\frac{d\widehat{G}_m(s)}{ds} - \frac{d\widehat{G}(s)}{ds}\right]ds}{p(p - ib)}\right|.$$

From the definition of the standard Fourier transform (5.1) we easily deduce that

$$\left|\frac{d\widehat{G}_m(p)}{dp} - \frac{d\widehat{G}(p)}{dp}\right| \le \frac{1}{\sqrt{2\pi}}\|xG_m(x) - xG(x)\|_{L^1(\mathbb{R})}.$$

This allows us to estimate the right side of (5.18) from above by

$$\frac{\|xG_m(x) - xG(x)\|_{L^1(\mathbb{R})}}{\sqrt{2\pi}|b|} \to 0, \quad m \to \infty,$$

as assumed, which proves (5.9) when a vanishes. Let us note that under our assumptions

$$N_{a,\,b} < \infty, \quad N_{a,\,b,\,m} < \infty, \quad m \in \mathbb{N}, \quad a \ge 0, \quad b \in \mathbb{R}, \quad b \ne 0$$

by means of the result of Lemma A1 above. We have inequalities (2.7) for $a > 0$ and (2.9) when $a = 0$. A trivial limiting argument using (5.11) and (5.12) gives us (5.13). □

Let the function $G(x) : I \to \mathbb{R}, \quad G(0) = G(2\pi)$ and its Fourier transform on the finite interval is given by

(5.19)
$$G_n := \int_0^{2\pi} G(x)\frac{e^{-inx}}{\sqrt{2\pi}}dx, \quad n \in \mathbb{Z}$$

and $G(x) = \sum_{n=-\infty}^{\infty} G_n \frac{e^{inx}}{\sqrt{2\pi}}$. Clearly, we have the upper bound

(5.20)
$$\|G_n\|_{l^\infty} \le \frac{1}{\sqrt{2\pi}}\|G(x)\|_{L^1(I)}.$$

Analogously to the whole real line case we define

(5.21)
$$\mathcal{N}_{a,\,b} := \max\left\{\left\|\frac{G_n}{n^2 - a - ibn}\right\|_{l^\infty}, \quad \left\|\frac{n^2 G_n}{n^2 - a - ibn}\right\|_{l^\infty}\right\}$$

for $a \ge 0,\ b \in \mathbb{R},\ b \ne 0$.

We have the following elementary proposition.

Lemma A3. *Let $G(x) : I \to \mathbb{R},\ G(x) \in L^\infty(I)$ and $G(0) = G(2\pi)$.*
a) *If $a > 0,\ b \in \mathbb{R},\ b \ne 0$ then $\mathcal{N}_{a,\,b} < \infty$.*
b) *If $a = 0,\ b \in \mathbb{R},\ b \ne 0$ then $\mathcal{N}_{0,\,b} < \infty$ if and only if*

(5.22)
$$(G(x), 1)_{L^2(I)} = 0.$$

Proof. Apparently, in both cases a) and b) of the lemma the boundedness of $\frac{G_n}{n^2 - a - ibn}$ yields the boundedness of $\frac{n^2 G_n}{n^2 - a - ibn}$. Indeed, $\frac{n^2 G_n}{n^2 - a - ibn}$ can be easily written as the following sum

(5.23)
$$G_n + a\frac{G_n}{n^2 - a - ibn} + ib\frac{nG_n}{n^2 - a - ibn}.$$

Clearly, the first term in (5.23) can be easily bounded from above via (5.20) for $G(x) \in$

$L^\infty(I) \subset L^1(I)$. The third term in (5.23) can be estimated from above using (5.20) as well, namely

$$|b|\frac{|n||G_n|}{\sqrt{(n^2-a)^2+b^2n^2}} \le |G_n| \le \frac{1}{\sqrt{2\pi}}\|G(x)\|_{L^1(I)} < \infty.$$

Therefore, $\frac{G_n}{n^2-a-ibn} \in l^\infty$ implies that $\frac{n^2G_n}{n^2-a-ibn} \in l^\infty$. To prove the statement of the part a) of the lemma, we need to treat

$$\frac{|G_n|}{\sqrt{(n^2-a)^2+b^2n^2}}. \tag{5.24}$$

Evidently, the denominator in (5.24) can be bounded below by a positive constant and the numerator in (5.24) can be easily treated by means of (5.20). Hence, $\mathcal{N}_{a,\,b} < \infty$ when $a > 0$. To establish the result of the part b), we observe that

$$\left|\frac{G_n}{n(n-ib)}\right| \tag{5.25}$$

is bounded if and only if $G_0 = 0$, which is equivalent to orthogonality condition (5.22). In this case (5.25) can be trivially estimated from above by

$$\frac{1}{\sqrt{2\pi}}\frac{\|G(x)\|_{L^1(I)}}{\sqrt{n^2+b^2}} \le \frac{1}{\sqrt{2\pi}}\frac{\|G(x)\|_{L^1(I)}}{|b|} < \infty$$

by virtue of (5.20) and the one of our assumptions.

□

In order to study equations (2.11), we introduce for the technical purposes

$$\mathcal{N}_{a,\,b,\,m} := \max\left\{\left\|\frac{G_{m,n}}{n^2-a-ibn}\right\|_{l^\infty}, \quad \left\|\frac{n^2G_{m,n}}{n^2-a-ibn}\right\|_{l^\infty}\right\} \tag{5.26}$$

for $a \ge 0,\ b \in \mathbb{R},\ b \ne 0$ and $m \in \mathbb{N}$. Our final technical proposition is as follows.

Lemma A4. *Let the assumptions of* Theorem 4 *hold,* $a \ge 0,\ b \in \mathbb{R},\ b \ne 0$. *Then*

$$\frac{G_{m,n}}{n^2-a-ibn} \to \frac{G_n}{n^2-a-ibn}, \quad m \to \infty, \tag{5.27}$$

$$\frac{n^2G_{m,n}}{n^2-a-ibn} \to \frac{n^2G_n}{n^2-a-ibn}, \quad m \to \infty \tag{5.28}$$

in l^∞*, such that*

$$\left\|\frac{G_{m,n}}{n^2-a-ibn}\right\|_{l^\infty} \to \left\|\frac{G_n}{n^2-a-ibn}\right\|_{l^\infty}, \quad m \to \infty, \tag{5.29}$$

$$\left\|\frac{n^2G_{m,n}}{n^2-a-ibn}\right\|_{l^\infty} \to \left\|\frac{n^2G_n}{n^2-a-ibn}\right\|_{l^\infty}, \quad m \to \infty. \tag{5.30}$$

Furthermore,

$$2\sqrt{\pi}\mathcal{N}_{a,\,b}l \le 1-\varepsilon. \tag{5.31}$$

Proof. Apparently, under our assumptions, the limiting kernel function $G(x)$ is periodic as well. Indeed, we have

$$|G(0)-G(2\pi)| \le |G(0)-G_m(0)| + |G_m(2\pi)-G(2\pi)| \le 2\|G_m(x)-G(x)\|_{L^\infty(I)} \to 0, \quad m\to\infty$$

as assumed, such that $G(0) = G(2\pi)$ holds. It is obvious that

$$\|G_{m,n}-G_n\|_{l^\infty} \le \frac{1}{\sqrt{2\pi}}\|G_m-G\|_{L^1(I)} \le \sqrt{2\pi}\|G_m-G\|_{L^\infty(I)} \to 0, \quad m\to\infty \tag{5.32}$$

as assumed. Let us note that the statements of (5.27) and (5.28) will imply (5.29) and (5.30) respectively via the triangle inequality. Let us show that (5.27) yields (5.28). We express $\dfrac{n^2[G_{m,n}-G_n]}{n^2-a-ibn}$ as the following sum

$$[G_{m,n}-G_n] + a\frac{G_{m,n}-G_n}{n^2-a-ibn} + ib\frac{n[G_{m,n}-G_n]}{n^2-a-ibn}. \tag{5.33}$$

The first term in (5.33) tends to zero in the l^∞ norm as $m\to\infty$ via estimate (5.32). The third term in (5.33) can be bounded from above in the absolute value as

$$\frac{|b||n||G_{m,n}-G_n|}{\sqrt{(n^2-a)^2+b^2n^2}} \le \|G_{m,n}-G_n\|_{l^\infty},$$

therefore it converges to zero as $m\to\infty$ in the l^∞ norm due to (5.32) as well. This proves that (5.27) implies (5.28).

Let us first establish (5.27) when $a > 0$. Then we need to consider the expression

$$\frac{|G_{m,n}-G_n|}{\sqrt{(n^2-a)^2+b^2n^2}}. \tag{5.34}$$

Evidently, the denominator of (5.34) can be bounded below by a positive constant and the numerator estimated from above by means of (5.32). This yields (5.27) for $a > 0$.

Then we proceed to establishing (5.27) in the case of $a = 0$. According to the one of our assumptions, we have orthogonality relations (2.13). Let us prove that the analogous condition holds in the limit. Indeed,

$$|(G(x),1)_{L^2(I)}| = |(G(x)-G_m(x),1)_{L^2(I)}| \le 2\pi\|G_m(x)-G(x)\|_{L^\infty(I)} \to 0, \quad m\to\infty$$

as assumed. Thus,

$$(G(x),1)_{L^2(I)} = 0,$$

or equivalently $G_0 = 0$. Note that $G_{m,0} = 0$, $m\in\mathbb{N}$ by virtue of orthogonality conditions (2.13). Then by means of (5.32), we have

$$\left|\frac{G_{m,n}-G_n}{n(n-ib)}\right| \le \frac{\sqrt{2\pi}\|G_m(x)-G(x)\|_{L^\infty(I)}}{|b|}.$$

Since the norm in the right side of this inequality tends to zero as $m\to\infty$, (5.27) holds when $a = 0$ as well. Let us note that under our assumptions

$$\mathcal{N}_{a,\,b} < \infty, \quad \mathcal{N}_{a,\,b,\,m} < \infty, \quad m\in\mathbb{N}, \quad a\ge 0, \quad b\in\mathbb{R}, \quad b\ne 0$$

by virtue of the result of Lemma A3 above. We have bounds (2.12) when $a > 0$ and (2.14) for $a = 0$. An elementary limiting argument using (5.29) and (5.30) gives us (5.31). □

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